The effect of initial stress on the propagation of surface waves in a layered half-space

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Abstract

In this paper the propagation of small amplitude surface waves guided by a layer with a finite thickness on an incompressible half-space is studied. The layer and half-space are both assumed to be initially stressed. The combined effect of initial stress and finite deformation on the speed of Rayleigh waves is analyzed and illustrated graphically. With a suitable simple choice of constitutive law that includes initial stress, it is shown that in many cases, as is to be expected, the effect of a finite deformation (with an associated pre-stress) is very similar to that of an initial stress (without an accompanying finite deformation). However, by contrast, when the finite deformation and initial stress are considered together independently with a judicious choice of material parameters different features are found that don't appear in the separate finite deformation or initial stress situations on their own.

Keywords: nonlinear elasticity, initial stress, surface waves, secular equation

1 Introduction

Guided wave propagation provides an important non-destructive method for assessing material properties and weaknesses in many engineering structures. In the absence of initial stress (residual stress or pre-stress) the classical theory of linear elasticity has been applied successfully in the analysis of such structures. One problem of special interest is the propagation of surface waves in an isotropic linearly elastic layered half-space, and for a treatment of this problem we refer to the classic text Ewing et al. (1957) for detailed discussion and the papers by Achenbach and Keshava (1967), Achenbach and Epstein (1967), Tiersten (1969) and Farnell and Adler (1972).

For a layered half-space of incompressible isotropic elastic material subject to a pure homogeneous finite deformation and an accompanying stress (a so-called pre-stress) the propagation of Rayleigh-type surface waves in a principal plane of the underlying deformation was examined in detail in Ogden and Sotiropoulos (1995) on the basis of the linearized theory of incremental deformations superimposed on a finite deformation. In the special case of the Murnaghan theory of second-order elasticity Akbarov and Ozisik (2004) also examined the effect of pre-stress on the propagation of surface waves. Surface waves for a half-space with an elastic material boundary without bending stiffness were studied by Murdoch (1976) and generalized to include bending stiffness by Ogden and Steigmann (2002) following the theory of intrinsic boundary elasticity developed by Steigmann and Ogden (1997).

For a half-space without a layer subject to a pure homogeneous finite deformation the propagation of Rayleigh surface waves was first studied by Hayes and Rivlin (1961), who, with particular attention to the second-order theory of elasticity, obtained the secular equation for the speed of surface waves first for compressible isotropic materials and then, by specialization, for incompressible materials. Focussing on the incompressible theory for an isotropic material Dowaikh and Ogden (1990) analyzed the propagation of surface waves in a principal plane of a deformed half-space and the limiting case of surface instability for which the wave speed is zero and obtained the secular equation in respect of a general form of strain-energy function. The corresponding problem for a compressible material was treated in Dowaikh and Ogden (1991a).

For references to the Barnett–Lothe–Stroh approach to the analysis of surface waves in pre-stressed elastic materials we refer to Chadwick and Jarvis (1979a) and Chadwick (1997) in which papers compressible and incompressible materials, respectively, were considered. In contrast to the situation of a half-space subject to finite deformation and a pre-stress associated with it through a constitutive law, for materials with an initial stress parallel to the half-space surface, surface waves were analyzed recently by Shams and Ogden (2014) for an incompressible material, and it is an extension of this development to the case of a layered half-space that is the subject of the present paper. The layer is taken to have a uniform finite thickness and material properties different from those of the half-space, and the initial stress is assumed to be different in the layer and half-space. In the presence of the initial stress (in the reference configuration) the strain-energy function depends on the initial stress as well as on the deformation from the reference configuration.

The basic equations required for the study are presented in Section 2, including development of the constitutive law for an initially stressed elastic material in terms of invariants, as described in Shams and Ogden (2014), and its specialization to the case of a plane strain deformation. Section 3 provides the incremental equations of motion based on the theory of linearized incremental deformations superimposed on a finite deformation, and expressions for the elasticity tensor of an initially stressed material are given in general form and then explicitly in the case of plane strain for a general form of strain-energy function.

Section 4 applies general incremental equations to the expressions that govern two-dimensional motions in the plane of a (pure homogeneous) plane strain, a principal plane which is also a principal plane of the considered uniform initial stress. In Section 5, these equations are applied to the analysis of surface waves in a homogeneously deformed half-space covered by a layer with a uniform uniaxial initial stress that is parallel to the direction of the wave to obtain the general dispersion equation. The complex form of the dispersion equation derived in Section 5 for a general form of strain-energy function is typical for problems involving pre-stressed media, and it is only by careful choice of notation that it is possible to obtain meaningful information from the equation without using an entirely numerical approach. In Section 6 the general dispersion equation is solved numerically in respect of a simple form of strain-energy function which extends the basic neo-Hookean material model to include the initial stress. The results are illustrated graphically for several values of the parameters associated with the underlying configuration (initial stress, stretches relative to the reference configuration in layer and half-space, and material parameters).

As a final illustration we exemplify results corresponding to vanishing of the surface wave speed, which corresponds to the emergence of static incremental deformations at critical values of the parameters involved and signals instability of the underlying homogeneous configuration, leading to undulations of layer/half-space structure that decay with depth in the half-space. Such undulations are also referred to as wrinkles, and we refer to the recent paper by Diab and Kim (2014) for a discussion of wrinkling stability patterns in a graded stiffness half-space.

2 Basic equations

2.1 Kinematics and stress

Consider an elastic material occupying some configuration in which there is a known initial (Cauchy) stress $\boldsymbol{\tau}$ which is not specified by a constitutive law. Deformations of the material are measured from this configuration, which is designated as the *reference configuration*. This is denoted by $\mathcal{B}_\mathrm{r}$ and its boundary by $\partial\mathcal{B}_\mathrm{r}$. The initial stress satisfies the equilibrium equation $\mathrm{Div}\,\boldsymbol{\tau} = \mathbf{0}$ in the absence of body forces, and is *symmetric* in the absence of intrinsic couple stresses, Div being the divergence operator on $\mathcal{B}_\mathrm{r}$. If the initial stress is a *residual* stress, in the sense of Hoger (1985), then it also satisfies the zero traction boundary condition $\boldsymbol{\tau}\mathbf{N} = \mathbf{0}$ on $\partial\mathcal{B}_\mathrm{r}$, where $\mathbf{N}$ is the unit outward normal to $\partial\mathcal{B}_\mathrm{r}$. According to this definition residual stresses are necessarily inhomogeneous, and they have a strong influence on the material response relative to $\mathcal{B}_\mathrm{r}$. For references to the literature on the inclusion of residual stress in the constitutive law we refer to Merodio et al. (2013). In this paper, however, only initial stresses that are homogeneous will be considered. These also have a significant effect on the material response relative to $\mathcal{B}_\mathrm{r}$.

The material is deformed relative to $\mathcal{B}_\mathrm{r}$ so that it occupies the deformed configuration $\mathcal{B}$, with boundary $\partial\mathcal{B}$. In standard notation the deformation is described in terms of the vector function $\boldsymbol{\chi}$ according to $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$, $\mathbf{X} \in \mathcal{B}_\mathrm{r}$, where $\mathbf{x}$ is the position vector in $\mathcal{B}$ of a material point that had position vector $\mathbf{X}$ in $\mathcal{B}_\mathrm{r}$. The deformation gradient tensor $\mathbf{F}$ is defined by $\mathbf{F} = \mathrm{Grad}\,\boldsymbol{\chi}$, where Grad is the gradient operator defined on $\mathcal{B}_\mathrm{r}$. We note, in particular, the polar decomposition $\mathbf{F} = \mathbf{VR}$ which will be used subsequently, where the so-called stretch tensor $\mathbf{V}$ is symmetric and positive definite and $\mathbf{R}$ is a proper orthogonal tensor. We shall also make use of the (symmetric) left and right Cauchy–Green deformation tensors, which are given by $\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$ and $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$, respectively.

We denote by $\boldsymbol{\sigma}$ the Cauchy stress tensor in the configuration $\mathcal{B}$ and by $\mathbf{S}$ the associated nominal stress tensor relative to $\mathcal{B}_{\mathrm{r}}$, which is given by $\mathbf{S} = J\mathbf{F}^{-1}\boldsymbol{\sigma}$, where $J = \det \mathbf{F}$. We assume that there are no couple stresses, so that $\boldsymbol{\sigma}$ is symmetric. In general, however, the nominal stress tensor is not symmetric, but it follows from the symmetry of $\boldsymbol{\sigma}$ that $\mathbf{FS} = \mathbf{S}^{\mathrm{T}}\mathbf{F}^{\mathrm{T}}$. Body forces are not considered in this paper, so the equilibrium equations to be satisfied by $\boldsymbol{\sigma}$ and $\mathbf{S}$ are $\operatorname{div}\boldsymbol{\sigma} = \mathbf{0}$ and $\operatorname{Div}\mathbf{S} = \mathbf{0}$, respectively, div being the divergence operator on $\mathcal{B}$.

2.2 The strain-energy function

In the presence of an initial stress $\boldsymbol{\tau}$ the material response relative to $\mathcal{B}_{\mathrm{r}}$ is strongly influenced by $\boldsymbol{\tau}$, and this is reflected in inclusion of $\boldsymbol{\tau}$ in the constitutive law. It can be regarded as a form of structure tensor similar to, but more general than, the structure tensor associated with a preferred direction in $\mathcal{B}_{\mathrm{r}}$. In the present work we consider the material properties to be characterized by a strain-energy function W, which is defined per unit volume in $\mathcal{B}_{\mathrm{r}}$. In the absence of initial stress W depends on the deformation gradient $\mathbf{F}$, but here it depends also on $\boldsymbol{\tau}$ and we write $W = W(\mathbf{F}, \boldsymbol{\tau})$.

For incompressible materials, on which we focus in this paper, the constraint $J \equiv \det \mathbf{F} = 1$ must be satisfied for all deformations, and the nominal and Cauchy stress tensors are given by

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{F}}(\mathbf{F}, \boldsymbol{\tau}) - p\mathbf{F}^{-1}, \quad \boldsymbol{\sigma} = \mathbf{FS} = \mathbf{F}\frac{\partial W}{\partial \mathbf{F}}(\mathbf{F}, \boldsymbol{\tau}) - p\mathbf{I}, \tag{1}$$

where p is a Lagrange multiplier associated with the constraint and $\mathbf{I}$ is the identity tensor in $\mathcal{B}$.

2.3 Invariant formulation

For full details of the constitutive formulation based on invariants, we refer to Shams et al. (2011) and Shams and Ogden (2014). Here we provide a summary of the equations that are needed in the following sections. Since the material is considered to be incompressible there are only two independent invariants of $\mathbf{C}$. We take these to be the standard invariants I_1 and I_2 defined by

$$I_1 = \mathrm{tr}\,(\mathbf{C}), \quad I_2 = \frac{1}{2}(I_1^2 - \mathrm{tr}\,(\mathbf{C}^2)). \tag{2}$$

For $\boldsymbol{\tau}$ three invariants are required in general. These are independent of $\mathbf{C}$ and it is convenient to collect these together as I_4 according to

$$I_4 \equiv \{I_{41}, I_{42}, I_{43}\}, \quad I_{41} = \mathrm{tr}\,\boldsymbol{\tau}, \quad I_{42} = \mathrm{tr}\,(\boldsymbol{\tau}^2), \quad I_{43} = \mathrm{tr}\,(\boldsymbol{\tau}^3). \tag{3}$$

The set of invariants is completed by four independent invariants that depend on both $\mathbf{C}$ and $\boldsymbol{\tau}$, which we define by

$$I_5 = \mathrm{tr}\,(\mathbf{C}\boldsymbol{\tau}), \quad I_6 = \mathrm{tr}\,(\mathbf{C}^2\boldsymbol{\tau}), \quad I_7 = \mathrm{tr}\,(\mathbf{C}\boldsymbol{\tau}^2), \quad I_8 = \mathrm{tr}\,(\mathbf{C}^2\boldsymbol{\tau}^2). \tag{4}$$

Note that in the reference configuration (2) and (4) reduce to

$$I_1 = I_2 = 3, \quad I_5 = I_6 = \mathrm{tr}\,\boldsymbol{\tau}, \quad I_7 = I_8 = \mathrm{tr}\,(\boldsymbol{\tau}^2). \tag{5}$$

With W regarded as a function of $I_1, I_2, I_4, I_5, I_6, I_7, I_8$ the Cauchy stress tensor given by $(1)_2$ can be expanded out as

$$\begin{aligned}\boldsymbol{\sigma} = 2W_1\mathbf{B} + 2W_2(I_1\mathbf{B} - \mathbf{B}^2) &+ 2W_5\boldsymbol{\Sigma} + 2W_6(\boldsymbol{\Sigma}\mathbf{B} + \mathbf{B}\boldsymbol{\Sigma}) \\ &+ 2W_7\boldsymbol{\Xi} + 2W_8(\boldsymbol{\Xi}\mathbf{B} + \mathbf{B}\boldsymbol{\Xi}) - p\mathbf{I},\end{aligned} \tag{6}$$

where $W_r = \partial W/\partial I_r$, $r \in \{1, 2, 5, 6, 7, 8\}$, $\boldsymbol{\Sigma} = \mathbf{F}\boldsymbol{\tau}\mathbf{F}^{\mathrm{T}} = \mathbf{V}\mathbf{R}\boldsymbol{\tau}\mathbf{R}^{\mathrm{T}}\mathbf{V}$ and $\boldsymbol{\Xi} = \mathbf{F}\boldsymbol{\tau}^2\mathbf{F}^{\mathrm{T}} = \mathbf{V}\mathbf{R}\boldsymbol{\tau}^2\mathbf{R}^{\mathrm{T}}\mathbf{V}$.

In the reference configuration, equation (6) reduces to

$$\boldsymbol{\tau} = (2W_1 + 4W_2 - p^{(\mathrm{r})})\mathbf{I}_{\mathrm{r}} + 2(W_5 + 2W_6)\boldsymbol{\tau} + 2(W_7 + 2W_8)\boldsymbol{\tau}^2, \tag{7}$$

where $\mathbf{I}_{\mathrm{r}}$ is the identity tensor in $\mathcal{B}_{\mathrm{r}}$, $p^{(\mathrm{r})}$ is the value of p in $\mathcal{B}_{\mathrm{r}}$, and all the derivatives of W are evaluated in $\mathcal{B}_{\mathrm{r}}$, where the invariants are given by (5). Following Shams et al. (2011), but in a slightly different notation, we therefore deduce that

$$2W_1 + 4W_2 - p^{(\mathrm{r})} = 0, \quad 2(W_5 + 2W_6) = 1, \quad 2(W_7 + 2W_8) = 0 \quad \text{in } \mathcal{B}_{\mathrm{r}}. \tag{8}$$

Specializations of these restrictions will be used later.

Suppose that $\mathbf{F}$ now corresponds to a pure homogeneous strain defined by

$$x_1 = \lambda_1 X_1, \quad x_2 = \lambda_2 X_2, \quad x_3 = \lambda_3 X_3, \tag{9}$$

where (X_1, X_2, X_3) and (x_1, x_2, x_3) are Cartesian coordinates in $\mathcal{B}_\mathrm{r}$ and $\mathcal{B}$, respectively, and $\lambda_1, \lambda_2, \lambda_3$ are the (uniform) principal stretches. By incompressibility, $\lambda_1\lambda_2\lambda_3 = 1$. Let τ_{ij}, $i, j \in \{1, 2, 3\}$, denote the components of $\boldsymbol{\tau}$ for the considered deformation. Then, referred to the principal axes of the left Cauchy–Green tensor $\mathbf{B}$, which coincide with the Cartesian axes for the pure homogeneous strain, $\Sigma_{ij} = \lambda_i\lambda_j\tau_{ij}$ and $\Xi_{ij} = \lambda_i\lambda_j(\lambda_i^2 + \lambda_j^2)\sum_{k=1}^{3}\tau_{ik}\tau_{jk}$. The component form of equation (6) is then given by

$$\begin{aligned}\sigma_{ij} = 2W_1\lambda_i^2\delta_{ij} &+ 2W_2(I_1 - \lambda_i^2)\lambda_i^2\delta_{ij} + 2[W_5 + W_6(\lambda_i^2 + \lambda_j^2)]\lambda_i\lambda_j\tau_{ij} \\ &+ 2[W_7 + (\lambda_i^2 + \lambda_j^2)W_8]\lambda_i\lambda_j\sum_{k=1}^{3}\tau_{ik}\tau_{jk} - p\delta_{ij}.\end{aligned} \tag{10}$$

2.4 Plane strain specialization

Subsequently, we shall specialize to plane strain (in the $1, 2$ plane with in-plane principal stretches λ_1, λ_2 and $\lambda_3 = 1$) and with the initial stress confined to this plane, i.e. with $\tau_{i3} = 0$ for $i = 1, 2, 3$. Then, in addition to the standard plane-strain connection $I_2 = I_1$, the connections

$$I_6 = (I_1 - 1)I_5 - (\tau_{11} + \tau_{22}), \tag{11}$$

$$I_7 = (\tau_{11} + \tau_{22})I_5 - (\tau_{11}\tau_{22} - \tau_{12}^2)(I_1 - 1), \tag{12}$$

$$I_8 = (I_1 - 1)I_7 - (\tau_{11}^2 + \tau_{22}^2 + 2\tau_{12}^2) \tag{13}$$

can be established. Thus, only two independent invariants that depend on the deformation remain, and we take these to be I_1 and I_5. We now write the energy function restricted to plane strain as $\hat{W}(I_1, I_5)$ and leave implicit the dependence on the invariants of $\boldsymbol{\tau}$ that do not depend on the deformation.

The in-plane Cauchy stress then takes on the simple form

$$\boldsymbol{\sigma} = 2\hat{W}_1\mathbf{B} + 2\hat{W}_5\boldsymbol{\Sigma} - \hat{p}\mathbf{I}, \tag{14}$$

wherein all the tensors are two dimensional (in the $1, 2$ plane) and $\mathbf{B}$ satisfies the two-dimensional Cayley–Hamilton theorem $\mathbf{B}^2 - (I_1 - 1)\mathbf{B} + \mathbf{I} = \mathbf{O}$, the zero tensor, remembering that we are considering incompressibility. Note that $\hat{p}$ is different from the p in (6).

The conditions (8) reduce to

$$2\hat{W}_1 - \hat{p}^{(\mathrm{r})} = 0, \quad 2\hat{W}_5 = 1. \tag{15}$$

3 Incremental equations

In terms of the nominal stress tensor $\mathbf{S}$ the equilibrium equation $\mathrm{Div}\,\mathbf{S} = \mathbf{0}$ is now written in Cartesian component form as

$$\mathcal{A}_{\alpha i \beta j} \frac{\partial^2 x_j}{\partial X_\alpha \partial X_\beta} - \frac{\partial p}{\partial x_i} = 0, \tag{16}$$

where $\mathcal{A}_{\alpha i \beta j}$ are the components of the elasticity tensor $\boldsymbol{\mathcal{A}} = \boldsymbol{\mathcal{A}}(\mathbf{F}, \boldsymbol{\tau})$. The tensor and component forms are defined by

$$\boldsymbol{\mathcal{A}} = \frac{\partial^2 W}{\partial \mathbf{F} \partial \mathbf{F}}, \quad \mathcal{A}_{\alpha i \beta j} = \frac{\partial^2 W}{\partial F_{i\alpha} \partial F_{j\beta}}, \tag{17}$$

with Greek and Roman indices relating to $\mathcal{B}_{\mathrm{r}}$ and $\mathcal{B}$, respectively.

We now consider a small incremental deformation superimposed on the finite deformation $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$. Let this be denoted by $\dot{\mathbf{x}} = \dot{\boldsymbol{\chi}}(\mathbf{X}, t)$ and its gradient by $\mathrm{Grad}\,\dot{\mathbf{x}} \equiv \dot{\mathbf{F}}$. Here and in the following a superposed dot indicates an increment in the considered quantity.

Based on the nominal stress the linearized incremental constitutive equation and the corresponding incremental incompressibility condition are

$$\dot{\mathbf{S}} = \boldsymbol{\mathcal{A}}\dot{\mathbf{F}} - \dot{p}\mathbf{F}^{-1} + p\mathbf{F}^{-1}\dot{\mathbf{F}}\mathbf{F}^{-1}, \quad \mathrm{tr}\,(\dot{\mathbf{F}}\mathbf{F}^{-1}) = 0. \tag{18}$$

where $\dot{p}$ is the linearized incremental form of p.

The incremental equation of motion for an initial homogeneous deformation (with $\boldsymbol{\mathcal{A}}$ and p constants) is then

$$\mathrm{Div}\,\dot{\mathbf{S}} = \mathrm{Div}\,(\boldsymbol{\mathcal{A}}\dot{\mathbf{F}}) - \mathbf{F}^{-\mathrm{T}}\mathrm{Grad}\,\dot{p} = \rho \dot{\mathbf{x}}_{,tt}, \tag{19}$$

where a subscript t following a comma indicates the material time derivative and ρ is the mass density of the material. In components this becomes

$$\mathcal{A}_{\alpha i \beta j} \frac{\partial^2 \dot{x}_j}{\partial X_\alpha \partial X_\beta} - \frac{\partial \dot{p}}{\partial x_i} = \rho \dot{x}_{i,tt}. \tag{20}$$

Also required is the incremental form of the symmetry condition $\mathbf{F}\mathbf{S} = \mathbf{S}^{\mathrm{T}}\mathbf{F}^{\mathrm{T}}$, i.e.

$$\mathbf{F}\dot{\mathbf{S}} + \dot{\mathbf{F}}\mathbf{S} = \dot{\mathbf{S}}^{\mathrm{T}}\mathbf{F}^{\mathrm{T}} + \mathbf{S}^{\mathrm{T}}\dot{\mathbf{F}}^{\mathrm{T}}. \tag{21}$$

Following Shams et al. (2011) and Shams and Ogden (2014) it is convenient to update the reference configuration so that it coincides with the configuration corresponding to the finite homogeneous deformation with all incremental quantities treated as functions of $\mathbf{x}$ and t instead of $\mathbf{X}$ and t. The incremental deformation (displacement) is denoted $\mathbf{u}$ and defined by $\mathbf{u}(\mathbf{x}, t) = \dot{\boldsymbol{\chi}}(\boldsymbol{\chi}^{-1}(\mathbf{x}), t)$, and all other updated incremental quantities are identified by a zero subscript. In particular, we have $\dot{\mathbf{F}}_0 = \dot{\mathbf{F}}\mathbf{F}^{-1} = \operatorname{grad}\mathbf{u}$ and $\dot{\mathbf{S}}_0 = \mathbf{F}\dot{\mathbf{S}}$, where grad is the gradient operator in $\mathcal{B}$, while $\boldsymbol{\mathcal{A}}_0$ denotes the updated form of $\boldsymbol{\mathcal{A}}$. In component form we have the connection $\mathcal{A}_{0piqj} = F_{p\alpha}F_{q\beta}\mathcal{A}_{\alpha i\beta j}$ (Ogden, 1984).

The updated forms of the incremental equation of motion and incompressibility condition are then, in component form,

$$\mathcal{A}_{0piqj}u_{j,pq} - \dot{p}_{,i} = \rho u_{i,tt}, \quad u_{p,p} = 0, \tag{22}$$

in which the notations $u_{i,j} = \partial u_i/\partial x_j$, $u_{i,jk} = \partial^2 u_i/\partial x_j \partial x_k$ have been adopted.

The updated form of equation (21) yields

$$\mathcal{A}_{0ijkl} + \delta_{il}(\sigma_{jk} + p\delta_{jk}) = \mathcal{A}_{0jikl} + \delta_{jl}(\sigma_{ik} + p\delta_{ik}), \tag{23}$$

as given in Shams et al. (2011).

At this point we record the *strong ellipticity condition* on the coefficients $\mathcal{A}_{0piqj}$, which states that

$$\mathcal{A}_{0piqj}n_p n_q m_i m_j > 0 \tag{24}$$

for all non-zero $\mathbf{m}, \mathbf{n}$ such that $\mathbf{m} \cdot \mathbf{n} = 0$ (this orthogonality follows from incompressibility), m_i and n_i, $i = 1, 2, 3$, being the components of $\mathbf{m}$ and $\mathbf{n}$, respectively. In terms of the acoustic tensor $\mathbf{Q}(\mathbf{n})$ defined in component form by $Q_{ij} = \mathcal{A}_{0piqj}n_p n_q$, strong ellipticity ensures that $[\mathbf{Q}(\mathbf{n})\mathbf{m}] \cdot \mathbf{m} > 0$ subject to the stated restrictions on $\mathbf{m}$ and $\mathbf{n}$.

The updated elasticity tensor can be expanded in its component form as

$$\mathcal{A}_{0piqj} = \sum_{r \in \mathcal{I}} W_r F_{p\alpha}F_{q\beta}\frac{\partial^2 I_r}{\partial F_{i\alpha}\partial F_{j\beta}} + \sum_{r,s \in \mathcal{I}} W_{rs}F_{p\alpha}F_{q\beta}\frac{\partial I_r}{\partial F_{i\alpha}}\frac{\partial I_s}{\partial F_{j\beta}}, \tag{25}$$

where $W_{rs} = \partial^2 W/\partial I_r \partial I_s$ and $\mathcal{I}$ is the index set $\{1, 2, 5, 6, 7, 8\}$. Expressions for the derivatives of the invariants which appear in (25) and the resulting lengthy expression for $\mathcal{A}_{0piqj}$ are given in Shams et al. (2011) and are not repeated here. We need only their plane strain specializations, which will be provided in the following.

3.1 Plane strain case

Considerable simplification arises in the plane strain specialization considered in Section 2.4, for then equation (25) applies with the reduced index set $\mathcal{I} = \{1, 5\}$. Then the only derivatives of the invariants required are simply

$$\frac{\partial I_1}{\partial F_{i\alpha}} = 2F_{i\alpha}, \quad \frac{\partial I_5}{\partial F_{i\alpha}} = 2\tau_{\alpha\beta}F_{i\beta}, \quad \frac{\partial^2 I_1}{\partial F_{i\alpha}\partial F_{j\beta}} = 2\delta_{\alpha\beta}\delta_{ij}, \quad \frac{\partial^2 I_5}{\partial F_{i\alpha}\partial F_{j\beta}} = 2\tau_{\alpha\beta}\delta_{ij}. \tag{26}$$

From (25) we then obtain

$$\begin{aligned}\mathcal{A}_{0piqj} = 2\hat{W}_1 B_{pq}\delta_{ij} + 2\hat{W}_5\Sigma_{pq}\delta_{ij} + 4\hat{W}_{11}B_{pi}B_{qj} \\ + 4\hat{W}_{15}(B_{pi}\Sigma_{qj} + B_{qj}\Sigma_{pi}) + 4\hat{W}_{55}\Sigma_{pi}\Sigma_{qj},\end{aligned} \tag{27}$$

with p, i, q, j taking values 1 and 2.

When specialized to the reference configuration $\mathcal{A}_{0piqj}$ is denoted $\mathcal{C}_{piqj}$, which is given by

$$\mathcal{C}_{piqj} = 2\hat{W}_1\delta_{pq}\delta_{ij} + \tau_{pq}\delta_{ij} + 4\hat{W}_{11}\delta_{pi}\delta_{qj} + 4\hat{W}_{15}(\delta_{pi}\tau_{qj} + \tau_{pi}\delta_{qj}) + 4\hat{W}_{55}\tau_{pi}\tau_{qj}, \tag{28}$$

wherein $\hat{W}_1$, $\hat{W}_{11}$, $\hat{W}_{15}$ and $\hat{W}_{55}$ are evaluated for $I_1 = 3$ and $I_5 = \tau_{11} + \tau_{22}$ and we have used $(15)_2$.

4 Plane incremental motions

We now illustrate the general theory by specializing the underlying configuration to one consisting of a pure homogeneous strain and focus attention on incremental motions in the (x_1, x_2) principal plane, so that the incremental displacement $\mathbf{u}$ has components

$$u_1(x_1, x_2, t), \quad u_2(x_1, x_2, t), \quad u_3 = 0. \tag{29}$$

We also take the initial stress to be uniform and confined to the (x_1, x_2) plane, so that $\tau_{i3} = 0$, $i = 1, 2, 3$. Moreover, the incremental incompressibility condition $(22)_2$ allows the components u_1 and u_2 to be expressed in the form

$$u_1 = \psi_{,2}, \quad u_2 = -\psi_{,1}, \tag{30}$$

where $\psi = \psi(x_1, x_2, t)$ is a scalar function. Elimination of $\dot{p}$ from the two resulting non-trivial components of the incremental equation of motion $(22)_1$, as detailed in Shams and Ogden (2014), leads to an equation for ψ, namely

$$\alpha\psi_{,1111} + 2\delta\psi_{,1112} + 2\beta\psi_{,1122} + 2\varepsilon\psi_{,1222} + \gamma\psi_{,2222} = \rho(\psi_{,11tt} + \psi_{,22tt}), \tag{31}$$

in which the (constant) coefficients are defined by

$$\begin{aligned} &\alpha = \mathcal{A}_{01212}, \quad 2\beta = \mathcal{A}_{01111} + \mathcal{A}_{02222} - 2\mathcal{A}_{01122} - 2\mathcal{A}_{02112}, \quad \gamma = \mathcal{A}_{02121}, \\ &\delta = \mathcal{A}_{01222} - \mathcal{A}_{01112}, \quad \varepsilon = \mathcal{A}_{01121} - \mathcal{A}_{02122}. \end{aligned} \tag{32}$$

Given that $\tau_{i3} = 0$, $i = 1, 2, 3$, we now assume additionally that $\tau_{12} = 0$. It follows that $\Sigma_{12} = 0$ and $\delta = \varepsilon = 0$, and from (27) that the coefficients α, β and γ are given by

$$\alpha = 2\hat{W}_1\lambda_1^2 + 2\hat{W}_5\Sigma_{11}, \quad \gamma = 2\hat{W}_1\lambda_2^2 + 2\hat{W}_5\Sigma_{22}, \tag{33}$$

$$2\beta = \alpha + \gamma + 4\hat{W}_{11}(\lambda_1^2 - \lambda_2^2)^2 + 8\hat{W}_{15}(\lambda_1^2 - \lambda_2^2)(\Sigma_{11} - \Sigma_{22}) + 4\hat{W}_{55}(\Sigma_{11} - \Sigma_{22})^2. \tag{34}$$

In the reference configuration these reduce to

$$\alpha = 2\hat{W}_1 + 2\hat{W}_5\tau_{11}, \quad \gamma = 2\hat{W}_1 + 2\hat{W}_5\tau_{22}, \quad \beta = 2\hat{W}_1 + \hat{W}_5(\tau_{11} + \tau_{22}) + 2\hat{W}_{55}(\tau_{11} - \tau_{22})^2 \tag{35}$$

with $2\hat{W}_5 = 1$.

For the considered plane strain the strong ellipticity condition (24) specializes to

$$\alpha n_1^4 + 2\beta n_1^2 n_2^2 + \gamma n_2^4 > 0, \quad (n_1, n_2) \neq (0,0), \tag{36}$$

where $\mathbf{m} = (-n_2, n_1, 0)$, $\mathbf{n} = (n_1, n_2, 0)$. With different values of α, β and γ necessary and sufficient conditions for (36) to hold were given by Dowaikh and Ogden (1990) as

$$\alpha > 0, \quad \gamma > 0, \quad \beta > -\sqrt{\alpha\gamma}. \tag{37}$$

5 Surface waves in a layered half-space

In this section we consider Rayleigh-type elastic surface waves guided by a layer bonded to the surface of a half-space, the layer being of a different material than that of the half-space. Let us consider an initially stressed half-space that is subjected to a pure homogeneous strain with principal stretches $\lambda_1, \lambda_2, \lambda_3$ so that the deformed half-space is defined by $x_2 < 0$ with boundary $x_2 = 0$ and we focus attention on the (x_1, x_2) principal plane. The initial stress is also taken to be uniform, and we have already assumed that $\tau_{ij} = 0$, $i \neq j$. The layer has uniform thickness h in the deformed configuration and is defined by $0 \leq x_2 \leq h$. The (planar) invariants for the material of half space are I_1, I_5, while the notations I_1^* and I_5^* are used for the layer. The (plane strain) elasticity tensor for the half-space is given by (27) and the corresponding elasticity tensor for the layer has a similar form but with $\hat{W}$, $\mathbf{B}$ and $\mathbf{\Sigma}$ replaced by $\hat{W}^*$, $\mathbf{B}^*$ and $\mathbf{\Sigma}^*$.

On specializing equation (14) we then obtain the only non-zero Cauchy stress components as

$$\sigma_{ii} = 2\hat{W}_1\lambda_i^2 + 2\hat{W}_5\lambda_i^2\tau_{ii} - p, \quad i = 1,2,3 \text{ (no summation)} \tag{38}$$

for the half-space, and similarly for the layer:

$$\sigma_{ii}^* = 2\hat{W}_1^*\lambda_i^{*2} + 2\hat{W}_5^*\lambda_i^{*2}\tau_{ii}^* - p^*, \quad i = 1,2,3 \text{ (no summation)}. \tag{39}$$

Now consider plane incremental motions within the half-space and layer with incremental displacements $\mathbf{u}$ and $\mathbf{u}^*$, respectively, having components

$$u_1(x_1, x_2, t), \quad u_2(x_1, x_2, t), \quad u_1^*(x_1, x_2, t), \quad u_2^*(x_1, x_2, t), \tag{40}$$

with $u_3 = u_3^* = 0$.

We assume that the boundary $x_2 = h$ of the layer is free of incremental traction, so that

$$\dot{S}_{021}^* = 0, \quad \dot{S}_{022}^* = 0 \quad \text{on} \quad x_2 = h. \tag{41}$$

We also consider both the displacement and the incremental traction to be continuous at the interface $x_2 = 0$, so that

$$u_1 = u_1^*, \quad u_2 = u_2^*, \quad \dot{S}_{021}^* = \dot{S}_{021}, \quad \dot{S}_{022}^* = \dot{S}_{022} \quad \text{on} \quad x_2 = 0. \tag{42}$$

The non-trivial components of the incremental traction per unit area of the surface and layer, respectively, are $\dot{S}_{02i}$ and $\dot{S}_{02i}^*$, $i = 1, 2$, which are given by

$$\dot{S}_{02i} = \mathcal{A}_{02ilk} u_{k,l} + p u_{2,i} - \dot{p}\delta_{2i}, \quad i = 1, 2, \tag{43}$$

$$\dot{S}_{02i}^* = \mathcal{A}_{02ilk}^* u_{k,l}^* + p^* u_{2,i}^* - \dot{p}^*\delta_{2i}, \quad i = 1, 2. \tag{44}$$

By differentiating (43) and (44) for $i = 2$ with respect to x_1 and eliminating $\dot{p}_{,1}$ using the first component of the equation of motion (as in Shams and Ogden, 2014 for the half-space problem), and similarly for $\dot{p}_{,1}^*$, the incremental traction continuity conditions $(42)_{3,4}$ are expressed in terms of ψ and its counterpart ψ^* for the layer as

$$(\sigma_{22} - \gamma)\psi_{,11} + \gamma\psi_{,22} = (\sigma_{22}^* - \gamma^*)\psi_{,11}^* + \gamma^*\psi_{,22}^*, \tag{45}$$

$$\rho\psi_{,2tt} - (2\beta + \gamma - \sigma_{22})\psi_{,112} - \gamma\psi_{,222} = \rho^*\psi_{,2tt}^* - (2\beta^* + \gamma^* - \sigma_{22}^*)\psi_{,112}^* - \gamma^*\psi_{,222}^*, \tag{46}$$

on $x_2 = 0$, the latter corresponding to $\dot{S}_{022,1} = \dot{S}_{022,1}^*$. Note that by continuity of the underlying configuration $\sigma_{22}^* = \sigma_{22}$. The zero incremental traction boundary conditions on $x_2 = h$ are

$$\dot{S}_{021}^* = (\sigma_{22}^* - \gamma^*)\psi_{,11}^* + \gamma^*\psi_{,22}^* = 0, \tag{47}$$

$$\dot{S}_{022,1}^* = \rho^*\psi_{,2tt}^* - (2\beta^* + \gamma^* - \sigma_{22}^*)\psi_{,112}^* - \gamma^*\psi_{,222}^* = 0. \tag{48}$$

Here we have used the connection

$$\mathcal{A}_{0ijij} - \mathcal{A}_{0ijji} = \sigma_{ii} + p, \quad i \neq j, \tag{49}$$

which can be obtained from (23), and the corresponding one for the layer.

We now specialize the initial stress so that it has just one non-zero component, namely τ_{11}, τ^*_{11}, in the half-space and layer, respectively. We also assume that there is no traction on the boundary $x_2 = 0$ associated with the underlying configuration, so that $\sigma_{22} = 0$ and $\sigma^*_{22} = 0$.

We consider surface waves propagating along the x_1 axis, which forms with x_2 a pair of principal axes of the underlying deformation so that the displacement components are given by (40). We take the surface wave to have the form

$$\psi = A \exp[skx_2 - \mathrm{i}k(x_1 - ct)], \quad \psi^* = A^* \exp[s^*kx_2 - \mathrm{i}k(x_1 - ct)], \tag{50}$$

in the half-space and layer, respectively, where A, A^* are constants, k is the wave number, c is the wave speed, and s, s^* are to be determined. Using equation (50) in the equation of motion (31), we obtain

$$\gamma s^4 - (2\beta - \rho c^2)s^2 + (\alpha - \rho c^2) = 0 \tag{51}$$

and

$$\gamma^* s^{*4} - (2\beta^* - \rho^* c^2)s^{*2} + (\alpha^* - \rho^* c^2) = 0 \tag{52}$$

for the half-space and layer, respectively.

For the half-space the solutions have to decay as $x_2 \to -\infty$, which requires that the relevant solutions of (51) for s should have positive real parts. Let s_1 and s_2 be those solutions. Since $-s^*$ is a solution of (52) whenever s^* is, let s^*_1, s^*_2, $-s^*_1$ and $-s^*_2$ denote the roots. The general solutions for ψ and ψ^* of the considered type may then be written in the form

$$\psi = (A_1 \mathrm{e}^{s_1 kx_2} + A_2 \mathrm{e}^{s_2 kx_2}) \exp[\mathrm{i}k(ct - x_1)], \tag{53}$$

and

$$\psi^* = (A^*_1 \mathrm{e}^{s^*_1 kx_2} + A^*_2 \mathrm{e}^{s^*_2 kx_2} + A^*_3 \mathrm{e}^{-s^*_1 kx_2} + A^*_4 \mathrm{e}^{-s^*_2 kx_2}) \exp[\mathrm{i}k(ct - x_1)], \tag{54}$$

where A_i, $i = 1, 2$, and A^*_i, $i = 1, ..., 4$, are constants.

Following the arguments in Dowaikh and Ogden (1990) and Ogden and Sotiropoulos (1995) we may deduce that there is an upper bound on the wave speed according to

$$0 \le \rho c^2 \le \rho c_{\mathrm{L}}^2 = \begin{cases} \alpha & \text{if} \quad 2\beta \ge \alpha \\ 2\beta - 2\gamma + 2\sqrt{\gamma}\sqrt{\alpha + \gamma - 2\beta} & \text{if} \quad 2\beta \le \alpha, \end{cases} \tag{55}$$

where c_{L} (> 0) is the *limiting speed*. The limiting value c_{L} for the case when $2\beta \ge \alpha$ is the speed of a plane shear wave propagating in the x_1-direction with displacement in the x_2-direction in an unbounded body subjected to the same homogeneous pure strain and initial stress. It does not correspond to a surface wave, and it is straightforward to show that $2\beta - 2\gamma + 2\sqrt{\gamma}\sqrt{\alpha + \gamma - 2\beta} \le \alpha$, with equality if $2\beta = \alpha$.

At this point it is convenient to define the notation

$$\eta = [(\alpha - \rho c^2)/\gamma]^{1/2}, \quad \eta^2 = s_1^2 s_2^2 = (\alpha - \rho c^2)/\gamma. \tag{56}$$

In order to qualify as a surface wave in the half-space $s_1^2 s_2^2$ must be positive and hence, without loss of generality, we may take $\eta = s_1 s_2 > 0$, so that s_1 and s_2 must either both be real and positive or be complex conjugates. If they are real

$$(s_1 + s_2)^2 = \eta^2 + 2\eta + 2\bar{\beta} - \bar{\alpha} > 0, \quad (s_1 - s_2)^2 = \eta^2 - 2\eta + 2\bar{\beta} - \bar{\alpha} > 0, \tag{57}$$

while if they are complex conjugates

$$(s_1 + s_2)^2 = \eta^2 + 2\eta + 2\bar{\beta} - \bar{\alpha} > 0, \quad (s_1 - s_2)^2 = \eta^2 - 2\eta + 2\bar{\beta} - \bar{\alpha} < 0, \tag{58}$$

within which we have defined the notation

$$\bar{\alpha} = \alpha/\gamma, \quad \bar{\beta} = \beta/\gamma. \tag{59}$$

The counterpart of (55) in respect of η is then

$$\sqrt{\bar{\alpha}} \ge \eta \ge \eta_{\mathrm{L}} = \begin{cases} 0 \quad \text{if} \quad 2\beta \ge \alpha \\ (1 + \bar{\alpha} - 2\bar{\beta})^{1/2} - 1 \quad \text{if} \quad 2\beta \le \alpha, \end{cases} \tag{60}$$

wherein η_{L}, the lower limiting value of η, is defined.

Similarly, we define

$$\eta^* = [(\alpha^* - \rho^* c^2)/\gamma^*]^{1/2}, \quad \eta^{*2} = s_1^{*2} s_2^{*2} = (\alpha^* - \rho^* c^2)/\gamma^*, \tag{61}$$

but we note that in contrast to η^2, η^{*2} may be either positive or negative, and hence η^* may be real or pure imaginary. We now consider these two possibilities separately, with the notation

$$\bar{\alpha}^* = \alpha^*/\gamma^*, \quad \bar{\beta}^* = \beta^*/\gamma^*. \tag{62}$$

Case (a): $\eta^{*2} < 0$.

In this case s_1^{*2} and s_2^{*2} cannot be complex conjugates and so must be real and have opposite signs. If

$$s_1^{*2} + s_2^{*2} < 0 \quad \text{then} \quad \rho^* c^2/\gamma^* > \max\{\bar{\alpha}^*, 2\bar{\beta}^*\}, \tag{63}$$

while if

$$s_1^{*2} + s_2^{*2} > 0 \quad \text{then} \quad \bar{\alpha}^* < \rho^* c^2/\gamma^* < 2\bar{\beta}^*. \tag{64}$$

On setting $s_1^* s_2^* = \eta^*$ we also note that

$$(s_1^* + s_2^*)^2 = \eta^{*2} + 2\eta^* + 2\bar{\beta}^* - \bar{\alpha}^* \quad \text{and} \quad (s_1^* - s_2^*)^2 = \eta^{*2} - 2\eta^* + 2\bar{\beta}^* - \bar{\alpha}^* \tag{65}$$

are complex conjugates.

Case (b): $\eta^{*2} > 0$.

Without loss of generality we take $s_1^* s_2^* = \eta^* > 0$. Let us first consider the situation in which s_1^{*2} and s_2^{*2} are real. Then, we have either

$$s_1^{*2} > 0 \text{ and } s_2^{*2} > 0 \quad \text{with} \quad \rho^* c^2/\gamma^* < \min\{\bar{\alpha}^*, 2\bar{\beta}^*\} \tag{66}$$

and

$$\eta^{*2} \pm 2\eta^* + 2\bar{\beta}^* - \bar{\alpha}^* > 0, \tag{67}$$

or

$$s_1^{*2} < 0 \text{ and } s_2^{*2} < 0 \quad \text{with} \quad 2\bar{\beta}^* < \rho^* c^2/\gamma^* < \bar{\alpha}^* \tag{68}$$

and

$$\eta^{*2} \pm 2\eta^* + 2\bar{\beta}^* - \bar{\alpha}^* < 0. \tag{69}$$

On the other hand, if s_1^{*2} and s_2^{*2} are complex conjugates

$$\eta^{*2} + 2\eta^* + 2\bar{\beta}^* - \bar{\alpha}^* > 0 > \eta^{*2} - 2\eta^* + 2\bar{\beta}^* - \bar{\alpha}^*, \tag{70}$$

with $2\bar{\beta}^* - \rho^* c^2/\gamma^*$ having either sign:

$$2\bar{\beta}^* < \rho^* c^2/\gamma^* < \bar{\alpha}^* \quad \text{or} \quad \rho^* c^2/\gamma^* < \min\{\bar{\alpha}^*, 2\bar{\beta}^*\}. \tag{71}$$

Each of the possibilities in Case (a) and Case (b) arises within the numerical examples illustrated in the following section.

Substituting (53) and (54) into the boundary conditions (41) and (42), expressed through (45)–(48), with use of (51) and (52) we obtain

$$(A_1^* \mathrm{e}^{s_1^* kh} + A_3^* \mathrm{e}^{-s_1^* kh})(1 + s_1^{*2}) + (A_2^* \mathrm{e}^{s_2^* kh} + A_4^* \mathrm{e}^{-s_2^* kh})(1 + s_2^{*2}) = 0, \tag{72}$$

$$(A_1^* \mathrm{e}^{s_1^* kh} - A_3^* \mathrm{e}^{-s_1^* kh}) s_1^* (1 + s_2^{*2}) + (A_2^* \mathrm{e}^{s_2^* kh} - A_4^* \mathrm{e}^{-s_2^* kh}) s_2^* (1 + s_1^{*2}) = 0, \tag{73}$$

$$A_1 + A_2 - A_1^* - A_2^* - A_3^* - A_4^* = 0, \tag{74}$$

$$A_1 s_1 + A_2 s_2 - (A_1^* - A_3^*) s_1^* - (A_2^* - A_4^*) s_2^* = 0, \tag{75}$$

$$[A_1(s_1^2 + 1) + A_2(s_2^2 + 1)]\gamma - [(A_1^* + A_3^*)(s_1^{*2} + 1) + (A_2^* + A_4^*)(s_2^{*2} + 1)]\gamma^* = 0, \tag{76}$$

$$[A_1 s_1 (1 + s_2^2) + A_2 s_2 (1 + s_1^2)]\gamma - [(A_1^* - A_3^*) s_1^* (1 + s_2^{*2}) - (A_2^* - A_4^*) s_2^* (1 + s_1^{*2})]\gamma^* = 0. \tag{77}$$

The set of equations (72)–(77) can be written in the matrix form

$$\boldsymbol{\mathcal{M}} \mathbf{A} = \mathbf{0}, \tag{78}$$

where $\mathbf{A} = (A_1, A_2, A_1^*, A_2^*, A_3^*, A_4^*)$ and $\boldsymbol{\mathcal{M}}$ is the 6×6 matrix

$$\boldsymbol{\mathcal{M}} = \begin{bmatrix} 1 & 1 & -1 & -1 & -1 & -1 \\ s_1 & s_2 & -s_1^* & -s_2^* & s_1^* & s_2^* \\ \gamma s_1 S_2 & \gamma s_2 S_1 & -\gamma^* s_1^* S_2^* & -\gamma^* s_2^* S_1^* & \gamma^* s_1^* S_2^* & \gamma^* s_2^* S_1^* \\ \gamma S_1 & \gamma S_2 & -\gamma^* S_1^* & -\gamma^* S_2^* & -\gamma^* S_1^* & -\gamma^* S_2^* \\ 0 & 0 & S_1^* e^{s_1^* kh} & S_2^* e^{s_2^* kh} & S_1^* e^{-s_1^* kh} & S_2^* e^{-s_2^* kh} \\ 0 & 0 & s_1^* S_2^* e^{s_1^* kh} & s_2^* S_1^* e^{s_2^* kh} & -s_1^* S_2^* e^{-s_1^* kh} & -s_2^* S_1^* e^{-s_2^* kh} \end{bmatrix} \tag{79}$$

within which we have used the notation $S_i = 1 + s_i^2, S_i^* = 1 + s_i^{*2}$, $i = 1, 2$.

For a nontrivial solution, the determinant of $\boldsymbol{\mathcal{M}}$ must vanish. After considerable manipulation it can be shown that $\det \boldsymbol{\mathcal{M}}$ can be written

$$\det \boldsymbol{\mathcal{M}} = -2(s_1 - s_2)(s_1^* - s_2^*)^2(s_1^* + s_2^*)^2\mathcal{N},$$

where $\mathcal{N}$ has the form

$$\begin{aligned}\mathcal{N} = {} & A(\eta,\eta^*)\frac{\sinh^2[\frac{1}{2}kh(s_1^* + s_2^*)]}{(s_1^* + s_2^*)^2} - A(\eta,-\eta^*)\frac{\sinh^2[\frac{1}{2}kh(s_1^* - s_2^*)]}{(s_1^* - s_2^*)^2} \\ & + B(\eta,\eta^*)\frac{\sinh[kh(s_1^* + s_2^*)]}{(s_1^* + s_2^*)} - B(\eta,-\eta^*)\frac{\sinh[kh(s_1^* - s_2^*)]}{(s_1^* - s_2^*)} + C(\eta,\eta^*), \qquad (80)\end{aligned}$$

in which, following Ogden and Sotiropoulos (1995), we have introduced the definitions

$$A(\eta,\eta^*) = 2f^*(\eta^*)[\gamma^2 f(\eta) + \gamma^{*2} f^*(\eta^*) + 2\gamma\gamma^*(\eta - 1)(\eta^* - 1)], \qquad (81)$$

$$B(\eta,\eta^*) = f^*(\eta^*)\gamma\gamma^*(\eta + \eta^*)\eta^{-1/2}[f(\eta) + (\eta - 1)^2]^{1/2}, \qquad (82)$$

$$C(\eta,\eta^*) = 2\gamma^2\eta^* f(\eta), \qquad (83)$$

with

$$f(\eta) = \eta^3 + \eta^2 + d\eta - 1, \quad f(\eta^*) = \eta^{*3} + \eta^{*2} + d^*\eta^* - 1. \qquad (84)$$

We have also defined

$$d = 2\bar{\beta} + 2 - \bar{\alpha}, \quad d^* = 2\bar{\beta}^* + 2 - \bar{\alpha}^*. \qquad (85)$$

Note that

$$(s_1 + s_2)^2 = \eta^2 + 2\eta + 2\bar{\beta} - \bar{\alpha} = \eta^{-1}[f(\eta) + (\eta - 1)^2] > 0$$

and

$$(s_1^* + s_2^*)^2 = \eta^{*2} + 2\eta^* + 2\bar{\beta}^* - \bar{\alpha}^* = \eta^{*-1}[f^*(\eta^*) + (\eta^* - 1)^2], \qquad (86)$$

the latter being complex in Case (a).

It is straightforward to show that $\mathcal{N}$ is real if $\eta^{*2} > 0$ and pure imaginary if $\eta^{*2} < 0$. The formula (80) is the same as one derived in Ogden and Sotiropoulos (1995) [equation (3.6) therein] apart from slight differences in notation. However, the content is different since the values of the material parameters α, β, γ and their starred counterparts are different. We focus first on some special cases of $\mathcal{N} = 0$ and then consider briefly the other factors in the expression for $\det \boldsymbol{\mathcal{M}}$.

First, the limiting case $kh \to 0$ corresponds to a half-space for which the secular equation

$$f(\eta) = \eta^3 + \eta^2 + d\eta - 1 = 0 \tag{87}$$

has been analyzed in detail in Shams and Ogden (2014).

Second, $kh \to \infty$ corresponds to the secular equation for interfacial (Stoneley-type) waves along the boundary between two half-spaces, having the equation

$$\begin{aligned}\gamma^{*2} f^*(\eta^*) &+ \gamma\gamma^*(\eta + \eta^*)\eta^{-1/2}\eta^{*-1/2}[f(\eta) + (\eta - 1)^2]^{1/2}[f^*(\eta^*) + (\eta^* - 1)^2]^{1/2} \\ &+ 2\gamma\gamma^*(\eta - 1)(\eta^* - 1) + \gamma^2 f(\eta) = 0.\end{aligned} \tag{88}$$

For the case in which the initial stress is a pre-stress associated with a finite deformation this equation was derived by Dowaikh and Ogden (1991b), but expressed in different notation.

The third special case corresponds to the absence of the half-space ($\gamma \to 0$), and the dispersion equation reduces to

$$[f^*(\eta^*)]^2 \frac{\sinh^2[\frac{1}{2}kh(s_1^* + s_2^*)]}{(s_1^* + s_2^*)^2} = [f^*(-\eta^*)]^2 \frac{\sinh^2[\frac{1}{2}kh(s_2^* - s_1^*)]}{(s_2^* - s_1^*)^2}. \tag{89}$$

This provides the dispersion equation for Lamb-type waves in a plate with uniform thickness h and of infinite extent in the lateral directions.

The final special case corresponds to $c = 0$ ($\eta = \sqrt{\bar{\alpha}}, \eta^* = \sqrt{\bar{\alpha}^*}$), in which case equation (80) specializes accordingly and provides a criterion for the existence of quasi-static incremental deformations.

Vanishing of either of the other factors $s_1 - s_2$, $s_1^* - s_2^*$ or $s_1^* + s_2^*$ may also lead to solutions of the secular equation, but each such solution that exists is independent of kh and arises as a special case of $\mathcal{N} = 0$ for specific ranges of values of $\bar{\alpha}$, $\bar{\beta}$, $\bar{\alpha}^*$ and $\bar{\beta}^*$.

6 Numerical illustrations

In order to illustrate the solutions of the secular equation $\mathcal{N} = 0$ we now select the simple prototype form of strain-energy function that was used in Shams et al. (2011) and given (in slightly different notation) by

$$W = \frac{1}{2}\mu(I_1 - 3) + \frac{1}{4}\mu_1[I_5 - \mathrm{tr}(\boldsymbol{\tau})]^2 + \frac{1}{2}[I_5 - \mathrm{tr}(\boldsymbol{\tau})], \tag{90}$$

where $\mu > 0$ is a material constant with the dimension of stress and μ_1 is a material constant with dimension of stress^{-1}. We allow μ_1 to be either positive or negative. The first term is the classical neo-Hookean model of rubber elasticity, while the second and third terms introduce the residual stress in a very simple form involving just the invariant I_5 (and its specialization to the reference configuration) and ensuring that the condition $(8)_2$ is satisfied. For the plane strain problem considered in the previous section we write (90) as $\hat{W}(I_1, I_5)$, the underlying deformation corresponding to $\lambda_1 = \lambda, \lambda_2 = \lambda^{-1}$ and $\lambda_3 = 1$. In the layer quantities are distinguished by an asterisk.

As we have already assumed the boundary $x_2 = h$ is free of traction in the underlying configuration. Thus, $\sigma_{22} = 0$ and correspondingly $\tau_{22} = 0$. The Cauchy stress components are then obtained by specializing (14) as

$$\sigma_{11} = \mu\lambda^2 + \lambda^2\tau + \mu_1(\lambda^2 - 1)\lambda^2\tau^2 - p, \quad 0 = \sigma_{22} = \mu\lambda^{-2} - p, \quad \sigma_{33} = \mu - p, \tag{91}$$

where τ_{11} has been written simply as τ. Hence, on elimination of p,

$$\sigma_{11} = \mu(\lambda^2 - \lambda^{-2}) + \lambda^2\tau + \mu_1(\lambda^2 - 1)\lambda^2\tau^2, \quad \sigma_{33} = \mu(1 - \lambda^{-2}), \tag{92}$$

the latter component being required to maintain the plane strain condition. Similarly, for the layer we have

$$\sigma^*_{11} = \mu^*(\lambda^{*2} - \lambda^{*-2}) + \lambda^{*2}\tau^* + \mu^*_1(\lambda^{*2} - 1)\lambda^{*2}\tau^{*2}, \quad \sigma^*_{33} = \mu^*(1 - \lambda^{*-2}). \tag{93}$$

At this point a comment on the effect of the term in μ_1 on the material response in the half-space is called for. In plane strain tension ($\lambda > 1$), for example, positive μ_1 increases the stiffness of the response, while negative μ_1 decreases the stiffness, the material softens on extension and the Cauchy stress reaches a maximum. Similarly for the layer.

For the model (90), the material coefficients are given by

$$\bar{\alpha} = \lambda^4[1 + \bar{\tau} + \bar{\mu}(\lambda^2 - 1)\bar{\tau}^2], \quad d = 2\bar{\beta} - \bar{\alpha} + 2 = 3 + 2\bar{\mu}\lambda^6\bar{\tau}^2, \tag{94}$$

$$\bar{\alpha}^* = \lambda^{*4}[1 + \bar{\tau}^* + \bar{\mu}^*(\lambda^{*2} - 1)\bar{\tau}^{*2}], \quad d^* = 2\bar{\beta}^* - \bar{\alpha}^* + 2 = 3 + 2\bar{\mu}^*\lambda^{*6}\bar{\tau}^{*2}, \tag{95}$$

where we have used $\Sigma_{11} = \lambda^2\tau$, $\Sigma^*_{11} = \lambda^{*2}\tau^*$ and introduced the dimensionless parameters $\bar{\tau} = \tau/\mu$, $\bar{\tau}^* = \tau^*/\mu^*$, $\bar{\mu} = \mu\mu_1$ and $\bar{\mu}^* = \mu^*\mu^*_1$.

We next consider the important special case $kh = 0$ corresponding to a half-space without a layer that was treated by Shams and Ogden (2014). For $kh = 0$ the secular equation reduces to $f(\eta) = 0$, where $f(\eta)$ is given by (87), which is the result obtained in Shams and Ogden (2014). Clearly $f(0) = -1$. Also, it is straightforward to show that $f(\eta_{\mathrm{L}}) < 0$. Hence, the requirement for the existence of a surface wave is $f(\bar{\alpha}^{1/2}) > 0$, and this gives

$$0 \le \eta_{\mathrm{L}} < \eta < \sqrt{\bar{\alpha}} = \lambda^2\sqrt{\epsilon}, \tag{96}$$

and

$$\xi \equiv \lambda^6\epsilon^{3/2} + \lambda^4\epsilon + (3 + 2\bar{\mu}\lambda^6\bar{\tau}^2)\lambda^2\epsilon^{1/2} - 1 > 0, \tag{97}$$

wherein we have defined ξ and introduced the notation

$$\epsilon = 1 + \bar{\tau} + \bar{\mu}(\lambda^2 - 1)\bar{\tau}^2 > 0. \tag{98}$$

Note, with reference to (37), that $f(\bar{\alpha}^{1/2}) > 0$ ensures that strong ellipticity holds since

$$f(\bar{\alpha}^{1/2}) = \bar{\alpha} + 2(\bar{\beta} + 1)\bar{\alpha}^{1/2} - 1 > 0$$

implies

$$(2\bar{\beta} + 2\bar{\alpha}^{1/2})\bar{\alpha}^{1/2} > (\bar{\alpha}^{1/2} - 1)^2 \ge 0.$$

As shown in Shams and Ogden (2014) for a half-space, when a surface wave exists it is unique. For the existence of a surface wave when $kh = 0$ we require both $\epsilon > \lambda^{-4}\eta_{\mathrm{L}}^2$ and $\xi > 0$. If $\bar{\mu} > 0$ then $\eta_{\mathrm{L}} = 0$, but if $\bar{\mu} < 0$ then η_{L} is only zero for certain ranges of values of λ and $\bar{\tau}$, as discussed in Shams and Ogden (2014). [Note that there is a typo in equation (6.26) of Shams and Ogden (2014) (1/8 should be 1/4).]

Examples of the region of $(\lambda, \bar{\tau})$ space for which $\epsilon > \lambda^{-4}\eta_{\mathrm{L}}^2$ and $\xi > 0$ are shown in Fig. 1 for both $\bar{\mu} > 0$ and $\bar{\mu} < 0$. The left-hand column of plots is for $\bar{\mu} > 0$ while the right-hand column is for $\bar{\mu} < 0$, in which case it is necessary to ensure that $\eta_{\mathrm{L}} = \sqrt{1 + \bar{\alpha} - 2\bar{\beta}} - 1 \ge 0$. In each case the region of $(\lambda, \bar{\tau})$ space in which a surface wave exists is marked with the + sign. In the left-hand column, Fig. 1 (a), (c), (e), the curves $\epsilon = 0$ (continuous) and $\xi = 0$ (dashed) are shown, and in Fig. 1 (b), (d), (f) the relevant curves are $\eta_{\mathrm{L}} = 0$ (dashed) and $\xi = 0$ (continuous). In the latter case η_{L} is positive only to the right of the curves $\eta_{\mathrm{L}} = 0$. In (b) ξ is positive between the two upper continuous curves and within the lower loop, while in (d) and (f) it is positive in between the three continuous curves.

We now provide a range of plots based on the solution of $\mathcal{N} = 0$ from equation (80) in respect of the energy function (90) in dimensionless form to obtain $\zeta = \rho c^2/\mu$ as a function of kh. These are based on a representative, but by no means exhaustive, set of values of $\bar{\mu}, \bar{\mu}^*, \bar{\tau}, \bar{\tau}^*, \lambda, \lambda^*$ and the ratios $R = \rho^*\mu/\rho\mu^*$ and $r = \mu^*/\mu$ that illustrate the main features that can arise.

First, in Figs. 2 and 3, for the classical incompressible linearly elastic case with no initial stress, we show how ζ changes with R and r. In Fig. 2 results for $R \leq 1$ are shown. In this case $\eta^{*2} = 1 - R\zeta$ is positive since $\zeta = 1$ is the upper limit for ζ. In each of the subfigures in Figs. 2(a)–(d) each of the curves passes through the classical limiting value $\zeta \approx 0.9126$ when $kh = 0$ (see Dowaikh and Ogden, 1990 for detailed discussion and references to the incompressible classical theory) and there is only one propagation mode. Except for $r < 1$ there is a cut-off value of kh above which waves do not propagate, while for $r < 1$ the wave speed is constant over a wide range of values of kh and tends to the interfacial wave speed between two half-spaces as $kh \to \infty$.

In Figs. 2(e) there are two modes for $r = 0.2$ and for only the first mode $\zeta \approx 0.9126$ when $kh = 0$, and the second mode emerges at a positive value of kh, a mode that has a cut off value of kh for low values of kh. For each of $r = 1$ and $r = 5$ there is only one mode. Finally, in Fig. 2(f), where $R = 1$, there are two modes for each $r \neq 1$, but there is no dependence on kh for $r = 1$ because the half-space and layer materials are then identical, and the result is that for a half-space (non-dispersive). The results for $r = 1$, $r < 1$ and $r > 1$ shown in Fig. 2 correspond to the continuous, thick continuous and dashed curves, respectively. No modes other than those shown appear at larger values of kh, and the general trend is the same for values of r other than those for which results are shown here.

In Fig. 3 corresponding results are illustrated for $R = 1$ and three values of $R > 1$. In this case $\eta^* = 0$ for $\zeta = 1/R$ and the dependence of ζ on kh when $R > 1$ separates into the regions $\zeta < 1/R$ ($\eta^{*2} > 0$) and $\zeta > 1/R$ ($\eta^{*2} < 0$). In each case the lower branch passes through $\zeta \approx 0.9126$ at $kh = 0$ for each value of r but multiple other branches

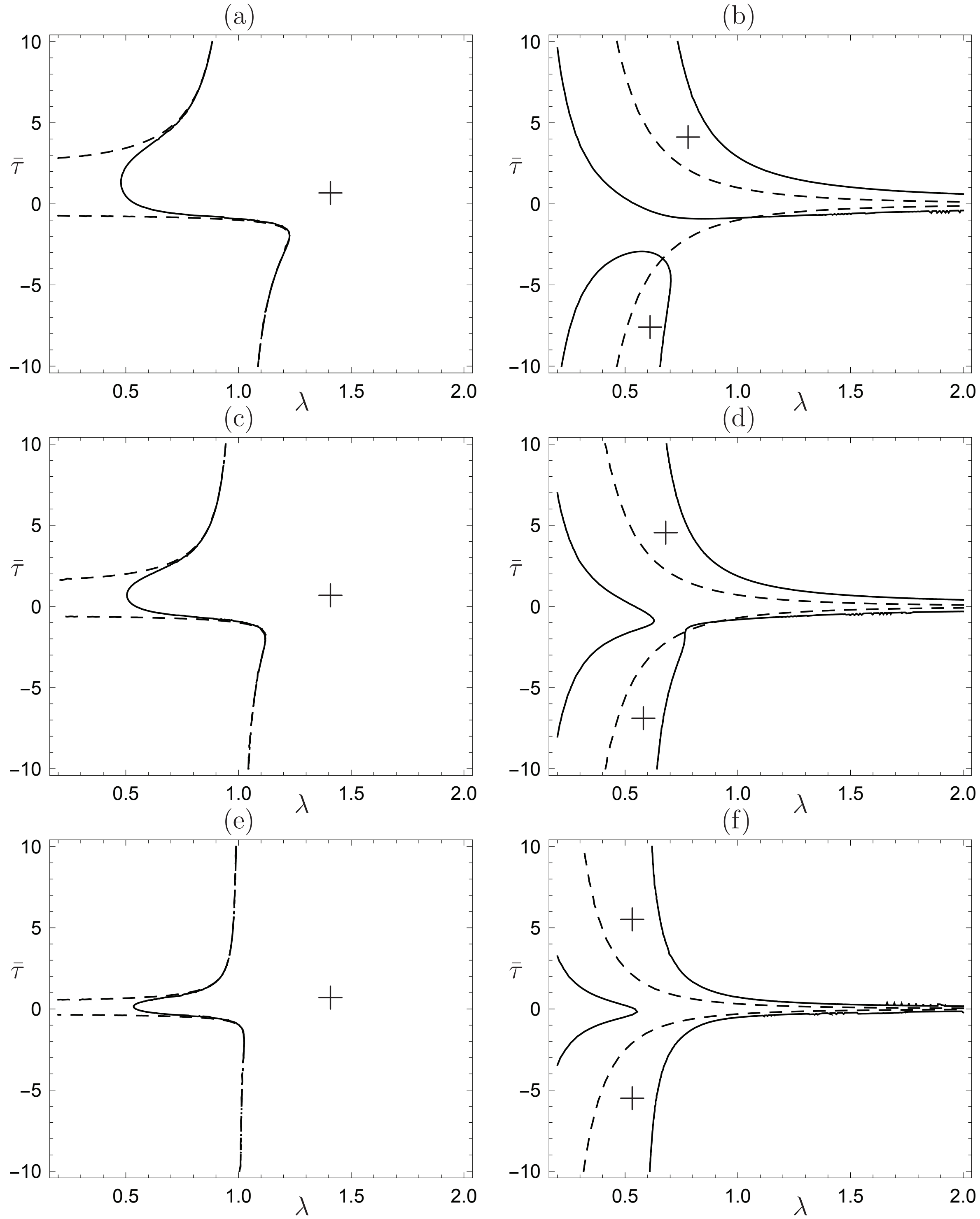

Figure 1: Plots of the curves $\epsilon = 0$ (dashed curves) and $\xi = 0$ (continuous curves) in $(\lambda, \bar{\tau})$ space for $\bar{\mu} =$ (a) 0.5, (c) 1, (e) 5. The + sign indicates the regions of values of λ and $\bar{\tau}$ for which surface waves exist and where $\xi > 0$. Plots of the curves $\eta_{\mathrm{L}} = 0$ (dashed) and $\xi = 0$ (continuous) for $\bar{\mu} =$ (b) -0.5, (d) -1, (f) -5. The + sign indicates the regions of values of λ and $\bar{\tau}$ for which surface waves exist and where $\eta_{\mathrm{L}} > 0$ and $\xi > 0$.

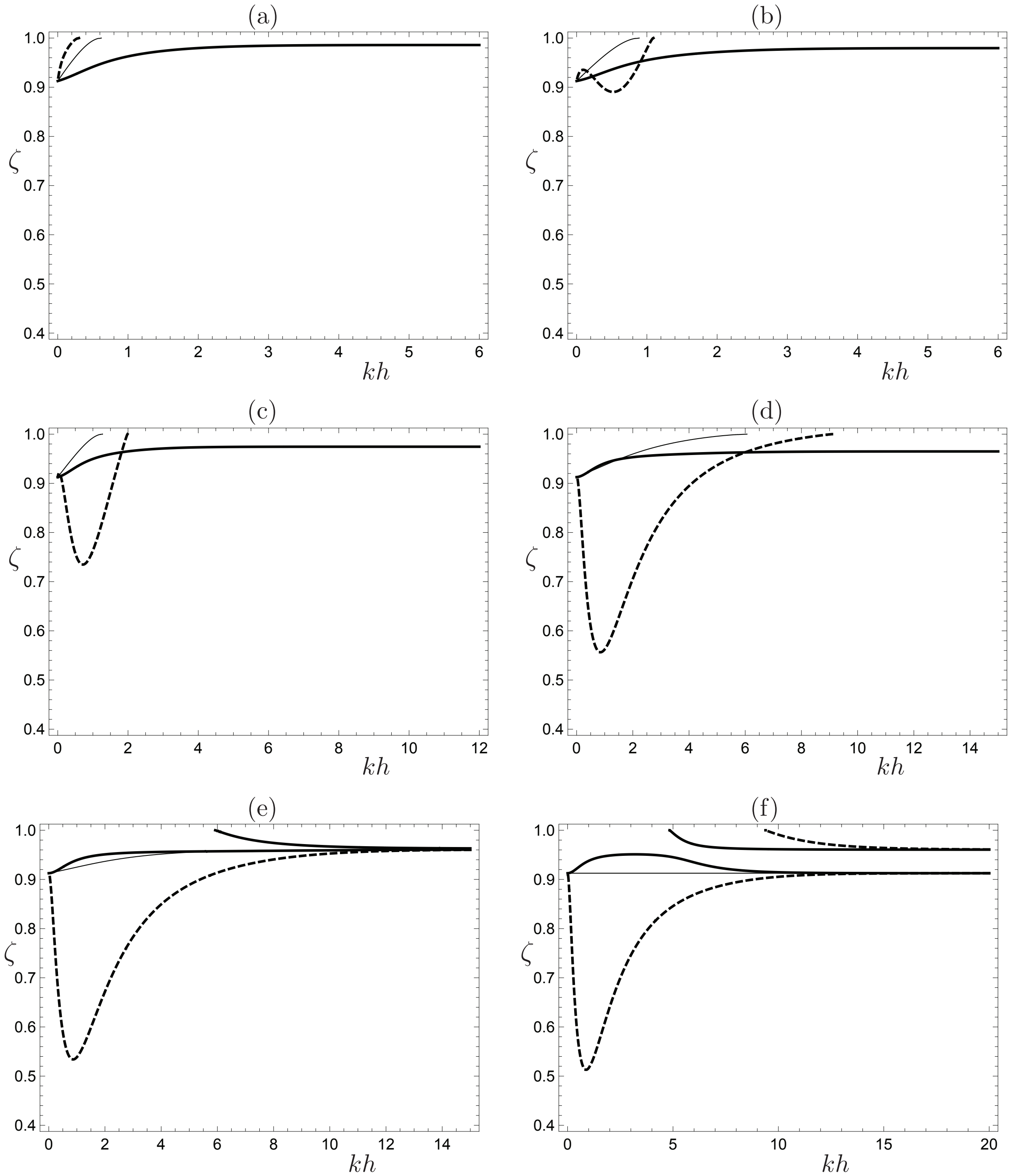

Figure 2: Plots of $\zeta = \rho c^2/\mu$ against kh with $\lambda = \lambda^* = 1, \bar{\tau} = \bar{\tau}^* = 0$ and $r = 0.2$ (thick continuous curves), $r = 1$ (continuous curves), $r = 5$ (dashed curves): (a) $R = 0.1$; (b) $R = 0.4$; (c) $R = 0.6$; (d) $R = 0.9$; (e) $R = 0.95$; (f) $R = 1$.

(modes) emerge at finite values of kh except for $R = 1$ in Fig. 3(a), which is the same as Fig. 2(f). Similar results are found for larger values of R.

Figure 4 serves to confirm that for the considered range of values used the effect of

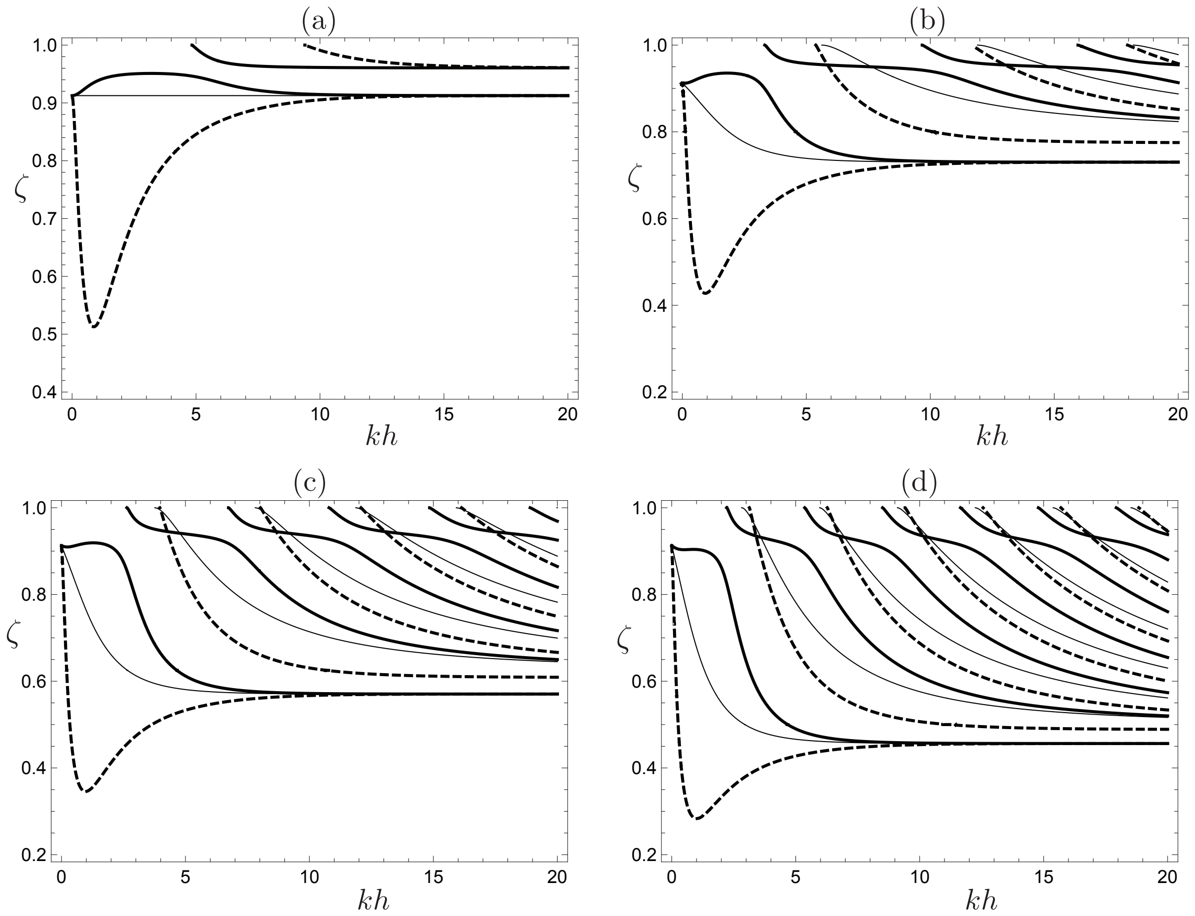

Figure 3: Plots of $\zeta = \rho c^2/\mu$ against kh with $\lambda = \lambda^* = 1, \bar{\tau} = \bar{\tau}^* = 0$ and $r = 0.2$ (thick continuous curves), $r = 1$ (continuous curves), $r = 5$ (dashed curves): (a) $R = 1$; (b) $R = 1.25$; (c) $R = 1.6$; (d) $R = 2$.

an initial stress is very similar to the effect of an initial stretch. Indeed, if the initial stress was calculated from a constitutive law for the same stretch then the effect would be identical. In each panel curves for three values of r, with $R = 1$, $\bar{\mu} = \bar{\mu}^* = 0$ are shown to illustrate the comparative effect of the relative stiffnesses of the layer and half-space. In Fig. 4(a) the stretches in the layer and half-space are set at unity ($\lambda = \lambda^* = 1$), the initial stress $\bar{\tau}^* = 0$ in the layer and that in the half-space negative ($\bar{\tau} = -0.2$), while Fig. 4(c) has $\lambda^* = 1$, $\bar{\tau} = \bar{\tau}^* = 0$ and $\lambda = 0.9$. Thus, the results show that the qualitative features of compressive initial stress and compressive stretch in the half-space are the same. Similarly, by comparing Figs. 4(b) and 4(d) the same applies when the compressive stretch and initial stress are in the layer instead. In particular, when there is a compressive stretch or initial stress in the half-space only one surface wave branch

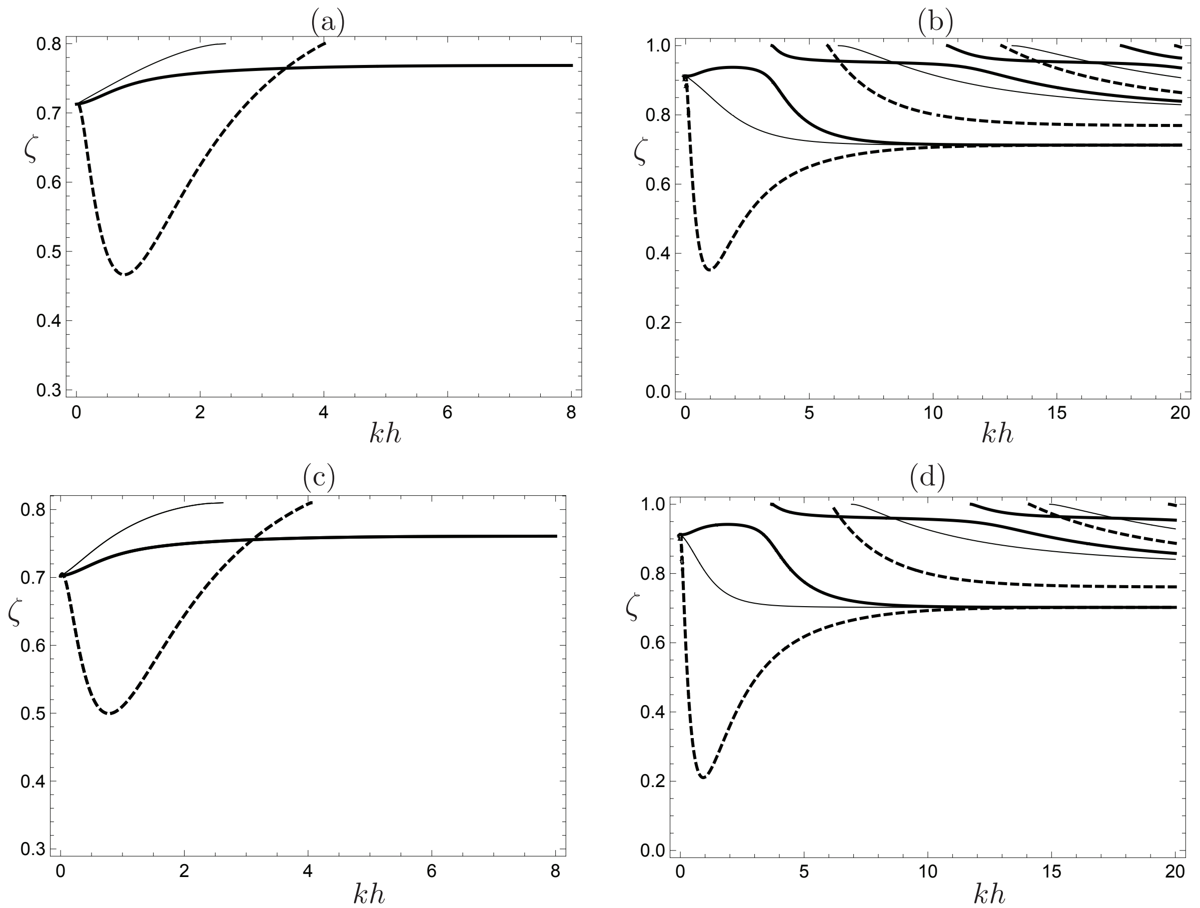

Figure 4: In each panel $\zeta = \rho c^2/\mu$ is plotted against kh for $R = 1$ and $r = 0.2$ (thick continuous curves), $r = 1$ (continuous curves), $r = 5$ (dashed curves): (a) $\lambda = \lambda^* = 1, \bar{\tau} = -0.2, \bar{\tau}^* = 0$; (b) $\lambda = \lambda^* = 1, \bar{\tau} = 0, \bar{\tau}^* = -0.2$; (c) $\lambda = 0.9, \lambda^* = 1, \bar{\tau} = 0, \bar{\tau}^* = 0$; (d) $\lambda = 1, \lambda^* = 0.9, \bar{\tau} = 0, \bar{\tau}^* = 0$.

exists, but if there is a compressive stretch or initial stress in the layer multiple modes are possible.

Figure 5 shows examples of corresponding results for tensile stretches and initial stresses. If these are in the half-space then multiple modes exist while for the layer only one mode arises.

These comparisons are limited to separate consideration of the stretches and the initial stresses, but when both stretches and initial stresses are included independently then the results can be significantly different. First, we note that if the material constants $\bar{\mu}$ and $\bar{\mu}^*$ are positive then their roles in (94) and (95) can be captured by varying $\bar{\tau}^2$ and $\bar{\tau}^{*2}$, respectively. Indeed, a range of plots for $\bar{\mu} = 1$ and $\bar{\mu}^* = 1$ does not reveal qualitative differences from those shown in Figs. 4 and 5, and we therefore focus on negative values of

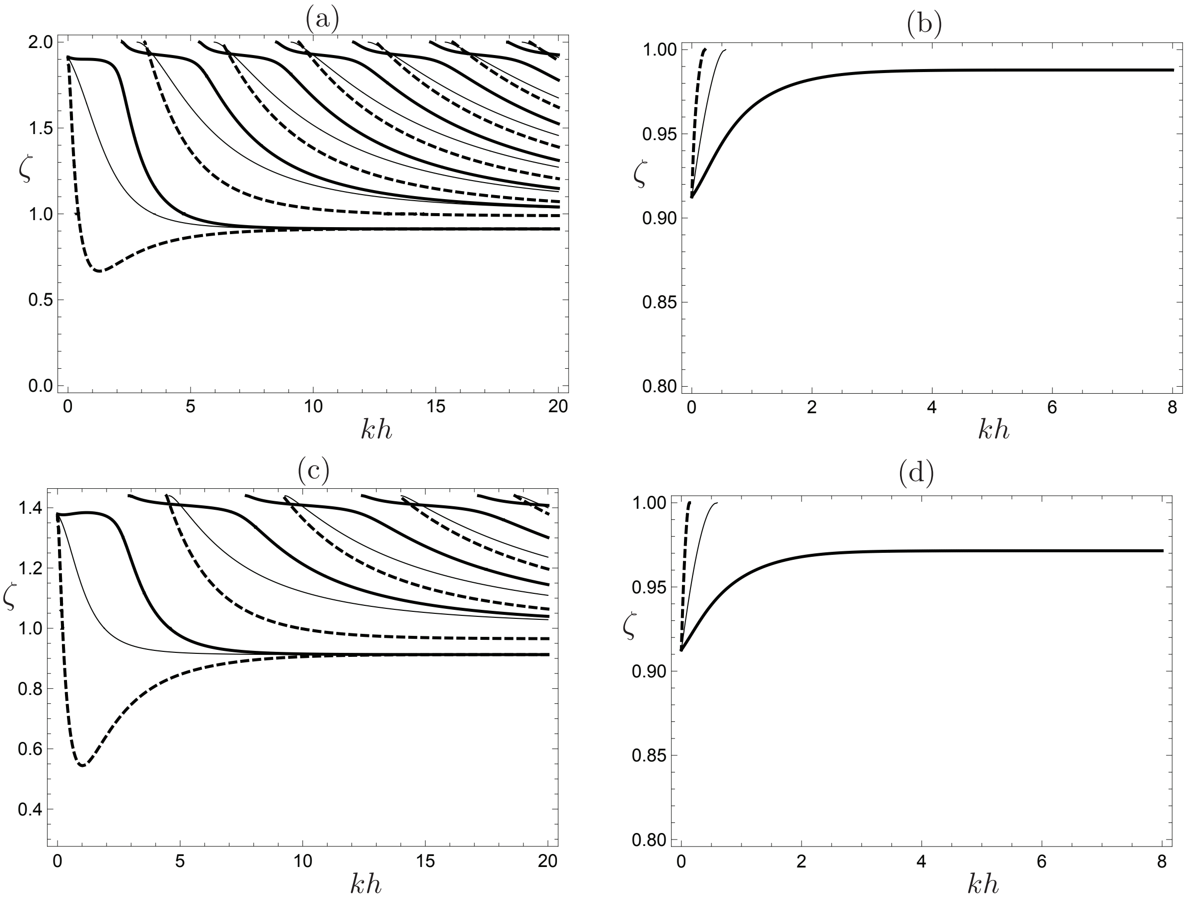

Figure 5: In each panel $\zeta = \rho c^2/\mu$ is plotted against kh for $R = 1$ and $r = 0.2$ (thick continuous curves), $r = 1$ (continuous curves), $r = 5$ (dashed curves):(a) $\lambda = \lambda^* = 1, \bar{\tau} = 1, \bar{\tau}^* = 0$; (b) $\lambda = \lambda^* = 1, \bar{\tau} = 0, \bar{\tau}^* = 1$; (c) $\lambda = 1.2, \lambda^* = 1, \bar{\tau} = 0, \bar{\tau}^* = 0$; (d) $\lambda = 1, \lambda^* = 1.4, \bar{\tau} = 0, \bar{\tau}^* = 0$.

$\bar{\mu}$ and/or $\bar{\mu}^*$, which can have a significant influence. To illustrate the effect of a negative $\bar{\mu}^*$, Fig. 6 shows results for $\bar{\mu}^* = -1.5$ with $\bar{\mu} = 0$, $\bar{\tau} = \bar{\tau}^* = 1$, $\lambda = \lambda^* = 1$ and $R = 1$, with $r = 0.1$ and $r = 10$ in the left and right figures, respectively. In this case multiple modes appear, the first corresponding to the half-space value at $kh = 0$, but they are different from the multiple modes seen in Figs. 3–5. Although, for each r, the secondary modes appear to meet, when seen on a larger scale they are clearly completely separate.

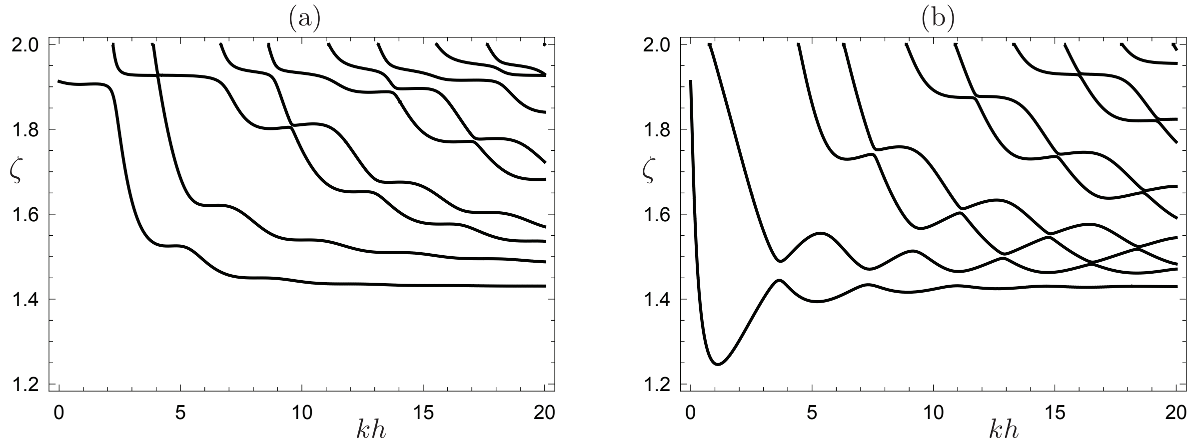

Figure 6: Plots of ζ against kh for $\lambda = \lambda^* = 1$, $R = 1$, $\bar{\mu} = 0$, $\bar{\tau} = 1$, $\bar{\mu}^* = -1.5$, $\bar{\tau}^* = 1$: (a) $r = 0.1$; (b) $r = 10$.

Figure 7 shows two further examples, which are quite different from those in Fig. 6. First, in Fig. 7(a) with $\lambda = 1$, $\lambda^* = 0.8$, $R = 1$, $\bar{\mu} = -1.5$, $\bar{\tau} = -0.5$, $\bar{\mu}^* = 0$, $\bar{\tau}^* = -0.5$, ζ is plotted against kh for three values of r: $0.2, 1, 2$. In this case there are multiple branches for each r, the first passing through the relevant half-space surface wave value at $kh = 0$ and the subsequent ones emerging at a non-zero value of kh. The new feature exemplified here is that as r increases (i.e. the layer becomes stiffer relative to the half-space) ζ becomes zero at two values of kh between which surface waves do not exist. A zero value of ζ is associated with the appearance of a static incremental mode of deformation arising at a point when the underlying configuration of the half-space/layer combination becomes unstable, resulting in surface undulations. The second example is shown in Fig. 7(b) with $\lambda = 1.2$, $\lambda^* = 0.8$, $R = 0.8$, $\bar{\mu} = -1$, $\bar{\tau} = 1$, $\bar{\mu}^* = 0$, $\bar{\tau}^* = -0.5$ and for $r = 0.2, 0.5, 2$ and again ζ is plotted against kh.

This leads, finally, to separate consideration of some situations in which $\zeta = 0$. Since the roles of λ and λ^* are in many cases similar to those of $\bar{\tau}$ and $\bar{\tau}^*$, in Fig. 8 we fix

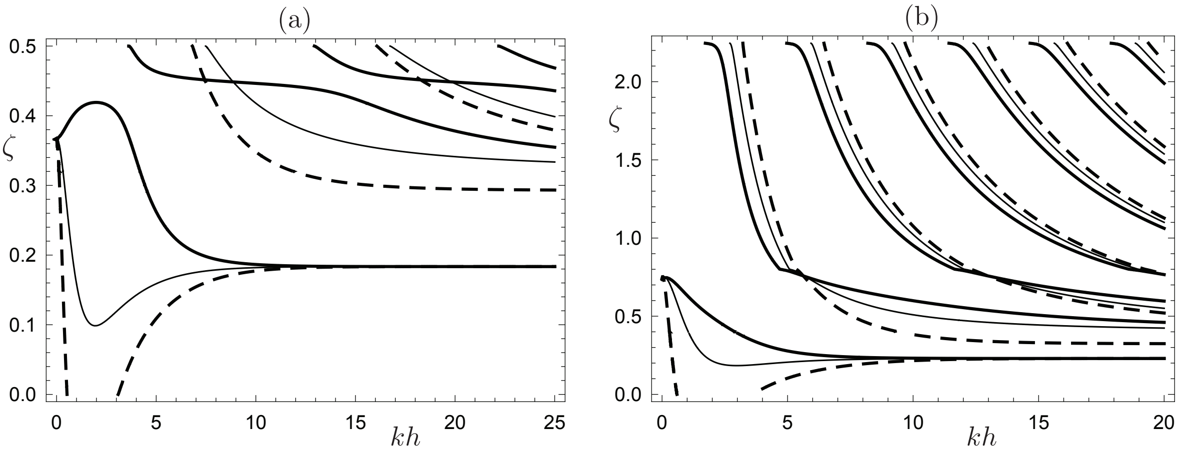

Figure 7: Plots of ζ against kh: (a) $\lambda = 1$, $\lambda^* = 0.8$, $R = 1$, $\bar{\mu} = -1.5$, $\bar{\tau} = -0.5$, $\bar{\mu}^* = 0$, $\bar{\tau}^* = -0.5$, and $r = 0.2, 1, 2$ corresponding to the thick continuous, continuous and dashed curves, respectively; (b) $\lambda = 1.2$, $\lambda^* = 0.8$, $R = 0.8$, $\bar{\mu} = -1$, $\bar{\tau} = 1$, $\bar{\mu}^* = 0$, $\bar{\tau}^* = -0.5$, and $r = 0.2, 0.5, 2$ corresponding to the thick continuous, continuous and dashed curves, respectively.

$\lambda = \lambda^* = 1$ and $\bar{\mu} = \bar{\mu}* = 0$, $R = 1$ and plot $\bar{\tau}^*$ against kh for a series of values of r with a negative and a positive $\bar{\tau}$ in the two panels. The curves in the two cases are very similar, with relatively small numerical differences. As the value of $\bar{\tau}^*$ is reduced from zero the uniform configuration remains stable until, for a given value of r, the appropriate curve is met, at which point surface undulations can appear that depend on kh. This occurs first for the larger values of r, i.e. for the stiffest layers.

On the other hand, in Fig. 9, instead of fixing $\lambda = \lambda^* = 1$, we fix $\bar{\tau} = \bar{\tau}^* = 0$ and plot λ against kh for four separate values of λ^* and, in each panel, several values of r. For the stiffest layers the results are very sensitive to values of compressive stretch, as Figs. 9(b), 9(c) and 9(d) demonstrate. In particular, a closed loop emerges for a small range of values of kh and as the compression advances this loop merges with the rest of the curve for $r = 6$ and expands upwards as the compression increases (not shown). Thus, for any given layer thickness the structure becomes very unstable for a range of wave numbers, and to prevent instability the stretch λ in half-space should therefore be sufficiently large.

Note that the value of λ at $kh = 0$ corresponds to the classical instability value for a compressed neo-Hookean half-space under plane strain (≈ 0.544) due to Biot and detailed in his book (Biot, 1965); see also Dowaikh and Ogden (1990) for further discussion.

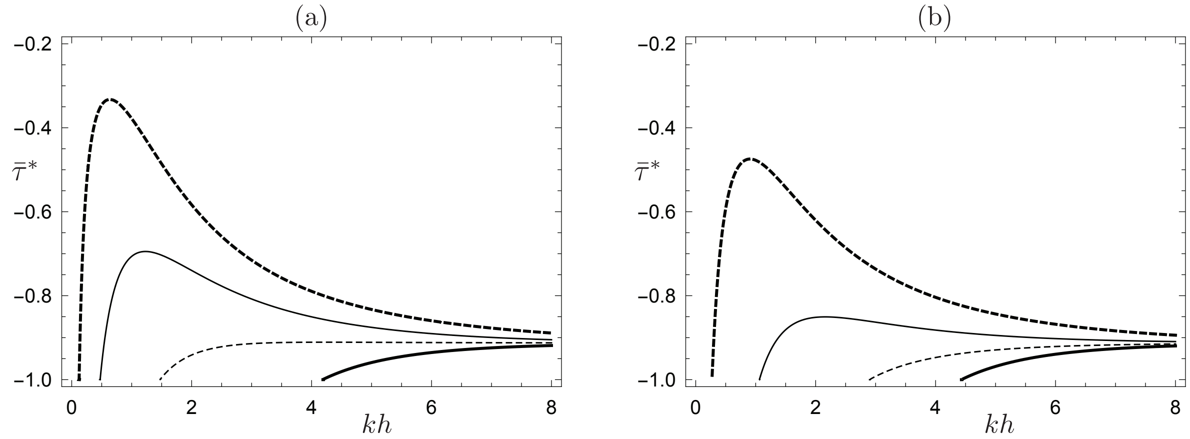

Figure 8: Plots of $\bar{\tau}^*$ against kh for $\bar{\mu} = \bar{\mu}^* = 0$, $\lambda = \lambda^* = 1$, $\zeta = 0$, $R = 1$: (a) $\bar{\tau} = -0.5$; (b) $\bar{\tau} = 0.5$. In each of (a) and (b) curves are shown for $r = 0.5, 1.5, 3, 10$, respectively the thick continuous, dashed, continuous and thick dashed curves.

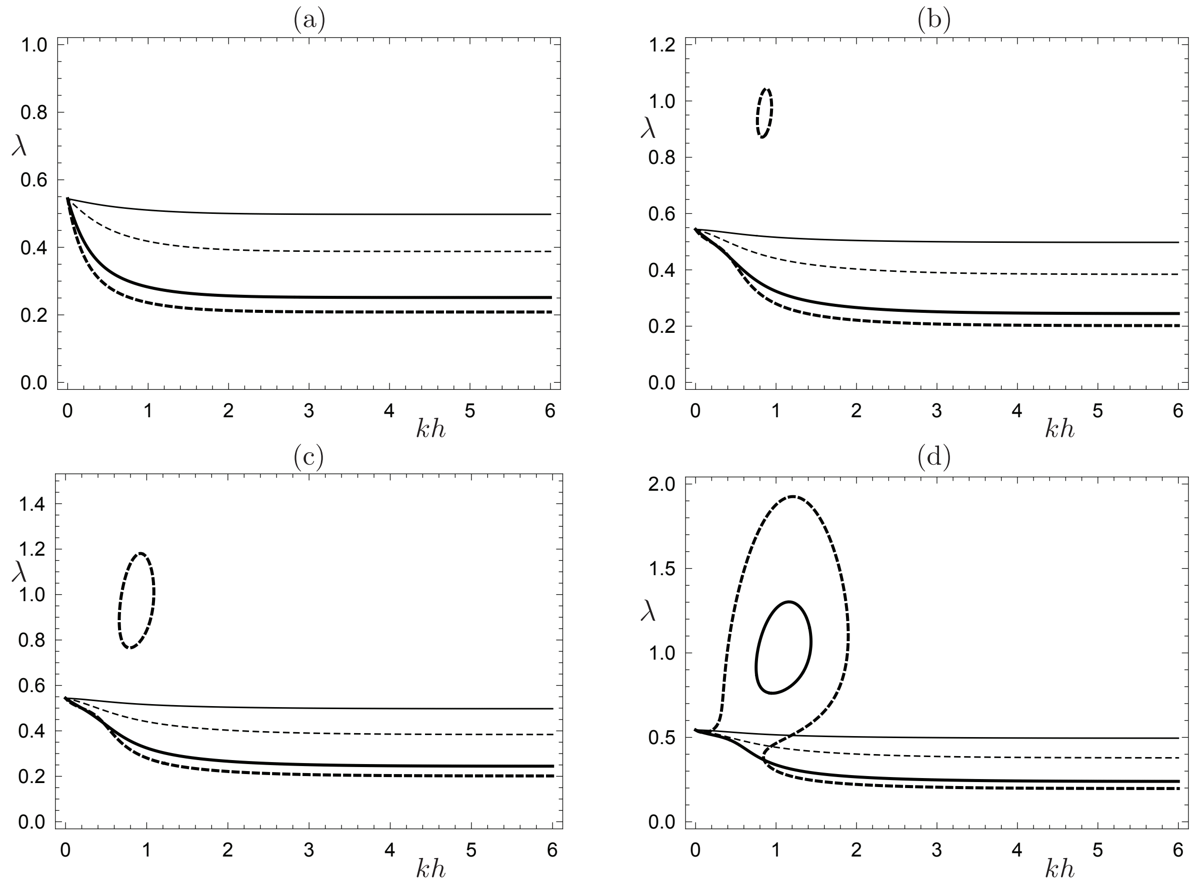

Figure 9: Plots of λ vs kh for $\bar{\mu} = \bar{\mu}^* = 0$, $\bar{\tau} = \bar{\tau}^* = 0$, $\zeta = 0, R = 1, r = \mu^*/\mu = 0.2, 1, 3.8, 6$ in each plot. The panels (a), (b), (c), (d) correspond to $\lambda^* = 1.4, 0.854, 0.85, 0.8$, respectively.

In this section we have selected particular values of the various stretch, initial stress and material parameters in order to illustrate the different features that can arise when considering the propagation of surface waves and loss of stability of a layered half-space. Clearly, many other possible combinations of these parameters could be adopted, but those we have chosen for illustration provide a representative range of possible results.

7 Concluding remarks

In the preceding sections we have analyzed the combined effect of a uniform initial stress and finite deformation on the propagation of harmonic waves of infinitesimal amplitude in a half-space of elastic material with an overlying layer of uniform thickness and different material which is also subject to a uniform initial stress and/or finite deformation. In particular, we have confirmed that when considered separately, not unexpectedly, the finite deformation and initial stress have similar consequences, but when they are both included independently the character of the waves is somewhat different.

We have also considered the special circumstances in which the wave speed vanishes, which corresponds to the emergence of small amplitude undulations of the layer and surface at critical values of the initial stress, deformation and material parameters. The analysis conducted here determines the point of bifurcation (or buckling) initiation, and we have not considered the post-bifurcation regime, which has also attracted recent attention but is more difficult to analyze in the general nonlinearly elastic context.

However, several authors have examined post-buckling using a perturbation approach, mainly by considering the bonding of an unstretched stiff thin film to a stretched compliant substrate, which is then relaxed so that the film buckles. For example, Song et al. (2008) adopted a beam model for the film and a compressible neo-Hookean model for the substrate. Beyond the initiation of these so-called wrinkling deformations, in the post-bifurcation regime, the wrinkles can develop into different structures, including period doubling, folding, and crease and ridge formation, as exemplified in the work of Sun et al. (2012), Cao and Hutchinson (2012), Jin et al. (2015) and references therein which involved combinations of theoretical analysis and finite element calculations supported by experimental observations of these characteristics.

A different aspect of the effect of pre-stress was examined by Bigoni et al. (2008) who studied waves that arise when a stiff periodic layer is bonded to a half-space of compressible elastic material. Using long-wave asymptotic methods their analysis revealed the existence of band gaps and found that the pre-stress can be used to tune the filtering properties of the structure.

Clearly, finite deformation, initial stress and material properties have a strong influence on the mechanical characteristics of different types of structure, as exemplified by the layer/half-space substrate structure considered here. Detailed fully nonlinear analysis for other structures is therefore desirable in order to determine critical conditions corresponding to the onset of bifurcation and the post-bifurcation continuation into the fascinating patterns illustrated in the papers cited above.

Acknowledgements

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