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On the sovereign debt paradox

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Abstract Bulow and Rogoff (1989) show that lending to small countries cannot be supported merely on the country's "reputation for repayment" if exclusion from future credit markets is the only consequence of default. Their arguments are valid under fairly general conditions but they do not go through when the output of the sovereign may vanish along a path of successive low productivity shocks, or when it may grow unboundedly along a path of successive high productivity shocks. We propose an alternative proof illustrating that their renowned sovereign debt paradox holds in full generality.

Keywords Sovereign risk · Lack of commitment · Reputation debt

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1 Introduction

To understand sovereign risk, it is necessary to understand borrowers' repayment incentives, or, equivalently, to understand why investors lend to sovereign countries. Since the legal enforcements in the case of sovereign debt are rather weak, repayment incentives should rely on a self-enforcing mechanism. The most obvious mechanism is the loss of reputation that may trigger a ban from future trading in international capital markets. But does this threat deter a country from defaulting?

The early studies in the sovereign debt literature (the classic reference is Eaton and Gersovitz (1981)) give an affirmative answer to this question in settings where the loss of reputation induces a permanent reversion to autarky (that is, the defaulting country is denied access to capital markets and looses any ability to smooth its expenditures). Bulow and Rogoff (1989) argue that given the recent developments in international capital markets (free entry), it is more realistic to assume that a defaulting country can, to some extend, smooth its expenditures by trading contracts that involve no credit (consumption-insurance contracts). Building on this insight, they provide a celebrated critique to the reputational mechanism by showing that exclusion from borrowing is too weak to sustain any debt repayment.[1]

The key assumptions that underpin the rationale of Bulow and Rogoff (1989)'s result are as follows: (1) the sovereign trades at the initial period contingent contracts that specify the net transfers to foreign investors in all future periods and events, (2) interest rates are higher than growth rates, in the sense that the prices of contingent contracts are such that the present value of the country's future net income is finite, and (3) upon default, the sovereign can purchase any consumption-insurance contract by paying cash in advance.[2] In this setting, the only traded contracts are those compatible with repayment incentives. Therefore, in addition to the investors' break-even condition (or equivalently, the standard Arrow–Debreu present value budget constraint), borrowers face participation constraints requiring that the continuation utility obtained by staying on the contract should be at least as high as that which can be obtained by defaulting and investing in consumption-insurance contracts.[3]

The proof of Bulow and Rogoff (1989) is very elegant and powerful. The key insight is that the debt levels associated to a budget feasible consumption contract are bounded by the natural debt limits defined as the present value

[1] Bulow and Rogoff (1989) led to a vast literature studying alternative mechanisms to answer why countries repay their debts in the absence of sanctions. It includes, among others, Cole and Kehoe (1995), Cole and Kehoe (1998), Kletzer and Wright (2000), Dutta and Kapur (2002), Kehoe and Perri (2002), Gul and Pesendorfer (2004), Krueger and Uhlig (2006), Amador (2012), and Popov (2014). We refer to Wright (2011) and Aguiar and Amador (2014) for a thorough discussion of this literature.

[2] A "cash-in-advance" contract is just a conventional insurance contract under which a country makes a payment up front in return for a state-contingent, non-negative future payment.

[3] In that respect the environment in Bulow and Rogoff (1989) is closely related to the one analyzed in Kehoe and Levine (1993) but with a different default option.

of future net income. When interest rates are sufficiently high, natural debt limits are well-defined and satisfy the following roll-over property: the natural debt limit at some event is larger that the present value of the natural debt limits at all immediate successor events. This roll-over property together with the access to all consumption-insurance contracts upon default induce strong default incentives that are not compatible with the participation constraints. Indeed, when the ratio of debt to natural debt limit attains its maximum, the sovereign can default and implement a replication strategy using consumption-insurance contracts which improves–at any contingency in which endowment is positive–upon the consumption plan that is subject to debt repayment. This "arbitrage" argument does not require any restriction on preferences except strict monotonicity. But, for the replication policy to be budget feasible, it is essential that–for any contingent contract satisfying the break even condition–the process of sovereign's debt to natural debt limit ratio attains its maximum in finite time. This turns out to be true, for instance, when the sovereign's net income is uniformly bounded from above and away from zero.

The first contribution of this paper is to provide examples to support the claim that, under the general assumptions Bulow and Rogoff (1989) impose on primitives, there are some contingent contracts for which the maximum expansion of debt relative to natural debt limits is not achieved in finite time. We show that this may happen in some plausible scenarios where the net income of the sovereign vanishes along a path of successive low productivity shocks (even if the probability of this path is zero), or when it grows unboundedly along a path of successive high productivity shocks. For these examples, the proof proposed by Bulow and Rogoff (1989) does not go through.

Recently, Hellwig and Lorenzoni (2009) provide an alternative proof of Bulow and Rogoff (1989)'s unsustainable debt result.[4] Their argument does not require any restriction on the net income process but it makes use of a characterization result of self-enforcing debt limits that requires additional assumptions on preferences. In particular, preferences are assumed to be additively separable and the Bernoulli function to be strictly concave, differentiable and bounded.

The second contribution of this paper amounts to show that the result in Bulow and Rogoff (1989) holds in full generality, free of any assumption imposed on the net income process or on preferences (apart from strict monotonicity).[5] We also show that, if preferences are dynamically consistent, its sequentially complete markets analogue proved in Hellwig and Lorenzoni (2009) follows as a direct corollary of Bulow and Rogoff (1989)'s original result.

[4] Hellwig and Lorenzoni (2009) analyze a slightly different (but equivalent) environment where agents trade sequentially a complete set of contingent bonds with self-enforcing debt limits in place of participation constraints. It is the same environment as in Alvarez and Jermann (2000) but with a different default option. In addition, they show that the risk-neutrality assumption and the ad-hoc separation between a small open economy and investors with "deep pockets" play no role for the impossibility result of Bulow and Rogoff (1989).

[5] Similarly to Hellwig and Lorenzoni (2009) we do not require investors to be risk-neutral.

Our proof shares some similarities with the one proposed in Bulow and Rogoff (1989). We also show for any budget feasible contingent contract involving some liabilities, there always exists an event at which the sovereign can default and initiate a replication strategy based on consumption-insurance contracts that improves upon the consumption plan that is subject to debt repayment. However, our approach differentiates from Bulow and Rogoff (1989) in two aspects: the date the country has incentives to default and the replication strategy that is implemented.

The layout of the paper is as follows. To get a more precise idea of the differences between our approach and the one proposed in Bulow and Rogoff (1989), we study in Section 2 a simple environment where the sovereign faces risk neutral foreign investors and there is no uncertainty. Section 3 describes the general environment. Section 4 presents the result in Bulow and Rogoff (1989) and shows the limitations of their approach. Our main result is proven in Section 5. Section 6 concludes with a discussion on related issues.

2 A simple environment: No uncertainty and risk-neutral investors

Assume that the price (in terms of time-0 consumption) of a contract delivering one unit of the good at period t is $(1+r)^{-t}$ where r is the world interest rate. Let $y = (y_t)_{t \geqslant 0}$ be the country's net income and denote by $c = (c_t)_{t \geqslant 0}$ the consumption sequence implemented by a contingent contract. Consider the sequence of the associated debt levels $D = (D_t)_{t \geqslant 0}$ where D_t is defined as the date-t present value of future net transfers $z_s := y_s - c_s$ for every $s \geqslant t$. If the interest rate is higher than the country's growth rates then the natural debt limit N_t, defined as the present value of future net income, is finite. Budget feasibility then implies that D_t is lower than N_t.[6] The key issue and what differentiates our approach from Bulow and Rogoff (1989) is the way we identify the date at which the country has incentives to default.

2.1 The approach in Bulow and Rogoff (1989)

We let κ be the supremum (over all dates t) of the debt to natural debt limit ratio D_t/N_t. If we assume that the contingent contract involves some positive level of debt at some date, then we must have $\kappa > 0$. Bulow and Rogoff (1989) show that the sovereign has incentives to default (i.e., the contract violates the participation constraint) at any date s satisfying $D_s \geqslant \kappa(N_s - y_s)$.[7] To understand why, first observe that the present value $(1+r)^{-1}\kappa N_{s+1}$ is lower than the current debt D_s. This is true since $N_s - y_s = (1+r)^{-1} N_{s+1}$. This in turn implies that, instead of complying with the terms of the contract by making the net transfer $D_s - (1+r)^{-1} D_{s+1}$ to foreign investors, the country

[6] Formally, we have $(1+r)^{-t} D_t := \sum_{s \geqslant t} (1+r)^{-s} z_s$ and $(1+r)^{-t} N_t := \sum_{s \geqslant t} (1+r)^{-s} y_s$.

[7] The problem is that such a date s may not exist under the general conditions Bulow and Rogoff (1989) impose on primitives. See Section 4 for details.

can default at date s, consume $\tilde{c}_s := c_s + D_s - (1+r)^{-1}\kappa N_{s+1}$ and save the difference $\theta_s := (1+r)^{-1}[\kappa N_{s+1} - D_{s+1}]$.[8] The definition of κ implies that $\theta_s \geqslant 0$. At date $s+1$, the return $(1+r)\theta_s = \kappa N_{s+1} - D_{s+1}$ on previous savings allows to implement the consumption $\tilde{c}_{s+1} := c_{s+1} + \kappa y_{s+1}$ and save the amount $\theta_{s+1} := (1+r)^{-1}[\kappa N_{s+2} - D_{s+2}]$. Similarly, the definition of κ implies $\theta_{s+1} \geqslant 0$. Recursively, we can show that, at any date $s+t$, the return $\kappa N_{s+t} - D_{s+t}$ on previous savings allows to implement consumption $\tilde{c}_{s+t} := c_{s+t} + \kappa y_{s+t}$ and save the amount $(1+r)^{-1}[\kappa N_{s+t+1} - D_{s+t+1}]$. This means that the consumption $(\tilde{c}_{s+t})_{t \geqslant 0}$ after date s can be implemented only through savings. Since $\tilde{c}_{s+t} \geqslant c_{s+t}$ for every $t \geqslant 0$ with a strict inequality for some t, this contradicts the participation constraint at date s and the sovereign has incentives to default on the terms of the contract.

2.2 Our approach

It is useful to first recall the concept of exact roll-over introduced by Hellwig and Lorenzoni (2009): a sequence $(M_t)_{t \geqslant 0}$ is said to satisfy exact roll-over if the time-0 discounted value $(1+r)^{-t}M_t$ is constant, or equivalently, if $M_t = (1+r)^{-1}M_{t+1}$ for any $t \geqslant 0$.

We show that it is possible to find a positive sequence $M = (M_t)_{t \geqslant 0}$ satisfying exact roll-over, and a date τ such that $D_\tau > M_\tau$ and $D_t \leqslant M_t$ for every $t > \tau$. Once we have identified the sequence M and the date τ satisfying the above conditions, it is straightforward to see that the country prefers to default at date τ. Indeed, instead of paying $D_\tau - (1+r)^{-1}D_{\tau+1}$ to foreign investors, the country can save $M_\tau - (1+r)^{-1}D_{\tau+1} \geqslant M_\tau - (1+r)^{-1}M_{\tau+1} = 0$ and consume, in addition to c_τ, the difference $D_\tau - M_\tau$. At date $\tau+1$, the return $(1+r)M_\tau - D_{\tau+1} = M_{\tau+1} - D_{\tau+1}$ allows the country to implement the consumption $c_{\tau+1}$ and save the difference $M_{\tau+1} - (1+r)^{-1}D_{\tau+2} = (1+r)^{-1}[M_{\tau+2} - D_{\tau+2}] \geqslant 0$. Recursively, we can show that for any $t > \tau$, the country implements the consumption c_t using the return $M_t - D_t$ of previous savings and saving $M_t - (1+r)^{-1}D_{t+1} = (1+r)^{-1}[M_{t+1} - D_{t+1}] \geqslant 0$. This implies that the country is strictly better off defaulting at date τ since it can enjoy the extra consumption $D_\tau - M_\tau$ at that date without decreasing consumption at all subsequent periods. The existence of the exact roll-over sequence $(M_t)_{t \geqslant 0}$ and of a date τ satisfying $D_\tau > M_\tau$ and $D_t \leqslant M_t$ for every $t > \tau$, follows from the property that the time-0 present value $(1+r)^{-t}N_t$ of the natural debt limit converges by assumption to zero. The formal details are presented in Section 5.

3 The general model

We consider a country whose government makes investment and consumption decisions over an infinite horizon. There is one good that can be either

[8] Recall that $y_s = c_s + D_s - (1+r)^{-1}D_{s+1}$.

consumed in the same period, or invested as capital for the next period. The production technology is subject to random shocks. We use an event tree Σ to describe time, uncertainty and the revelation of information over the infinite horizon (time and uncertainty are both discrete).

3.1 Uncertainty

There is a unique initial date-0 event $s^0 \in \Sigma$ and for each date $t \in \{0, 1, 2, \ldots\}$ there is a finite set $S^t \subset \Sigma$ of date-t events s^t. Each s^t has a unique predecessor $\sigma(s^t)$ in S^{t-1} and a finite number of successors $s^{t+1} \in S^{t+1}$ for which $\sigma(s^{t+1}) = s^t$. We use the notation $s^{t+1} \succ s^t$ to specify that s^{t+1} is a successor of s^t. Event $s^{t+\tau}$ is said to follow event s^t, also denoted $s^{t+\tau} \succ s^t$, if $\sigma^{(\tau)}(s^{t+\tau}) = s^t$. The set $S^{t+\tau}(s^t) := \{s^{t+\tau} \in S^{t+\tau} \; : \; s^{t+\tau} \succ s^t\}$ denotes the collection of all date-$(t+\tau)$ events following s^t. Abusing notation, we let $S^t(s^t) := \{s^t\}$. The subtree of all events starting from s^t is then

$$\Sigma(s^t) := \bigcup_{\tau \geqslant 0} S^{t+\tau}(s^t).$$

We use the notation $s^\tau \succeq s^t$ when $s^\tau \succ s^t$ or $s^\tau = s^t$. In particular, we have $\Sigma(s^t) = \{s^\tau \in \Sigma \; : \; s^\tau \succeq s^t\}$.

3.2 Consumption and preferences

If $k(s^t) \geqslant 0$ is the investment on capital at date-t event s^t, then the country's production at every successor event $s^{t+1} \succ s^t$ is denoted by $f(s^{t+1}, k(s^t))$. We do not impose any restriction on the production function $f(s^t, \cdot)$. A standard example is the neoclassical production technology $f(s^t, k) = A(s^t)k^\alpha + (1-\delta)k$ where uncertainty only affects the total factor productivity $A(s^t)$.

We denote by $y = (y(s^t))_{s^t \in \Sigma}$ the net income (or endowment) process derived from the capital investment $k = (k(s^t))_{s^t \in \Sigma}$ and defined by

$$y(s^t) := f(s^t, k(s^{t-1})) - k(s^t)$$

where $k(s^0) > 0$ is exogenously fixed. We follow Bulow and Rogoff (1989) and assume that current investment in capital is deduced from current production.

Assumption 3.1. For every event s^t, we have $y(s^t) \geqslant 0$.

The country can finance investment and consumption by trading a state contingent contract $z = (z(s^t))_{s^t \in \Sigma}$ where $z(s^t)$ specifies the net transfer from the country to foreign investors. Negative $z(s^t)$ indicates a payment from investors to the country. The consumption process $c \geqslant 0$ associated to the contract z is then given by

$$c(s^t) := f(s^t, k(s^{t-1})) - k(s^t) - z(s^t) \quad \text{or, equivalently,} \quad c := y - z.$$

At any event s^t, the country's preference relation over consumption c is defined by a "contingent utility function" $U(c|s^t)$. Following Bulow and Rogoff (1989) we only restrict preference relations to be strictly increasing.

Assumption 3.2. For every event s^t, the function $c \mapsto U(c|s^t)$ is strictly increasing with respect to the consumption $c(s^\tau)$ at any successor event $s^\tau \succeq s^t$.

Remark 3.1. The above assumption is very general. It allows to encompass preference relations that are not necessarily additively separable. In particular, it holds true when preferences satisfy the general recursive functional form

$$U(c|s^t) = W\left(c(s^t), \mathcal{M}_{s^t}\left[\left(U(c|s^{t+1})\right)_{s^{t+1} \succ s^t}\right]\right)$$

where the intertemporal aggregator $W : \mathbb{R}^2 \to \mathbb{R}$ is strictly increasing and the event s^t certainty equivalent $\mathcal{M}_{s^t}((U(c|s^{t+1}))_{s^{t+1} \succ s^t})$ of future continuation utility $U(c|s^{t+1})$ is also strictly increasing. If preference relations are additively separable, in the sense that

$$U(c|s^t) = u(c(s^t)) + \sum_{\tau \geqslant 1} \beta^\tau \sum_{s^{t+\tau} \succ s^t} \pi(s^{t+\tau}|s^t) u(c(s^{t+\tau}))$$

where $\beta \in (0,1)$ is the discount factor and $\pi(s^{t+\tau}|s^t)$ is the conditional probability of $s^{t+\tau}$ given s^t, then Assumption 3.2 is satisfied if the Bernoulli function u is strictly increasing. Observe that we do not need to assume that u is concave, differentiable or bounded.

3.3 Markets

We denote by $q(s^{t+1})$ the price, in terms of event s^t consumption, of the s^{t+1}-contingent bond. Given bond prices $q = (q(s^t))_{s^t \succ s^0}$ we denote by $p(s^t)$ the associated date-0 price of consumption at s^t defined recursively by $p(s^0) = 1$ and $p(s^t) = q(s^t)p(\sigma(s^t))$ for every $s^t \succ s^0$. We use $\mathrm{PV}(x|s^t)$ to denote the present value at date-t event s^t of a process x restricted to the subtree $\Sigma(s^t)$ and defined by

$$\mathrm{PV}(x|s^t) := \frac{1}{p(s^t)} \sum_{s^{t+\tau} \in \Sigma(s^t)} p(s^{t+\tau}) x(s^{t+\tau}).$$

Following Bulow and Rogoff (1989), we make the following assumption.

Assumption 3.3. The present value of the country's future net income is finite, i.e., $\mathrm{PV}(y|s^0) < \infty$.

Remark 3.2. We could assume, as in Bulow and Rogoff (1989), that foreign investors are risk-neutral. In this case,

$$q(s^{t+1}) = \frac{1}{1+r} \pi(s^{t+1}|s^t)$$

where r would be the world interest rate and $\pi(s^{t+1}|s^t)$ the conditional probability of s^{t+1} given s^t. However, the risk-neutrality assumption and the ad-hoc separation between a small open economy and investors with "deep pockets" play no role in Bulow and Rogoff (1989)'s analysis. The mechanism underlying their unsustainable debt result is valid even if we consider a general equilibrium version of their model where all agents are treated symmetrically (implying that lenders and borrowers are determined endogenously).

Foreign investors are willing to provide any state-contingent claim z if they break-even in present value terms, i.e.,[9]

$$\mathrm{PV}(z|s^0) \geqslant 0. \tag{3.1}$$

Given that the country's future net income has finite present value, the break-even condition (3.1) is equivalent to the standard Arrow–Debreu present value budget constraint $\mathrm{PV}(c|s^0) \leqslant \mathrm{PV}(y|s^0)$. In particular, we have the equivalent sequential formulation: for every $s^t \succeq s^0$,

$$c(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1})a(s^{t+1}) = y(s^t) + a(s^t) \tag{3.2}$$

where $a(s^{t+1}) := \mathrm{PV}(c - y|s^{t+1})$ is the country's holding of the bond issued at event s^t and contingent to the successor event s^{t+1}.[10]

Since consumption is non-negative, the bond holding process satisfies the natural debt limit constraint, i.e,

$$a(s^t) \geqslant -N(s^t) \quad \text{where} \quad N(s^t) := \mathrm{PV}(y|s^t).$$

Observe that the country's outstanding debt $D(s^t) := \mathrm{PV}(z|s^t)$ at event s^t satisfies $D(s^t) = -a(s^t)$. In particular, the debt $D(s^t)$ does not exceed the value of a claim to the country's entire future net income stream, i.e., $D(s^t) \leqslant N(s^t)$.

3.4 Self-enforcing contracts

A state-contingent contract z involves debt repayment if $\mathrm{PV}(z|s^t) > 0$ for some event s^t. The country keeps the promises associated to contract z if, and only if, for every event s^t, the continuation utility $U(c|s^t)$ associated to the consumption $c = y - z$ is not lower than the maximum utility $V(s^t)$ the country would get by defaulting. Following Bulow and Rogoff (1989) we assume that after default, the country is excluded from borrowing but keeps the ability to save by investing in a complete set of one-period contingent bonds.[11] This

[9] Observe that this condition restricts feasible contracts z to be such that the series $\mathrm{PV}(z|s^0)$ is well-defined.

[10] The term $a(s^0)$ can be interpreted as an initial transfer.

[11] This is equivalent to assuming that the sovereign has access to any consumption-insurance contract. Formally, at any event s^τ after default, the country can make the payment $\sum_{s^{\tau+1} \succ s^\tau} q(s^{\tau+1})\tilde{a}(s^{\tau+1})$ up front in return for any state-contingent, non-negative future payment $\tilde{a}(s^{\tau+1}) \geqslant 0$.

implies that the value $V(s^t)$ of the default option is the largest continuation utility $U(\tilde{c}|s^t)$ where $\tilde{c}$ is financed by a process $\tilde{a}$ of non-negative bond holdings, i.e.,

$$\forall s^\tau \succeq s^t, \quad \tilde{c}(s^\tau) + \sum_{s^{\tau+1} \succ s^\tau} q(s^{\tau+1})\tilde{a}(s^{\tau+1}) \leqslant y(s^\tau) + \tilde{a}(s^\tau) \tag{3.3}$$

where $\tilde{a}(s^t) = 0$ and $\tilde{a}(s^\tau) \geqslant 0$ for every strict successor $s^\tau \succ s^t$. A contract z is said to be *self-enforcing* if the incentive compatibility constraint

$$U(c|s^t) \geqslant V(s^t) \tag{3.4}$$

is satisfied for every event $s^t \in \Sigma$.

4 Bulow and Rogoff (1989)'s result

Consider a contract z satisfying the break-even condition (3.1). Bulow and Rogoff (1989) propose to show that if z involves positive levels of debt, i.e., $\mathrm{PV}(z|s^\tau) > 0$ for some event s^τ in Σ, then it cannot satisfy the incentive compatibility constraints (3.4).

To state formally their claim, we need to introduce the following notation. Let $\kappa(z) \in [0,1]$ be the lowest number satisfying $\mathrm{PV}(z|s^t) \leqslant \kappa(z)\,\mathrm{PV}(y|s^t)$ for every s^t. Such a number exists since $z \leqslant y$ (consumption c is non-negative) and the present value of net income is finite (which implies that $\mathrm{PV}(z|s^t) \leqslant \mathrm{PV}(y|s^t)$ for every event s^t).

Proposition 4.1 (Bulow and Rogoff (1989), Theorem 1)**.** Consider a contract z satisfying the break-even condition (3.1). Assume that the following condition is satisfied

$$\exists s^t \in \Sigma, \quad \mathrm{PV}(z|s^t) \geqslant \kappa(z)[\mathrm{PV}(y|s^t) - y(s^t)]. \tag{BR}$$

If the contract z involves some positive level of debt, then it cannot be self-enforcing (i.e., it must violate the incentive compatibility constraint (3.4)).

Bulow and Rogoff (1989) assume implicitly that condition (BR) is satisfied. Eaton and Fernandez (1995) claim that this condition follows from the definition of $\kappa(z)$. Observe that an alternative definition of $\kappa(z)$ is

$$\kappa(z) := \sup_{s^t \in \Sigma} \kappa(z|s^t)$$

where $\kappa(z|s^t) \in [0,1]$ is defined by the equation $\mathrm{PV}(z|s^t) = \kappa(z|s^t)\,\mathrm{PV}(y|s^t)$.[12] If the supremum in the definition of $\kappa(z)$ is attained at some event s^t where the net income $y(s^t)$ is strictly positive, then condition (BR) is satisfied (with a strict inequality). There are also other conditions under which (BR) is satisfied.

[12] If $\mathrm{PV}(y|s^t) = 0$, we pose $\kappa(z|s^t) = 0$.

Proposition 4.2. Assume that foreign investors are risk-neutral, i.e., $p(s^t) = \gamma^t \pi(s^t)$ where $\pi(s^t)$ is the unconditional probability of event s^t and $\gamma = (1+r)^{-1}$ where $r > 0$ is the risk-free interest rate. Assume moreover that there exist $y_{\mathrm{H}} > y_{\mathrm{L}} > 0$ such that $y(s^t) \in [y_{\mathrm{L}}, y_{\mathrm{H}}]$ for any event s^t. Then, for any contract z satisfying the break-even condition (3.1), condition (BR) is satisfied.

Proof. For any $s^t \in \Sigma$ we have

$$y(s^t) \geqslant y_{\mathrm{L}} \geqslant \varepsilon \frac{y_{\mathrm{H}}}{1-\gamma} \geqslant \varepsilon \,\mathrm{PV}(y|s^t)$$

where $\varepsilon := (1-\gamma) y_{\mathrm{L}}/y_{\mathrm{H}}$. This implies that

$$\forall s^t \in \Sigma, \quad \kappa(z)[\mathrm{PV}(y|s^t) - y(s^t)] \leqslant \kappa(z)[1-\varepsilon]\,\mathrm{PV}(y|s^t).$$

Therefore, there must exist some event s^t such that $\mathrm{PV}(z|s^t) > \kappa(z)[\mathrm{PV}(y|s^t) - y(s^t)]$. Otherwise, we get that $\mathrm{PV}(z|s^t) \leqslant \kappa(z)[1-\varepsilon]\,\mathrm{PV}(y|s^t)$ for all $s^t \in \Sigma$, which contradicts the definition of $\kappa(z)$. Q.E.D

If the net income process is not bounded away from zero, or, it is unbounded from above, then a contract that satisfies the break-even condition does not necessarily satisfy condition (BR). To prove our claim, we first provide an example of an economy for which the net income process may vanish along a path of successive negative shocks.

Example 4.1. At every date t, a shock $s_t \in \{\ell, h\}$ realizes. The initial event is $s^0 = \ell$ and a date-t event s^t is a history of shocks, $s^t = (s_1, \dots, s_t)$. A following event $s^{t+1} \succ s^t$ can be written as $s^{t+1} = (s^t, s_{t+1})$ with $s_{t+1} \in \{\ell, h\}$. The net income process of the country is defined by

$$y(s^0) := y_{\mathrm{L}} \quad \text{and} \quad y(s^t, s_{t+1}) := \begin{cases} y_{\mathrm{H}} & \text{if } s_{t+1} = h \\ \alpha y(s^t) & \text{if } s_{t+1} = \ell \end{cases}$$

where $\alpha \in (0,1)$ and $y_{\mathrm{H}} > y_{\mathrm{L}} > 0$. The conditional probability $\pi(s^{t+1}|s^t)$ is defined as follows:

$$\pi(s^{t+1}|s^t) = \begin{cases} 1 & \text{if } (s_t, s_{t+1}) = (h,h) \\ \pi_{\mathrm{L}} & \text{if } (s_t, s_{t+1}) = (\ell, \ell). \end{cases}$$

After a good shock $s_t = h$ at date t, the country's endowment switches to its high level y_{H} and remains high forever. After a bad shock $s_t = \ell$, the country's endowment is reduced by a fraction α. A good shock can be interpreted as innovation that allows to sustain a high productivity level.

Foreign investors are risk-neutral. That is, at every event s^t, we have $p(s^t) = \gamma^t \pi(s^t)$ where $\gamma = (1+r)^{-1}$ and the risk-free interest rate r is positive. The price of the bond contingent to next-period shock s_{t+1} is given by $q(s^{t+1}) = \pi(s^{t+1}|s^t)/(1+r)$.

We denote by ℓ^t the history of successive negative shocks $(\ell, \ell, \dots, \ell)$ up to date t. Consider the process $\lambda = (\lambda(s^t))_{s^t \in \Sigma}$ where $\lambda(s^t) = 0$ if $s^t \neq \ell^t$ and

$\lambda(\ell^t) = \kappa_t \in (0,1)$ where $(\kappa_t)_{t\in\mathbb{N}}$ is an increasing sequence of positive numbers satisfying $\lim_{t\to\infty} \kappa_t = 1$. Given the process λ, we can construct a contract z satisfying[13]

$$\mathrm{PV}(z|s^t) = \lambda(s^t)\,\mathrm{PV}(y|s^t). \tag{4.1}$$

Observe that $\kappa(z|s^t) = \lambda(s^t)$. This implies that

$$\kappa(z) = \lim_{t\to\infty} \kappa(z|\ell^t) = \lim_{t\to\infty} \kappa_t = 1$$

and $\kappa(z|s^t) < \kappa(z)$ for every event s^t. In particular, the supremum $\kappa(z)$ is not attained in finite time.

Proposition 4.3. For the economy considered in Example 4.1, the contingent contract z defined by Equation (4.1) does not satisfy condition (BR), i.e.,

$$\forall s^t \in \Sigma, \quad \mathrm{PV}(z|s^t) < \kappa(z)[\mathrm{PV}(y|s^t) - y(s^t)].$$

Proof. Given that $\mathrm{PV}(z|s^t) = \lambda(s^t)\,\mathrm{PV}(y|s^t)$ and $\kappa(z) = 1$, we have to show that

$$\forall s^t \in \Sigma, \quad \lambda(s^t)y(s^t) < [1-\lambda(s^t)] \sum_{s^{t+1}\succ s^t} q(s^{t+1})\,\mathrm{PV}(y|s^{t+1}). \tag{4.2}$$

This is obvious if $s^t \neq \ell^t$ since $\lambda(s^t) = 0$. To show that Equation (4.2) is valid for $s^t = \ell^t$, observe that

$$\sum_{s^{t+1}\succ s^t} q(s^{t+1})\,\mathrm{PV}(y|s^{t+1}) \geqslant \gamma(1-\pi_{\mathrm{L}})\,\mathrm{PV}(y|(\ell^t,h)) = \gamma(1-\pi_{\mathrm{L}})\frac{y_{\mathrm{H}}}{1-\gamma}.$$

Moreover, $\lambda(\ell^t) = \kappa_t$ and $y(\ell^t) = \alpha^t y_{\mathrm{L}}$. Therefore, Equation (4.2) is satisfied if

$$\forall t \geqslant 0, \quad \alpha^t y_{\mathrm{L}} < \frac{1-\kappa_t}{\kappa_t}\gamma(1-\pi_{\mathrm{L}})\frac{y_{\mathrm{H}}}{1-\gamma}. \tag{4.3}$$

It is always possible to choose the sequence $(\kappa_t)_{t\in\mathbb{N}}$ such that Equation (4.3) is satisfied. Take for instance $\kappa_t = \mu/(\alpha^t+\mu)$ where $\mu < [\gamma(1-\pi_{\mathrm{L}})y_{\mathrm{H}}]/[(1-\gamma)y_{\mathrm{L}}]$. Q.E.D

In Example 4.1, we have exploited the fact that the process of net income is not uniformly bounded away from zero to show that a contingent contract that satisfies the break-even condition (3.1) does not necessarily satisfy condition (BR). As suggested in Bulow and Rogoff (1989) (Footnote 5), we can replace the process y by a larger process $y^\star \geqslant y$ that also has finite present value (i.e., $\mathrm{PV}(y^\star|s^t) < \infty$). Consider the special case where foreign investors are risk-neutral. Given the result of Proposition 4.2, it sounds reasonable to let $y^\star(s^t) := y(s^t) + 1$. We get a process that is bounded away from zero. However, if foreign investors are not risk-neutral, it is not clear how the process y could be increased to some $y^\star$ such that any contingent contract z

[13] We can choose $z(s^t) := \lambda(s^t)\,\mathrm{PV}(y|s^t) - \sum_{s^{t+1}\succ s^t} \lambda(s^{t+1})q(s^{t+1})\,\mathrm{PV}(y|s^{t+1})$. Since $\lambda(s^{t+1}) \geqslant \lambda(s^t)$, we have that $z(s^t) \leqslant y(s^t)$.

that satisfies the break-even condition (3.1) would also satisfy condition (BR). More importantly, the process y may still be unbounded from above. We show next that, even if the process of net income is uniformly bounded away from zero, a contract that satisfies the break-even condition (3.1) may not satisfy condition (BR).

Proposition 4.4. Consider the same economy as in Example 4.1 except that the net income process is now defined as follows:

$$y(s^t, s_{t+1}) := \begin{cases} (1+g)^t & \text{if } s_{t+1} = h \\ 1 & \text{if } s_{t+1} = \ell \end{cases}$$

where $g \in (0, r)$. The contract z defined by Equation (4.1) does not satisfy condition (BR), i.e.,

$$\forall s^t \in \Sigma, \quad \mathrm{PV}(z|s^t) < \kappa(z)[\mathrm{PV}(y|s^t) - y(s^t)].$$

Proof. It suffices to show that Equation (4.2) is valid for $s^t = \ell^t$. Observe that

$$\sum_{s^{t+1} \succ s^t} q(s^{t+1}) \, \mathrm{PV}(y|s^{t+1}) \geqslant \gamma(1 - \pi_{\mathrm{L}}) \, \mathrm{PV}(y|(\ell^t, h)) \geqslant \gamma(1 - \pi_{\mathrm{L}}) \frac{(1+g)^{t+1}}{1 - \chi}$$

where $\chi := (1+g)/(1+r) < 1$. Moreover, $\lambda(\ell^t) = \kappa_t$ and $y(\ell^t) = 1$. Therefore, Equation (4.2) is satisfied if

$$\forall t \geqslant 0, \quad \kappa_t < (1 - \kappa_t)\gamma(1 - \pi_{\mathrm{L}}) \frac{(1+g)^{t+1}}{1 - \chi}. \tag{4.4}$$

It is always possible to choose the sequence $(\kappa_t)_{t \in \mathbb{N}}$ such that Equation (4.4) is satisfied. Take for instance $\kappa_t = \mu/((1+g)^{-t} + \mu)$ where $\mu < \gamma(1 - \pi_{\mathrm{L}})(1 + g)/(1 - \chi)$. Q.E.D

5 The general result

The main contribution of this paper is to show that condition (BR) is superfluous for the validity of the result stated in Bulow and Rogoff (1989).

Theorem 5.1. *Consider a contract z satisfying the break-even condition (3.1). If the contract z involves some positive level of debt, i.e., $\mathrm{PV}(z|s^t) > 0$ for some event s^t, then there exists a successor event $s^\tau \succeq s^t$ for which the incentive compatibility constraint $U(c|s^\tau) \geqslant V(s^\tau)$ is not satisfied.*

Before we proceed to prove the general result, we find it useful to illustrate how our approach applies to the deterministic case.

Proof for the deterministic case. Assume that the country's net income is deterministic. We then replace the notation s^t by t. In particular, the country's net income is now a sequence $(y_t)_{t\geqslant 0}$. Fix a contract $z = (z_t)_{t\geqslant 0}$ that satisfies the break-even condition (3.1) and denote by $D = (D_t)_{t\geqslant 0}$ the associated sequence of debt levels defined by $D_t := \mathrm{PV}(z|t)$. The consumption $c := y - z$ can be financed by trading sequentially one period contingent bonds. This is because we have

$$\forall t \geqslant 0, \quad c_t - q_{t+1}D_{t+1} = y_t - D_t. \tag{5.1}$$

It is important to notice that $D_t \leqslant N_t$ where N_t is the natural debt limit defined by $N_t := \mathrm{PV}(y|t)$.

Assume there exists a date $\xi \geqslant 0$ such that $D_\xi > 0$. Since the country's future net income is assumed to be finite, we have $\lim_{t\to\infty} p_t N_t = 0$. This implies that there exists a sufficiently large time period $\eta > \xi$ such that $p_\xi D_\xi > p_\eta N_\eta$. Let τ be the largest time period in $\{\xi, \dots, \eta\}$ satisfying

$$p_\tau D_\tau > p_\eta N_\eta. \tag{5.2}$$

We claim that

$$\forall t > \tau, \quad p_t D_t \leqslant p_\eta N_\eta. \tag{5.3}$$

If $t \in \{\tau + 1, \dots, \eta\}$, this follows from the definition of τ. If $t > \eta$, this follows from the following inequalities[14]

$$p_t D_t \leqslant p_t N_t \leqslant p_\eta N_\eta.$$

Adding $p_\eta N_\eta$ on both sides of the flow constraints (5.1) for any $t \geqslant \tau$, we get

$$p_\tau c_\tau + \underbrace{p_\tau D_\tau - p_\eta N_\eta}_{p_\tau \delta} + p_\eta N_\eta - p_{\tau+1}D_{\tau+1} = p_\tau y_\tau$$

and for every $t > \tau$

$$p_t c_t + p_\eta N_\eta - p_{t+1}D_{t+1} = p_t y_t + p_\eta N_\eta - p_t D_t.$$

If we let $(\tilde{a}_t)_{t>\tau}$ be defined by

$$p_t \tilde{a}_t := p_\eta N_\eta - p_t D_t$$

then we get that

$$c_\tau + \delta + q_{\tau+1}\tilde{a}_{\tau+1} = y_\tau$$

and for every $t > \tau$

$$c_t + q_{t+1}\tilde{a}_{t+1} = y_t + \tilde{a}_t.$$

The inequality (5.2) implies that $\delta > 0$. The inequality (5.3) implies that $\tilde{a}_t \geqslant 0$ for every $t > \tau$. It follows that if the sovereign defaults at date τ, he succeeds to finance the consumption sequence $(c_\tau + \delta, c_{\tau+1}, \dots)$ by a sequence of non-negative bond holdings $(\tilde{a})_{t>\tau}$. This means that $V(\tau) > U(c|\tau)$ and z cannot be incentive compatible. Q.E.D

[14] Recall that $p_{t-1}N_{t-1} = p_{t-1}y_{t-1} + p_t N_t \geqslant p_t N_t$.

Remark 5.1. Eaton and Fernandez (1995) propose a different proof for the deterministic case. They show that the sovereign has incentives to default at the date s for which the debt is maximum. To ensure that the sequence $(D_t)_{t\geqslant 0}$ admits a maximum, they assume that the sequence $(\mathrm{PV}(y|t))_{t\geqslant 0}$ is bounded.[15] This is stronger that the assumption imposed by Bulow and Rogoff (1989) who only assume that the value $\mathrm{PV}(y|t)$ of future net income discounted to any time t is finite. Indeed, assume that there is growth, say $y_t = (1+g)^t$ where $g < r$. Since interest rates are higher than growth rates, we get that $\mathrm{PV}(y|t) = (1+g)^t/(1-\chi)$ where $\chi := (1+g)/(1+r)$. The assumption of Bulow and Rogoff (1989) is then satisfied but we do have $\lim_{t\to\infty} \mathrm{PV}(y|t) = \infty$.

We now provide the details of the proof for the general environment where net income is stochastic.

Proof of Theorem 5.1. Let z be a contract that satisfies the break-even condition (3.1). We let $D(s^t) := \mathrm{PV}(z|s^t)$ for each date-t event $s^t \in \Sigma$. Since $z = y - c$ where the consumption process c is non-negative, we deduce that $D(s^t) \leqslant N(s^t)$ where $N(s^t) := \mathrm{PV}(y|s^t)$ is the natural debt limit.

Fix an arbitrary date $\eta \in \mathbb{N}$ and let $M^\eta = (M^\eta(s^t))_{s^t\in\Sigma}$ be a process satisfying the following properties.

(i) The process M^η satisfies exact roll-over in the sense that

$$\forall s^t \in \Sigma, \quad M^\eta(s^t) = \sum_{s^{t+1}\succ s^t} q(s^{t+1}) M^\eta(s^{t+1}). \tag{ER}$$

(ii) For every date-η event $s^\eta \in S^\eta$, we have $M^\eta(s^\eta) = N(s^\eta)$.

(iii) For every $t > \eta$ and every event $s^t \in S^t$, we have $M^\eta(s^t) \geqslant N(s^t)$.

Remark 5.2. Such a process always exists. Indeed, first pose $M^\eta(s^\eta) := N(s^\eta)$ for every date-η event s^η. Second, the exact roll-over condition (ER) defines in a unique way the process $M^\eta(s^t)$ at every event s^t with $t < \eta$. Third, we know that $N(s^\eta) \geqslant \sum_{s^{\eta+1}\succ s^\eta} q(s^{\eta+1})N(s^{\eta+1})$. This implies that we can find $\varepsilon(s^{\eta+1}) \geqslant 0$ such that $N(s^\eta) = \sum_{s^{\eta+1}\succ s^\eta} q(s^{\eta+1})[N(s^{\eta+1}) + \varepsilon(s^{\eta+1})]$. We can then let $M^\eta(s^{\eta+1}) := N(s^{\eta+1}) + \varepsilon(s^{\eta+1})$ for each $s^{\eta+1} \in S^{\eta+1}$. Since $N(s^{\eta+1}) + \varepsilon(s^{\eta+1}) \geqslant \sum_{s^{\eta+2}\succ s^{\eta+1}} q(s^{\eta+2})N(s^{\eta+2})$, we can find $\varepsilon(s^{\eta+2}) \geqslant 0$ satisfying the condition: $N(s^{\eta+1}) + \varepsilon(s^{\eta+1}) = \sum_{s^{\eta+2}\succ s^{\eta+1}} q(s^{\eta+2})[N(s^{\eta+2}) + \varepsilon(s^{\eta+2})]$. We can then let $M^\eta(s^{\eta+2}) := N(s^{\eta+2}) + \varepsilon(s^{\eta+2})$. Proceeding by induction, we can define $M^\eta(s^t)$ at every node s^t where $t > \eta$ such that (i) and (iii) are satisfied.

Assume that there exists a date-ξ event s^ξ such that $D(s^\xi) > 0$. Exact roll-over of M^t implies that for every $t > \xi$ we have[16]

$$p(s^\xi) M^t(s^\xi) = \sum_{s^t\in S^t(s^\xi)} p(s^t) N(s^t).$$

[15] Since $D_t \leqslant N_t := \mathrm{PV}(y|t)$, if the sequence $(\mathrm{PV}(y|t))_{t\geqslant 0}$ is bounded, then the sequence $(D_t)_{t\geqslant 0}$ is bounded from above and admits a least upper bound.

[16] Recall that $S^t(s^\xi)$ is the set of all possible date-t events following s^ξ.

Since the present value of the country's future net income is finite, we have

$$\lim_{t\to\infty} p(s^\xi)M^t(s^\xi) = \lim_{t\to\infty} \sum_{s^t\in S^t(s^\xi)} p(s^t)N(s^t) = 0.$$

Therefore, the set

$$\left\{t \in \mathbb{N} : t > \xi \quad \text{and} \quad D(s^\xi) > M^t(s^\xi)\right\}$$

must be non-empty. Let us denote by η the smallest element of the above set. We do have $D(s^\xi) > M^\eta(s^\xi)$.

Claim 5.1. There exists a date $\tau \leqslant \eta$ and an event $\sigma^\tau \succeq s^\xi$ such that
(a) $D(\sigma^\tau) > M^\eta(\sigma^\tau)$;
(b) for every $s^t \succ \sigma^\tau$, we have $D(s^t) \leqslant M^\eta(s^t)$.

Proof. Let $\mathcal{T}$ be the set of dates defined by

$$\mathcal{T} := \left\{t \in \{\xi, \dots, \eta\} : \exists s^t \in S^t(s^\xi), \quad D(s^t) > M^\eta(s^t)\right\}.$$

This set is non-empty since it contains ξ. Let τ be the largest element of $\mathcal{T}$ and σ^τ be any element of $S^\tau(s^\xi)$ such that $D(\sigma^\tau) > M^\eta(\sigma^\tau)$. Property (a) is trivially satisfied. We prove now property (b). Fix an arbitrary $s^t \succ \sigma^\tau$. If $t \in \{\tau+1, \dots, \eta\}$, then by the definition of date τ, we must have $D(s^t) \leqslant M^\eta(s^t)$. If $t > \eta$, then $D(s^t) \leqslant N(s^t) \leqslant M^\eta(s^t)$ by property (iii). Q.E.D

We claim that the sovereign has incentives to default at the event σ^τ. Indeed, if the sovereign honors the contract z, then the associated consumption process $c := y - z$ has to satisfy the following flow constraints

$$\forall s^t \succeq \sigma^\tau, \quad c(s^t) - \sum_{s^{t+1}\succ s^t} q(s^{t+1})D(s^{t+1}) \leqslant y(s^t) - D(s^t).$$

Suppose that, instead of fulfilling its promises, the sovereign chooses to default at event σ^τ. The sovereign can then increase its consumption by $D(\sigma^\tau) - M^\eta(\sigma^\tau) > 0$ at event σ^τ and opt for the asset holding $\tilde{a}(s^t) := M^\eta(s^t) - D(s^t) \geqslant 0$ at all successors $s^t \succ \sigma^\tau$. Letting $\tilde{c}(\sigma^\tau) := c(\sigma^\tau) + D(\sigma^\tau) - M^\eta(\sigma^\tau)$ and recalling that M^η satisfies exact roll-over, we have

$$\tilde{c}(\sigma^\tau) + \sum_{s^{\tau+1}\succ\sigma^\tau} q(s^{\tau+1})\tilde{a}(s^{\tau+1}) \leqslant y(\sigma^\tau).$$

That is, the sovereign can enjoy higher (due to property (a)) consumption at the event σ^τ. Posing $\tilde{c}(s^t) = c(s^t)$ for every strict successor $s^t \succ \sigma^\tau$, we also have that

$$\forall s^t \succ \sigma^\tau, \quad \tilde{c}(s^t) + \sum_{s^{t+1}\succ s^t} q(s^{t+1})\tilde{a}(s^{t+1}) \leqslant y(s^t) + \tilde{a}(s^t).$$

That is, the sovereign can support through savings (due to property (b) and the fact that M^η satisfies exact roll-over) the consumption $\tilde{c}(s^t) = c(s^t)$ specified by the contract z at all successor events $s^t \succ \sigma^\tau$. It follows that

$$V(\sigma^\tau) \geqslant U(\tilde{c}|\sigma^\tau) > U(c|\sigma^\tau)$$

and the incentive compatibility constraint is not satisfied at σ^τ. Q.E.D

6 Discussions

6.1 Capital adjustment after default

In the definition of the default option $V(s^t)$ we assume that the country does not modify its investment plan on the out-of-equilibrium path corresponding to default. This is an ad-hoc restriction. We should replace $V(s^t)$ by the higher opportunity cost $W(s^t)$ defined as the largest continuation utility $U(\tilde{c}|s^t)$ where $\tilde{c}$ is financed by a process $\tilde{a}$ of non-negative bond holdings and a process $\tilde{k}$ of capital investment, i.e., for every event $s^\tau \succeq s^t$,

$$\tilde{c}(s^\tau) + \tilde{k}(s^\tau) + \sum_{s^{\tau+1} \succ s^\tau} q(s^{\tau+1})\tilde{a}(s^{\tau+1}) \leqslant f(s^\tau, \tilde{k}(s^{\tau-1})) + \tilde{a}(s^\tau) \tag{6.1}$$

where $\tilde{k}(s^{t-1}) = k(s^{t-1})$, $\tilde{a}(s^t) = 0$ and $\tilde{a}(s^\tau) \geqslant 0$ for every strict successor $s^\tau \succ s^t$.

If a contract z satisfies the incentive compatibility constraints with respect to $W(s^t)$ then z also satisfies the incentive compatibility constraints with respect to $V(s^t)$. Therefore, since debt cannot be sustained when the outside option is $V(s^t)$, it can neither be sustained when the outside option is $W(s^t)$.

Eaton and Gersovitz (1981), Kehoe and Levine (1993) and Alvarez and Jermann (2000) assume that, after default, the sovereign can neither access international saving markets, nor modify its investment on capital. This leads to autarky and the default option is now $U(y|s^t)$. Such a punishment is severe enough to sustain some level of debt. However, it is hard to justify that the sovereign cannot modify its investment on capital after default. In the environment where a defaulting agent is excluded from borrowing and saving, we should then consider the default option $R(s^t)$ defined as the largest continuation utility $U(\tilde{c}|s^t)$ where $\tilde{c}$ is solely financed by a process $\tilde{k}$ of capital investment, i.e.,

$$\forall s^\tau \succeq s^t, \quad \tilde{c}(s^\tau) + \tilde{k}(s^\tau) \leqslant f(s^\tau, \tilde{k}(s^{\tau-1})) \tag{6.2}$$

where $\tilde{k}(s^{t-1}) = k(s^{t-1})$. Rosenthal (1991) shows that when productivity is deterministic and under some additional conditions on the initial capital holding, debt is never sustained under the default option $R(s^t)$.

6.2 Negative net income

To simplify the presentation, we assume (as in Bulow and Rogoff (1989)) that the current investment in capital is deduced from the current production, i.e., the net income $y(s^t) := f(s^t, k(s^{t-1})) - k(s^t)$ is non-negative. We can handle the general case where net income can be negative at some events if we strengthen Assumption 3.3 by requiring that

$$\mathrm{PV}(y^+|s^t) < \infty \tag{6.3}$$

where $y^+(s^t) := \max\{y(s^t), 0\}$ is the positive part of $y(s^t)$.

To encompass this case it suffices to pose $\widetilde{N}(s^t) := \mathrm{PV}(y^+|s^t)$ and observe that $D(s^t) \leqslant N(s^t) \leqslant \widetilde{N}(s^t)$. The proof of Theorem 5.1 follows verbatim replacing N by $\widetilde{N}$.

Observe that Condition (6.3) is satisfied if gross output (instead of net income) has finite present value, i.e.,

$$\sum_{s^t \in \Sigma} p(s^t) f(s^t, k(s^{t-1})) < \infty$$

where $k(s^{-1})$ is the given initial capital stock.

6.3 More severe punishment: loss of income

We follow Section III.B in Bulow and Rogoff (1989) and analyze debt repayment incentives when, in addition to the exclusion from borrowing, the defaulting country looses a part of its net income.[17] Formally, if the country defaults at event s^t, we denote by $\ell(s^\tau) \in [0, y(s^\tau)]$ the loss in net income at every event $s^\tau \succeq s^t$. The default option $V(s^t)$ is then replaced by $V^\ell(s^t)$ defined as the largest continuation utility $U(\tilde{c}|s^t)$ where $\tilde{c}$ is financed by a process $\tilde{a}$ of non-negative bond holdings and where net income $y(s^\tau)$ is replaced by $\tilde{y}(s^\tau) := y(s^\tau) - \ell(s^\tau)$, i.e.,

$$\forall s^\tau \succeq s^t, \quad \tilde{c}(s^\tau) + \sum_{s^{\tau+1} \succ s^\tau} q(s^{\tau+1}) \tilde{a}(s^{\tau+1}) \leqslant \tilde{y}(s^\tau|s^t) + \tilde{a}(s^\tau) \tag{6.4}$$

where $\tilde{a}(s^t) = 0$ and $\tilde{a}(s^\tau) \geqslant 0$ for every strict successor $s^\tau \succ s^t$.

As in Theorem 2 in Bulow and Rogoff (1989), we get as a direct corollary of Theorem 5.1 that debt can be sustained when default entails some loss of net income, but up to a level that does not exceed the present value of the loss.

Corollary 6.1. *Consider a contract $z = y - c$ that satisfies the break-even condition (3.1). If the debt level at some event s^t is strictly larger than the present value of the loss of net income, i.e.,*

$$\mathrm{PV}(z|s^t) > \mathrm{PV}(\ell|s^t)$$

then there exists a successor event $s^\tau \succeq s^t$ for which the incentive compatibility constraint $U(c|s^\tau) \geqslant V^\ell(s^\tau)$ is not satisfied.

[17] We also refer to Cole and Kehoe (2000), Dutta and Kapur (2002), Aguiar and Gopinath (2006), Arellano (2008), Bai and Zhang (2010), and Mendoza and Yue (2012) for models where default induces a loss in net income in addition to the exclusion from borrowing.

Proof. Consider a contract z satisfying the break-even condition (3.1) and denote by $c := y - z$ the associated consumption process. Assume there exists an event $s^t \succeq s^0$ such that $\mathrm{PV}(z|s^t) > \mathrm{PV}(\ell|s^t)$. Let $\tilde{z}$ be the contract defined on the subtree $\Sigma(s^t)$ by $\tilde{z}(s^\tau) := z(s^\tau) - \ell(s^\tau)$ for every $s^\tau \succeq s^t$. Observe that $\tilde{z} = \tilde{y} - c$ on the subtree $\Sigma(s^t)$ and $\mathrm{PV}(\tilde{z}|s^t) > 0$. We can apply Theorem 5.1 to the economy defined on the subtree $\Sigma(s^t)$ with net income $\tilde{y}$ and conclude that there exists some event $s^\tau \succeq s^t$ such that the corresponding participation constraints $U(c|s^\tau) \geqslant V^\ell(s^\tau)$ is violated. Q.E.D

6.4 Self-enforcing and not-too-tight debt limits

We have argued that the environment in Bulow and Rogoff (1989) can be seen as the analogue of Kehoe and Levine (1993) (i.e., constrained Arrow–Debreu contingent markets) but with a different default punishment.[18] Alvarez and Jermann (2000) study a sequential formulation of Kehoe and Levine (1993). Instead of assuming that agents can trade any contingent contract at the initial date, they consider an environment where agents trade sequentially a complete set of one period contingent bonds (Arrow securities). In this setting, the Arrow–Debreu present value constraint is replaced by a sequence of flow constraints, while the participation constraints are replaced by agent-specific debt limits that are compatible with maximal risk sharing subject to debt repayment being individually rational. Such limits on borrowing are referred in the literature as not-too-tight (self-enforcing) debt limits.

Formally, a non-negative process $A = (A(s^t))_{s^t \in \Sigma}$ of debt limits is said to be not-too-tight when

$$\forall s^t \in \Sigma, \quad J(A, -A(s^t)|s^t) = V(s^t)$$

where, given an initial financial claim b at event s^t, the value $J(A, b|s^t)$ is the largest continuation utility $U(c|s^t)$ where c is financed by a process a of bond holdings satisfying

$$\forall s^\tau \succeq s^t, \quad c(s^\tau) + \sum_{s^{\tau+1} \succ s^\tau} q(s^{\tau+1}) a(s^{\tau+1}) \leqslant y(s^\tau) + a(s^\tau) \tag{6.5}$$

with

$$a(s^t) = b \quad \text{and} \quad a(s^\tau) \geqslant -A(s^\tau), \quad \text{for every } s^\tau \succ s^t. \tag{6.6}$$

Hellwig and Lorenzoni (2009) (see Proposition 3 in their paper) show that if A is not-too-tight and bounded from above by the natural debt limits N, then we must have $A = 0$. We propose to show that this result can be obtained as a direct corollary of Theorem 5.1 provided that preferences, apart from being strictly increasing, are also dynamically consistent.

[18] In Bulow and Rogoff (1989) the outside option is $V(s^t)$ which is larger than the autarchic outside option $U(y|s^t)$ of Kehoe and Levine (1993).

Definition 6.1. Preferences are dynamically consistent if for every event $s^t \in \Sigma$, for every consumption processes c and $\tilde{c}$, the following conditions

$$\tilde{c}(s^t) = c(s^t) \quad \text{and} \quad U(\tilde{c}|s^{t+1}) \geqslant U(c|s^{t+1}), \quad \forall s^{t+1} \succ s^t$$

with at least one strict inequality for some event s^{t+1}, imply that $U(\tilde{c}|s^t) > U(c|s^t)$.

Remark 6.1. Assume that preferences satisfy the general recursive functional form of Remark 3.1 where the intertemporal aggregator $W : \mathbb{R}^2 \to \mathbb{R}$ is strictly increasing and the event s^t certainty equivalent $\mathcal{M}_{s^t}((U(c|s^{t+1}))_{s^{t+1} \succ s^t})$ of future continuation utility $U(c|s^{t+1})$ is strictly increasing. Then, such preferences are dynamically consistent and strictly increasing.

Proposition 6.1. Suppose that preferences are dynamically consistent and strictly increasing. If a process A of non-negative and not-too-tight debt limits is tighter than natural debt limits (i.e., $A \leqslant N$), then $A(s^t) = 0$ for every event $s^t \in \Sigma$.

Proof. For every event s^t and initial financial wealth $b \in \mathbb{R}$, we denote by $B(A, b|s^t)$ the budget set of pairs (c, a) of consumption and bond holdings satisfying (6.5) and (6.6). The demand set $d(A, b|s^t)$ is defined to be the set of optimal pairs in $B(A, b|s^t)$, i.e., $(c, a) \in d(A, b|s^t)$ if, and only if, $(c, a) \in B(A, b|s^t)$ and $U(c|s^t) = J(A, b|s^t)$.

We only prove that $A(s^0) = 0$. The result for an arbitrary event s^t is obtained replacing the whole tree Σ by the subtree $\Sigma(s^t)$. Let (c, a) be an optimal plan in the demand set $d(A, -A(s^0)|s^0)$. We first prove that the consumption process c is self-enforcing. Dynamic consistency of preferences implies that for every s^t, the plan (c, a) belongs to the demand set $d(A, a(s^t)|s^t)$. Given that $a(s^t) \geqslant -A(s^t)$, monotonicity implies that

$$U(c|s^t) = J(A, a(s^t)|s^t) \geqslant J(A, -A(s^t)|s^t) = V(s^t).$$

Let $\tilde{c}$ be the consumption process defined by $\tilde{c}(s^0) := c(s^0) + A(s^0)$ and $\tilde{c}(s^t) := c(s^t)$ for every strict successor $s^t \succ s^0$. Observe that $\tilde{c}$ satisfies the incentive compatibility constraints (3.4). We now prove that it satisfies the break-even condition (3.1). Summing the present value of the flow budget constraints, we have for every $\tau > 1$

$$\sum_{t=0}^{\tau-1} \sum_{s^t \in S^t} p(s^t)(c(s^t) - y(s^t)) - \sum_{s^\tau \in S^\tau} p(s^\tau)A(s^\tau) \leqslant -p(s^0)A(s^0).$$

Since $0 \leqslant A(s^\tau) \leqslant N(s^\tau)$ we can pass to the limit when $\tau \to \infty$ to get

$$\mathrm{PV}(c - y|s^0) \leqslant -A(s^0), \quad \text{or, equivalently,} \quad \mathrm{PV}(y - \tilde{c}|s^0) \geqslant 0.$$

Applying Theorem 5.1 to the contract $\tilde{z} := y - \tilde{c}$, we get that $\tilde{a}(s^t) := \mathrm{PV}(\tilde{c} - y|s^t) \geqslant 0$ for every $s^t \succ s^0$. It follows that the pair $(\tilde{c}, \tilde{a})$ belongs to the budget set $B(0, 0|s^0)$ implying that $V(s^0) \geqslant U(\tilde{c}|s^0)$. By definition of $A(s^0)$, we have $U(c|s^0) = J(A, -A(s^0)|s^0) = V(s^0)$. This implies that $U(c|s^0) \geqslant U(\tilde{c}|s^0)$. Since $\tilde{c}(s^0) = c(s^0) + A(s^0)$ and $\tilde{c}(s^t) = c(s^t)$ for every $s^t \succ s^0$, strict monotonicity implies that we must have $A(s^0) = 0$. Q.E.D

Remark 6.2. In order to prove that $A(s^0) = 0$, we can adapt in a straightforward manner the above proof to show that the assumption "the debt limits in A are non-negative and tighter than the natural debt limits" can be replaced by: $A(s^0) \geqslant 0$ and the process A satisfies the following transversality condition:

$$\lim_{\tau \to \infty} \sum_{s^\tau \in S^\tau} p(s^\tau) A(s^\tau) = 0.$$

We make use of this observation in the proof of Proposition 6.2.

We next explore the implications of our analysis to models where the continuation utility after default at event s^t is some arbitrary number $W(s^t)$. If the default punishment is stronger, in the sense that $W(s^t) \leqslant V(s^t)$, one expect that some level of self-enforcing debt can be sustained at equilibrium. The polar example is when $W(s^t) = U(y|s^t)$ as in Kehoe and Levine (1993) or Alvarez and Jermann (2000). Another example is temporary exclusion as in Azariadis and Kaas (2013).

Using Theorem 5.1, we can show that the maximum level of self-enforcing debt is unique.

Proposition 6.2. Suppose that preferences are dynamically consistent and strictly increasing. If A and A' are two processes of non-negative, not-too-tight debt limits that are tighter than natural debt limits, then $A = A'$.

Proof. We only prove that $A(s^0) = A'(s^0)$. The result for an arbitrary event s^t can be obtained replacing the whole tree Σ by the subtree $\Sigma(s^t)$. We can assume without any loss of generality that $A(s^0) \geqslant A'(s^0)$. We let y' be the process defined by

$$y'(s^t) := y(s^t) - A'(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) A'(s^{t+1}).$$

Fix an event s^t and let (c', a') be an optimal plan in the set $d(A', -A'(s^t)|s^t)$. Since $c'(s^t) \geqslant 0$, we must have $y'(s^t) \geqslant 0$. We denote by $\mathcal{E}'$ the economy where the income process y is replaced by y'. Observe that the plan $(c', a' + A')$ belongs to the set $d'(0, 0|s^t)$ where d' is the demand of the economy $\mathcal{E}'$. In particular, we have $W(s^t) = V'(s^t)$ where $V'(s^t) := J'(0, 0|s^t)$ is the Bulow–Rogoff's default continuation utility in the economy $\mathcal{E}'$. It follows from a straightforward translation invariance of the budget constraints (6.5) and (6.6) that the process $D := A - A'$ satisfies

$$J(A, -A(s^t)|s^t) = J'(D, -D(s^t)|s^t), \quad \text{for all } s^t.$$

Even if D is not necessarily non-negative, the right-hand side of the above inequality is well-defined. Observe that the above equality means that D is not-too-tight in the economy $\mathcal{E}'$. Moreover, D satisfies the conditions of Remark 6.2

since $D(s^0) \geqslant 0$ and $|D| \leqslant 2N$. We can then apply Proposition 6.1 to get that $D(s^0) = 0$. Q.E.D

Proposition 6.2 extends Proposition 3 in Bidian and Bejan (2015) in the same way Proposition 6.1 extends Proposition 3 in Hellwig and Lorenzoni (2009). Since these results follow from an arbitrage argument, we do not need to assume that preferences are additively separable with a strictly concave and differentiable Bernoulli function. We only assume preference relations to be dynamically consistent and strictly increasing.

7 Conclusion

Bulow and Rogoff (1989) consider a competitive environment where creditors have no legal recourse in the event of a default. They show that if debtors have access to a rich set of deposit contracts to put savings abroad rather than repaying creditors, then debt cannot be sustained at equilibrium. Their argument is surprisingly simple and applies under very general assumptions. Formally, when the ratio of debt to natural debt limit attains it maximum value, the sovereign is better off not repaying this debt, but rather use the scheduled payments to buy cash-in-advance contracts as a form of self-insurance. However, Bulow and Rogoff (1989)'s argument may fail unless we impose additional restrictions on the sovereign's net income. Indeed, we show, by means of examples, that if there is a path along which the sovereign's net income may vanish or grow unboundedly, then there may exist feasible contracts for which the ratio of the debt to the natural debt limit does not achieve a maximum at any contingency, in which case their proof fails. We provide an alternative proof which restores unconditionally the validity of Bulow and Rogoff (1989)'s no-borrowing result.

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