EQUILIBRIUM COORDINATION WITH DISCRETIONARY POLICYMAKING — Online Appendix

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A Coordination mechanisms

A.1 Eductive learning and expectational stability

Recall that a Markov-perfect equilibrium is characterised by $\{\mathbf{H},\ \mathbf{F},\ \mathbf{M},\ \mathbf{V},\ d\}$. Because $\mathbf{M}$ and d follow immediately and uniquely from $\mathbf{F}$, $\mathbf{H}$, and $\mathbf{V}$, we implement the partitioning $\{\{\mathbf{H}, \mathbf{F}, \mathbf{V}\}, \{\mathbf{M}, d\}\}$ and focus on $\{\mathbf{H}, \mathbf{F}, \mathbf{V}\}$ in what follows. Specifically, we consider:

1. Private sector learning, where we analyse whether private agents can learn $\mathbf{H}$, conditional on $\{\mathbf{F}, \mathbf{V}\}$.
2. Policymaker learning, where we analyse whether the policymaker can learn $\{\mathbf{F}, \mathbf{V}\}$, conditional on $\{\mathbf{H}\}$.
3. Joint learning, where we analyse whether private agents and the policymaker can learn $\{\mathbf{H}, \mathbf{F}, \mathbf{V}\}$ jointly.

A.1.1 Preliminaries

To place the three learning problems in a unified framework, let us denote by $\mathbf{\Phi}$ the object(s) to be learned. Thus, in the case where only private agents are learning $\mathbf{\Phi} = \{\mathbf{H}\}$. Then, to determine whether $\mathbf{\Phi}$ is learnable we construct and analyse the $\mathbb{T}$-map that relates a perception of $\mathbf{\Phi}$, denoted $\overline{\mathbf{\Phi}}$, to an actual $\mathbf{\Phi}$, $\mathbf{\Phi} = \mathbb{T}\left(\overline{\mathbf{\Phi}}\right)$.

Definition 1 *A fix-point, $\mathbf{\Phi}^*$, of the $\mathbb{T}$-map, $\mathbf{\Phi} = \mathbb{T}\left(\overline{\mathbf{\Phi}}\right)$, is said to be IE-stable if*

$$\lim_{k\uparrow\infty} \mathbb{T}^k \left(\overline{\mathbf{\Phi}}\right) = \mathbf{\Phi}^*,$$

for all $\overline{\mathbf{\Phi}} \neq \mathbf{\Phi}^*$.

It follows that $\mathbf{\Phi}^*$ is IE-stable if and only if it is a stable fix-point of the difference equation

$$\mathbf{\Phi}_{k+1} = \mathbb{T}\left(\mathbf{\Phi}_k\right), \tag{1}$$

where index k denotes the step of the updating process. Similarly,

Definition 2 *A fix-point, $\mathbf{\Phi}^*$, of the $\mathbb{T}$-map, $\mathbf{\Phi} = \mathbb{T}\left(\overline{\mathbf{\Phi}}\right)$, is said to be locally IE-stable if*

$$\lim_{k\uparrow\infty} \mathbb{T}^k\left(\overline{\mathbf{\Phi}}\right) = \mathbf{\Phi}^*,$$

for all $\overline{\mathbf{\Phi}}$ about a neighborhood of $\mathbf{\Phi}^*$.

Let the derivative of the $\mathbb{T}$-map be denoted $\mathbb{DT}(\mathbf{\Phi}^*)$, then it is straightforward to prove the following Lemma.

Lemma 1 *Assume that the derivative map, $\mathbb{DT}(\mathbf{\Phi}^*)$, has no eigenvalues with modulus equal to 1. A fix-point, $\mathbf{\Phi}^*$, of the $\mathbb{T}$-map, $\mathbf{\Phi} = \mathbb{T}\left(\overline{\mathbf{\Phi}}\right)$, is locally IE-stable if and only if all eigenvalues of the derivative map, $\mathbb{DT}(\mathbf{\Phi}^*)$, have modulus less than 1.*

Proof. Following Evans (1985), to analyse the local stability of equation (1) we linearise the equation about $\mathbf{\Phi}^*$. Using matrix calculus results from Magnus and Neudecker (1999), we obtain

$$d\left(vec\left(\mathbf{\Phi}_{k+1}\right)\right) = \mathbb{DT}\left(\mathbf{\Phi}^*\right) d\left(vec\left(\mathbf{\Phi}_k\right)\right)$$

where $\mathbb{DT}(\mathbf{\Phi}^*) = \partial\left(vec\left(\mathbb{T}\left(\mathbf{\Phi}^*\right)\right)\right)/\partial\left(vec\left(\mathbf{\Phi}\right)\right)'$. Applying standard results for linear difference equations, if all of the eigenvalues of $\mathbb{DT}(\mathbf{\Phi}^*)$ have modulus less than one, then $\mathbf{\Phi}^*$ is locally stable. In contrast, if one or more of the eigenvalues of $\mathbb{DT}(\mathbf{\Phi}^*)$ have modulus greater than one, then $\mathbf{\Phi}^*$ is not locally stable. ■

A.1.2 Eductive learning by private agents

We begin with the case in which only private agents are learning and examine whether private agents can learn $\mathbf{H}$, given $\{\mathbf{F}, \mathbf{V}\}$. For a given policy rule, $\mathbf{u}_t = \mathbf{F}\mathbf{x}_t$, and a postulated private sector decision rule

$$\mathbf{y}_t = \overline{\mathbf{H}}\mathbf{x}_t,$$

the actual private sector decision rule takes the form

$$\mathbf{y}_t = \mathbf{H}\mathbf{x}_t.$$

The perceived low-of-motion will be consistent with a rational expectations equilibrium if it is supported by the evolution of the economy. This yields

$$\mathbf{H} = \left(\overline{\mathbf{H}}\mathbf{A}_{12} - \mathbf{A}_{22}\right)^{-1}\left[\mathbf{A}_{21} + \mathbf{B}_2\mathbf{F} - \overline{\mathbf{H}}\left(\mathbf{A}_{11} + \mathbf{B}_1\mathbf{F}\right)\right]. \tag{2}$$

Equation (2) describes the $\mathbb{T}$-map, $T(\overline{\mathbf{H}})$, from $\overline{\mathbf{H}}$ to $\mathbf{H}$.

Lemma 2 *A Markov-perfect equilibrium is locally IE-stable under private sector learning if and only if all eigenvalues of*

$$-\left[\mathbf{I}\otimes\left(\mathbf{H}\mathbf{A}_{12} - \mathbf{A}_{22}\right)\right]^{-1}\left[\left(\mathbf{A}_{11} + \mathbf{A}_{12}\mathbf{H} + \mathbf{B}_1\mathbf{F}\right)' \otimes \mathbf{I}\right]$$

have modulus less than 1.

Proof. Applying standard matrix calculus rules to equation (2), the total differential can be written as

$$(\mathbf{HA}_{12}-\mathbf{A}_{22})\,d\,(\mathbf{H})+d\left(\overline{\mathbf{H}}\right)\mathbf{A}_{12}\mathbf{H}+d\left(\overline{\mathbf{H}}\right)(\mathbf{A}_{11}+\mathbf{B}_1\mathbf{F})=\mathbf{0},$$

which after vectorising can be rearranged to give

$$vec\,[d\,(\mathbf{H})]=-\left[\mathbf{I}\otimes(\mathbf{HA}_{12}-\mathbf{A}_{22})\right]^{-1}\left[(\mathbf{A}_{11}+\mathbf{A}_{12}\mathbf{H}+\mathbf{B}_1\mathbf{F})^{'}\otimes\mathbf{I}\right]vec\left[d\left(\overline{\mathbf{H}}\right)\right].$$

We apply Lemma 1 to obtain the required result. Note that invertability of $(\mathbf{HA}_{12}-\mathbf{A}_{22})$ is virtually ensured by the assumption that $\mathbf{A}_{22}$ has full rank. ■

Because the eigenvalues of $\mathbf{M}=\mathbf{A}_{11}+\mathbf{A}_{12}\mathbf{H}+\mathbf{B}_1\mathbf{F}$ are all strictly less than $\beta^{-\frac{1}{2}}$, equilibria that are not locally IE-stable under private sector learning are those for which $(\mathbf{HA}_{12}-\mathbf{A}_{22})$ is close to equaling the null matrix.

A.1.3 Eductive learning by the leader

We now turn to the case where the policymaker is learning, but private agents are not. Here we examine whether the policymaker can learn $\{\mathbf{F},\mathbf{V}\}$, given $\{\mathbf{H}\}$. We show that although learning by policymakers is interesting and important in many contexts, here this local IE-stability criterion cannot discriminate among equilibria.

For a given private sector decision rule, $\mathbf{y}_t=\mathbf{H}\mathbf{x}_t$, and a postulated policy rule

$$\mathbf{u}_t=\overline{\mathbf{F}}\mathbf{x}_t,$$

and a postulated value function matrix $\overline{\mathbf{V}}$, the T-map $T(\overline{\mathbf{F}},\overline{\mathbf{V}})$ from $\left\{\overline{\mathbf{F}},\overline{\mathbf{V}}\right\}$ to $\{\mathbf{F},\mathbf{V}\}$, consistent with implementing the best response to the private sector's reaction function, is described by the following updating relationships

$$\begin{aligned}\mathbf{F} &= -\left(\widehat{\mathbf{Q}}+\beta\widehat{\mathbf{B}}^{'}\overline{\mathbf{V}}\widehat{\mathbf{B}}\right)^{-1}\left(\widehat{\mathbf{U}}^{'}+\beta\widehat{\mathbf{B}}^{'}\overline{\mathbf{V}}\widehat{\mathbf{A}}\right), &(3)\\ \mathbf{V} &= \widehat{\mathbf{W}}+2\widehat{\mathbf{U}}\mathbf{F}+\mathbf{F}^{'}\widehat{\mathbf{Q}}\mathbf{F}+\beta\left(\widehat{\mathbf{A}}+\widehat{\mathbf{B}}\mathbf{F}\right)^{'}\overline{\mathbf{V}}\left(\widehat{\mathbf{A}}+\widehat{\mathbf{B}}\mathbf{F}\right), &(4)\end{aligned}$$

where $\widehat{\mathbf{W}}$, $\widehat{\mathbf{U}}$, $\widehat{\mathbf{Q}}$, $\widehat{\mathbf{A}}$, and $\widehat{\mathbf{B}}$ are defined by

$$\begin{aligned}\widehat{\mathbf{W}} &= \mathbf{W}_{11}+\mathbf{W}_{12}\mathbf{J}+\mathbf{J}^{'}\mathbf{W}_{21}+\mathbf{J}^{'}\mathbf{W}_{22}\mathbf{J}, &(5)\\ \widehat{\mathbf{U}} &= \mathbf{W}_{12}\mathbf{K}+\mathbf{J}^{'}\mathbf{W}_{22}\mathbf{K}+\mathbf{U}_1+\mathbf{J}^{'}\mathbf{U}_2, &(6)\\ \widehat{\mathbf{Q}} &= \mathbf{Q}+\mathbf{K}^{'}\mathbf{W}_{22}\mathbf{K}+2\mathbf{K}^{'}\mathbf{U}_2, &(7)\\ \widehat{\mathbf{A}} &= \mathbf{A}_{11}+\mathbf{A}_{12}\mathbf{J}, &(8)\\ \widehat{\mathbf{B}} &= \mathbf{B}_1+\mathbf{A}_{12}\mathbf{K}. &(9)\end{aligned}$$

where

$$\begin{aligned}\mathbf{J} &= (\mathbf{A}_{22}-\mathbf{HA}_{12})^{-1}(\mathbf{HA}_{11}-\mathbf{A}_{21}),\\ \mathbf{K} &= (\mathbf{A}_{22}-\mathbf{HA}_{12})^{-1}(\mathbf{HB}_1-\mathbf{B}_2).\end{aligned}$$

so that they do not depend on $\mathbf{F}$ or $\mathbf{V}$ (or on $\overline{\mathbf{F}}$ or $\overline{\mathbf{V}}$). Notice, that $\mathbf{F}$, given $\mathbf{H}$, is uniquely determined by $\mathbf{V}$, so the key to learning $\mathbf{F}$ is to learn $\mathbf{V}$. As a consequence, without loss of generality we can substitute equation (3) into equation (4) and analyse the learning problem using the concentrated T-map $T(\overline{\mathbf{V}}) = \mathbf{V}$.

Lemma 3 *All Markov-perfect equilibria are locally IE-stable under policymaker learning.*

Proof. Applying standard matrix calculus rules to equations (3) and (4), total differentials are given by

$$\begin{aligned} \left(\widehat{\mathbf{Q}} + \beta\widehat{\mathbf{B}}'\mathbf{V}\widehat{\mathbf{B}}\right) d(\mathbf{F}) + \beta\widehat{\mathbf{B}}' d\left(\overline{\mathbf{V}}\right)\left(\widehat{\mathbf{A}} + \widehat{\mathbf{B}}\mathbf{F}\right) &= \mathbf{0}, \quad (10) \\ 2\left[\widehat{\mathbf{U}} + \mathbf{F}'\widehat{\mathbf{Q}} + \beta\left(\widehat{\mathbf{A}} + \widehat{\mathbf{B}}\mathbf{F}\right)'\mathbf{V}\widehat{\mathbf{B}}\right] d(\mathbf{F}) + \beta\left(\widehat{\mathbf{A}} + \widehat{\mathbf{B}}\mathbf{F}\right)' d\left(\overline{\mathbf{V}}\right)\left(\widehat{\mathbf{A}} + \widehat{\mathbf{B}}\mathbf{F}\right) &= \mathbf{I}d(\mathbf{V}). \quad (11) \end{aligned}$$

Using equation (10) to solve for $d(\mathbf{F})$ and substituting the resulting expression into equation (11) yields, upon rearranging,

$$\beta\left[-2\left(\widehat{\mathbf{U}} + \beta\widehat{\mathbf{A}}'\mathbf{V}\mathbf{B}\right)\left(\widehat{\mathbf{Q}} + \beta\widehat{\mathbf{B}}'\mathbf{V}\widehat{\mathbf{B}}\right)^{-1}\widehat{\mathbf{B}}' - 2\mathbf{F}'\widehat{\mathbf{B}}' + \left(\widehat{\mathbf{A}} + \widehat{\mathbf{B}}\mathbf{F}\right)'\right] d\left(\overline{\mathbf{V}}\right)\left(\widehat{\mathbf{A}} + \widehat{\mathbf{B}}\mathbf{F}\right) = \mathbf{I}d(\mathbf{V}),$$

which, given equation (3), collapses to

$$\beta\left(\widehat{\mathbf{A}} + \widehat{\mathbf{B}}\mathbf{F}\right)' d\left(\overline{\mathbf{V}}\right)\left(\widehat{\mathbf{A}} + \widehat{\mathbf{B}}\mathbf{F}\right) = \mathbf{I}d(\mathbf{V}). \quad (12)$$

After vectorising and recognising that $\mathbf{M} = \widehat{\mathbf{A}} + \widehat{\mathbf{B}}\mathbf{F}$, equation (12) can be written as

$$vec[d(\mathbf{V})] = \beta\left(\mathbf{M}' \otimes \mathbf{M}'\right) vec\left[d\left(\overline{\mathbf{V}}\right)\right].$$

The matrix $\beta\left(\mathbf{M}' \otimes \mathbf{M}'\right)$ defines the derivative map $\mathbb{DT}(\mathbf{V})$. Applying Lemma 1, a Markov-perfect equilibria $\{\mathbf{H}, \mathbf{F}, \mathbf{M}, \mathbf{V}, d\}$ is a local IE-stable policy equilibrium if and only if all of the eigenvalues of $\mathbb{DT}(\mathbf{V})$ have modulus less than 1. Because the eigenvalues of $\mathbf{M}$ all have modulus less than $\beta^{-\frac{1}{2}}$ in all Markov-perfect equilibria the result follows. ■

A.1.4 Eductive joint learning

Finally, we analyse the case in which both private agents and the policymaker are learning. The postulated policy and decision rules are

$$\begin{aligned} \mathbf{y}_t &= \overline{\mathbf{H}}\mathbf{x}_t, \\ \mathbf{u}_t &= \overline{\mathbf{F}}\mathbf{x}_t, \end{aligned}$$

and the postulated value function matrix is $\overline{\mathbf{V}}$. Then the actual policy and decision rules, which are consistent with evolution of the economy and with implementing the best response to the private sector's reaction function, are given by

$$\mathbf{H} = \mathbf{J} + \mathbf{KF}, \tag{13}$$

$$\mathbf{F} = -\left(\widehat{\mathbf{Q}} + \beta\widehat{\mathbf{B}}'\overline{\mathbf{V}}\widehat{\mathbf{B}}\right)^{-1}\left(\widehat{\mathbf{U}} + \beta\widehat{\mathbf{B}}'\overline{\mathbf{V}}\widehat{\mathbf{A}}\right), \tag{14}$$

$$\mathbf{V} = \widehat{\mathbf{W}} + 2\widehat{\mathbf{U}}\mathbf{F} + \mathbf{F}'\widehat{\mathbf{Q}}\mathbf{F} + \beta\left(\widehat{\mathbf{A}} + \widehat{\mathbf{B}}\mathbf{F}\right)'\overline{\mathbf{V}}\left(\widehat{\mathbf{A}} + \widehat{\mathbf{B}}\mathbf{F}\right), \tag{15}$$

where

$$\mathbf{J} = \left(\mathbf{A}_{22} - \overline{\mathbf{H}}\mathbf{A}_{12}\right)^{-1}\left(\overline{\mathbf{H}}\mathbf{A}_{11} - \mathbf{A}_{21}\right), \tag{16}$$

$$\mathbf{K} = \left(\mathbf{A}_{22} - \overline{\mathbf{H}}\mathbf{A}_{12}\right)^{-1}\left(\overline{\mathbf{H}}\mathbf{B}_{1} - \mathbf{B}_{2}\right), \tag{17}$$

and $\widehat{\mathbf{W}}$, $\widehat{\mathbf{U}}$, $\widehat{\mathbf{Q}}$, $\widehat{\mathbf{A}}$, and $\widehat{\mathbf{B}}$ are defined by equations (5)—(9) and are functions of $\mathbf{J}$ and $\mathbf{K}$.

Given equations (16) and (17), equations (13)—(15) describe the $\mathbb{T}$-map, $T\left(\overline{\mathbf{H}}, \overline{\mathbf{F}}, \overline{\mathbf{V}}\right)$, from $\left\{\overline{\mathbf{H}}, \overline{\mathbf{F}}, \overline{\mathbf{V}}\right\}$, to $\{\mathbf{H}, \mathbf{F}, \mathbf{V}\}$.

Lemma 4 *A Markov-perfect equilibrium is locally IE-stable under joint learning if and only if all eigenvalues of the matrix* $\mathbf{P}^{-1}\mathbf{L}$ *in*

$$vec\left[d\left(\mathbf{G}\right)\right] = \mathbf{P}^{-1}\ \mathbf{L}vec\left[d\left(\overline{\mathbf{G}}\right)\right],$$

where $vec\left[d\left(\mathbf{G}\right)\right] = \left[\begin{array}{ccc} vec\left[d\left(\mathbf{H}\right)\right]' & vec\left[d\left(\mathbf{F}\right)\right]' & vec\left[d\left(\mathbf{V}\right)\right]' \end{array}\right]'$ *and* $\mathbf{P}$ *and* $\mathbf{L}$ *are characterised below, have modulus less than* 1.

Proof. Total differentials of equations (13)—(17) about the point $\{\mathbf{H}, \mathbf{F}, \mathbf{V}, \mathbf{J}, \mathbf{K}\}$ are given by

$$\mathbf{0} = d\left(\mathbf{J}\right) + d\left(\mathbf{K}\right)\mathbf{F} + \mathbf{K}d\left(\mathbf{F}\right) - d\left(\mathbf{H}\right), \tag{18}$$

$$\mathbf{0} = d\left(\overline{\mathbf{H}}\right)\widehat{\mathbf{A}} - \left(\mathbf{A}_{22} - \mathbf{H}\mathbf{A}_{12}\right)d\left(\mathbf{J}\right), \tag{19}$$

$$\mathbf{0} = d\left(\overline{\mathbf{H}}\right)\widehat{\mathbf{B}} - \left(\mathbf{A}_{22} - \mathbf{H}\mathbf{A}_{12}\right)d\left(\mathbf{K}\right), \tag{20}$$

$$\begin{aligned}\mathbf{0} = {} & \beta\widehat{\mathbf{B}}'d\left(\overline{\mathbf{V}}\right)\mathbf{M} + \left(\widehat{\mathbf{Q}} + \beta\widehat{\mathbf{B}}'\mathbf{V}\widehat{\mathbf{B}}\right)d\left(\mathbf{F}\right) + 2\left(\mathbf{K}'\mathbf{W}_{22} + \mathbf{U}_2' + \beta\widehat{\mathbf{B}}'\mathbf{V}\mathbf{A}_{12}\right)d\left(\mathbf{K}\right)\mathbf{F} \\ & + \left(\mathbf{W}_{12} + \mathbf{J}'\mathbf{W}_{22} + \beta\widehat{\mathbf{A}}'\mathbf{V}\mathbf{A}_{12}\right)d\left(\mathbf{K}\right) + \left(\mathbf{K}'\mathbf{W}_{22} + \mathbf{U}_2' + \beta\widehat{\mathbf{B}}'\mathbf{V}\mathbf{A}_{12}\right)d\left(\mathbf{J}\right),\end{aligned} \tag{21}$$

$$\begin{aligned}\mathbf{0} = {} & 2\left(\widehat{\mathbf{U}} + \mathbf{F}'\widehat{\mathbf{Q}} + \beta\mathbf{M}'\mathbf{V}\widehat{\mathbf{B}}\right)d\left(\mathbf{F}\right) + 2\left(\mathbf{W}_{12} + \mathbf{H}'\mathbf{W}_{22} + \mathbf{F}'\mathbf{U}_2' + \beta\mathbf{M}'\mathbf{V}\mathbf{A}_{12}\right)d\left(\mathbf{J}\right) \\ & + 2\left(\mathbf{W}_{12} + \mathbf{H}'\mathbf{W}_{22} + \mathbf{F}'\mathbf{U}_2' + \beta\mathbf{M}'\mathbf{V}\mathbf{A}_{12}\right)d\left(\mathbf{K}\right)\mathbf{F} + \beta\mathbf{M}'d\left(\overline{\mathbf{V}}\right)\mathbf{M} - d\left(\mathbf{V}\right).\end{aligned} \tag{22}$$

Now, using equations (19) and (20) to solve for $d\left(\mathbf{J}\right)$ and $d\left(\mathbf{K}\right)$, respectively, and substituting these expressions into equations (18), (21), and (22) produces

$$\begin{aligned}
\mathbf{0} &= \mathbf{K}d(\mathbf{F}) + (\mathbf{A}_{22} - \mathbf{H}\mathbf{A}_{12})^{-1} d\left(\overline{\mathbf{H}}\right)\mathbf{M} - d(\mathbf{H}), \qquad (23) \\
\mathbf{0} &= \beta\widehat{\mathbf{B}}' d\left(\overline{\mathbf{V}}\right)\mathbf{M} + \left(\widehat{\mathbf{Q}} + \beta\widehat{\mathbf{B}}'\mathbf{V}\widehat{\mathbf{B}}\right) d(\mathbf{F}) \\
&\quad + \left(\mathbf{W}_{12} + \mathbf{J}'\mathbf{W}_{22} + \beta\widehat{\mathbf{A}}'\mathbf{V}\mathbf{A}_{12}\right)(\mathbf{A}_{22} - \mathbf{H}\mathbf{A}_{12})^{-1} d\left(\overline{\mathbf{H}}\right)\widehat{\mathbf{B}} \\
&\quad + 2\left(\mathbf{K}'\mathbf{W}_{22} + \mathbf{U}_2' + \beta\widehat{\mathbf{B}}'\mathbf{V}\mathbf{A}_{12}\right)(\mathbf{A}_{22} - \mathbf{H}\mathbf{A}_{12})^{-1} d\left(\overline{\mathbf{H}}\right)\widehat{\mathbf{B}}\mathbf{F} \\
&\quad + \left(\mathbf{K}'\mathbf{W}_{22} + \mathbf{U}_2' + \beta\widehat{\mathbf{B}}'\mathbf{V}\mathbf{A}_{12}\right)(\mathbf{A}_{22} - \mathbf{H}\mathbf{A}_{12})^{-1} d\left(\overline{\mathbf{H}}\right)\widehat{\mathbf{A}} \qquad (24) \\
\mathbf{0} &= 2\left(\widehat{\mathbf{U}} + \mathbf{F}'\widehat{\mathbf{Q}} + \beta\mathbf{M}'\mathbf{V}\widehat{\mathbf{B}}\right) d(\mathbf{F}) + \beta\mathbf{M}' d\left(\overline{\mathbf{V}}\right)\mathbf{M} - d(\mathbf{V}) \\
&\quad + 2\left(\mathbf{W}_{12} + \mathbf{H}'\mathbf{W}_{22} + \mathbf{F}'\mathbf{U}_2' + \beta\mathbf{M}'\mathbf{V}\mathbf{A}_{12}\right)(\mathbf{A}_{22} - \mathbf{H}\mathbf{A}_{12})^{-1} d\left(\overline{\mathbf{H}}\right)\mathbf{M}, \qquad (25)
\end{aligned}$$

where, again, the invertability of $(\mathbf{A}_{22} - \mathbf{H}\mathbf{A}_{12})$ is virtually ensured by the assumption that $\mathbf{A}_{22}$ has full rank. By vectorising and stacking equations (23)—(25) they can be written in the form

$$\mathbf{P}vec\,[d(\mathbf{G})] = \mathbf{L}vec\left[d\left(\overline{\mathbf{G}}\right)\right],$$

where

$$\mathbf{P} = \begin{bmatrix} \mathbf{I} & -\mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\left(\widehat{\mathbf{Q}} + \beta\widehat{\mathbf{B}}'\mathbf{V}\widehat{\mathbf{B}}\right) & \mathbf{0} \\ \mathbf{0} & -2\left(\widehat{\mathbf{U}} + \mathbf{F}'\widehat{\mathbf{Q}} + \beta\mathbf{M}'\mathbf{V}\widehat{\mathbf{B}}\right) & \mathbf{I} \end{bmatrix},$$

and $\mathbf{L}$ is defined implicitly by equations (23)—(25). Because $\left(\widehat{\mathbf{Q}} + \beta\widehat{\mathbf{B}}'\mathbf{V}\widehat{\mathbf{B}}\right)$ has full rank in any Markov-perfect equilibrium, $\mathbf{P}$ too has full rank. The result follows. ■

Lemma 5 *The equilibrium identified by Oudiz and Sachs (1985) and all equilibria identified by Backus and Driffill (1986) are IE-stable under joint learning.*

Proof. The iterative numerical schemes employed by the Backus and Driffill (1986) and Oudiz and Sachs (1985) solution methods coincide with the learning scheme described by the $\mathbb{T}$-map (13)—(15). As a consequence, these numerical solution methods apply direct numerical iterations on the non-linear T-map. If these numerical solution methods converge to a fix-point, then, by construction, the resulting equilibrium is IE-stable under joint learning. ■

A.2 Coalition formation

Assume that the model has N Markov-perfect equilibria. Because the economic environment is one in which there is complete and perfect information, the existence and nature of all N equilibria is known to all agents. Moreover, the N equilibria can (invariably) be welfare ranked and, as a consequence, agents are not indifferent to which equilibrium prevails.

Treating the policy rules associated with the N equilibria as a set of policy actions, because the equilibria are Nash, if policymakers in periods $s = t+1, ..., \infty$ are expected to play $\overline{\mathbf{F}}_j$,

$j = 1, ..., N$, then the period-t policymaker's best response is to also play $\mathbf{F}_j$. However, although it is never beneficial for the period-t policymaker to unilaterally deviate from Nash play, the period-t policymaker can potentially benefit from deviations that involve multiple policymakers. With this in mind, we introduce the possibility that a "small" coalition of policymakers could form that may deviate from the play prescribed in equilibrium j. The coalitions that we envisage are motivated by the fact that policymakers have tenures spanning multiple decision periods and, as a consequence, we model them in terms of sequential players.

Let $(p_j + 1)$ represent the number of sequential players in a potential coalition and consider the period-t policymaker's best response where the predicted future play is given by $\{\mathbf{F}_i^{t+1}, ..., \mathbf{F}_i^{t+p_j}, \mathbf{F}_j^{t+p_j+1}, \mathbf{F}_j^{t+p_j+2}, ...\}$, $j \neq i$, with private agents in periods $s = t, ..., \infty$ responding according to their reaction function. In this scenario, during periods $s = t + p_j + 1, ..., \infty$ the policy rule and private-sector decision rules are given by $\mathbf{F}_j$ and $\mathbf{H}_j$, respectively. However, during periods $s = t, ..., t+p_j$ the policy rule is given by $\mathbf{F}_i$ and private agents respond according to their reaction function,

$$\mathbf{H}^s = \left(\mathbf{H}^{s+1}\mathbf{A}_{12} - \mathbf{A}_{22}\right)^{-1}\left[\mathbf{A}_{21} + \mathbf{B}_2\mathbf{F}_i - \mathbf{H}^{s+1}\left(\mathbf{A}_{11} + \mathbf{B}_1\mathbf{F}_i\right)\right]. \tag{26}$$

Given equation (26), the law-of-motion for the state vector during periods $s = t, ..., t + p_j$ is

$$\mathbf{M}^s = \mathbf{A}_{11} + \mathbf{A}_{12}\mathbf{H}^s + \mathbf{B}_1\mathbf{F}_i.$$

We know that if $p_j = 0$, then the period-t policymaker's best response is to play $\mathbf{F}_j$. However, as p_j increases, the period-t policymaker's best response can switch from $\mathbf{F}_j$ to $\mathbf{F}_i$. For each $\mathbf{F}_j$, we calculate the number of periods of multilateral deviation p_j required to switch the period-t policymaker's best response from $\mathbf{F}_j$ to $\mathbf{F}_i$. Of course, although the period-t policymaker's best response may switch from $\mathbf{F}_j$ to $\mathbf{F}_i$ as p_j increases, it need not. In fact, whether the period-t policymaker's best response switches from $\mathbf{F}_j$ to $\mathbf{F}_i$ as p_j increases turns on whether equilibrium i is Pareto-preferred to equilibrium j and on whether equilibrium i is locally IE-stable under private sector learning.

Lemma 6 *The period-t policymaker's best response will switch from $\mathbf{F}_j$ to $\mathbf{F}_i$ in the limit as $p_j \uparrow \infty$ if and only if equilibrium i is Pareto-preferred to equilibrium j and equilibrium i is locally IE-stable under private sector learning.*

Proof. Consider equation (26). If equilibrium i is locally IE-stable under private sector learning, then, $\mathbf{H}^s \to \mathbf{H}^i$ in the limit as $p_j \uparrow \infty$, which implies $\mathbf{M}^s \to \mathbf{M}^i$ and $\mathbf{V}^s \to \mathbf{V}^i$. Because equilibrium i Pareto-dominates equilibrium j, the period-t policymaker's best response must switch from $\mathbf{F}_j$ to $\mathbf{F}_i$. On the contrary, if equilibrium i is not locally IE-stable under private sector learning, then although $\mathbf{H}^s$ may converge to $\widetilde{\mathbf{H}} \neq \mathbf{H}^i$ in the limit as $p_j \uparrow \infty$, because $\widetilde{\mathbf{H}} \neq \mathbf{H}^i$ the period-t policymaker's best response cannot be $\mathbf{F}_i$. ■

An additional issue that we consider is whether coalition forming can generate a switch from the prevailing equilibrium to the Pareto-preferred equilibrium and, if so, how large of a coalition is required to generate such a switch. It follows from Lemma 6 that the Pareto-preferred equilibrium must be locally IE-stable under private sector learning if such a switch is to occur.

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References

Backus, D., and Driffill, J. (1986). 'The Consistency of Optimal Policy in Stochastic Rational Expectations Models', Centre for Economic Policy Research Discussion Paper #124.

Evans, G. (1985). 'Expectational Stability and the Multiple Equilibria Problem in Linear Rational Expectations Models', *The Quarterly Journal of Economics*, vol. 100, 4, pp. 1217–33.

Magnus, J., and Neudecker, H. (1988). *Matrix Differential Calculus with Applications to Statistics and Econometrics*, New York: John Wiley and Sons.

Oudiz, G., and Sachs, J. (1985). 'International Policy Coordination in Dynamic Macroeconomic Models', in Buiter, W., and R. Marston, (eds), *International Economic Policy Coordination*, Cambridge University Press, Cambridge, pp. 275–319.