STRUCTURAL PROPERTIES OF CLOSE II_1 FACTORS

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Abstract. We show that a number of key structural properties transfer between sufficiently close II_1 factors, including solidity, strong solidity, uniqueness of Cartan masas and property Γ. We also examine II_1 factors close to tensor product factors, showing that such factors also factorise as a tensor product in a fashion close to the original.

1. Introduction

In [20], Kadison and Kastler equipped the collection of all operator algebras acting on a Hilbert space with a metric which measures how close the unit balls of two algebras are in operator norm. Using the operator norm in this fashion makes closeness a very strong condition on a pair of operator algebras, leading Kadison and Kastler to conjecture that sufficiently close algebras should be spatially isomorphic. Strong results for amenable von Neumann algebras were obtained in the late 1970's in [7, 36, 9]: sufficiently close amenable von Neumann algebras must arise from small unitary perturbations. A few years ago corresponding results for separable nuclear C^*-algebras were obtained in [13] (examples of Johnson from [18] show that one can only expect a small unitary perturbation in the point norm topology in the C^*-setting). In [3] we examined nonamenable algebras, providing the first nonamenable von Neumann algebras which satisfy the Kadison-Kastler conjecture (an expository account of this work can be found in [1]).

The driving theme of this paper is the transfer of structural properties between close von Neumann algebras. This was the focus of the original paper [20], which shows that close von Neumann algebras M and N have the same nonzero summands in their type decomposition, and further that the corresponding summands are again close. Subsequently close C^*-algebras were shown to have isomorphic ideal lattices (and correspondingly close ideals) by Phillips in [28], and C^*-algebras which remain close under all matrix amplifications were shown to have isomorphic K-theories by Khoshkam in [22]. Recently questions of this nature have been explored for more refined C^*-algebra invariants in [12] (which demonstrates a strong connection between close operator algebras and Kadison's similarity problem from [19], which the authors extended in [2]) and [27]. In this paper we turn to von Neumann algebras, and more precisely II_1 factors, showing how the methods developed in [3] can be used to examine properties such as (strong) solidity [24, 25] and uniqueness of Cartan masas [25], which have come to the forefront as part of the revolutionary progress in the structure theory of II_1 factors made over the last fifteen years. We also consider Murray and von Neumann's property Γ and tensorial decompositions, transferring these properties to sufficiently close factors, and examine the structure of masas within close factors.

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Before proceeding, we recall the definitions of the Kadison-Kastler metric and the closely related notion of near containments from [20] and [9] respectively. Note that the metric is not quite obtained from symmetrising the notion of near inclusion.

Definition 1.1 (Kadison-Kastler, Christensen)**.** Let M and N be von Neumann algebras acting nondegenerately on a Hilbert space $\mathcal{H}$. The distance, $d(M,N)$ is the infimum of those $\gamma > 0$ such that for every operator x in the unit ball of one algebra, there exists y in the unit ball of the other algebra with $\|x-y\| < \gamma$. A near containment $M \subseteq_\gamma N$ arises when for every $x \in M$, there exists $y \in N$ with $\|x-y\| \leq \gamma\|x\|$. Write $M \subset_\gamma N$ when there exists $\gamma' < \gamma$ with $M \subseteq_{\gamma'} N$.

We note that there is no assumption that $\|y\| \leq \|x\|$ in the definition of a near containment. Consequently, the composition of near containments $P \subset_\alpha Q \subset_\beta R$ becomes

$$P \subset_{(\alpha+\beta+\alpha\beta)} R, \tag{1.1}$$

easily obtained from the triangle inequality.

It is also natural to consider 'completely bounded' versions of the above notions. Let $d_{cb}(M,N) = \sup_n d(M \otimes \mathbb{M}_n, N \otimes \mathbb{M}_n)$, where one measures the distance between $M \otimes \mathbb{M}_n$ and $N \otimes \mathbb{M}_n$ on $\mathcal{H} \otimes \mathbb{C}^n$. Similarly, write $M \subset_{cb,\gamma} N$ when $M \otimes \mathbb{M}_n \subset_\gamma N \otimes \mathbb{M}_n$ for all $n \in \mathbb{N}$.

A key tool in the study of close von Neumann algebras is the embedding theorem for a near containment of an amenable von Neumann algebra from [9, Theorem 4.3, Corollary 4.4]. This is used repeatedly in this paper and so we recall the statement here for the reader's convenience.

Theorem 1.2 (Christensen)**.** *Let P be an amenable von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$ containing $I_{\mathcal{H}}$. Suppose that B is another von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$ and $P \subset_\gamma B$ for a constant $\gamma < 1/100$. Then there exists a unitary $u \in (P \cup B)''$ with $uPu^* \subseteq B$, $\|I_{\mathcal{H}} - u\| \leq 150\gamma$ and $\|uxu^* - x\| \leq 100\gamma\|x\|$ for $x \in P$. If, in addition, $\gamma < 1/101$ and $B \subset_\gamma P$, then $uPu^* = B$.*

In the next section we consider the structure of close masas, providing a one-to-one correspondence between unitary equivalence classes of Cartan masas, and transfer property Γ, solidity and strong solidity to close factors. These results were originally given in the preprint version of [3] on the arXiv, but were removed from the publication version. In Section 3 we consider tensor product decompositions, and the paper ends with a short list of open problems in Section 4.

2. Masas, solidity and property Γ

We start with the structure of maximal abelian subalgebras (masas) in close II_1 factors. Recall that in [15] Diximer introduced a rough classification of masas A in a II_1 factor M through their *normalisers*, namely those unitaries $u \in M$ with $uAu^* = A$. The collection of all normalisers is denoted $\mathcal{N}(A \subseteq M)$ and A is said to be *Cartan* when $\mathcal{N}(A \subseteq M)$ generates M as a von Neumann algebra (general subalgebras P of M with $\mathcal{N}(P \subseteq M)'' = M$ are called regular). At the other extreme, A is said to be *singular* when $\mathcal{N}(A \subseteq M) \subseteq A$. The transfer of normalisers between close pairs of inclusions provided a key tool in [3], which we use here to examine the behaviour of close masas in close algebras.

Since the breakthrough paper [25], there has been considerable interest in how many Cartan masas a II_1 factor contains, up to unitary conjugacy: [25] gives the first class of factors with a unique Cartan masa up to unitary conjugacy, [14] provides the first examples of factors with two Cartan masas which are not even conjugate by an automorphism, and [26] presents more examples of factors with unique Cartan masas and also new factors with at least two Cartan masas. More recently, large classes of crossed products have been shown to have unique Cartan masas [5, 32, 33]. At the other end of the spectrum, [38] provides a II_1 factor with unclassifiably many Cartan masas up to conjugacy by an automorphism. Our first objective is to show that close II_1 factors have the same Cartan masa structure. Given a II_1 factor M, let $\mathrm{Cartan}(M)$ be the collection of equivalence classes of Cartan masas in M under the relation $A_1 \sim A_2$ if and only if there is a unitary $u \in M$ with $uA_1u^* = A_2$.

Theorem 2.1. *Let M and N be II_1 factors with separable preduals acting nondegenerately on a Hilbert space $\mathcal{H}$ with $M \subset_\gamma N$ and $N \subset_\gamma M$ for a constant $\gamma < 5.7 \times 10^{-16}$.*

(i) Suppose $P \subseteq M$ is an amenable regular von Neumann subalgebra with $P' \cap M \subseteq P$ and $Q \subseteq N$ is a von Neumann subalgebra with $P \subset_\delta Q \subset_\delta P$ for some $\delta \geq 0$ such that $300\gamma + \delta < 1/8$. Then Q is regular in N and satisfies $Q' \cap N \subseteq Q$.

(ii) If A is a Cartan masa in M, then there exists a Cartan masa B in N satisfying $d(A, B) < 100\gamma$.

(iii) There exists a canonical bijective map $\Theta : \mathrm{Cartan}(M) \to \mathrm{Cartan}(N)$, which is defined by $\theta([A]) = [B]$ where $A \subseteq M$ and $B \subseteq N$ are Cartan masas with $d(A, B) < 100\gamma$.

(iv) If M has a unique Cartan masa up to unitary conjugacy, then the same is true for N.

Proof. (i). Since $\gamma < 1/100$, we may apply the embedding theorem (Theorem 1.2) to obtain a unitary $u \in (P \cup N)''$ satisfying $\|u - I_{\mathcal{H}}\| \leq 150\gamma$, $uPu^* \subseteq N$, and $d(P, uPu^*) \leq 100\gamma$. Define $N_1 = u^*Nu$, so that $P \subseteq M \cap N_1$ and $M \subset_{\gamma_1} N_1 \subset_{\gamma_1} M$ where $\gamma_1 = 301\gamma$. Then the bound on γ gives $\gamma_1 < 1.74 \times 10^{-13}$, so we may apply [3, Lemma 4.10] to conclude that P is regular in N_1 and $P' \cap N_1 \subseteq P$. Thus $Q_1 := uPu^*$ is regular in N and satisfies $Q_1' \cap N \subseteq Q_1$. Now by [3, Equation (2.1)], $Q_1 \subset_\eta Q \subset_\eta Q_1$ where $\eta = 300\gamma + \delta < 1/8$. By [8, Theorem 4.1], Q and Q_1 are unitarily conjugate inside N (strictly speaking, the statement of [8, Theorem 4.1] requires the hypothesis $d(Q, Q_1) < \frac{1}{8}$ but, as noted in [3, Section 3], the proof only needs the hypothesis in terms of near inclusions). Thus Q inherits the desired properties from Q_1.

(ii). Given a Cartan masa A in M, Theorem 1.2 gives a unitary $u \in (A \cup N)''$ such that the algebra $B := uAu^*$ lies in N and satisfies $d(A, B) < 100\gamma$. Then B is a masa in N by [3, Lemma 2.17] and so is Cartan by (i).

(iii). From (ii), we may associate to each Cartan masa A in M a Cartan masa B in N so that $d(A, B) < 100\gamma$. Let A_1 be another Cartan masa in M and choose a Cartan masa B_1 in N with $d(A_1, B_1) < 100\gamma$. If there exists a unitary $u \in M$ such that $A_1 = uAu^*$, then by [3, Lemma 2.12 (i)], there is a unitary $v \in N$ with $\|u - v\| < \sqrt{2}\gamma$. Then

$$\begin{aligned} d(B_1, vBv^*) &\leq d(B_1, uBu^*) + 2\|u - v\| < d(B_1, uAu^*) + 2\sqrt{2}\gamma + 100\gamma \\ &= d(B_1, A_1) + (100 + 2\sqrt{2})\gamma < (200 + 2\sqrt{2})\gamma < 1/8. \end{aligned} \tag{2.1}$$

Thus B_1 and vBv^* are unitarily conjugate in N by [8, Theorem 4.1], and hence B_1 and B are unitarily conjugate in N. This shows that there is a well defined map $\Theta : \mathrm{Cartan}(M) \to \mathrm{Cartan}(N)$, defined on $[A]$ by choosing a Cartan masa B as above and letting $\Theta([A]) = [B]$. In the same way there is a map $\Phi : \mathrm{Cartan}(N) \to \mathrm{Cartan}(M)$ so that for each Cartan masa B in N, $\Phi([B]) = [A]$ where $A \subseteq M$ is chosen so that $d(B, A) < 100\gamma$. By construction Φ is the inverse of Θ so Θ is bijective.

(iv). This is immediate from (iii). ∎

At the other end of the spectrum, one has the singular masas. Various ad hoc methods have been used to determine whether certain explicit singular masas are conjugate via an automorphism of the underlying factor; perhaps the most successful is Pukànszky's invariant, originating in [34], which associates to a masa $A \subseteq M$ a nonempty subset of $\mathbb{N} \cup \{\infty\}$ as follows: the relative commutant of A inside the basic construction algebra $\langle M, e_A\rangle$ gives a type I von Neumann algebra $A' \cap \langle M, e_A\rangle = (A \cup J_M A J_M)'$. This always has a type I_1 summand, as e_A is central in $A' \cap \langle M, e_A\rangle$ with $e_A(A' \cap \langle M, e_A\rangle) = e_A A$. The Pukánszky invariant $\mathrm{Puk}(A \subseteq M)$ consists of those $n \in \mathbb{N} \cup \{\infty\}$ such that $(1 - e_A)(A' \cap \langle M, e_A\rangle)$ has a nonzero type I_n component. See [37, Chapter 7] for more information on the Pukánszky invariant (including proofs of the facts above). In the next result we do not need the precise definitions of the basic construction, just that the Pukánszky invariant is obtained from the relative positions of e_A, A and $\langle M, e_A\rangle$. Note that the embedding theorem can be used to provide algebras $B \subseteq N$ satisfying the estimates of the next proposition, when δ is sufficiently small.

Proposition 2.2. *Let M and N be II_1 factors with separable preduals acting nondegenerately on a Hilbert space $\mathcal{H}$ with $M \subset_\gamma N$ and $N \subset_\gamma M$ for a constant γ. Let $A \subseteq M$ be a masa in M.*

(i) Suppose that $\delta > 0$ satisfies $(4+2\sqrt{2})(\gamma+24\delta) < 1$. If A is singular, then any subalgebra $B \subseteq N$ with $d(A, B) < \delta$ is a singular masa in N.

(ii) Suppose that $\delta > 0$ satisfies $(\gamma+24\delta) < 1.74\times10^{-13}$. Then any von Neumann subalgebra $B \subseteq N$ with $d(A, B) < \delta$ is a masa in N satisfying

$$\mathrm{Puk}(A \subseteq M) = \mathrm{Puk}(B \subseteq N). \tag{2.2}$$

Proof. First note that [6, Lemma 2.3] shows that B is abelian. Then [3, Lemma 2.16(i)] gives $B' \cap N \subset_{2\sqrt{2}\delta+\gamma} A' \cap M = A \subset_\delta B$ so that $B' \cap N \subset_\eta B \subseteq B' \cap N$, where $\eta = 2\sqrt{2}\delta + \gamma + (1 + \gamma + 2\sqrt{2}\delta)\delta$ using (1.1). Since the hypothesis of (i) implies that $\eta < 1$, we have $B = B' \cap N$ (see [13, Proposition 2.4]). Thus B is a masa in N. As both A and B are amenable, by [9, Corollary 4.2(c)] there exists a unitary $u \in (A \cup B)''$ with $\|u-1\| < 12\delta$ and $uAu^* = B$. Write $N_1 = u^*Nu$ so that A is a masa in N_1 and the near inclusions $M \subset_{\gamma+24\delta} N_1$ and $N_1 \subset_{\gamma+24\delta} M$ hold.

Now suppose $A \subseteq M$ is singular. Given any unitary normaliser $v \in \mathcal{N}(A \subseteq N_1)$, [3, Lemma 3.4(iii)] provides a unitary normaliser $v' \in \mathcal{N}(A \subseteq M)$ with $\|v-v'\| \leq (4+2\sqrt{2})(\gamma+24\delta) < 1$. By [3, Proposition 3.2], we have $v' = vu_1u_2$ for unitaries $u_1 \in A$ and $u_2 \in A' \cap \mathcal{B}(\mathcal{H})$. Thus $vxv^* = v'xv'^*$ for all $x \in A$. Since $A \subseteq M$ is singular, it follows that $vxv^* = x$ for all $x \in A$, and so $v \in A$ since A is a masa in N_1. Thus A is singular in N_1, and so B is singular in N, proving (i).

For (ii), as $M \subset_{\gamma_1} N_1$ and $N_1 \subset_{\gamma_1} M$ for $\gamma_1 = (\gamma + 24\delta) < 1.74 \times 10^{-13}$, we can use [3, Lemma 4.10] (with $P = A$) to simultaneously represent M and N_1 on a new Hilbert space $\mathcal{K}$ such that both these algebras are simultaneously in standard form with respect to the same trace vector, and have equal basic constructions $\langle M, e_A\rangle = \langle N_1, e_A\rangle$. It follows that $\mathrm{Puk}(A \subseteq M) = \mathrm{Puk}(A \subseteq N_1)$, and hence $\mathrm{Puk}(A \subseteq M) = \mathrm{Puk}(B \subseteq N)$. □

Recall from [24] that a II_1 factor is said to be *solid* when every diffuse unital von Neumann subalgebra $P \subseteq M$ has an amenable relative commutant $P' \cap M$. Note that to establish solidity of M it suffices to show that $P' \cap M$ is amenable when P is diffuse and amenable (or abelian), as given a general diffuse subalgebra P of M, take a maximal abelian subalgebra

P_0 of P. This will be diffuse and $P' \cap M \subseteq P_0' \cap M$ so that $P' \cap M$ will inherit amenability from $P_0' \cap M$ (since M is finite).

Subsequently Ozawa and Popa generalised the concept of solidity further in [25]: a II_1 factor M is said to be *strongly solid* if every unital diffuse amenable subalgebra $B \subseteq M$ has an amenable normalizing algebra $\mathcal{N}(B \subseteq M)''$. Both these properties transfer to sufficiently close factors, as we now show.

Proposition 2.3. *Let M and N be II_1 factors acting nondegenerately on a Hilbert space $\mathcal{H}$ with $d(M,N) < 1/3200$. Then:*

(i) *M is solid if and only if N is solid;*
(ii) *M is strongly solid if and only if N is strongly solid.*

Proof. Let M and N be II_1 factors acting nondegenerately on a Hilbert space $\mathcal{H}$ with $d(M,N) < \gamma < 1/3200$. We will assume that N is solid, or strongly solid, and show that M has the same property, so take a diffuse unital amenable subalgebra P of M. By Theorem 1.2, there exists a unital von Neumann subalgebra $Q \subseteq N$ isomorphic to P such that $d(P,Q) \leq 100\gamma$. When N is strongly solid, let $Q_1 = \mathcal{N}(Q \subseteq N)'' \subseteq N$, and when N is solid, let $Q_1 = (Q \cup (Q' \cap N))'' \subseteq N$. In both cases Q_1 is amenable. This is the hypothesis of strong solidity in the first case, while when N is solid, $Q' \cap N$ is amenable, which implies that Q_1 is amenable as it is the von Neumann algebra generated by two commuting amenable subalgebras. (One way to see this is via the equivalence of injectivity and hyperfiniteness, since certainly two commuting finite dimensional algebras generate another finite dimensional algebra).

Applying Theorem 1.2 again gives a unitary $u \in (Q_1 \cup M)''$ such that $uQ_1u^* \subseteq M$, $\|u - I_{\mathcal{H}}\| < 150\gamma$ and $d(uQ_1u^*, Q_1) \leq 100\gamma$. Thus $d(P, uQu^*) \leq d(P,Q) + 2\|u - I_{\mathcal{H}}\| \leq 400\gamma$. Since $400\gamma < 1/8$, [8, Theorem 4.1] gives a unitary $u_1 \in (P \cup uQu^*)'' \subseteq M$ satisfying $u_1Pu_1^* = uQu^*$ and $\|u_1 - I_{\mathcal{H}}\| \leq 7d(P, uQu^*) \leq 2800\gamma$ (here we have crudely estimated the function δ appearing in [8, Theorem 4.1]).

Now write $N_1 = u_1^*uNu^*u_1$ so that $P = u_1^*uQu^*u_1$ is a subalgebra of N_1. Since $P' \cap N_1 = u_1^*u(Q' \cap N)u^*u_1$, $Q' \cap N \subseteq Q_1$, and $u_1 \in M$, we have $P' \cap N_1 \subseteq P' \cap M$. As $u_1 \in M$,

$$\begin{aligned} d(M, N_1) &= d(u_1Mu_1^*, uNu^*) = d(M, uNu^*) \\ &\leq d(M,N) + 2\|u - I_{\mathcal{H}}\| < 301\gamma. \end{aligned} \tag{2.3}$$

By [3, Lemma 2.16 (i)] (with $\delta = 0$) we have $P' \cap M \subseteq_{301\gamma} P' \cap N_1$. So, as $301\gamma < 1$, $P' \cap M = P' \cap N_1$ (this is a folklore Banach space argument, see [13, Proposition 2.4] for the precise statement being used). In the case when N is solid, $Q' \cap N$ is amenable, and hence $P' \cap M = P' \cap N_1 = u_1^*u(Q' \cap N)u^*u_1$ is amenable. This proves that M is solid.

In the case when N is strongly solid, note that

$$\mathcal{N}(P \subseteq N_1)'' = u_1^*u\mathcal{N}(Q \subseteq N)''u^*u_1 = u_1^*uQ_1u^*u_1 \subseteq M. \tag{2.4}$$

Now take a unitary $v \in \mathcal{N}(P \subseteq M)$. As $301\gamma < 2^{-3/2}$, [3, Lemma 3.4 (iii)] provides $v' \in \mathcal{N}(P \subseteq N_1) \subseteq M$ with $\|v - v'\| \leq (4 + 2\sqrt{2})301\gamma$. We have $(4 + 2\sqrt{2})301\gamma < 1$, so [3, Proposition 3.2] gives unitaries $w \in P$ and $w' \in P' \cap \mathcal{B}(\mathcal{H})$ satisfying $v' = vww'$. Then $w' = w^*v^*v' \in P' \cap M$ since w, v, and v' all lie in M. Thus $w' \in P' \cap N_1 \subseteq \mathcal{N}(P \subseteq N_1)''$. Then $v = v'w'^*w^* \in \mathcal{N}(P \subseteq N_1)$, so that $\mathcal{N}(P \subseteq M) \subseteq \mathcal{N}(P \subseteq N_1)$. Since $\mathcal{N}(P \subseteq N_1)''$ is amenable so too is its subalgebra $\mathcal{N}(P \subseteq M)''$. Thus M is strongly solid. $\square$

To conclude this section, we turn to Murray and von Neumann's property Γ. Recall that a II_1 factor M with trace τ has property Γ if for any finite set $\{x_1, \cdots, x_n\}$ in M and $\varepsilon > 0$, there exists a unitary $u \in M$ with $\tau(u) = 0$ and $\|[x_i, u]\|_2 < \varepsilon$ (as is usual, $\|\cdot\|_2$ denotes the norm induced by the trace: $\|x\|_2 = \tau(x^*x)^{1/2}$). Equivalently (in the presence of a separable predual), property Γ is characterised by the nontriviality of the central sequence algebra $M^\omega \cap M'$, where ω is a free ultrafilter (see [40, Theorem XIV.4.7]). For II_1 factors with nonseparable preduals this equivalence no longer holds (see [17, Section 3]) and instead one must work with ultrafilters on sets of larger cardinality. For simplicity, we restrict to the separable predual situation here. However the argument can be modified to handle the nonseparable situation (with the same constants). To reach our stability result we need an extension of [3, Lemma 2.15].

Lemma 2.4. *Let M and N be II_1 factors represented nondegenerately on a Hilbert space $\mathcal{H}$ and let γ and η be positive constants. Suppose that $d(M,N) < \gamma < 1$ and that we have x_1, x_2 in the unit ball of M and y_1, y_2 in the unit ball of N with $\|x_i - y_i\| \le \eta$, $i = 1, 2$. Then*

$$\|y_1 - y_2\|_{2,N}^2 \le \|x_1 - x_2\|_{2,M}^2 + 8\eta + (8\sqrt{2} + 8)\gamma. \tag{2.5}$$

Proof. Define $s = y_1 - y_2 \in N$ and $t = x_1 - x_2 \in M$, so that $\|s\|, \|t\| \le 2$ and $\|s - t\| \le 2\eta$. Let Φ be a state on $\mathcal{B}(\mathcal{H})$ extending τ_M. Then [3, Lemma 2.15] gives

$$|\tau_N(s^*s) - \Phi(s^*s)| \le (2\sqrt{2} + 2)\gamma\|s^*s\| \le (8\sqrt{2} + 8)\gamma. \tag{2.6}$$

We also have

$$|\Phi(s^*s) - \Phi(t^*t)| \le \|(s^* - t^*)s + t^*(s - t)\| \le 8\eta, \tag{2.7}$$

so

$$\begin{aligned} \|s\|_{2,N}^2 = \tau_N(s^*s) &\le |\Phi(s^*s)| + |\tau_N(s^*s) - \Phi(s^*s)| \\ &\le |\Phi(s^*s) - \Phi(t^*t)| + \Phi(t^*t) + (8\sqrt{2} + 8)\gamma \\ &\le \|t\|_{2,M}^2 + 8\eta + (8\sqrt{2} + 8)\gamma, \end{aligned} \tag{2.8}$$

since Φ and τ_M agree on M. This is (2.5). $\square$

Proposition 2.5. *Let M and N be II_1 factors with separable preduals acting nondegenerately on a Hilbert space $\mathcal{H}$ with $d(M,N) < \gamma$ for a constant $\gamma < 1/190$. Suppose that M has property Γ. Then N also has property Γ.*

Proof. Suppose that M has property Γ and fix a free ultrafilter ω on $\mathbb{N}$. By definition, there is a sequence $(u_n)_{n=1}^\infty$ of trace zero unitaries such that $u = (u_n)$ defines an element in $M^\omega \cap M'$. For each n, use [3, Lemma 2.12] to find a unitary $v_n \in N$ with $\|u_n - v_n\| < \sqrt{2}\gamma$ and let v denote the class of (v_n) in N^ω. Let Φ denote a state on $\mathcal{B}(\mathcal{H})$ extending τ_N. Then [3, Lemma 2.15] gives the estimate

$$|\tau_M(u_n) - \Phi(u_n)| \le (2\sqrt{2} + 2)\gamma, \quad n \in \mathbb{N}, \tag{2.9}$$

so that

$$|\tau_M(u_n) - \tau_N(v_n)| \le |\tau_M(u_n) - \Phi(u_n)| + |\Phi(u_n) - \Phi(v_n)| \le (3\sqrt{2} + 2)\gamma. \tag{2.10}$$

Thus

$$|\tau_{N^\omega}(v)| \le (3\sqrt{2} + 2)\gamma. \tag{2.11}$$

Given a unitary $w \in N$, use [3, Lemma 2.12] to find a unitary $w' \in M$ with $\|w' - w\| < \sqrt{2}\gamma$. Then

$$\|w'u_n - wv_n\| \leq \|(w' - w)u_n\| + \|w(u_n - v_n)\| \leq 2\sqrt{2}\gamma \tag{2.12}$$

and similarly $\|u_n w' - v_n w\| \leq 2\sqrt{2}\gamma$. Taking $\eta = 2\sqrt{2}\gamma$ in Lemma 2.4 with $x_1 = w'u_n$, $x_2 = u_n w'$, $y_1 = wv_n$ and $y_2 = v_n w$ gives

$$\|wv_n - v_n w\|_{2,N}^2 \leq \|w'u_n - u_n w'\|_{2,M}^2 + (24\sqrt{2} + 8)\gamma. \tag{2.13}$$

Since $\lim_{n\to\omega} \|w'u_n - u_n w'\|_{2,M} = 0$, we have the estimate

$$\|wvw^* - v\|_{2,N^\omega}^2 = \|wv - vw\|_{2,N^\omega}^2 \leq (24\sqrt{2} + 8)\gamma \tag{2.14}$$

in N^ω. Let y be the unique element of minimal $\|\cdot\|_{2,N^\omega}$-norm in $\overline{\mathrm{conv}}^{2,N^\omega}\{wvw^* : w \in \mathcal{U}(N)\}$. This lies in N^ω and uniqueness ensures that $y \in N' \cap N^\omega$. It remains to check that y is nontrivial.

The estimate (2.14) gives

$$\|y - v\|_{2,N^\omega}^2 \leq (24\sqrt{2} + 8)\gamma, \tag{2.15}$$

and so

$$\|y\|_{2,N^\omega} \geq 1 - ((24\sqrt{2} + 8)\gamma)^{1/2} \tag{2.16}$$

as $\|v\|_{2,N^\omega} = 1$. We can estimate

$$\begin{aligned} |\tau_{N^\omega}(y)| &\leq |\tau_{N^\omega}(v)| + |\tau_{N^\omega}(y - v)| \leq (3\sqrt{2} + 2)\gamma + \|y - v\|_{2,N^\omega} \\ &\leq (3\sqrt{2} + 2)\gamma + ((24\sqrt{2} + 8)\gamma)^{1/2}, \end{aligned} \tag{2.17}$$

using (2.11), (2.15) and the Cauchy-Schwarz inequality. If $y \in \mathbb{C}I_{N^\omega}$, then $y = \tau_{N^\omega}(y)I_{N^\omega}$ so $\|y\|_{2,N^\omega} = |\tau_{N^\omega}(y)|$, and it follows that

$$1 - ((24\sqrt{2} + 8)\gamma)^{1/2} \leq \|y\|_{2,N^\omega} \leq (3\sqrt{2} + 2)\gamma + ((24\sqrt{2} + 8)\gamma)^{1/2}. \tag{2.18}$$

Direct computations show that this is a contradiction when $\gamma < 1/190$, so that y is a nontrivial element of $N' \cap N^\omega$. Therefore N has property Γ. ☐

Remark 2.6. As a consequence of the results of this section, factors close to free group factors inherit a number of their properties. Assume $d(M, L\mathbb{F}_2)$ is sufficiently small. Then M is strongly solid by Proposition 2.3 and [25], and every masa in M has infinite multiplicity (i.e. unbounded Pukánszky invariant) by Proposition 2.2 and [16]. Further, there are masas A_1 and A_2 in M close to the generator masas in $L\mathbb{F}_2$ and a masa B close to the radial masa in $L\mathbb{F}_2$. These are singular with Pukánzksy invariant $\{\infty\}$ by Proposition 2.2 and [15, 35, 37]. The embedding theorem was used in Proposition 2.3 and can be employed in a similar way to establish maximal injectivity of A_1, A_2 and B in M since their counterparts in $L\mathbb{F}_2$ are known to be maximal injective [31, 4].

3. Tensor products

In [3, Section 5] we considered McDuff factors (those which absorb the hyperfinite II$_1$ factor tensorially), showing that this property transfers to sufficiently close factors. In this section, we examine general tensor product factorisations, transferring these to close factors. If P and Q are II$_1$ factors, then $M := P \mathbin{\overline{\otimes}} Q$ is generated by two commuting infinite dimensional subalgebras. As shown in [23], this characterises the property of being isomorphic to a tensor product: if M is a II_1 factor generated by two commuting infinite dimensional von Neumann subalgebras S and T, then M is isomorphic to $S\,\overline{\otimes}\,T$, and S and T are automatically II_1 factors. This result will prove useful below.

We begin with a technical observation.

Lemma 3.1. *Let $\gamma > 0$ and suppose that $M, N \subseteq \mathcal{B}(\mathcal{H})$ are von Neumann algebras acting nondegenerately on a Hilbert space $\mathcal{H}$ such that $d(M,N) < \gamma$. Let $A \subseteq M' \cap N'$ be an abelian von Neumann algebra. Then $d((M \cup A)'', (N \cup A)'') < \gamma$.*

Proof. Choose γ' to satisfy $d(M,N) < \gamma' < \gamma$. Let $B \subseteq A$ be the span of the projections in A, so that B is a $*$-subalgebra of A. If $x \in \mathrm{Alg}(M \cup B)$, $\|x\| \leq 1$, then there exist orthogonal projections $p_1, \ldots, p_n \in B$ and elements $x_1, \ldots, x_n \in M$ with $\|x_i\| \leq 1$ so that $x = \sum_{i=1}^n x_i p_i$. Choose elements $y_1, \ldots, y_n \in N$ so that $\|y_i\| \leq 1$ and $\|x_i - y_i\| \leq \gamma'$, and let $y = \sum_{i=1}^n y_i p_i \in \mathrm{Alg}(N \cup B)$. Then $\|y\| \leq 1$ and

$$\|x - y\| = \|\sum_{i=1}^n (x_i - y_i)p_i\| = \max\{\|x_i - y_i\| : 1 \leq i \leq n\} \leq \gamma'. \tag{3.1}$$

The argument is symmetric in M and N, so $d(C^*(M \cup B), C^*(N \cup B)) \leq \gamma'$. The result follows from the Kaplansky density theorem via [20, Lemma 5]. □

The next lemma takes a tensor product factor $M = P\,\overline{\otimes}\,Q$ acting on $\mathcal{H}_1 \otimes \mathcal{H}_2$ and considers close factors N generated by commuting II_1 factors S and T which are assumed close to P and Q respectively. The lemma shows that, provided we have a reverse near containment of M' into N', then we can make a small unitary perturbation of N, S and T so that S can be viewed as acting on $\mathcal{H}_1$ and T on $\mathcal{H}_2$.

Lemma 3.2. *Let $P \subseteq \mathcal{B}(\mathcal{H}_1)$ and $Q \subseteq \mathcal{B}(\mathcal{H}_2)$ be II_1 factors, let $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, and let $M = P\,\overline{\otimes}\,Q$. Suppose that $N \subseteq \mathcal{B}(\mathcal{H})$ is a II_1 factor and has two commuting subfactors S and T so that*

$$d(M,N),\ d(P \otimes I_{\mathcal{H}_2}, S),\ d(I_{\mathcal{H}_1} \otimes Q, T) < \lambda, \quad M' \subset_{k\lambda} N', \tag{3.2}$$

for constants $\lambda, k > 0$ satisfying

$$(90301 + 27180600k)\lambda < 1/100. \tag{3.3}$$

Then there exists a unitary $u \in \mathcal{B}(\mathcal{H})$ such that

$$\|I_{\mathcal{H}} - u\| < 150(90602 + 27271202k)\lambda, \tag{3.4}$$

$u^*Su \subseteq \mathcal{B}(\mathcal{H}_1) \otimes I_{\mathcal{H}_2}$, $u^*Tu \subseteq I_{\mathcal{H}_1} \otimes \mathcal{B}(\mathcal{H}_2)$, and

$$d(M, u^*Nu),\ d(P, u^*Su),\ d(Q, u^*Tu) \leq (27180601 + 8181360600k)\lambda. \tag{3.5}$$

Proof. Let A be a masa in $(P' \cap \mathcal{B}(\mathcal{H}_1)) \otimes I_{\mathcal{H}_2}$. Then $A \subseteq M' \subset_{k\lambda} N'$. Since $k\lambda < 1/100$ there exists, by the embedding theorem (Theorem 1.2), a unitary $u_1 \in (A \cup N')''$ such that $u_1 A u_1^* \subseteq N'$ and $\|I_{\mathcal{H}} - u_1\| \leq 150k\lambda$. Let $N_1 = u_1^* N u_1$, $S_1 = u_1^* S u_1$ and $T_1 = u_1^* T u_1$. Then $A \subseteq (P \otimes I_{\mathcal{H}_2})' \cap S_1'$ and

$$d(M, N_1),\ d(P \otimes I_{\mathcal{H}_2}, S_1),\ d(I_{\mathcal{H}_1} \otimes Q, T_1) < (1 + 300k)\lambda, \quad M' \subset_{301k\lambda} N_1'. \tag{3.6}$$

By Lemma 3.1,

$$d(((P \otimes I_{\mathcal{H}_2}) \cup A)'', (S_1 \cup A)'') < (1 + 300k)\lambda, \tag{3.7}$$

and $((P\otimes I_{\mathcal{H}_2})\cup A)''$ is amenable since $((P\otimes I_{\mathcal{H}_2})\cup A)'=(A\cup(I_{\mathcal{H}_1}\otimes\mathcal{B}(\mathcal{H}_2)))''$, which is amenable. By the embedding theorem (Theorem 1.2) there is a unitary $u_2\in(((P\otimes I_{\mathcal{H}_2})\cup A)''\cup(S_1\cup A)'')''$ such that $u_2((P\otimes I_{\mathcal{H}_2})\cup A)''u_2^*=(S_1\cup A)''$ and $\|I_{\mathcal{H}}-u_2\|\leq 150(1+300k)\lambda$. Let $N_2=u_2^*N_1u_2$, $S_2=u_2^*S_1u_2$ and $T_2=u_2^*T_1u_2$. Then

$$d(M,N_2),\ d(P\otimes I_{\mathcal{H}_2},S_2),\ d(I_{\mathcal{H}_1}\otimes Q,T_2)\leq 301(1+300k)\lambda,\quad M'\subset_{(300+90301k)\lambda}N_2'. \tag{3.8}$$

Moreover,

$$S_2\subseteq u_2^*(S_1\cup A)''u_2=((P\otimes I_{\mathcal{H}_2})\cup A)''\subseteq\mathcal{B}(\mathcal{H}_1)\otimes I_{\mathcal{H}_2}. \tag{3.9}$$

Now choose a masa $B\subseteq I_{\mathcal{H}_1}\otimes(Q'\cap\mathcal{B}(\mathcal{H}_2))$. Then $B\subseteq M'\subset_{(300+90301k)\lambda}N_2'$. The estimate (3.3) allows the embedding theorem (Theorem 1.2) to be applied to give a unitary $u_3\in(B\cup N_2')''$ so that $u_3Bu_3^*\subseteq N_2'$ and $\|I_{\mathcal{H}}-u_3\|\leq 150(300+90301k)\lambda$. Let $N_3=u_3^*N_2u_3$, and $T_3=u_3^*T_2u_3$, and note that $S_2=u_3^*S_2u_3$ since S_2 commutes with B and N_2'. We also have the estimates

$$d(M,N_3),\ d(Q,T_3)\leq(90301+27180600k)\lambda. \tag{3.10}$$

By construction, $B\subseteq(I_{\mathcal{H}_1}\otimes Q)'\cap T_3'$, so by Lemma 3.1 and the inequality (3.3),

$$d(((I_{\mathcal{H}_1}\otimes Q)\cup B)'',(T_3\cup B)'')\leq(90301+27180600k)\lambda<1/100. \tag{3.11}$$

As $((I_{\mathcal{H}_1}\otimes Q)\cup B)''$ is amenable (it is the commutant of the amenable algebra $((\mathcal{B}(\mathcal{H}_1)\otimes I_{\mathcal{H}_2})\cup B)''$), Theorem 1.2 gives a unitary $u_4\in(((I_{\mathcal{H}_1}\otimes Q)\cup B)''\cup(T_3\cup B)'')''$ with the property that $u_4((I_{\mathcal{H}_1}\otimes Q)\cup B)''u_4^*=(T_3\cup B)''$ and $\|I_{\mathcal{H}}-u_4\|\leq 150(90301+27180600k)\lambda$. Since S_2 commutes with $I_{\mathcal{H}_1}\otimes Q$, B and T_3, we see that $u_4^*S_2u_4=S_2$. Also

$$u_4^*T_3u_4\subseteq((I_{\mathcal{H}_1}\otimes Q)\cup B)''\subseteq I_{\mathcal{H}_1}\otimes\mathcal{B}(\mathcal{H}_2). \tag{3.12}$$

Consequently the desired unitary u is $u_1u_2u_3u_4$, and

$$\|I_{\mathcal{H}}-u\|\leq\sum_{i=1}^{4}\|I_{\mathcal{H}}-u_i\|\leq 150(90602+27271202k)\lambda \tag{3.13}$$

from previous estimates, while

$$d(M,u^*Nu),\ d(P\otimes I_{\mathcal{H}_2},u^*Su),\ d(I_{\mathcal{H}_1}\otimes Q,u^*Tu)\leq(27180601+8181360600k)\lambda. \tag{3.14}$$

We have $u^*Su\subseteq\mathcal{B}(\mathcal{H}_1)\otimes 1_{\mathcal{H}_2}$ from (3.9) as u_3 and u_4 commute with $S_2=u_2^*u_1^*Su_1u_2$ and $u^*Tu\subseteq I_{\mathcal{H}_1}\otimes\mathcal{B}(\mathcal{H}_2)$ from (3.12). $\square$

Lemma 2.16 of [3] considers near containments of relative commutants. We will use the following version of this lemma in the context of distance estimates. The proof is identical to part (i) of [3, Lemma 2.16], noting that if the $y\in N$ in the proof of that lemma lies in the unit ball, then this is also true for the approximating elements $E_{Q'\cap N}(y)$.

Lemma 3.3 (c.f. [3, Lemma 2.16(i)])**.** *Let M and N be II_1 factors acting nondegenerately on a Hilbert space and suppose that $P\subseteq M$ and $Q\subseteq N$ are unital von Neumann subalgebras. Then $d(P'\cap M,Q'\cap N)\leq d(M,N)+2\sqrt{2}d(P,Q)$.*

Lemma 3.4. *Suppose that M and N are II_1 factors acting nondegenerately on a Hilbert space $\mathcal{H}$ and that $\dim_M\mathcal{H}=1$. If $d(M,N)<1/(301\times 136209)=1/40998909$, then $\dim_N\mathcal{H}=1$.*

Proof. Choose γ to satisfy $d(M,N) < \gamma < 1/40998909$ and choose a masa $A \subseteq M$. Then $A \subseteq_\gamma N$ and $\gamma < 1/100$, so by the embedding theorem (Theorem 1.2), there exists a unitary $u \in (A \cup N)''$ with $\|u - I_{\mathcal{H}}\| \leq 150\gamma$ so that $uAu^* \subseteq N$. Let $N_1 = u^*Nu$, so that $d(M, N_1) \leq 301\gamma < 1/136209$. Since $A \subseteq M \cap N_1$, we may apply [3, Proposition 4.6] to M and N_1 to conclude that $\dim_{N_1} \mathcal{H} = 1$. Since N is unitarily conjugate to N_1, it follows that $\dim_N \mathcal{H} = 1$ as required. ◻

We now turn to the tensor product decomposition in a II_1 factor N close to a tensor product $M \cong P \,\overline{\otimes}\, Q$, using the reduction to standard form technique of [3, Section 4]. We do this first under the assumption that both factors M and N contain a suitable hyperfinite subfactor; this assumption is removed in the subsequent theorem by means of the embedding theorem.

Lemma 3.5. *Let M and N be II_1 factors acting nondegenerately on a Hilbert space $\mathcal{H}$ with $d(M,N) < \gamma$. Suppose that M is generated by two commuting II_1 factors P and Q and that there are hyperfinite II_1 factors $R_1 \subseteq P$ and $R_2 \subseteq Q$ with $R_1' \cap P = \mathbb{C}I_P$ and $R_2' \cap Q = \mathbb{C}I_Q$ which further satisfy $(R_1 \cup R_2)'' \subseteq N$. Write $S = R_2' \cap N$ and $T = R_1' \cap N$. Then the following statements hold:*

(i) $d(P,S) < \gamma$ and $d(Q,T) < \gamma$;
(ii) if $\gamma < \frac{1}{2\sqrt{2}+2}$, then $S' \cap N = T$ and $T' \cap N = S$;
(iii) if $\gamma < 10^{-39}$, then N is generated by the commuting II_1 factors S and T.

Proof. By [23] we may view M as $P \,\overline{\otimes}\, Q$, and it follows from Tomita's commutation theorem (see [39, Theorem IV.5.9 and Corollary IV.5.10]) that $P = R_2' \cap M$. Similarly, $Q = R_1' \cap M$. Part (i) then follows from Lemma 3.3.

For (ii), note that $R_1 \subseteq S$, so that $S' \cap N \subseteq R_1' \cap N = T$. Applying Lemma 3.3 to the close pairs (M,N) and (S,P) gives

$$Q = P' \cap M \subseteq_{(2\sqrt{2}+1)\gamma} S' \cap N \subseteq T. \tag{3.15}$$

Since $d(T,Q) < \gamma$, it follows from (3.15) that

$$T \subset_{(2\sqrt{2}+2)\gamma} S' \cap N \subseteq T. \tag{3.16}$$

By hypothesis, $(2\sqrt{2}+2)\gamma < 1$ and this ensures that $S' \cap N = T$ (see [13, Proposition 2.4]). The identity $T' \cap N = S$ is obtained similarly.

Now we turn to (iii). Since $\gamma < 1/87$, [3, Lemma 4.8] gives an integer n, a nonzero projection $e \in M'$, and a unitary $u \in (M' \cup N')''$ such that $e \in (u^*Nu)'$,

$$\|u - I_{\mathcal{H}}\| \leq 12\sqrt{2}(1+\sqrt{2})\gamma + 4\sqrt{2}((1+\sqrt{2})\gamma)^{1/2}, \tag{3.17}$$

and $\dim_{Me}(e\mathcal{H}) = 1/n$. Let $\mathcal{K} = (e\mathcal{H}) \otimes \mathbb{C}^n$, and define factors by $M_1 = (Me) \otimes I_{\mathbb{C}^n}$, $P_1 = (Pe) \otimes I_{\mathbb{C}^n}$, $Q_1 = (Qe) \otimes I_{\mathbb{C}^n}$, $R_3 = (R_1e) \otimes I_{\mathbb{C}^n}$, $R_4 = (R_2e) \otimes I_{\mathbb{C}^n}$, and also let $N_1 = ((u^*Nu)e) \otimes I_{\mathbb{C}^n}$, $S_1 = ((u^*Su)e) \otimes I_{\mathbb{C}^n}$, $T_1 = ((u^*Tu)e) \otimes I_{\mathbb{C}^n}$. Then M_1 and N_1 are faithful normal representations of M and N respectively on $\mathcal{K}$, and $(R_3 \cup R_4)'' \subseteq N_1$ since u commutes with $M \cap N$.

Combining (3.17) and the inequality $\gamma < 10^{-19}\gamma^{1/2}$, we have the estimate

$$d(M_1, N_1) \leq \gamma + 2\|u - I_{\mathcal{H}}\| \leq (49 + 24\sqrt{2})\gamma + 8\sqrt{2}((1+\sqrt{2})\gamma)^{1/2} < 18\gamma^{1/2}. \tag{3.18}$$

Let γ_1 denote the last term in (3.18), so that $d(M_1, N_1) \leq \gamma_1$. By construction, $\dim_{M_1} \mathcal{K} = 1$ so M_1 is in standard position on $\mathcal{K}$. As the bound on γ ensures that $\gamma_1 < 1/40998909$, Lemma 3.4 shows that N_1 is also in standard position on $\mathcal{K}$. If we represent P_1 and Q_1 in standard position on Hilbert spaces $\mathcal{K}_1$ and $\mathcal{K}_2$ respectively, then $P_1 \overline{\otimes} Q_1 \cong M_1$ is in standard position on $\mathcal{K}_1 \otimes \mathcal{K}_2$. This allows us to assume that $\mathcal{K} = \mathcal{K}_1 \otimes \mathcal{K}_2$ and to identify P_1 with $P_1 \otimes I_{\mathcal{K}_2}$ and Q_1 with $I_{\mathcal{K}_1} \otimes Q_1$. As both M_1 and N_1 are in standard position, [3, Lemma 4.1(i)] gives

$$M_1' \subset_{2(1+\sqrt{2})\gamma_1} N_1', \qquad N_1' \subset_{2(1+\sqrt{2})\gamma_1} M_1'. \tag{3.19}$$

The hypotheses of Lemma 3.2 are now met by taking $k = 2(1+\sqrt{2})$ and $\lambda = \gamma_1 = 18\gamma^{1/2}$. Thus there exists a unitary $u_1 \in \mathcal{B}(\mathcal{K})$ such that

$$\|I_{\mathcal{K}} - u_1\| < 150(90602 + 54542404(1+\sqrt{2}))\gamma_1, \tag{3.20}$$

and if we define $N_2 = u_1^* N_1 u_1$, $S_2 = u_1^* S_1 u_1$, and $T_2 = u_1^* T_1 u_1$, then $S_2 \subseteq \mathcal{B}(\mathcal{K}_1)$, $T_2 \subseteq \mathcal{B}(\mathcal{K}_2)$, and

$$\begin{aligned} d(M_1, N_2),\ d(P_1, S_2),\ d(Q_1, T_2) &< (27180601 + 16362721200(1+\sqrt{2}))\gamma_1 \\ &< 1/40998909, \end{aligned} \tag{3.21}$$

from the choice of the bound on γ. By Lemma 3.4, S_2 is in standard position on $\mathcal{K}_1$ and similarly T_2 is in standard position on $\mathcal{K}_2$. It follows that $(S_2 \cup T_2)''$, which is canonically identified with $S_2 \overline{\otimes} T_2$ with respect to $\mathcal{K} = \mathcal{K}_1 \otimes \mathcal{K}_2$, is also in standard position on $\mathcal{K}$. Since $(S_2 \cup T_2)'' \subseteq N_2$ and $\dim_{N_2} \mathcal{K} = 1$, we conclude that $(S_2 \cup T_2)'' = N_2$, and hence also that $(S \cup T)'' = N$. □

We are now in a position to show that tensorial decompositions can be transferred between close II_1 factors.

Theorem 3.6. *Let M and N be II_1 factors with separable preduals, acting nondegenerately on a Hilbert space $\mathcal{H}$. If M is generated by two commuting II_1 factors P and Q and $d(M,N) < \gamma < 3.3 \times 10^{-42}$, then there exist commuting II_1 subfactors S and T which generate N and satisfy*

$$d(P,S),\ d(Q,T) < (200\sqrt{2}+1)\gamma < 284\gamma, \quad d_{cb}(P,S),\ d_{cb}(Q,T) \leq 601\,\gamma. \tag{3.22}$$

Proof. By [30], choose amenable subfactors $R_1 \subseteq P$ and $R_2 \subseteq Q$ with trivial relative commutants. Then $(R_1 \cup R_2)''$ is also amenable, and we denote this factor by R. Since $R \subset_\gamma N$, we may choose a unitary $v \in (R \cup N)''$ with $\|v - I_{\mathcal{H}}\| \leq 150\gamma$, $\|x - vxv^*\| \leq 100\gamma\|x\|$ for $x \in R$ and $vRv^* \subseteq N$ by the embedding theorem (Theorem 1.2). Write $N_1 = v^* N v$ so that $R \subseteq M \cap N_1$ and $d(M, N_1) < \gamma_1 = 301\gamma$. Since $301\gamma < 10^{-39}$, Lemma 3.5 (iii) can be applied to conclude that N_1 is generated by the commuting subfactors $S_1 = R_2' \cap N_1$ and $T_1 = R_1' \cap N_1$. Hence N is generated by the commuting subfactors $S := (vR_2v^*)' \cap N = vS_1v^*$ and $T := (vR_1v^*)' \cap N = vT_1v^*$. Since $d(R_1, vR_1v^*), d(R_2, vR_2v^*) \leq 100\gamma$, Lemma 3.3 shows that $d(P,S) \leq (200\sqrt{2}+1)\gamma$ and similarly $d(Q,T) \leq (200\sqrt{2}+1)\gamma$.

We now estimate the cb-distance between P and S, so fix $n \in \mathbb{N}$ and let F denote a unital subalgebra of R_2 isomorphic to a copy of the $n \times n$ matrices $\mathbb{M}_n$. By construction, $F \subseteq R_2 \subseteq Q \cap T_1$, so there are induced factorizations $Q \cong F \otimes Q_0$, $T_1 \cong F \otimes T_0$ and $R_2 \cong F \otimes R_0$ where $Q_0 = F' \cap Q$, $T_0 = F' \cap T_1$ and $R_0 = F' \cap R_2$. Thus M is generated by the two commuting factors $P_0 = (P \cup F)''$ and Q_0 (amounting to taking a copy of $\mathbb{M}_n$ from Q and attaching it to P) and N_1 by the commuting factors $S_0 = (S_1 \cup F)''$ and T_0.

We note that $R_0 \subseteq Q_0 \cap T_0$ and has trivial relative commutants in Q_0 and T_0. In this way $(P \cup F)'' = R_0' \cap M$ and $(S_1 \cup F)'' = R_0' \cap N_1$. Another application of Lemma 3.3 gives

$$d((S_1 \cup F)'', (P \cup F)'') = d(R_0' \cap N_1, R_0' \cap M) \leq d(M, N_1) \leq 301\,\gamma. \tag{3.23}$$

Since F is a factor, there is an isometric $*$-isomorphism between $(F' \cup F)'' \subseteq \mathcal{B}(\mathcal{H})$ and $F' \otimes F \cong F' \otimes \mathbb{M}_n \subseteq \mathcal{B}(\mathcal{H}) \otimes \mathbb{M}_n$, defined on generators by $f'f \mapsto f' \otimes f$, which carries $(P \cup F)''$ and $(S_1 \cup F)''$ respectively to $P \otimes \mathbb{M}_n$ and $S_1 \otimes \mathbb{M}_n$. In this way (3.23) gives $d_{cb}(P, S_1) \leq 301\gamma$. As $S = vS_1v^*$ where v is a unitary satisfying $\|v - I_{\mathcal{H}}\| \leq 150\,\gamma$, it follows that $d_{cb}(S, S_1) \leq 300\gamma$, whence the triangle inequality gives $d_{cb}(P, S) \leq 601\gamma$. The estimate on $d_{cb}(Q, T)$ is proved in the same way. $\square$

The following corollary is a rewording of the last theorem.

Corollary 3.7. *Let M and N be II_1 factors with separable preduals, acting nondegenerately on a Hilbert space $\mathcal{H}$, and suppose that $d(M, N) < 3.3 \times 10^{-42}$. If M is prime, then so too is N.*

In [3], we encapsulated the weakest form of the Kadison-Kastler conjecture by defining a II_1 factor M to be weakly Kadison-Kastler stable if there exists $\varepsilon > 0$ so that if $\pi : M \to \mathcal{B}(\mathcal{H})$ is a normal representation and $N \subseteq \mathcal{B}(\mathcal{H})$ is a II_1 factor satisfying $d(\pi(M), N) < \varepsilon$, then $\pi(M)$ and N are $*$-isomorphic. Theorem 3.6 shows that this property is preserved under taking tensor products.

Corollary 3.8. *Let P and Q be II_1 factors with separable preduals and suppose that both are weakly Kadison-Kastler stable. Then so too is $M := P \overline{\otimes} Q$.*

Proof. Let $\varepsilon > 0$ be small enough to satisfy the definition of weak Kadison-Kastler stability of both P and Q. Suppose that M and N are represented on some Hilbert space. When $d(M, N)$ is sufficiently small, Theorem 3.6 shows that N is generated by two commuting II_1 factors S and T such that $d(P, S) < \varepsilon$ and $d(Q, T) < \varepsilon$. Thus $P \cong S$ and $Q \cong T$, from which it follows that $M \cong P \overline{\otimes} Q \cong S \overline{\otimes} T \cong N$. Hence M is weakly Kadison-Kastler stable. $\square$

We now turn to the strongest form of the Kadison-Kastler conjecture, which asks that close von Neumann algebras arise from small unitary perturbations. As in [3], we say that a II_1 factor M is *strongly Kadison-Kastler stable* if, given $\varepsilon > 0$, there exists $\delta > 0$ with the following property: if $\pi : M \to \mathcal{B}(\mathcal{H})$ is a normal representation and $N \subseteq \mathcal{B}(\mathcal{H})$ is a II_1 factor satisfying $d(\pi(M), N) < \delta$, then there exists a unitary $u \in \mathcal{B}(\mathcal{H})$ with $\|I_{\mathcal{H}} - u\| < \varepsilon$ such that $u\pi(M)u^* = N$. We need a standard observation regarding representations of tensor products.

Lemma 3.9. *Let M and N be II_1 factors and let $\pi : M \overline{\otimes} N \to \mathcal{B}(\mathcal{H})$ be a normal representation on a Hilbert space $\mathcal{H}$. Then there exists a type I_∞ factor P such that*

$$\pi(M \otimes 1_N) \subseteq P \subseteq \pi(1_M \otimes N)'. \tag{3.24}$$

Proof. Let λ denote the standard representation of $M \overline{\otimes} N$ on $L^2(M) \otimes L^2(N)$. From the general form of normal representations of von Neumann algebras, there exists a Hilbert space $\mathcal{K}$ and a projection $p \in \lambda(M \overline{\otimes} N)' \overline{\otimes} \mathcal{B}(\mathcal{K}) = J_M M J_M \overline{\otimes} J_N N J_N \overline{\otimes} \mathcal{B}(\mathcal{K})$ so that π is unitarily equivalent to the representation $p(\lambda(x) \otimes 1_{\mathcal{K}})$, for $x \in M \overline{\otimes} N$. Since $\lambda(M \overline{\otimes} N)' \overline{\otimes} \mathbb{B}(\mathcal{K})$ is a type II factor, the projection p is Murray-von Neumann equivalent to $p_1 \otimes q_1 \otimes f$, where the projections p_1, q_1 and f lie in $J_M M J_M$, $J_N N J_N$ and $\mathcal{B}(\mathcal{K})$ respectively. A partial isometry implementing this equivalence provides a unitary conjugacy between the representations $p\lambda(\cdot)$ and $(p_1 \otimes q_1 \otimes f)\lambda(\cdot)$, so π is unitarily equivalent to $\pi_1 = (p_1 \otimes q_1 \otimes f)\lambda(\cdot)$. For this latter representation we can verify the statement of the lemma with the I_∞ factor $P_1 = p_1(\mathcal{B}(L^2(M))p_1 \overline{\otimes} q_1 \overline{\otimes} f\mathcal{B}(\mathcal{K})f$, and hence an appropriate unitary conjugation provides the required P for the representation π. $\square$

By the results of [2], strong Kadison-Kastler stability for a II$_1$ factor M implies that M has a positive solution to Kadison's similarity problem. To our knowledge, it is not known whether a tensor product of two II$_1$ factors with the similarity property necessarily also has the similarity property, so to obtain preservation results for strongly Kadison-Kastler stable factors we need to impose an additional hypothesis to take care of the similarity property. In the next lemma, this is the condition that $\pi(M)' \subset_\delta N'$.

Lemma 3.10. *Let P and Q be strongly Kadison-Kastler stable II$_1$ factors with separable preduals and let $M = P \overline{\otimes} Q$. Then, for all $\varepsilon > 0$, there exists $\delta > 0$ with the following property: if $\pi : M \to \mathcal{B}(\mathcal{H})$ is a normal representation and $N \subseteq \mathcal{B}(\mathcal{H})$ is a von Neumann algebra with $d(\pi(M), N) < \delta$ and $\pi(M)' \subset_\delta N'$, then there exists a unitary $u \in \mathcal{B}(\mathcal{H})$ with $\|u - I_{\mathcal{H}}\| < \varepsilon$ such that $u\pi(M)u^* = N$.*

Proof. Fix $\varepsilon < 1/50$. If we apply the strong stability hypothesis to P and Q with ε replaced by $\varepsilon/4$, then there exists $\delta_0 > 0$ with the following property: if $\sigma : P \to \mathcal{B}(\mathcal{H})$ is a normal representation and $S \subseteq \mathcal{B}(\mathcal{H})$ satisfies $d(\sigma(P), S) < \delta_0$, then $\sigma(P)$ and S are unitarily conjugate by a unitary $u \in \mathcal{B}(\mathcal{H})$ satisfying $\|I_{\mathcal{H}} - u\| < \varepsilon/4$, with a similar statement for Q. Now choose $\delta > 0$ so small that the following three inequalities are satisfied:

$$\delta < 3.3 \times 10^{-42}, \tag{3.25}$$

$$150(90602 + 27271202) \times 284\,\delta < \varepsilon/2 < 1/100, \tag{3.26}$$

$$(27180601 + 8181360600) \times 284\,\delta < \delta_0. \tag{3.27}$$

Let $\pi : M \to \mathcal{B}(\mathcal{H})$ be a normal representation and let $N \subseteq \mathcal{B}(\mathcal{H})$ be such that $d(\pi(M), N) < \delta$. Let us write $M_1 = \pi(M)$, $P_1 = \pi(P \otimes I_Q)$ and $Q_1 = \pi(I_P \otimes Q)$. Since $\delta < 3.3 \times 10^{-42}$, Theorem 3.6 shows that N is generated by two commuting subfactors S and T satisfying

$$d(P_1, S),\ d(Q_1, T) < \delta_1 := 284\,\delta. \tag{3.28}$$

By Lemma 3.9, there is a type I_∞ factor lying between P_1 and Q_1'. By [21, Theorem 9.3.2] there is, up to unitary equivalence, a decomposition of $\mathcal{H}$ as $\mathcal{H}_1 \otimes \mathcal{H}_2$ such that this type I_∞ factor is $\mathcal{B}(\mathcal{H}_1) \otimes I_{\mathcal{H}_2}$, whereupon $P_1 \subseteq \mathcal{B}(\mathcal{H}_1) \otimes I_{\mathcal{H}_2}$ and $Q_1 \subseteq I_{\mathcal{H}_1} \otimes \mathcal{B}(\mathcal{H}_2)$. From (3.26), the inequalities in the hypotheses of Lemma 3.2 are satisfied for $\lambda = \delta_1$ and $k = 1$, so by that lemma there exists a unitary $u_1 \in \mathcal{B}(\mathcal{H})$ with the following properties. The inequality

$$\|I_{\mathcal{H}} - u_1\| < 150(90602 + 27271202)\delta_1 \tag{3.29}$$

holds and, upon setting $N_1 = u_1^* N u_1$, $S_1 = u_1^* S u_1$, $T_1 = u_1^* T u_1$, we have $S_1 \subseteq \mathcal{B}(\mathcal{H}_1) \otimes I_{\mathcal{H}_2}$ and $T_1 \subseteq I_{\mathcal{H}_1} \otimes \mathcal{B}(\mathcal{H}_2)$. Moreover, the estimates

$$d(M_1, N_1),\ d(P_1, S_1),\ d(Q_1, T_1) \le (27180601 + 8181360600)\delta_1 < \delta_0 \tag{3.30}$$

are valid, where the last inequality is (3.27). Thus there exist unitaries $v \in \mathcal{B}(\mathcal{H}_1)$ and $w \in \mathcal{B}(\mathcal{H}_2)$ such that $(v \otimes I_{\mathcal{H}_2})P_1(v \otimes I_{\mathcal{H}_2})^* = S_1$, $(I_{\mathcal{H}_1} \otimes w)Q_1(I_{\mathcal{H}_1} \otimes w)^* = T_1$ and the inequalities $\|I_{\mathcal{H}_1}-v\|,\ \|I_{\mathcal{H}_2}-w\|<\varepsilon/4$ hold. Let $u_2=v\otimes w$ and observe that $\|I_{\mathcal{H}}-u_2\|<\varepsilon/2$. If we define $u=u_2u_1$, then $u\pi(M)u^*=N$ and

$$\begin{aligned}\|I_{\mathcal{H}}-u\| &\leq \|I_{\mathcal{H}}-u_1\|+\|I_{\mathcal{H}}-u_2\|\\ &< 150(90602+27271202)\delta_1+\varepsilon/2<\varepsilon \end{aligned}\tag{3.31}$$

from (3.29) and (3.26). □

The most general class of II_1 factors known to have the similarity property ([10, 29, 11]) are those with Murray and von Neumann's property Γ. By definition, property Γ passes to tensor products, yielding the following result.

Theorem 3.11. *Let P and Q be II_1 factors with separable preduals and suppose that both are strongly Kadison-Kastler stable. Suppose further that at least one has property Γ. Then $M:=P\mathbin{\overline{\otimes}}Q$ is strongly Kadison-Kastler stable.*

Proof. Let $\pi:M\to\mathcal{B}(\mathcal{H})$ be a faithful normal representation. Let $N\subseteq\mathcal{B}(\mathcal{H})$ be another II_1 factor with $d(\pi(M),N)<1/190$ so that N inherits property Γ from $M=P\mathbin{\overline{\otimes}}Q$ from Proposition 2.5. Proposition 2.4(ii) of [3] shows that if $d(\pi(M),N)<\gamma$, then the near inclusion $\pi(M)'\subset_{5\gamma}N'$ holds. Strong Kadison-Kastler stability now follows from Lemma 3.10. □

Remark 3.12. The examples of strongly Kadison-Kastler stable II_1 factors constructed in [3] all have the form $(P\rtimes_\alpha G)\mathbin{\overline{\otimes}}R$ where P is amenable and G is $SL_n(\mathbb{Z})$ for $n\geq 3$, and these have property Γ since they are McDuff. These satisfy the hypothesis of Theorem 3.11 and thus new examples of strongly Kadison-Kastler stable factors can be generated by taking finite tensor products of the existing ones of [3]. □

4. Open questions

We end with a short list of open problems.

(1) Does property (T) transfer to sufficiently close subalgebras?
(2) What can be said about the fundamental group, or outer automorphism group of close II_1 factors?
(3) How do nonamenable subalgebras of close II_1 factors, such as subfactors behave? If M and N are sufficiently close II_1 factors and M has an index 2 subfactor, must N also have an index 2 subfactor?
(4) Does a non-prime II_1 factor have the similarity property? Less generally, does the tensor product of two II_1 factors with the similarity property have the similarity property?

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