

Hom, J., Lidman, T. and Watson, L. (2017) The Alexander module, Seifert forms, and categorification. *Journal of Topology*, 10(1), pp. 22-100. (doi:[10.1112/topo.12001](https://doi.org/10.1112/topo.12001))

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Deposited on: 27 January 2016

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Referee Report

Title: The Alexander module, Seifert forms and categorification

Authors: Jennifer Hom, Tye Lidman and Liam Watson

1. OVERVIEW

The primary aim of the paper under review is to show that bordered Floer homology provides a categorification of a TQFT defined by Donaldson. In particular, this paper shows that both the Alexander module and Seifert form of a knot are determined by Heegaard Floer theory.

Heegaard Floer theory is an incredibly powerful package of invariants of 3 and 4-manifolds and their subspaces. In its most basic incarnation, this theory assigns to a closed 3-manifold Y a filtered chain complex $CF^\infty(Y)$ whose filtered homotopy type is an invariant of Y . A knot $K \subset Y$ induces a second filtration on $CF^\infty(Y)$ and the doubly-filtered homotopy type of this complex is an invariant of the pair (Y, K) (called “knot Floer homology”). Over the years, this package of invariants has proven a powerful tools for answering problems in low-dimensional topology.

As a general rule, Heegaard Floer invariants are well-adapted to questions in low-dimensional topology which have strong geometric components. For instance, Heegaard Floer invariants determine the Thurston norm of a 3-manifold. An example more relevant to the paper under review is provided by knot Floer homology, which categorifies the Alexander polynomial. Though this polynomial has many definitions, it is perhaps best understood as the total order of the Alexander module, which is the homology of the infinite cyclic cover of the given knot’s complement. It is precisely this relationship that motivates to the following question.

Question: Can the Alexander module and Seifert form of a knot be recovered from Heegaard Floer theory?

The paper under review answers this question resoundingly in the affirmative by showing that the bordered Floer invariants of the complement of a Seifert surface categorify a TQFT defined by Donaldson which determines the Alexander module and Seifert Form of a knot. Over the years, many have been motivated by this question and a patchwork of mostly negative results emerged. For instance, Kanenobu discovered an infinite family of examples of knots with distinct Alexander modules which (using work of Petkova) are seen to have filtered chain homotopic knot Floer chain complexes. Additional examples are provide by Horn who identified an infinite family of knots with isomorphic (hat) knot Floer homology groups which could be distinguished by higher-order analogues of the Alexander polynomial. A positive results in this general direction is provided by Hedden, Juhasz and Sarkar who showed that sutured Floer homology is capable of distinguishing Seifert surfaces of knots.

This is an exceptional paper containing a great number of interesting and significant results. It is incredibly well-written, especially for a paper of its length. The authors made several strategic decisions (like relegating proofs related to gradings to

an Appendix) that significantly enhance the readability of the paper. I enthusiastically recommend it for publication in the Journal of Topology. After carefully working thorough the paper I found no substantive errors, mathematically or otherwise. Listed below are some optional suggestions, which I believe could possibly be used to improve the overall exposition. The suggestions are mostly stylistic and some would require a significant reorganization of the paper. The recommendation to publish stands even if none of these suggestions are incorporated into the current manuscript.

2. GENERAL COMMENTS

1: In general, the filtration on knot Floer homology depends on a choice of Seifert surface. The authors play a bit fast and loose with this in terms of notation and the overall discussion. This does not impact the truth of the various Theorems, Corollaries, Lemmas and Propositions in the paper.

2: Many/Most papers which use bordered Floer technology include absurdly long definitions or result statements which are difficult to internalize on the first few passes. The authors do a good job including plain-language versions near some of the more technical definitions and result statements. That said, I still encourage them to add plain-language wherever possible. As an example, there is an nine-line definition of $\mathcal{Z}_1 \# \mathcal{Z}_2$ in the middle of Page 12 that basically boils down to “stack \mathcal{Z}_1 on top of \mathcal{Z}_2 ”.

3: Be sure to update the bibliography to reflect changes in the status of the various references (e.g., The title of [LOT08] is not correct).

3. SPECIFIC COMMENTS

Page 3, Line 7: Suggested rewording: “...Floer chain complex known as the Alexander filtration and the homology...”

Page 4, Line -12: Suggested replacement: $(Y \setminus \nu(F^\circ)) \cup D \times I \rightarrow (Y \setminus \nu(F^\circ)) \cup (D \times I)$.

Page 8, First Paragraph: To the referee’s knowledge, elements of $\bigwedge *G$ which can be written $g_1 \wedge \cdots \wedge g_k$ are usually called “decomposable”.

Page 9, Line -4: $(t^t + t^{-i}) \rightarrow (t^i + t^{-i})$.

Page 12, Paragraph 3: This discussion is a bit off. When the authors say “attaching two-dimensional 1-handles to Z along...”, what they really mean is “attaching two-dimensional 1-handles to $Z = \partial D^2$ along...”.

Page 13, Remark 3.3: For the unfamiliar reader, it would be nice to include a reference for the AZ diagram.

Page 28, Line 4: Suggested rewording: “...have homotopy equivalent CFK^- ...”

Page 35, Line -13: Should this be a functor from $\mathcal{A}(-\mathcal{Z})\mathbf{Mod}$ to the homotopy category of $\mathbb{Z}/2$ -graded complexes as opposed to the category of $\mathbb{Z}/2$ -graded complexes?