EXTENDED AFFINE WEYL GROUPS OF BCD-TYPE: THEIR FROBENIUS MANIFOLDS AND LANDAU–GINZBURG SUPERPOTENTIALS

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ABSTRACT. For the root systems of type B_l, C_l and D_l, we generalize the result of [7] by showing the existence of Frobenius manifold structures on the orbit spaces of the extended affine Weyl groups that correspond to any vertex of the Dynkin diagram instead of a particular choice made in [7]. It also depends on certain additional data. We also construct Landau–Ginzburg superpotentials for these Frobenius manifold structures.

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1. Introduction

1.1. **Background.** In [7], the precursor to this paper, the Frobenius manifold structures on the orbit spaces $\mathbb{C}^{l+1}/\widetilde{W}$, where $\widetilde{W}$ are certain extended affine Weyl groups, were constructed. In particular, the construction depended on a choice of a specific node on the Dynkin diagram of the underlying Weyl group W. In the case $W \cong A_l$ this marked node could be taken to be an arbitrary node, but for the remaining cases a specific node was used. In addition, again for the case $W \cong A_l$, a Landau–Ginzburg (or LG) superpotential construction of the Frobenius manifold structure was given.

The aim of this paper is to complete the construction for the underlying Weyl groups $W \cong B_l, C_l, D_l$, providing:

- a construction of the Frobenius manifold structure on the orbit space for an *arbitrary* marked node;
- a LG superpotential construction for all these manifolds.

A remaining problem is to extend the construction to the few remaining exceptional Weyl groups, and we hope to return to this problem in a later paper.

The construction rests on the proof of a Chevalley-type theorem for invariant polynomials for the extended-affine Weyl groups, together with a Saito-construction for the flat coordinates on the orbit space. Besides this, there is another construction, which is provided by a LG superpotential construction, and involves a refinement of the genus zero Hurwitz space theory to the case of cosine-Laurent polynomials. These two separate constructions can then be shown to result in isomorphic Frobenius manifolds.

1.2. **The main results.** Let R be an irreducible reduced root system defined in an l-dimensional Euclidean space V with the Euclidean inner product $(\ ,\ )$. We fix a basis of simple roots $\alpha_1, \ldots, \alpha_l$ and denote by $\alpha_j^\vee,\ j = 1, 2, \cdots, l$ the corresponding coroots. The Weyl group W is generated by the reflections

$$\mathbf{x} \mapsto \mathbf{x} - (\alpha_j^\vee, \mathbf{x})\alpha_j, \quad \forall\, \mathbf{x} \in V,\ j = 1, \ldots, l. \tag{1.1}$$

Recall that the *Cartan matrix* of the root system has integer entries $A_{ij} = \left(\alpha_i, \alpha_j^\vee\right)$ satisfying $A_{ii} = 2$, $A_{ij} \le 0$ for $i \ne j$. The semi-direct product of W by the lattice of coroots yields the affine Weyl group W_a that acts on V by the affine transformations

$$\mathbf{x} \mapsto w(\mathbf{x}) + \sum_{j=1}^{l} m_j \alpha_j^\vee, \quad w \in W,\ m_j \in \mathbb{Z}. \tag{1.2}$$

We denote by $\omega_1, \dots, \omega_l$ the fundamental weights defined by the relations

$$(\omega_i, \alpha_j^\vee) = \delta_{ij}, \quad i, j = 1, \dots, l. \tag{1.3}$$

Note that the root system R is one of the type $A_l, B_l, C_l, D_l, E_6, E_7, E_8,\ F_4, G_2$. In what follows the Euclidean space V and the basis $\alpha_1, \dots, \alpha_l$ of the simple roots will be defined as in Plate I-IX of [2]. Let us fix a simple root α_k and define an extension of the affine Weyl group W_a in a similar way as was done in [7].

Definition 1.1. *The extended affine Weyl group* $\widetilde{W} = \widetilde{W}^{(k)}(R)$ *acts on the extended space*

$$\widetilde{V} = V \oplus \mathbb{R}\,,$$

and is generated by the transformations

$$x = (\mathbf{x}, x_{l+1}) \mapsto (w(\mathbf{x}) + \sum_{j=1}^{l} m_j \alpha_j^\vee,\ x_{l+1}), \quad w \in W,\ m_j \in \mathbb{Z}, \tag{1.4}$$

and

$$x = (\mathbf{x}, x_{l+1}) \mapsto (\mathbf{x} + \gamma\, \omega_k,\ x_{l+1} - \gamma). \tag{1.5}$$

Here $1 \le k \le l$, $\gamma = 1$ *except for the cases when* $R = B_l, k = l$ *and* $R = F_4, k = 3$ *or* $k = 4$, *in these three cases* $\gamma = 2$.

The above definition of the extended affine Weyl group coincides with the one given in [7] for the particular choice of α_k made there. We note that in the cases for which $\gamma = 1$ the numbers $\frac{1}{2}(\alpha_k, \alpha_k)$ are integers, while for the three exceptional cases $\frac{1}{2}(\alpha_k, \alpha_k) = \frac{1}{2}$. Since we will only be considering the non-exceptional cases, we will take $\gamma = 1$ for the remainder of the paper.

Coordinates $x_1,\dots,x_l$ may be introduced on the space V via the expression

$$\mathbf{x} = x_1\alpha_1^\vee + \cdots + x_l\alpha_l^\vee. \tag{1.6}$$

Let $f=\det(A_{ij})$, the determinant of the Cartan matrix of the root system R.

Definition 1.2 ([7]). *$\mathcal{A}=\mathcal{A}^{(k)}(R)$ is the ring of all $\widetilde{W}$-invariant Fourier polynomials of the form*

$$\sum_{m_1,\dots,m_{l+1}\in\mathbb{Z}} a_{m_1,\dots,m_{l+1}} e^{2\pi i(m_1x_1+\cdots+m_lx_l+\frac{1}{f}m_{l+1}x_{l+1})}$$

bounded in the limit

$$\mathbf{x}=\mathbf{x}^0 - i\omega_k\tau, \quad x_{l+1}=x^0_{l+1}+i\tau, \quad \tau\to+\infty \tag{1.7}$$

for any $x^0=(\mathbf{x}^0, x^0_{l+1})$.

Condition (1.7) is essential for this construction.[1] Constraints also appear in the more abstract constructions in [14, 18, 20] and an open problem is to relate these seemingly different sets of constraints.

We introduce a set of numbers

$$d_j=(\omega_j,\omega_k), \quad j=1,\dots,l \tag{1.8}$$

and define the following Fourier polynomials [7]

$$\tilde{y}_j(x) = e^{2\pi i d_j x_{l+1}} y_j(\mathbf{x}), \quad j=1,\dots,l, \tag{1.9}$$

$$\tilde{y}_{l+1}(x) = e^{\frac{2\pi i}{\gamma}x_{l+1}}. \tag{1.10}$$

Here $y_1(\mathbf{x}),\dots,y_l(\mathbf{x})$ are the basic W_a-invariant Fourier polynomials defined by

$$y_j(\mathbf{x}) = \frac{1}{n_j}\sum_{w\in W} e^{2\pi i(\omega_j, w(\mathbf{x}))}, \quad n_j=\#\{w\in W | e^{2\pi i(\omega_j,w(\mathbf{x}))}=e^{2\pi i(\omega_j,\mathbf{x})}\}. \tag{1.11}$$

It was shown in [7] that for certain particular choices of the simple root α_k, a Chevalley-type theorem holds true for the ring $\mathcal{A}$, i.e., it is isomorphic to the

[1] From the invariance with respect to $\widetilde{W}$, it easily follows (using theorem [B] in [7, 2]) that any $f(x)$ can be represented as a polynomial in $\tilde{y}_1(x),\dots,\tilde{y}_{l+1}(x),\tilde{y}_{l+1}(x)^{-1}$. The boundedness condition (1.7) is equivalent to the restriction to invariants which extend over the locus $\tilde{y}_{l+1}=0$. With this condition, $\mathcal{M}=\mathrm{Spec}(\mathcal{A})$ is a partial compactification of the full complex orbit space.

polynomial ring generated by $\tilde{y}_1, \ldots, \tilde{y}_{l+1}$, and thus the orbit space of the extended affine Weyl group $\widetilde{W}$ *defined* as $\mathcal{M} = \operatorname{Spec} \mathcal{A}$ is an affine algebraic variety of dimension $l+1$. In [7] it was further proved that on such an orbit space there exists a Frobenius manifold structure whose potential is a polynomial of $t^1, \ldots, t^{l+1}, e^{t^{l+1}}$. Here $t^1, \ldots, t^{l+1}$ are the flat coordinates of the Frobenius manifold. For the root system of type A_l, there is in fact no restrictions on the choice of α_k. However, for the root systems of type $B_l, C_l, D_l, E_6, E_7, E_8, F_4, G_2$ there is only one choice for each system.

In [18] Slodowy pointed out that the Chevalley-type theorem of [7] is a consequence of the results of Looijenga and Wirthmüller [13, 14, 20], and in fact it holds true for any choice of the base element α_k, or equivalently, for any fixed vertex of the Dynkin diagram. Hence:

Theorem 1.3 ([18, 20, 13, 14])**.** *The ring $\mathcal{A}$ is isomorphic to the ring of polynomials of $\tilde{y}_1(x)$, $\cdots$, $\tilde{y}_{l+1}(x)$.*

A natural question, as was raised in [7, 18], is *whether the geometric structures revealed in* [7] *also exist on the orbit spaces of the extended affine Weyl groups for an arbitrary choice of the root α_k?* The purpose of the present paper is to give an affirmative answer to this question for the root systems of type B_l, C_l and D_l (recall that for the root system of type A_l, the question was already answered affirmatively in [7]). We will therefore concentrate on these cases, the proofs of which turn out to work in a very similar manner. In Sec.2 we give an elementary proof of Theorem 1.3 for the root systems of type B_l, C_l and D_l that is based on the proof of the Chevalley type theorem given in [7].

Let $\mathcal{M}$ be the orbit space of the extended affine Weyl group $\widetilde{W}^{(k)}(C_l)$ and $\widetilde{\mathcal{M}}$ the universal covering of $\mathcal{M} \setminus \{\tilde{y}_{l+1} = 0\}$. In Sec.3, firstly we introduce an indefinite metric [2] $(\ ,\ )^{\sim}$ on $T^*\widetilde{V}_{\mathbb{C}}$, the complexification $\widetilde{V}_{\mathbb{C}} = \widetilde{V} \otimes_{\mathbb{R}} \mathbb{C}$ of the extended space $\widetilde{V} = V \oplus \mathbb{R}$. This is defined by

$$(dx_s, dx_n)^{\sim} = \frac{s}{4\pi^2}, \quad (dx_s, dx_{l+1})^{\sim} = 0, \quad (dx_{l+1}, dx_{l+1})^{\sim} = -\frac{1}{4k\pi^2} \qquad (1.12)$$

[2] As is common in the Frobenius manifold literature, we use the word metric to denote a complex-valued, symmetric, non-degenerate, bilinear form.

for $1 \le s \le n \le l$. The projection

$$\mathrm{Pr}: \widetilde{V} \to \widetilde{\mathcal{M}}, \qquad (x_1, \cdots, x_{l+1}) \mapsto (y^1, \cdots, y^{l+1})$$

induces a symmetric bilinear form on $T^*\widetilde{\mathcal{M}}$ defined by the matrix

$$g^{ij}(y) := \sum_{a,b=1}^{l+1} \frac{\partial y^i}{\partial x_a} \frac{\partial y^j}{\partial x_b} (dx_a, dx_b)^{\sim}, \tag{1.13}$$

in the coordinates $y^1 = \tilde{y}_1, \dots, y^l = \tilde{y}_l,\ y^{l+1} = \log \tilde{y}_{l+1} = 2\pi i x_{l+1}$. We will show in Sec. 3 that the functions $g^{ij}(y)$ are homogenous polynomials of $y^1, \dots, y^l, e^{y^{l+1}}$, so they are well defined on $\mathcal{M}$. Denote

$$\Sigma = \{(y^1, \dots, y^l, e^{y^{l+1}}) \in \mathcal{M} \mid \det(g^{ij}(y)) = 0\}. \tag{1.14}$$

Then Σ called the *discriminant* of the extended affine Weyl group $\widetilde{W}^{(k)}(C_l)$ is an algebraic subvariety of $\mathcal{M}$. It consists of the orbits of points $(\mathbf{x}, x_{l+1}) \in \widetilde{V}$ on the *mirrors* of the group with $(\beta, \mathbf{x}) \in \mathbb{Z}$ for some positive root β, the details of which will be given in Section 3 below. Afterwards, we will construct another symmetric bilinear form on $T^*\mathcal{M}$ by

$$\eta^{ij}(y) := \mathcal{L}_e g^{ij}(y). \tag{1.15}$$

Here the vector field e has the form

$$e = \sum_{j=k}^{l} c_j \frac{\partial}{\partial y^j}, \tag{1.16}$$

it depends on the choice of an integer m in the range $0 \le m \le l-k$. Namely, for a given m the coefficients $c_k, \dots, c_l$ are defined by the generating function

$$\sum_{j=k}^{l} c_j u^{l-j} = (u+2)^m (u-2)^{l-k-m}.$$

The symmetric bilinear forms (η^{ij}) is non-degenerate on $\mathcal{M} \setminus \Sigma_1 \cup \Sigma_2$, where

$$\Sigma_1 = \{f_1 = 0\} \subset \mathcal{M}, \quad \Sigma_2 = \{f_2 = 0\} \subset \mathcal{M}$$

are the loci [3] of zeros of the following $\widetilde{W}^{(k)}(C_l)$-invariant polynomials

$$f_1 = e^{2\pi i k\, x_{l+1}} \prod_{j=1}^{l} \cos^2 \pi(x_j - x_{j-1}), \quad f_2 = e^{2\pi i k\, x_{l+1}} \prod_{j=1}^{l} \sin^2 \pi(x_j - x_{j-1}) \quad (1.17)$$

with $x_0 = 0$, and it gives the flat metric of the Frobenius manifold structure that we are to construct. Denote

$$\mathcal{M}_{k,m}(C_l) := \begin{cases} \mathcal{M} \setminus \{\tilde{y}_{l+1} = 0\} \cup \Sigma_1 \cup \Sigma_2, & \text{when } 0 < m < l-k; \\ \mathcal{M} \setminus \{\tilde{y}_{l+1} = 0\} \cup \Sigma_1, & \text{when } m = 0; \\ \mathcal{M} \setminus \{\tilde{y}_{l+1} = 0\} \cup \Sigma_2, & \text{when } m = l-k. \end{cases}$$

Then we will show the following result.

Main Theorem 1. *For any fixed integer $0 \le m \le l-k$, there exists a unique Frobenius manifold structure of charge $d = 1$ on the orbit space $\mathcal{M}_{k,m}(C_l)$ of $\widetilde{W}^{(k)}(C_l)$ such that*

(1) *the invariant flat metric and the intersection form of the Frobenius manifold structure coincide with the metrics $(\eta^{ij}(y))$ (see eq. (1.15)) and $(g^{ij}(y))$ (see eq. (1.13)) respectively;*

(2) *the unity and the Euler vector fields have the form*

$$e = \sum_{j=k}^{l} c_j \frac{\partial}{\partial y^j} \quad (1.18)$$

and

$$E = \sum_{\alpha=1}^{l} \frac{d_\alpha}{d_k} y^\alpha \frac{\partial}{\partial y^\alpha} + \frac{1}{d_k} \frac{\partial}{\partial y^{l+1}}, \quad (1.19)$$

where $d_1, \ldots, d_l$ are defined in (2.7);

(3) *in the flat coordinates $t^1, \ldots, t^{l+1}$ of the metric (1.15) defined on certain covering of $\mathcal{M}_{k,m}(C_l)$ the Frobenius manifold structure is polynomial in*

[3] Vanishing of f_1 or f_2 is equivalent to the condition $2(x_j - x_{j-1}) \in \mathbb{Z}$ for some j. Thus the loci Σ_1, Σ_2 consist of the orbits of points belonging to the mirrors of the affine Weyl group corresponding to the long roots. Hence $\Sigma_1 \cup \Sigma_2 \subset \Sigma$. We are grateful to the anonymous referee for this observation.

$t^1, \cdots, t^{l+1}, \frac{1}{t^{l-m}}, \frac{1}{t^l}, e^{t^{l+1}}$. In these coordinates

$$e = \frac{\partial}{\partial t^k} \tag{1.20}$$

and

$$E = \sum_{\alpha=1}^{l} \tilde{d}_\alpha t^\alpha \frac{\partial}{\partial t^\alpha} + \frac{1}{k}\frac{\partial}{\partial t^{l+1}}, \tag{1.21}$$

where $\tilde{d}_1, \dots, \tilde{d}_l$ are defined in (3.63)–(3.65).

We prove the above theorem in Sec. 4. Let us note that the monodromy group of the Frobenius manifold $\mathcal{M}_{k,m}(C_l)$ (for the definition see [6]) is isomorphic to $\widetilde{W}^{(k)}(C_l)$.

In Sec.4 we further show that for the root systems of type B_l and D_l we can apply a similar construction as the one for the root system of type C_l. The resulting Frobenius manifolds are isomorphic to those obtained from the root system of type C_l.

For the case of A_l an alternative construction of the Frobenius manifold structure was given in [7]. This structure was given in terms of a LG superpotential construction. In particular, it was shown that the extended affine Weyl group $\widetilde{W}^{(k)}(A_l)$ describes the monodromy of roots of trigonometric polynomials - the superpotential - with a given bidegree being of the form

$$\lambda(\varphi) = e^{ik\varphi} + a_1 e^{i(k-1)\varphi} + \cdots + a_{l+1} e^{i(k-l-1)\varphi}, \quad a_{l+1} \neq 0.$$

A natural question is *does there exist a similar construction for the root systems of type B_l, C_l and D_l?* In Sec.5, let us denote by $\mathfrak{M}_{k,m,n}$ the space of a particular class of cosine Laurent series or superpotentials of one variable with a given tri-degree $(2k, 2m, 2n)$ being of the form

$$\lambda(\varphi) = \left(\cos^2(\varphi) - 1\right)^{-m} \sum_{j=0}^{k+m+n} a_j \cos^{2(k+m-j)}(\varphi),$$

where all $a_j \in \mathbb{C}$, $m, n \in \mathbb{Z}_{\geq 0}$, $k \in \mathbb{N}$, and the coefficients $a_0, \dots, a_{k+m+n}$ satisfy the conditions given in (5.2)–(5.4). The space $\mathfrak{M}_{k,m,n}$ carries a natural structure of Frobenius manifold. Its invariant inner product η and the intersection form

g of two vectors ∂', ∂'' tangent to $\mathfrak{M}_{k,m,n}$ at a point $\lambda(\varphi)$ can be defined by the formulae (5.5) and (5.6). We will show that (see Theorem 5.6)

Main Theorem 2. *The Frobenius manifolds $\mathcal{M}_{k,m}(C_{k+m+n})$ and $\mathfrak{M}_{k,m,n}$ are locally isomorphic.*

A function involved in the representation of the form (5.5), (5.6) of the flat pencil of metrics on the Frobenius manifold is called a *LG superpotential* of the Frobenius manifold. Observe that the multiplication law on the tangent spaces to the Frobenius manifold can also be expressed in terms of the LG superpotential (see eq. (5.8) below).

Some concluding remarks are given in the last section.

2. The proof of Theorem 1.3 for the root systems of type B_l, C_l, D_l

In this section, we give an elementary proof of the Theorem 1.3 for the root systems of type B_l, C_l and D_l for any fixed vertex of the Dynkin diagram. To this end, we first write down the explicit expressions of the invariant Fourier polynomials $\tilde{y}_j(x)$ that are defined by (1.9), (1.10) for these root systems with the fixed simple root α_k. We then prove the theorem by using an approach that is similar to the one used in [7].

For the root system of type B_l, the numbers d_j defined in (1.8) have the values

$$d_i = i,\ 1 \le i \le k, \quad d_j = k,\ k+1 \le j \le l-1, \quad d_l = \frac{k}{2}, \tag{2.1}$$

for $k < l$ and

$$d_i = \frac{i}{2},\ 1 \le i \le l-1, \quad d_k = \frac{l}{4} \tag{2.2}$$

for $k = l$. The W_a-invariant Fourier polynomials $y_1(\mathbf{x}), \dots, y_l(\mathbf{x})$ defined in (1.11) have the expressions [12]

$$y_j(\mathbf{x}) = \sigma_j(\xi_1, \cdots, \xi_l), \quad j = 1, \dots, l-1, \tag{2.3}$$

$$y_l(\mathbf{x}) = \prod_{j=1}^{l} \left(e^{i\pi v_j} + e^{-i\pi v_j}\right), \tag{2.4}$$

where

$$\begin{aligned} &v_1 = x_1, \quad v_m = x_m - x_{m-1}, \quad 2 \le m \le l-1, \\ &v_l = 2x_l - x_{l-1}, \\ &\xi_j = e^{2\,i\pi v_j} + e^{-2\,i\pi v_j}, \quad 1 \le j \le l. \end{aligned} \tag{2.5}$$

Here and henceforth the functions $\sigma_j(\xi_1, \dots, \xi_l)$ denote the j-th elementary symmetric polynomial of $\xi_1, \cdots, \xi_l$ defined by

$$\prod_{j=1}^{l}(z + \xi_j) = \sum_{j=0}^{l} \sigma_j(\xi_1, \dots, \xi_l) z^{l-j}. \tag{2.6}$$

For the root system of type C_l, the numbers d_j are given by

$$d_1 = 1, \dots, d_{k-1} = k-1, \; d_j = k, \quad k \le j \le l. \tag{2.7}$$

The W_a-invariant Fourier polynomials $y_1(\mathbf{x}), \dots, y_l(\mathbf{x})$ defined in (1.11) have the expressions

$$y_j(\mathbf{x}) = \sigma_j(\xi_1, \cdots, \xi_l). \tag{2.8}$$

Here ξ_j are defined by

$$\xi_j = e^{2i\pi(x_j - x_{j-1})} + e^{-2i\pi(x_j - x_{j-1})}, \quad x_0 = 0, \; 1 \le j \le l.$$

For the root system of type D_l, we have

i)

$$d_j = j, \; 1 \le j \le k, \quad d_j = k, \; k+1 \le j \le l-2, \tag{2.9}$$

$$d_j = \frac{k}{2}, \; j = l-1, l \tag{2.10}$$

for $k \le l-2$; and

ii)

$$d_j = \frac{j}{2}, \; 1 \le j \le l-2, \quad d_{l-1} = \frac{l}{4}, \quad d_l = \frac{l-2}{4} \tag{2.11}$$

for $k = l-1$; and

iii)

$$d_j = \frac{j}{2}, \; 1 \le j \le l-2, \quad d_{l-1} = \frac{l-2}{4}, \quad d_l = \frac{l}{4} \tag{2.12}$$

for $k=l$. The basis of the W_a-invariant Fourier polynomials defined in (1.11) has the form

$$\begin{aligned} y_j(\mathbf{x}) &= \sigma_j(\xi_1,\cdots,\xi_l), \quad j=1,\ldots,l-2,\\ y_{l-1}(\mathbf{x}) &= \frac{1}{2}\left(\prod_{j=1}^{l}\left(e^{i\pi v_j}+e^{-i\pi v_j}\right)+\prod_{j=1}^{l}\left(e^{i\pi v_j}-e^{-i\pi v_j}\right)\right), \\ y_l(\mathbf{x}) &= \frac{1}{2}\left(\prod_{j=1}^{l}\left(e^{i\pi v_j}+e^{-i\pi v_j}\right)-\prod_{j=1}^{l}\left(e^{i\pi v_j}-e^{-i\pi v_j}\right)\right), \end{aligned} \tag{2.13}$$

where

$$\begin{aligned} & v_1 = x_1, \quad v_m = x_m - x_{m-1}, \quad 2\le m\le l-2,\\ & v_{l-1} = x_l + x_{l-1} - x_{l-2}, \quad v_l = x_{l-1} - x_l,\\ & \xi_j = e^{2i\pi v_j} + e^{-2i\pi v_j}, \quad 1\le j\le l. \end{aligned} \tag{2.14}$$

Proof of Theorem 1.3 for the root system $R=B_l, C_l, D_l$. From the explicit expressions of the Fourier polynomials $\tilde y_1(x),\ldots,\tilde y_{l+1}(x)$, it is not difficult to see that they are $\widetilde{W}^{(k)}(R)$-invariant. So in order to prove the theorem, we only need to show that any element $f(x)$ of the ring $\mathcal{A}$ can be expressed as a polynomial of $\tilde y_1(x),\ldots,\tilde y_{l+1}(x)$. By using the fact that the ring of W_a-invariant Fourier polynomials is isomorphic to the polynomial ring generated by $y_1(\mathbf{x}),\ldots,y_l(\mathbf{x})$ and by using the $\widetilde{W}$-invariance of the function $f(x)\in\mathcal{A}$, we can represent it as a polynomial of $\tilde y_1(x),\ldots,\tilde y_l(x),\tilde y_{l+1}(x),\tilde y_{l+1}^{-1}$. Assume

$$f(x)=\sum_{n\ge -N}\tilde y_{l+1}^n P_n(\tilde y_1(x),\ldots,\tilde y_l(x)),$$

and that the polynomial $P_{-N}(\tilde y_1(x),\ldots,\tilde y_l(x))$ does not vanish identically for a certain positive integer N. From the definition of the functions $\tilde y_j(x)$ we know that in the limit (1.7) we have

$$y_j(\mathbf{x}) = e^{2\pi d_j\tau}[y_j^0(\mathbf{x}^0)+\mathcal{O}(e^{-2\alpha\pi\tau})], \quad j=1,\ldots,l, \tag{2.15}$$

where α is a certain positive integer and the expressions of the functions $y_j^0(\mathbf{x}^0)$ will be given below. So in the limit (1.7) the function $f(x)$ behaves as

$$f(x) = e^{\frac{2\pi}{\gamma}N\tau - \frac{2\pi i}{\gamma}Nx_{l+1}^0}[P_{-N}(\tilde{y}_1^0(x^0), \ldots, \tilde{y}_l^0(x^0)) + \mathcal{O}(e^{-2\beta\pi\tau})]$$

for a certain positive integer β and

$$\tilde{y}_j^0(x^0) = e^{2\pi i d_j x_{l+1}^0}\; y_j^0(\mathbf{x}^0), \quad j = 1, \cdots, l.$$

Since the function $f(x)$ is bounded for $\tau \to +\infty$, we must have

$$P_{-N}(\tilde{y}_1^0(x^0), \ldots, \tilde{y}_l^0(x^0)) \equiv 0$$

for any $x^0 = (\mathbf{x}^0, x_{l+1}^0)$. This leads to a contradiction to the algebraic independence of the functions $\tilde{y}_1^0, \ldots, \tilde{y}_l^0$, a fact that we will now prove, case-by-case, for the root systems of type B_l, C_l and D_l.

i) For the root system of type B_l with $1 \le k \le l-1$,

$$\begin{aligned} y_j^0(\mathbf{x}^0) &= \rho_j, \quad j = 1, \cdots, k, \\ y_s^0(\mathbf{x}^0) &= \rho_k \rho_s, \quad s = k+1, \cdots, l-1, \\ y_l^0(\mathbf{x}^0) &= \sqrt{\rho_k}\rho_l\,, \end{aligned}$$

where the functions ρ_i are defined by

$$\begin{aligned} \rho_j &= \sigma_j(e^{2\pi i v_1^0}, \cdots, e^{2\pi i v_k^0}), \quad j = 1, \cdots, k, \\ \rho_s &= \sigma_{s-k}(\xi_{k+1}^0, \cdots, \xi_l^0), \quad s = k+1, \cdots, l-1, \\ \rho_l &= \prod_{s=k+1}^{l} \left(e^{i\pi v_s^0} + e^{-i\pi v_s^0}\right) \end{aligned}$$

with

$$\begin{aligned} &\xi_m^0 = e^{2\pi i v_m^0} + e^{-2\pi i v_m^0}, \quad m = 1, \cdots, l, \\ &v_1^0 = x_1^0, \quad v_j^0 = x_j^0 - x_{j-1}^0,\ 2 \le j \le l-1, \quad v_l^0 = 2x_l^0 - x_{l-1}^0. \end{aligned}$$

With these we obtain

$$\det\left(\frac{\partial y_i^0(\mathbf{x}^0)}{\partial \rho_j}\right) = \rho_k^{l-k-1}\sqrt{\rho_k}. \tag{2.16}$$

When $k = l$, we have

$$y_j^0(\mathbf{x}^0) = \rho_j = \sigma_j(e^{2\pi i v_1^0}, \cdots, e^{2\pi i v_l^0}), \quad j = 1, \cdots, l-1,$$
$$y_l^0(\mathbf{x}^0) = \rho_l = \prod_{s=1}^{l} e^{\pi i v_s^0},$$
$$\det(\frac{\partial y_i^0(\mathbf{x}^0)}{\partial \rho_j}) = 1. \tag{2.17}$$

ii) For the root system of type C_l,

$$y_j^0(\mathbf{x}^0) = \rho_j, \quad j = 1, \cdots, k,$$
$$y_s^0(\mathbf{x}^0) = \rho_k \rho_s, \quad s = k+1, \cdots, l,$$

where the functions ρ_j are defined by

$$\rho_j = \sigma_j(e^{2\pi i v_1^0}, \cdots, e^{2\pi i v_k^0}), \quad j = 1, \cdots, k,$$
$$\rho_s = \sigma_{s-k}(\xi_{k+1}^0, \cdots, \xi_l^0), \quad s = k+1, \cdots, l$$

with

$$\xi_m^0 = e^{2\pi i v_m^0} + e^{-2\pi i v_m^0},$$
$$v_1^0 = x_1^0,\ v_m^0 = x_m^0 - x_{m-1}^0, \quad m = 2, \cdots, l.$$

With these we obtain

$$\det\left(\frac{\partial y_i^0(\mathbf{x}^0)}{\partial \rho_j}\right) = (\rho_k)^{l-k}. \tag{2.18}$$

iii) For the root system of type D_l with $k \le l-2$,

$$y_j^0(\mathbf{x}^0) = \rho_j, \quad j = 1, \cdots, k,$$
$$y_s^0(\mathbf{x}^0) = \rho_k \rho_s, \quad s = k+1, \cdots, l-2,$$
$$y_{l-1}^0(\mathbf{x}^0) = \frac{1}{2}\sqrt{\rho_k}\left(\rho_l + \rho_{l-1}\right), \quad y_l^0(\mathbf{x}^0) = \frac{1}{2}\sqrt{\rho_k}\left(\rho_l - \rho_{l-1}\right)$$

where the functions ρ_j are given by

$$\rho_j = \sigma_j(e^{2\pi i v_1^0}, \cdots, e^{2\pi i v_k^0}), \quad j = 1, \cdots, k,$$
$$\rho_s = \sigma_{s-k}(\xi_{k+1}^0, \cdots, \xi_l^0), \quad s = k+1, \cdots, l-2,$$
$$\rho_{l-1} = \prod_{s=k+1}^{l} \left(e^{i\pi v_s^0} + e^{-i\pi v_s^0}\right), \quad \rho_l = \prod_{s=k+1}^{l} \left(e^{i\pi v_s^0} - e^{-i\pi v_s^0}\right)$$

with

$$\xi_m^0 = e^{2\pi i v_m^0} + e^{-2\pi i v_m^0},$$
$$v_1^0 = x_1^0,\ v_m^0 = x_m^0 - x_{m-1}^0, \quad m = 2, \cdots, l-2,$$
$$v_{l-1}^0 = x_l^0 + x_{l-1}^0 - x_{l-2}^0, \quad v_l^0 = x_{l-1}^0 - x_l^0.$$

With these we obtain

$$\det\left(\frac{\partial y_i^0(\mathbf{x}^0)}{\partial \rho_j}\right) = \frac{1}{2}(\rho_k)^{l-k-1}. \tag{2.19}$$

iv) For the case D_l with $k = l-1$ we have

$$y_m^0(\mathbf{x}^0) = \rho_m, \quad m = 1, \cdots, l-2,$$
$$y_{l-1}^0(\mathbf{x}^0) = \rho_l, \quad y_l^0(\mathbf{x}^0) = \frac{\rho_{l-1}}{\rho_l},$$

where the functions ρ_j are defined by

$$\rho_j = \sigma_j(e^{2\pi i v_1^0}, \cdots, e^{2\pi i v_l^0}), \quad j = 1, \cdots, l-1, \quad \rho_l = \prod_{s=1}^{l} e^{\pi i v_s^0}$$

with

$$v_1^0 = x_1^0,\ v_m^0 = x_m^0 - x_{m-1}^0, \quad 2 \le m \le l-2,$$
$$v_{l-1}^0 = x_l^0 + x_{l-1}^0 - x_{l-2}^0,\ v_l^0 = x_{l-1}^0 - x_l^0.$$

With these we obtain

$$\det\left(\frac{\partial y_i^0(\mathbf{x}^0)}{\partial \rho_j}\right) = -\frac{1}{\rho_l}. \tag{2.20}$$

v) For the case D_l with $k = l$ the functions $y_j^0(\mathbf{x}^0)$ and ρ_j are defined in the same way as in the above case iv), except

$$y_{l-1}^0(\mathbf{x}^0) = \frac{\rho_{l-1}}{\rho_l}, \quad y_l^0(\mathbf{x}^0) = \rho_l, \quad v_l^0 = x_l^0 - x_{l-1}^0.$$

With these we obtain

$$\det\left(\frac{\partial y_i^0(\mathbf{x}^0)}{\partial \rho_j}\right) = \frac{1}{\rho_l}. \tag{2.21}$$

From the above calculation of the Jacobian $\det\left(\frac{\partial y_i^0(\mathbf{x}^0)}{\partial \rho_j}\right)$ and from the algebraic independence of the functions $\rho_1, \ldots, \rho_l$ we deduce the algebraic independence of the functions $y_1^0(\mathbf{x}^0),\cdots,y_l^0(\mathbf{x}^0)$. This completes the proof of the theorem. □

3. Frobenius manifold structures on the orbit space of $\widetilde{W}^{(k)}(C_l)$

3.1. **Flat pencils of metrics on the orbit space of $\widetilde{W}^{(k)}(C_l)$.** Let $\mathcal{M}$ be the orbit space defined as Spec$\mathcal{A}$ of the extended affine Weyl group $\widetilde{W}^{(k)}(C_l)$ for any fixed $1 \le k \le l$. Following [7] we define an indefinite metric $(\ ,\ )^{\sim}$ on $\widetilde{V}_{\mathbb{C}} = \widetilde{V} \otimes_{\mathbb{R}} \mathbb{C}$ where $\widetilde{V}$ is the orthogonal direct sum of V and $\mathbb{R}$. Here V is endowed with the W-invariant Euclidean metric

$$(dx_s, dx_n)^{\sim} = \frac{s}{4\pi^2}, \quad 1 \le s \le n \le l \tag{3.1}$$

and $\mathbb{R}$ is endowed with the metric

$$(dx_{l+1}, dx_{l+1})^{\sim} = -\frac{1}{4k\pi^2}. \tag{3.2}$$

The set of generators for the ring $\mathcal{A} = \mathcal{A}^{(k)}(C_l)$ are defined by (1.9), (1.10), (2.8) with $\gamma = 1$. They form a system of global coordinates on $\mathcal{M}$. We now introduce a system of local coordinates on $\mathcal{M}$ as follows

$$y^1 = \tilde{y}_1, \ldots, y^l = \tilde{y}_l,\ y^{l+1} = \log \tilde{y}_{l+1} = 2\pi i x_{l+1}. \tag{3.3}$$

They live on the universal covering $\widetilde{\mathcal{M}}$ of $\mathcal{M} \setminus \{\tilde{y}_{l+1} = 0\}$. The projection

$$\mathrm{Pr} : \widetilde{V} \to \widetilde{\mathcal{M}} \tag{3.4}$$

induces a symmetric bilinear form on $T^*\mathcal{M}$

$$(dy^i, dy^j)^{\sim} \equiv g^{ij}(y) := \sum_{a,b=1}^{l+1} \frac{\partial y^i}{\partial x_a}\frac{\partial y^j}{\partial x_b}(dx_a, dx_b)^{\sim}. \tag{3.5}$$

Proposition 3.1. *The functions $g^{ij}(y)$ and $\Gamma_n^{ij}(y)$ defined by* (3.5) *and*

$$\sum_{n=1}^{l+1} \Gamma_n^{ij}(y) dy^n = \sum_{p,q,r=1}^{l+1} \frac{\partial y^i}{\partial x_p}\frac{\partial^2 y^j}{\partial x_q \partial x_r}(dx_p, dx_q)^{\sim} dx_r \tag{3.6}$$

are weighted homogeneous polynomials in $y^1, \cdots, y^l$, $e^{y^{l+1}}$ *of the degree*

$$\deg g^{ij}(y) = \deg y^i + \deg y^j, \tag{3.7}$$

$$\deg \Gamma^{ij}_n(y) = \deg y^i + \deg y^j - \deg y^n, \tag{3.8}$$

where $\deg y^j = d_j$ *and* $\deg y^{l+1} = d_{l+1} = 0$.

Proof. The proposition follows from Theorem 1.3. □

From the above proposition we know that $\det(g^{ij})$ is a polynomial of $y^1, \ldots, y^l$ and $e^{y^{l+1}}$, so the discriminant Σ of the extended affine Weyl group is an algebraic subvariety of $\mathcal{M}$. It was shown in [7] that Σ is the Pr-image of the hyperplanes

$$\{(\mathbf{x}, x_{l+1}) | (\beta, \mathbf{x}) = r \in \mathbb{Z},\ x_{l+1} = \text{arbitrary}\}, \quad \beta \in \Phi^+, \tag{3.9}$$

where Φ^+ is the set of all positive roots. The matrix (g^{ij}) is invertible on $\mathcal{M} \setminus \Sigma$ and the inverse matrix $(g^{ij})^{-1}$ defines a flat metric on $\mathcal{M}\setminus\Sigma$. The functions $\Gamma^{ij}_n(y)$ defined by (3.6) coincide with the contravariant components

$$-\sum_{s=1}^{l+1} g^{is}(y)\Gamma^j_{sn}(y)$$

of the Levi-Civita connection of (g^{ij}) on $\mathcal{M} \setminus \Sigma$.

We now proceed to look for other flat metrics on a certain subvariety in $\mathcal{M}$ that are compatible with the metric $(g^{ij})^{-1}$. To this end, let us introduce the following new coordinates on $\mathcal{M}$:

$$\theta^j = \begin{cases} e^{k y^{l+1}}, & j = 0, \\ y^j e^{(k-j) y^{l+1}}, & j = 1, \cdots, k-1, \\ y^j, & j = k, \cdots, l \end{cases} \tag{3.10}$$

and denote

$$\mu_j = 2\pi i (x_j - x_{j-1}), \quad \mu_{l+1} = y^{l+1} = 2\pi i x_{l+1}, \quad j = 1, \cdots, l. \tag{3.11}$$

In the coordinates $\mu_1, \ldots, \mu_{l+1}$ the indefinite metric on $\widetilde{V}$ has the form

$$\left((d\mu_i, d\mu_j)^{\sim}\right) = \operatorname{diag}(-1, \ldots, -1, \frac{1}{k}). \tag{3.12}$$

Define

$$P(u) := \sum_{j=0}^{l} u^{l-j}\theta^j = e^{k\mu_{l+1}} \prod_{j=1}^{l}(u+\xi_j). \tag{3.13}$$

We can easily verify that the function $P(u)$ satisfies

$$\frac{\partial P(u)}{\partial \mu_a} = \frac{1}{u+\xi_a} P(u)(e^{\mu_a} - e^{-\mu_a}), \quad 1 \le a \le l; \tag{3.14}$$

$$\frac{\partial P(u)}{\partial \mu_{l+1}} = kP(u), \quad P'(u) := \frac{\partial P(u)}{\partial u} = P(u)\sum_{a=1}^{l} \frac{1}{u+\xi_a}. \tag{3.15}$$

Lemma 3.2. *The following formulae hold true for the generating functions of the metric* (g^{ij}) *and the contravariant components of its Levi-Civita connection* Γ_k^{ij} *in the coordinates* $\theta^0, \dots, \theta^l$*:*

$$\begin{aligned}&\sum_{i,j=0}^{l} (d\theta^i, d\theta^j)^{\sim} u^{l-i} v^{l-j} = (dP(u), dP(v))^{\sim} \\ &= (k-l)P(u)P(v) + \frac{u^2-4}{u-v}P'(u)P(v) - \frac{v^2-4}{u-v}P(u)P'(v),\end{aligned} \tag{3.16}$$

$$\begin{aligned}\sum_{i,j,r=0}^{l} \Gamma_r^{ij}(\theta) d\theta^r u^{l-i} v^{l-j} &= \sum_{a,b,r=1}^{l+1} \frac{\partial P(u)}{\partial \mu_a}\frac{\partial^2 P(v)}{\partial \mu_b \partial \mu_r} d\mu_r (d\mu_a, d\mu_b)^{\sim} \\ &= (k-l)P(u)dP(v) + \frac{u^2-4}{u-v}P'(u)dP(v) - \frac{v^2-4}{u-v}P(u)dP'(v) \\ &\quad + \frac{uv-4}{(u-v)^2}P(v)dP(u) - \frac{uv-4}{(u-v)^2}P(u)dP(v).\end{aligned} \tag{3.17}$$

Here $\Gamma_r^{ij}(\theta)$ *are the contravariant components of the Levi-Civita connection of* (g^{ij}) *represented in the coordinates* $\theta^0, \theta^1, \dots, \theta^l$*.*

Proof. By using (3.14) and (3.15), we have

$$\begin{aligned}(dP(u), dP(v))^{\sim} &= \frac{1}{k}\frac{\partial P(u)}{\partial \mu_{l+1}}\frac{\partial P(v)}{\partial \mu_{l+1}} - \sum_{a=1}^{l}\frac{\partial P(u)}{\partial \mu_a}\frac{\partial P(v)}{\partial \mu_a} \\ &= kP(u)P(v) - \sum_{a=1}^{l} P(u)P(v)\frac{\xi_a^2-4}{(u+\xi_a)(v+\xi_a)} \\ &= kP(u)P(v) - \sum_{s=1}^{l} P(u)P(v)\left(1 - \frac{u^2-4}{u-v}\frac{1}{u+\xi_a} + \frac{v^2-4}{u-v}\frac{1}{v+\xi_a}\right) \\ &= (k-l)P(u)P(v) + \frac{u^2-4}{u-v}P'(u)P(v) - \frac{v^2-4}{u-v}P(u)P'(v).\end{aligned}$$

So we proved the first formula, the second formula can be proved in the same way. The lemma is proved. □

The above lemma shows that in the coordinates $\theta^0, \dots, \theta^l$ the functions $g^{ij}(\theta)$ are quadratic polynomials, and the contravariant components Γ^{ij}_s are homogeneous linear functions[4]. To find flat metrics that are compatible with this quadratic metric $g^{ij}(\theta)$, we need the following lemma.

Lemma 3.3. *If there is a set of constants $\{c_0, \dots, c_l\}$ such that*

(i) the functions

$$g^{ij}(\theta^0 + c_0\lambda, \theta^1 + c_1\lambda, \dots, \theta^l + c_l\lambda),$$
$$\Gamma^{ij}_s(\theta^0 + c_0\lambda, \theta^1 + c_1\lambda, \dots, \theta^l + c_l\lambda)$$

are linear in the parameter λ for $1 \le i, j, s \le l+1$, and

(ii) the matrix (η^{ij}) with

$$\eta^{ij} = \mathcal{L}_e g^{ij}, \quad e = \sum_{j=0}^{l} c_j \frac{\partial}{\partial \theta^j} \tag{3.18}$$

is nondegenerate on certain open subset $\mathcal{U}$ of $\mathcal{M}$.

Then the metrics $(g^{ij}), (\eta^{ij})$ form a flat pencil, i.e., the linear combination $(g^{ij} + \lambda\eta^{ij})$ yields a flat metric on $\mathcal{U}$ for any λ satisfying $\det(g^{ij} + \lambda\eta^{ij}) \neq 0$, and the contravariant components of the Levi-Civita connection for this metric equal to

$$\Gamma^{ij}_s + \lambda\,\gamma^{ij}_s. \tag{3.19}$$

Here γ^{ij}_s are the contravariant components of the Levi-Civita connection for the metric (η^{ij}) which can be evaluated by $\gamma^{ij}_s = \mathcal{L}_e \Gamma^{ij}_s$.

Proof. For the proof of this lemma, see Appendix D of [6]. □

[4] These metrics give rise to a quadratic Poisson structure on the space of "loops" $\{S^1 \to M\}$ (see [5] for the details):

$$\{\theta^i(a), \theta^j(b)\} = g^{ij}(\theta(a))\delta'(a-b) + \Gamma^{ij}_s(\theta(a))\theta^s_a\delta(a-b).$$

We plan to study such important class of quadratic metrics and Poisson structures in a separate publication.

Theorem 3.4. *For any fixed integer $0 \leq m \leq l-k$ there is a flat pencil of metrics $(g^{ij}), (\eta^{ij})$ on a certain open subset $\mathcal{U}$ of $\mathcal{M}$ with (g^{ij}) given by (3.5) and $\eta^{ij} = \mathcal{L}_e g^{ij}$. Here the vector field e has the form*

$$e := \sum_{j=k}^{l} c_j \frac{\partial}{\partial \theta^j} = \sum_{j=k}^{l} c_j \frac{\partial}{\partial y^j} \tag{3.20}$$

with the constants $c_k, \cdots, c_l$ defined by the generating function

$$P_0(u) = \sum_{j=k}^{l} c_j u^{l-j} = (u+2)^m (u-2)^{l-k-m}. \tag{3.21}$$

Explicitly, $c_j = (-2)^{j-k} \sum_{s=0}^{m} (-1)^{m-s} \binom{m}{s} \binom{l-k-m}{l-j-s}$ *for* $j = k, \cdots, l$.

Proof. Firstly we want to find the constants $c_0, \ldots, c_l$ satisfying the condition (i) in Lemma 3.3. It suffices to find a polynomial $P_0(u) = \sum_{j=0}^{l} c_j u^{l-j}$ such that after the shift

$$P(u) \mapsto P(u) + \lambda P_0(u), \quad P(v) \mapsto P(v) + \lambda P_0(v),$$

the right hand side of (3.16) and (3.17) are linear in λ. This yields that $P_0(u)$ and $P_0(v)$ must satisfy

$$(k-l) P_0(u) P_0(v) + \frac{u^2-4}{u-v} P_0'(u) P_0(v) - \frac{v^2-4}{u-v} P_0(u) P_0'(v) = 0. \tag{3.22}$$

Separating the variables and integrating one obtains

$$P_0(u) = a \left(\frac{u-2}{u+2} \right)^b [(u-2)(u+2)]^{\frac{l-k}{2}} = (u-2)^{\frac{l-k}{2}+b} (u+2)^{\frac{l-k}{2}-b}$$

for some constants a, b. This is a polynomial *if and only if* $m := \frac{l-k}{2} - b$ is a non-negative integer. Hence any polynomial solution to eq. (3.22) must have the form $P_0(u) = a(u+2)^m (u-2)^{l-k-m}$ for an integer where $0 \leq m \leq l-k$. Thus, up to a common factor, the constants $c_0, \ldots, c_l$ are determined by

$$\sum_{j=0}^{l} c_j u^{l-j} = (u+2)^m (u-2)^{l-k-m}.$$

Actually, by comparing the degrees of u, we know $c_j = 0$ for $j = 0, \cdots, k-1$.

Next we want to check the condition (ii) in Lemma 3.3. In order to do this, taking any fixed integer $0 \le m \le l-k$ we consider the following linear change of coordinates

$$(y^1, \ldots, y^{l+1}) \mapsto (\tau^1, \ldots, \tau^{l+1})$$

defined by the relations $\tau^{l+1} = y^{l+1}$ and

$$\sum_{j=0}^{l} \theta^j u^{l-j} = \sum_{j=0}^{l-m} \varpi^j \, (u+2)^m \, (u-2)^{l-m-j} - \sum_{j=l-m+1}^{l} \varpi^j \, (u+2)^{l-j} \, (u-2)^{j-k-1}\,, \tag{3.23}$$

where

$$\varpi^j = \begin{cases} e^{k\,\tau^{l+1}}, & j=0, \\ \tau^j e^{(k-j)\tau^{l+1}}, & j=1,\cdots,k-1, \\ \tau^j, & j=k,\cdots,l. \end{cases} \tag{3.24}$$

Then,

$$\sum_{j=0}^{l} \frac{\partial \theta^j}{\partial \tau^k} u^{l-j} = (u+2)^m (u-2)^{l-k-m} = \sum_{j=0}^{l} c_j u^{l-j}.$$

This means that in terms of the new coordinates τ^i the vector field e defined in (3.20) has the expression

$$e = \sum_{j=0}^{l} \frac{\partial \theta^j}{\partial \tau^k} \frac{\partial}{\partial \theta^j} = \frac{\partial}{\partial \tau^k}.$$

Furthermore, observe that the left hand side of (3.23) coincides with the polynomial $P(u)$. By substituting the expressions of $P(u), P(v)$ given by the right hand side of (3.23) into both sides of (3.16), we get an identity which on the right hand side has at most linear terms in τ^k due to the definition of $P_0(u)$ and $P_0(v)$. Differentiating both sides by τ^k and dividing by $P_0(u)$ and $P_0(v)$, we obtain an identity relating two rational functions in u (v is kept as a parameter) with poles at $u=2$ and $u=-2$. Comparing the regular parts and the polar parts at $u=2$ and at $u=-2$ we get explicit formulae for $\eta(d\varpi^i, d\varpi^j)$. Finally, with the use of (3.24) we obtain explicit formulae for the matrix $(\eta^{ij}(\tau))$ with entries

$$\eta^{ij}(\tau) = \mathcal{L}_e g^{ij}(\tau) \tag{3.25}$$

which has the block form

$$\begin{pmatrix} & W_1 & \begin{matrix} P_1 \\ \vdots \\ P_{k-1} \end{matrix} & & & \\ P_1 & \cdots P_{k-1} & P_k & 0 & 0 & 1 \\ & & 0 & W_2 & 0 & 0 \\ & & 0 & 0 & W_3 & 0 \\ & & 1 & 0 & 0 & 0 \end{pmatrix}, \tag{3.26}$$

where W_i are triangular blocks

$$W_1 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & k \\ 0 & 0 & 0 & \cdots & k & R_1 \\ 0 & 0 & 0 & \cdots & R_1 & R_2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ k & R_1 & R_2 & \cdots & & R_{k-2} \end{pmatrix}, \tag{3.27}$$

$$W_2 = \begin{pmatrix} Q_1 & Q_2 & \cdots & Q_{l-k-m} \\ Q_2 & Q_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 \\ Q_{l-k-m} & 0 & \cdots & 0 \end{pmatrix}, \quad W_3 = \begin{pmatrix} S_1 & S_2 & \cdots & S_m \\ S_2 & S_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 \\ S_m & 0 & \cdots & 0 \end{pmatrix} \tag{3.28}$$

with entries

$$\begin{aligned} &R_j = 4(k-j+1)\tau^{j-1}e^{\tau^{l+1}} + (k-j)\tau^j, \quad \tau^0 = 1, \\ &P_j = 4(k-j+1)\tau^{j-1}e^{\tau^{l+1}}, \\ &Q_s = 4s\tau^{k+s} + (1-\delta_{s,l-k-m})(s+1)\tau^{k+s+1}, \\ &S_r = 4r\tau^{l-m+r} - 4(1-\delta_{r,m})r\tau^{l-m+r+1}, \\ &1 \le j \le k, \quad 1 \le r \le m, \quad 1 \le s \le l-k-m. \end{aligned} \tag{3.29}$$

A simple computation gives

$$\det(\eta^{ij}) = (-1)^l k^{k-1} 4^{l-k} m^m (l-k-m)^{l-k-m} (\tau^{l-m})^{l-k-m} (\tau^l)^m. \tag{3.30}$$

So the metrix $(\eta^{ij}(\tau))$ does not degenerate on $\mathcal{M}\setminus\{\tau^l=0\}\cup\{\tau^{l-m}=0\}$ when $m\neq 0, l-k$, and on $\mathcal{M}\setminus\{\tau^l=0\}$ when $m=0$ or $m=l-k$. This completes the proof of the theorem. $\square$

Remark 3.5. *(1). The block W_2 or W_3 does not appears in the matrix (3.26) when $m=l-k$ or $m=0$ (e.g.[11]). (2). The flat pencil of metrics that corresponds to a fixed integer m is equivalent to the one that corresponds to the integer $l-k-m$, this is due to the fact that under replacement $u\mapsto -u$ the polynomial $P_0(u)=(u+2)^m(u-2)^{l-k-m}$ is transformed to the polynomial $(-1)^{l-k}(u+2)^{l-k-m}(u-2)^m$.*

Remark 3.6. *From* (3.30) *it follows that the zero loci of* $\det(\eta^{ij}(\tau))$ *are given by that of τ^l, τ^{l-m} when $m\neq 0, l-k$, and by that of τ^l when $m=0$ or $m=l-k$. By using* (3.13), (3.24) *and* (3.23) *we know that*

$$4^m\tau^{l-m}=P(2)=4^l\widetilde{y}^k_{l+1}\prod_{j=1}^{l}\cos^2\left(\pi(x_j-x_{j-1})\right)=4^lf_1$$

when $m\neq l-k$, and

$$-(-4)^{l-k-1}\tau^l=P(-2)=(-4)^l\widetilde{y}^k_{l+1}\prod_{j=1}^{l}\sin^2\left(\pi(x_j-x_{j-1})\right)=(-4)^lf_2$$

when $m\neq 0$, here f_1, f_2 are defined in (1.17). *We note that $\tau^l=4^lf_1$ when $m=0$, and $\tau^l=-(-4)^{k+1}f_2$ when $m=l-k$.*

Corollary 3.7. *In the coordinates $\tau^1,\dots,\tau^{l+1}$ the components $g^{ij}(\tau)$, $\Gamma^{ij}_m(\tau)$ of the metric (3.5) and its Levi-Civita connection are weighted homogeneous polynomials with degrees*

$$\deg g^{ij}=d_i+d_j,\quad \deg\Gamma^{ij}_s(\tau)=d_i+d_j-d_s. \tag{3.31}$$

They are at most linear in τ^k. Here $\deg\tau^j=d_j$ and $\deg\tau^{l+1}=d_{l+1}=0$.

3.2. **Flat coordinates of the metric** (η^{ij})**.** In this subsection, we will show that the flat coordinates of the metric (η^{ij}) defined in the last subsection are algebraic functions of $\tau^1,\dots,\tau^{l+1}, e^{\tau^{l+1}}$. To this end, we first perform changes of coordinates to simplify the matrix $(\eta^{ij}(\tau))$.

Lemma 3.8. *There exists a system of coordinates $z^1, \dots, z^{l+1}$ of the form*

$$z^j = \tau^j + p_j(\tau^1, \dots, \tau^{j-1}, e^{\tau^{l+1}}),\ 1 \le j \le k, \tag{3.32}$$

$$z^j = \tau^j + \sum_{s=j+1}^{l-m} c^j_s \tau^s, \quad k+1 \le j \le l-m-1, \tag{3.33}$$

$$z^j = \tau^j + \sum_{s=j+1}^{l} h^j_s \tau^s, \quad l-m+1 \le j \le l-1, \tag{3.34}$$

$$z^{l-m} = \tau^{l-m}, \quad z^l = \tau^l, \quad z^{l+1} = \tau^{l+1}, \tag{3.35}$$

where c^j_s and h^j_s are some constants and p_j are homogeneous polynomials of degree d_j such that in the new coordinates z^i the components of the metric (η^{ij}) can still been encoded into a block diagonal matrix of the form (3.26)–(3.28) with the entries replaced by

$$R_j = 0, \quad P_j = 0, \quad Q_s = 4sz^{k+s}, S_r = 4rz^{l-m+r}, \tag{3.36}$$
$$1 \le j \le k, \quad 1 \le s \le l-k-m, \quad 1 \le r \le m.$$

Proof. Let us first note that the $(k+1)\times(k+1)$ matrix $(\tilde{\eta}^{ij})$ which has entries

$$\tilde{\eta}^{ij} = \eta^{ij}(\tau),\ \tilde{\eta}^{k+1,m} = \tilde{\eta}^{m,k+1} = \delta_{m,k}, \quad 1 \le i,j \le k,\ 1 \le m \le k+1 \tag{3.37}$$

coincides, under renaming of the label of coordinate $\tau^{l+1} \mapsto \tau^{k+1}$, with the matrix $(\eta^{ij}(\tau))_{(k+1)\times(k+1)}$ that was constructed in the last subsection with respect to the extended affine Weyl group $\widetilde{W}^{(k)}(C_k)$. Thus by using the results of [7] we can find homogeneous polynomials $p_j, 1 \le j \le k$ such that under the change of coordinates (3.32) and $z^j = \tau^j$, $k+1 \le j \le l+1$ the matrix $(\eta^{ij}(z))$ has the form (3.26)–(3.28) with entries

$$R_j = 0, \quad P_j = 0, \quad Q_s = 4sz^{k+s} + (1-\delta_{s,l-k-m})(s+1)z^{k+s+1},$$
$$S_r = 4rz^{l-m+r} - 4(1-\delta_{m,r})rz^{l-m+r+1},$$
$$1 \le j \le k, \quad 1 \le r \le m, \quad 1 \le s \le l-k-m.$$

To finish the proof of the lemma, we need to perform a second change of coordinates. To this end, denote by Ψ an $n \times n$ matrix with entries as linear functions of $a^1, \dots, a^n$

$$\psi^{ij}(a) = 4(i+j-1)a^{i+j-1} + \kappa(i,j)a^{i+j}, \quad i, j \geq 1, \tag{3.38}$$

$$\kappa(i,j) = i+j, \quad or \quad -4(i+j-1). \tag{3.39}$$

Here $a^s = 0$ for $s \geq n+1$. We require a linear transformation of the triangular form

$$a^j = \sum_{\alpha=j}^{n} B^j_\alpha b^\alpha, \quad B^j_j = 1, \ j \geq 1 \tag{3.40}$$

such that

$$\sum_{r,s=1}^{n} 4(r+s-1)b^{r+s-1}\frac{\partial a^i}{\partial b^r}\frac{\partial a^j}{\partial b^s} = \psi^{ij}(a), \tag{3.41}$$

where $b^s = 0$ for $s \geq n+1$. Equivalently, the constants B^i_j must satisfy the relations

$$\begin{aligned} &4(i+j-1)B^{i+j-1}_\gamma + \kappa(i,j)B^{i+j}_\gamma = 4\gamma \sum_{\alpha+\beta=\gamma+1} B^i_\alpha B^j_\beta, \\ &\quad i+j \leq \gamma \leq n. \end{aligned} \tag{3.42}$$

Consider the generating functions

$$f^i(t) = \sum_{\alpha \geq 0} B^i_{i+\alpha} t^\alpha, \quad i = 1, 2, \dots. \tag{3.43}$$

Then the relations in (3.42) can be encoded into the following equations:

$$4(i+j-1)t^{i+j-2}f^{i+j-1} + \kappa(i,j)t^{i+j-1}f^{i+j} = 4\frac{d}{dt}\left(t^{i+j-1}f^i f^j\right). \tag{3.44}$$

When $\kappa(i,j) = i+j$ and $\kappa(i,j) = -4(i+j-1)$, this system of equations has, respectively, the following solutions:

$$f^i(t) = \cosh\left(\frac{\sqrt{t}}{2}\right)\left(\frac{2\sinh\left(\frac{\sqrt{t}}{2}\right)}{\sqrt{t}}\right)^{2i-1}, \tag{3.45}$$

and

$$f^i(t) = \left(\frac{\tanh(\sqrt{t})}{\sqrt{t}}\right)^{2i-1}. \tag{3.46}$$

From the above result we know the existence of constants c_s^j and h_s^j such that under the change of coordinates

$$
\begin{aligned}
&z^i \mapsto z^i, \ i = 1, \ldots, k, l-m, l, l+1, \\
&z^j \mapsto z^j + \sum_{s=j+1}^{l-m} c_s^j\, z^s, \quad k+1 \le j \le l-m-1, \\
&z^j \mapsto z^j + \sum_{s=j+1}^{l} h_s^j\, z^s, \quad l-m+1 \le j \le l-1,
\end{aligned}
$$

the matrix $(\eta^{ij}(z))$ has the form (3.26)–(3.28) and with entries given by (3.36). The lemma is proved. □

Lemma 3.9. *Under the change of coordinates*

$$w^i = z^i, \quad i = 1, \ldots, k, \ l+1, \tag{3.47}$$

$$w^{k+1} = z^{k+1}(z^{l-m})^{-\frac{1}{2(l-m-k)}}, \tag{3.48}$$

$$w^s = z^s(z^{l-m})^{-\frac{s-k}{l-m-k}}, \ s = k+2, \cdots, l-m-1, \tag{3.49}$$

$$w^{l-m} = (z^{l-m})^{\frac{1}{2(l-m-k)}}, \tag{3.50}$$

$$w^{l-m+1} = z^{l-m+1}(z^l)^{-\frac{1}{2m}}, \tag{3.51}$$

$$w^r = z^r(z^l)^{-\frac{r+m-l}{m}}, \ r = l-m+2, \cdots, l-1, \tag{3.52}$$

$$w^l = (z^l)^{\frac{1}{2m}}, \tag{3.53}$$

the components of the metric $(\eta^{ij}(z))$ are transformed to the form

$$\begin{pmatrix} A & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & B_1 & 0 & 0 \\ 0 & 0 & 0 & B_2 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \tag{3.54}$$

where the matrix $A = A_{(k-1)\times(k-1)}$ *has entries* $A^{ij} = \delta_{i,k-j}k$ *and the upper triangular matrices* B_1 *and* B_2 *have the form*

$$B_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 2 \\ 0 & H_{k+3} & H_{k+4} & \cdots & H_{l-m-1} & H_{l-m} & \\ 0 & H_{k+4} & H_{k+5} & \cdots & H_{l-m} & & \\ \vdots & \vdots & \vdots & & & & \\ 0 & H_{l-m} & & & & & \\ 2 & & & & & & \end{pmatrix} \tag{3.55}$$

and

$$B_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 2 \\ 0 & H_{l-m+3} & H_{l-m+4} & \cdots & H_{l-1} & H_l & \\ 0 & H_{l-m+4} & H_{l-m+5} & \cdots & H_l & & \\ \vdots & \vdots & \vdots & & & & \\ 0 & H_l & & & & & \\ 2 & & & & & & \end{pmatrix} \tag{3.56}$$

with

$$\begin{aligned} &H_{k+s} = 4s(w^{l-m})^{-2}w^{k+s},\ H_{l-m} = 4(l-m-k)\left(w^{l-m}\right)^{-2}, \\ &H_{l-m+j} = 4j(w^l)^{-2}w^{l-m+j},\ H_l = 4m\left(w^l\right)^{-2}, \\ &3 \le s \le l-m-k-1, \quad 3 \le j \le m-1. \end{aligned} \tag{3.57}$$

Proof. By a straightforward calculation. □

Remark 3.10. *When* $m = l-k$, *the matrix* B_1 *does not appear in (3.54), i.e., the matrix given in (3.54) has the form*

$$\begin{pmatrix} A & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & B_2 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

In this case we use the formulae (3.47), (3.51)–(3.53) for the change of coordinates. When $m = l-k-1$, *we have* $B_1 = 1$, *and we use the formulae (3.47),* *(3.50)–(3.53) to define the new coordinates. When $m = l-k-2$, the matrix B_1 has the form* $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$. *We understand the above lemma in a similar way as we did for the cases when $m = 0, 1, 2$.*

Theorem 3.11. *We can choose the flat coordinates of the metric $(\eta^{ij}(w))$ in the form*

$$\begin{aligned}
&t^1 = w^1, \dots, t^k = w^k,\ t^{l+1} = w^{l+1},\\
&t^{k+1} = w^{k+1} + w^{l-m}\, h_{k+1}(w^{k+2}, \dots, w^{l-m-1}),\\
&t^j = w^{l-m}(w^j + h_j(w^{j+1}, \dots, w^{l-m-1})),\ k+2 \le j \le l-m-1,\\
&t^{l-m} = w^{l-m},\\
&t^{l-m+1} = w^{l-m+1} + w^l\, h_{l-m+1}(w^{l-m+2}, \dots, w^{l-1}),\\
&t^s = w^l(w^s + h_s(w^{s+1}, \dots, w^{l-1})),\ l-m+2 \le s \le l-1,\\
&t^l = w^l.
\end{aligned}$$

Here $h_{l-m-1} = h_{l-1} = 0$, h_j are weighted homogeneous polynomials of degree $\frac{k\,(l-m-j)}{l-m-k}$ for $j = k+1, \dots, l-m-2$ and h_s are weighted homogeneous polynomials of degree $\frac{k\,(l-s)}{m}$ for $s = l-m+2, \dots, l-1$. The degrees of the coordinates w^i are defined in a natural way through the degrees of y^i given in (2.7).

Proof. From the block diagonal form (3.54) of the matrix $(\eta^{ij}(w))$ and the definition (3.55)–(3.57) of its entries, we know that the flat coordinates can be chosen to have the form

$$t^i = w^i,\ 1 \le i \le k,\ i = l+1, \tag{3.58}$$

$$t^j = t^j(w^{k+1}, \dots, w^{l-m}), \quad k+1 \le j \le l-m \tag{3.59}$$

$$t^s = t^s(w^{l-m+1}, \dots, w^l), \quad l-m+1 \le s \le l. \tag{3.60}$$

Since the matrices B_1 and B_2 have the same form, and B_1 becomes constant when $m = l-k$ or $m = l-k-1$, we only need to consider the flat coordinates (3.59) for the metric that corresponds to the matrix B_1 defined in (3.55) with $m \le l-k-3$.

The functions $t^j = t^j(w^{k+1}, \dots, w^{l-m})$ must satisfy the following system of PDEs

$$\frac{\partial^2 t}{\partial w^a \partial w^b} - \sum_{c=k+1}^{l-m} \gamma^c_{ab} \frac{\partial t}{\partial w^c} = 0, \quad a, b = k+1, \dots, l-m, \tag{3.61}$$

where γ^c_{ab} are the Christoffel symbols with respect to the metric B_1. Let us introduce the $(l-m-k) \times (l-m-k)$ matrix

$$\Phi = (\phi^i_j), \quad \phi^i_j = \frac{\partial t^{k+i}}{\partial w^{k+j}}, \quad 1 \le i, j \le l-m-k,$$

where in the notation ϕ^i_j the upper (resp. lower) index denotes the row (resp. column) number of Φ. Then the system (3.61) can be written in the form

$$\partial_s \Phi = \Phi A_s, \quad \partial_s = \frac{\partial}{\partial w^s}, \quad s = k+1, \dots, l-m, \tag{3.62}$$

where the entries of the coefficient matrices A_s are rational functions of $w^{k+1}, \dots, w^{l-m}$. It follows from the simple expressions of the entries of the matrix B_1 that the systems (3.62) are regular at $\mathbf{w} =(w^{k+1}, \dots, w^{l-m}) = 0$ except for case when $s = l-m$, in this case the coefficient matrix has the form

$$A_{l-m} = \mathrm{diag}(0, \frac{1}{w^{l-m}}, \dots, \frac{1}{w^{l-m}}, 0).$$

Note for all the cases with $m = k+1, \dots, l-m-1$ the entries of the matrices A_s are weighted homogeneous polynomials of $w^{k+1}, \dots, w^{l-m}$.

On writingΦ in the form

$$\Phi = \Psi\, \mathrm{diag}(1, w^{l-m}, \dots, w^{l-m}, 1),$$

the systems in (3.62) are converted to

$$\partial_s \Psi = \Psi B_s, \quad \partial_{l-m} \Psi = 0, \quad s = k+1, \dots, l-m-1.$$

The entries of the coefficient matrices B_s are now weighted homogeneous polynomials of $w^{k+1}, \dots, w^{l-m}$, thus we can find a unique solution Ψ of the above systems such that it is analytic at $\mathbf{w} = 0$ and

$$\Psi|_{\mathbf{w}=0} = \mathrm{diag}(1, \dots, 1).$$

From the weighted homogeneity of the coefficient matrices B_s it follows that the elements of Ψ are also weighted homogeneous. Since $\deg w^j > 0$ for $j = k+$

$1, \dots, l-m$ we know that they are in fact polynomials of $w^{k+1}, \dots, w^{l-m}$, and thus the results of the theorem follow. The theorem is proved. $\square$

Due to the above construction, we can associate the following natural degrees to the flat coordinates

$$\tilde{d}_j = \deg t^j := \frac{j}{k}, \quad 1 \le j \le k, \tag{3.63}$$

$$\tilde{d}_s = \deg t^s := \frac{2l-2m-2s+1}{2(l-m-k)}, \quad k+1 \le s \le l-m, \tag{3.64}$$

$$\tilde{d}_\alpha = \deg t^\alpha := \frac{2l-2\alpha+1}{2m}, \quad l-m+1 \le \alpha \le l, \tag{3.65}$$

$$\tilde{d}_{l+1} = \deg t^{l+1} := 0, \quad \deg e^{t^{l+1}} := \frac{1}{k}, \tag{3.66}$$

and we readily have the following corollary.

Corollary 3.12. *In the flat coordinates $t^1, \dots, t^{l+1}$, the nonzero entries of the matrix $(\eta^{ij}(t))$ are given by*

$$\eta^{ij} = \begin{cases} k, & j = k-i, & 1 \le i \le k-1, \\ 1, & i = l+1, j = k & \text{or } i = k,\ j = l+1, \\ 4(l-m-k), & j = l-m+k-i+1, & k+2 \le i \le l-m-1, \\ 2, & i = l-m, j = k+1 & \text{or } i = k+1,\ j = l-m, \\ 4m, & j = 2l-m-i+1, & l-m+2 \le i \le l-1, \\ 2, & i = l, j = l-m+1 & \text{or } i = l-m+1,\ j = l. \end{cases} \tag{3.67}$$

The entries of the matrix $(g^{ij}(t))$ and the Christoffel symbols $\Gamma^{ij}_m(t)$ are weighted homogeneous polynomials of $t^1, \dots, t^l, \frac{1}{t^{l-m}}, \frac{1}{t^l}, e^{t^{l+1}}$ of degrees $\tilde{d}_i + \tilde{d}_j$ and $\tilde{d}_i + \tilde{d}_j - \tilde{d}_m$ respectively. In particular,

$$\begin{aligned} g^{s,l+1} &= \tilde{d}_s t^s, \qquad 1 \le s \le l, \quad g^{l+1,l+1} = \frac{1}{k}, \\ \Gamma^{l+1,i}_j &= \tilde{d}_j \delta_{i,j}, \qquad 1 \le i, j \le l+1. \end{aligned} \tag{3.68}$$

The numbers $\tilde{d}_1, \dots, \tilde{d}_{l+1}$ satisfy a duality relation that is similar to that of [7]. To describe this duality relation, let us delete the k-th vertex of the Dynkin diagram $\mathcal{R}$. We then obtain two components $\mathcal{R} \setminus \alpha_k = \mathcal{R}_1 \cup \mathcal{R}_2$. For any given integer $0 \leq m \leq l-k$, we denote $\mathcal{R}_2 = \mathcal{R}_{21} \cup \mathcal{R}_{22}$, where $\mathcal{R}_{21} = \{\alpha_{k+1}, \cdots, \alpha_{l-m}\}$ and $\mathcal{R}_{22} = \{\alpha_{l-m+1}, \cdots, \alpha_l\}$. On each component we have an involution $i \mapsto i^*$ given by the reflection with respect to the center of the component. Define

$$k^* = l+1, \quad (l+1)^* = k, \tag{3.69}$$

then we have

$$\tilde{d}_i + \tilde{d}_{i^*} = 1, \quad i = 1, \dots, l+1, \tag{3.70}$$

and from the above corollary we see that η^{ij} is a nonzero constant iff $j = i^*$.

3.3. **Frobenius manifold structures on the orbit space of $\widetilde{W}^{(k)}(C_l)$.** Now we are ready to describe the Frobenius manifold structures on the orbit space of the extended affine Weyl group $\widetilde{W}^{(k)}(C_l)$. Let us first recall the definition of Frobenius manifold, see [6] for details.

Definition 3.13. *A Frobenius algebra is a pair $(A, < \, , \, >)$ where A is a commutative associative algebra with a unity e over a field $\mathcal{K}$ (in our case $\mathcal{K} = \mathbb{C}$) and $< \, , \, >$ is a $\mathcal{K}$-bilinear symmetric nondegenerate invariant form on A, i.e.,*

$$< x \cdot y, z > = < x, y \cdot z >, \quad \forall\, x, y, z \in A.$$

Definition 3.14. *A Frobenius structure of charge d on an n-dimensional manifold M is a structure of Frobenius algebra on the tangent spaces $T_tM = (A_t, < \, , \, >_t)$ depending (smoothly, analytically etc.) on the point t. This structure satisfies the following axioms:*

- FM1. *The metric $< \, , \, >_t$ on M is flat, and the unity vector field e is covariantly constant, i.e., $\nabla e = 0$. Here we denote ∇ the Levi-Civita connection for this flat metric.*
- FM2. *Let c be the 3-tensor $c(x, y, z) := < x \cdot y, z >$, $x, y, z \in T_tM$. Then the 4-tensor $(\nabla_w c)(x, y, z)$ is symmetric in $x, y, z, w \in T_tM$.*
- FM3. *The existence on M of a vector field E, called the Euler vector field, which satisfies the conditions $\nabla\nabla E = 0$ and*

$$[E, x \cdot y] - [E, x] \cdot y - x \cdot [E, y] = x \cdot y,$$

$$E < x, y > - < [E, x], y > - < x, [E, y] >= (2 - d) < x, y >$$

for any vector fields x, y on M.

A manifold M equipped with a Frobenius structure on it is called a Frobenius manifold.

Let us choose local flat coordinates $t^1, \cdots t^n$ for the invariant flat metric, then locally there exists a function $F(t^1, \cdots, t^n)$, called the *potential* of the Frobenius manifold, such that

$$< u \cdot v, w >= u^i v^j w^s \frac{\partial^3 F}{\partial t^i \partial t^j \partial t^s} \tag{3.71}$$

for any three vector fields $u = u^i \frac{\partial}{\partial t^i}$, $v = v^j \frac{\partial}{\partial t^j}$, $w = w^s \frac{\partial}{\partial t^s}$. Here and in what follows summations over repeated indices are assumed. By definition, we can also choose the coordinates t^1 such that $e = \frac{\partial}{\partial t^1}$. Then in the flat coordinates the components of of the flat metric $< \frac{\partial}{\partial t^i}, \frac{\partial}{\partial t^j} >$ can be expressed in the form

$$\frac{\partial^3 F}{\partial t^1 \partial t^i \partial t^j} = \eta_{ij}, \quad i, j = 1, \dots, n. \tag{3.72}$$

The associativity of the Frobenius algebras is equivalent to the following overdetermined system of equations for the function F

$$\frac{\partial^3 F}{\partial t^i \partial t^j \partial t^\lambda} \eta^{\lambda\mu} \frac{\partial^3 F}{\partial t^\mu \partial t^k \partial t^m} = \frac{\partial^3 F}{\partial t^k \partial t^j \partial t^\lambda} \eta^{\lambda\mu} \frac{\partial^3 F}{\partial t^\mu \partial t^i \partial t^m} \tag{3.73}$$

for arbitrary indices i, j, k, m from 1 to n.

We assume that flat coordinates have been chosen so that the Euler vector field E has the form

$$E = \sum_{i=1}^{n} (\hat{d}_i t^i + r_i) \frac{\partial}{\partial t^i} \tag{3.74}$$

for some constants $\hat{d}_i, r_i$, $i = 1, \dots, n$ which satisfy $\hat{d}_1 = 1, r_1 = 0$. From the axiom FM3, it follows that the potential F satisfies the quasi-homogeneity condition

$$\mathcal{L}_E F = (3 - d)F + \text{quadratic polynomial in t.} \tag{3.75}$$

The system (3.72)–(3.75) is called the $WDVV$ *equations of associativity* which is equivalent to the above definition of Frobenius manifold in the chosen system of local coordinates.

Let us also recall an important geometrical structure on a Frobenius manifold M, the *intersection form* of M. This is a symmetric bilinear form $(\ ,\ )^*$ on T^*M defined by the formula

$$(w_1, w_2)^* = i_E(w_1 \cdot w_2), \tag{3.76}$$

here the product of two 1-forms w_1, w_2 at a point $t \in M$ is defined by using the algebra structure on T_tM and the isomorphism

$$T_tM \to T_t^*M \tag{3.77}$$

established by the invariant flat metric $<\ ,\ >$. In the flat coordinates $t^1, \cdots, t^n$ of the invariant metric, the intersection form can be represented by

$$(dt^i, dt^j)^* = \mathcal{L}_E F^{ij} = (d - 1 + \hat{d}_i + \hat{d}_j) F^{ij}, \tag{3.78}$$

where

$$F^{ij} = \eta^{ii'} \eta^{jj'} \frac{\partial^2 F}{\partial t^{i'} \partial t^{j'}} \tag{3.79}$$

and $F(t)$ is the potential of the Frobenius manifold. Denote by $\Sigma_0 \subset M$ the *discriminant* of M on which the intersection form degenerates, then an important property of the intersection form is that on $M \setminus \Sigma_0$ its inverse defines a new flat metric.

Proof of the Main Theorem 1. From Theorem 3.4 and Theorem 3.11 we already know the existence of a flat metric (η^{ij}) and its flat local coordinates $t^1, \ldots, t^{l+1}$ on $\mathcal{M}_{k,m}(C_l)$. By following the lines of the proof of Lemma 2.6 given in [7] we can show the existence of a unique weighted homogeneous polynomial

$$G(t) := G(t^1, \ldots, t^{k-1}, t^{k+1}, \ldots, t^l, \frac{1}{t^{l-m}}, \frac{1}{t^l}, e^{t^{l+1}})$$

of degree 2 such that the function

$$F(t) = \frac{1}{2}(t^k)^2 t^{l+1} + \frac{1}{2} t^k \sum_{i,j \neq k} \eta_{ij}\, t^i t^j + G(t) \tag{3.80}$$

satisfies the equations

$$g^{ij}(t) = \mathcal{L}_E F^{ij}(t), \quad \Gamma^{ij}_m(t) = \tilde{d}_j\, c^{ij}_m(t), \quad i, j, m = 1, \ldots, l+1, \tag{3.81}$$

where $c_m^{ij}(t) = \frac{\partial F^{ij}(t)}{\partial t^m}$. Obviously, the function F satisfies the equations

$$\frac{\partial^3 F(t)}{\partial t^k \partial t^i \partial t^j} = \eta_{ij}, \quad i, j = 1, \ldots, l+1 \tag{3.82}$$

and the quasi-homogeneity condition

$$\mathcal{L}_E F = 2F + A_{\alpha\beta} t^\alpha t^\beta + B_\alpha t^\alpha + C, \tag{3.83}$$

where $A_{\alpha\beta}, B_\alpha, C$ are certain constants. From the properties of a flat pencil of metrics [6] it follows that F also satisfies the associativity equations

$$c_m^{ij}(t)\, c_q^{mp}(t) = c_m^{ip}(t)\, c_q^{mj}(t) \tag{3.84}$$

for any set of fixed indices i, j, p, q. From the definition of the Euler vector field we also have

$$\mathcal{L}_E e = -e.$$

Thus we have constructed locally the Frobenius manifold structure on $\mathcal{M}_{k,m}(C_l)$.

It follows from the results of [5] that the above locally defined Frobenius manifold structure is actually globally defined on $\mathcal{M}_{k,m}(C_l)$. In fact, by using the definition of the vector fields e, E given respectively in (1.20), (1.21) and Corollary 3.12, we know that $(g_1^{ij}) = (g^{ij}(y))$, $(g_2^{ij}) = (\eta^{ij}(y))$ form a quasi-homogeneous flat pencil of metrics of degree $d = 1$ in the sense of [5] with $\tau = y^{l+1}$. By using Theorem 2.1 and the remark given before Example 2.1 of [5], we know that the multiplication rule of the Frobenius manifold structure that we defined above, in terms of the local flat coordinates $t^1, \ldots, t^{l+1}$, can be represented in a coordinate free form by using the flat metric η, the intersection form g and the Euler vector fields E. From Proposition 3.1 and the formulae (3.30), (3.35) we know that in the coordinates $\tilde{y}_1 = y^1, \ldots, \tilde{y}_l = y^l, \tilde{y}_{l+1} = e^{y^{l+1}}$ the components of the intersection form g are polynomials of these coordinates, while the components of the $\eta_{\alpha\beta}(\tilde{y})$ of the flat metric η are polynomials of

$$\tilde{y}_1, \ldots, \tilde{y}_{l+1}, \frac{1}{\tilde{y}_{l+1}}, \frac{1}{\tau^l}, \frac{1}{\tau^{l-m}}$$

when $m \neq 0, l-k$, and are polynomials of

$$\tilde{y}_1, \ldots, \tilde{y}_{l+1}, \frac{1}{\tilde{y}_{l+1}}, \frac{1}{\tau^l}$$

when $m = 0$ or $m = l - k$. Thus it follows from Remark 3.6 that the Frobenius manifold structure is globally defined on $\mathcal{M}_{k,m}(C_l)$. The theorem is proved. □

Remark 3.15. *By using Lemma 3.2 we know that we can also represent the Frobenius manifold structure globally in the coordinates $\theta^0, \dots, \theta^l$ introduced in 3.10, which correspond to the coordinates $a_0, \dots, a_l$ of $\mathfrak{M}_{k,m,n}$ under the map $\mathfrak{h}$ given in Theorem 5.6.*

Remark 3.16. *It follows from Remark 3.5 that the Frobenius manifold structures which correspond to the integers m and $l - k - m$ are equivalent. From the above construction we see that the potential F is in general a polynomial of $t^1, \dots, t^{l+1}, \frac{1}{t^{l-m}}, \frac{1}{t^l}, e^{t^{l+1}}$, in the particular cases when $m = 1$ and $m = l - k - 1$ it does not depend on $\frac{1}{t^l}$ and $\frac{1}{t^{l-1}}$ respectively. When $k = l$, $m = 0$, the Frobenius manifold structure coincides with the one that is constructed in* [7].

3.4. **Examples.** To end this section we give some examples to illustrate the above construction of Frobenius manifold structures. For notational convenience, instead of $t^1, \dots, t^{l+1}$ we will denote the flat coordinates of the metric η^{ij} by $t_1, \dots, t_{l+1}$, and we will also denote $\partial_i = \frac{\partial}{\partial t_i}$ in the the following examples.

Example 3.17. $[C_3, k = 1]$ *Let R be the root system of type C_3, take $k = 1$, then $d_1 = d_2 = d_3 = 1$, and*

$$\begin{aligned}
y^1 &= e^{2i\pi x_4}\left(\xi_1 + \xi_2 + \xi_3\right), \\
y^2 &= e^{2i\pi x_4}\left(\xi_1\xi_2 + \xi_1\xi_3 + \xi_2\xi_3\right), \\
y^3 &= e^{2i\pi x_4}\xi_1\xi_2\xi_3, \\
y^4 &= 2i\pi x_4,
\end{aligned}$$

where $\xi_j = e^{2i\pi(x_j - x_{j-1})} + e^{-2i\pi(x_j - x_{j-1})}$ and $x_0 = 0$, $j = 1, 2, 3$. The metric $(\ ,\ )^\sim$ has the form

$$((dx_i, dx_j)^\sim) = \frac{1}{4\pi^2}\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 2 & 0 \\ 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Case I. $m = 0$, *i.e.*, $e = \frac{\partial}{\partial y^1} - 4\frac{\partial}{\partial y^2} + 4\frac{\partial}{\partial y^3}$.
We first introduce the variables

$$z^1 = y^1 + 6\, e^{y^4},\ z^2 = y^2 + 4\, y^1 + 12\, e^{y^4},$$
$$z^3 = y^3 + 2\, y^2 + 4\, y^1 + 8\, e^{y^4},\ z^4 = y^4.$$

Then the flat coordinates are given by

$$t_1 = z^1 - 2\, e^{z_4},\ t_2 = (z^2 - \frac{1}{6}\, z^3)(z^3)^{-\frac{1}{4}},\ t_3 = (z^3)^{\frac{1}{4}},\ t_4 = z^4$$

and the intersection form has the expression

$$g^{11} = 2\, t_2 t_3\, e^{t_4} + \frac{1}{3}\, {t_3}^4 e^{t_4} + 4\, e^{2t_4},$$
$$g^{12} = \frac{7}{3}\, {t_3}^3 e^{t_4} + \frac{7}{2}\, t_2 e^{t_4},\ g^{13} = \frac{5}{2}\, t_3 e^{t_4},\ g^{14} = t_1,$$
$$g^{22} = 12\, {t_3}^2 e^{t_4} - \frac{1}{4}\, {t_2}^2 + \frac{1}{12}\, {t_3}^3 t_2 - \frac{1}{108}\, {t_3}^6 + \frac{1}{4}\, \frac{{t_2}^3}{{t_3}^3},$$
$$g^{23} = 2\, t_1 + 4\, e^{t_4} - \frac{1}{3}\, t_2 t_3 + \frac{1}{72}\, {t_3}^4 - \frac{1}{4}\, \frac{{t_2}^2}{{t_3}^2},$$
$$g^{24} = \frac{3}{4}\, t_2,\ g^{33} = \frac{1}{4}\, \frac{t_2}{t_3} - \frac{1}{12}\, {t_3}^2,\ g^{34} = \frac{1}{4}\, t_3,\ g^{44} = 1.$$

The potential has the form

$$F = \frac{1}{2}\, {t_1}^2 t_4 + \frac{1}{2}\, t_1 t_2 t_3 - \frac{1}{48}\, {t_2}^2 {t_3}^2 + \frac{1}{1440}\, t_2 {t_3}^5 - \frac{1}{36288}\, {t_3}^8$$
$$+ t_2 t_3 e^{t_4} + \frac{1}{6}\, {t_3}^4 e^{t_4} + \frac{1}{2}\, e^{2t_4} + \frac{1}{48}\, \frac{{t_2}^3}{t_3}$$

and the Euler vector field is given by

$$E = t_1 \partial_1 + \frac{3}{4} t_2 \partial_2 + \frac{1}{4} t_3 \partial_3 + \partial_4.$$

Case II. $m = 1$, *i.e.*, $e = \frac{\partial}{\partial y^1} - 4\frac{\partial}{\partial y^3}$.
Define

$$z^1 = y^1 + 2\, e^{y^4},\ z^2 = \frac{1}{2}\, y^2 + \frac{1}{4}\, y^3 + y^1 + 2\, e^{y^4},$$
$$z^3 = \frac{1}{4}\, y^3 - \frac{1}{2}\, y^2 + y^1 - 2\, e^{y^4},\ z^4 = y^4.$$

Then the flat coordinates are

$$t_1 = z^1 - 2e^{z^4},\ t_2 = \sqrt{z^2},\ t_3 = \sqrt{z^3},\ t_4 = z^4$$

and the intersection form is given by

$$\begin{aligned}
g^{11} &= 2\,{t_2}^2 e^{t_4} - 2\,{t_3}^2 e^{t_4} + 4\,e^{2t_4},\\
g^{12} &= 3\,t_2 e^{t_4},\ g^{13} = -3\,t_3 e^{t_4},\ g^{14} = t_1,\\
g^{22} &= 2\,e^{t_4} + t_1 - \frac{1}{4}\,{t_3}^2 - \frac{1}{4}\,{t_2}^2,\ g^{23} = -\frac{1}{2}\,t_2 t_3,\\
g^{33} &= -2\,e^{t_4} + t_1 - \frac{1}{4}\,{t_2}^2 - \frac{1}{4}\,{t_3}^2,\\
g^{24} &= \frac{1}{2}\,t_2,\ g^{34} = \frac{1}{2}\,t_3,\ g^{44} = 1.
\end{aligned}$$

The potential has the expression

$$\begin{aligned}
F = \frac{1}{2}\,t_1 {t_2}^2 + \frac{1}{2}\,t_1 {t_3}^2 + \frac{1}{2}{t_1}^2\,t_4 - \frac{1}{48}\,{t_2}^4 \\
-\frac{1}{48}\,{t_3}^4 - \frac{1}{8}\,{t_2}^2 {t_3}^2 + {t_2}^2 e^{t_4} - {t_3}^2 e^{t_4} + \frac{1}{2}\,e^{2t_4}
\end{aligned}$$

and the Euler vector field is given by

$$E = t_1\partial_1 + \frac{1}{2}t_2\partial_2 + \frac{1}{2}t_3\partial_3 + \partial_4.$$

The Frobenius manifold structure that we obtain for this case is isomorphic to the one given in Example 2.6 [$A_3, k = 2$] of [7].

Example 3.18. [$C_3, k = 2$] *Let R be the root system of type C_3, take $k = 2$, then $d_1 = 1, d_2 = d_3 = 2$, and*

$$\begin{aligned}
y^1 &= e^{2i\pi x_4}\left(\xi_1 + \xi_2 + \xi_3\right),\\
y^2 &= e^{2i\pi x_4}\left(\xi_1\xi_2 + \xi_1\xi_3 + \xi_2\xi_3\right),\\
y^3 &= e^{2i\pi x_4}\xi_1\xi_2\xi_3,\\
y^4 &= 2i\pi x_4,
\end{aligned}$$

where $\xi_j = e^{2i\pi(x_j - x_{j-1})} + e^{-2i\pi(x_j - x_{j-1})}$ *and* $x_0 = 0$, $j = 1, 2, 3$. *The metric* $(\ , \ )^{\sim}$ *has the form*

$$((dx_i, dx_j)^{\sim}) = \frac{1}{4\pi^2}\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 2 & 0 \\ 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix}.$$

Case I. $m = 0$, *i.e.,* $e = \frac{\partial}{\partial y^2} - 2\frac{\partial}{\partial y^3}$. *The Frobenius manifold structure that we obtain for this case is isomorphic to the one given in Example 2.7 [*$B_3, k = 2$*] of* [7].

Case II. $m = 1$, *i.e.,* $e = \frac{\partial}{\partial y^2} + 2\frac{\partial}{\partial y^3}$.

We first introduce the following variables

$$z^1 = y^1 + 2\, e^{y^4}, \ z^2 = y^2 + 4e^{2y^4},$$
$$z^3 = 2\, y^2 - 4\, y^1 e^{y_4} - y^3 + 8\, e^{2y^4}, \ z^4 = y^4.$$

Then the flat coordinates given by

$$t_1 = z^1 - 4e^{z^4}, \ t_2 = z^2 - 2\, z^1 e^{z^4} + 6\, e^{2\, z^4}, \ t_3 = \sqrt{z^3}, \ t_4 = z^4.$$

The potential has the expression

$$F = \frac{1}{2}\, t_2 {t_3}^2 + \frac{1}{4}\, {t_1}^2 t_2 + \frac{1}{2}\, {t_2}^2 t_4 - \frac{1}{48}\, {t_3}^4 - \frac{1}{96}\, {t_1}^4 + {t_3}^2 e^{2t_4} - {t_3}^2 t_1 e^{t_4} + \frac{1}{2}\, {t_1}^2 e^{2t_4} + \frac{1}{4}\, e^{4t_4}$$

and the Euler vector field is given by

$$E = \frac{1}{2} t_1 \partial_1 + t_2 \partial_2 + \frac{1}{2} t_3 \partial_3 + \frac{1}{2} \partial_4.$$

This Frobenius manifold structure is exactly the one given in Example 2.7 [$B_3, k = 2$*] of* [7].

Example 3.19. $[C_4, k=1, m=0]$ *Let* R *be the root system of type* C_4*, take* $k=1$*, then* $d_1=d_2=d_3=d_4=1$*, and*

$$
\begin{aligned}
y^1 &= e^{2\,i\pi\,x_5}\left(\xi_1+\xi_2+\xi_3+\xi_4\right),\\
y^2 &= e^{2\,i\pi\,x_5}\sum_{1\le a<b\le 4}\xi_a\xi_b,\\
y^3 &= e^{2\,i\pi\,x_5}\sum_{1\le a<b<c\le 4}\xi_a\xi_b\xi_c,\\
y^4 &= e^{2\,i\pi\,x_5}\xi_1\xi_2\xi_3\xi_4,\\
y^5 &= 2\,i\pi\,x_5,
\end{aligned}
$$

where $\xi_j = e^{2i\pi(x_j-x_{j-1})}+e^{-2i\pi(x_j-x_{j-1})}$ *and* $x_0=0$*,* $j=1,2,3,4$*. The metric* $(\ ,\ )^{\sim}$ *has the form*

$$
((dx_i, dx_j)^{\sim}) = \frac{1}{4\pi^2}\begin{pmatrix} 1 & 1 & 1 & 1 & 0\\ 1 & 2 & 2 & 2 & 0\\ 1 & 2 & 3 & 3 & 0\\ 1 & 2 & 3 & 4 & 0\\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}.
$$

Introduce the variables

$$
\begin{aligned}
&z^1 = y^1+8\,e^{y^5},\ z^2 = y^2+6\,y^1+24\,e^{y^5},\\
&z^3 = y^3+4\,y^2+12\,y^1+32\,e^{y^5},\ z^5 = y^5,\\
&z^4 = y^4+2\,y^3+8\,y^1+4\,y^2+16\,e^{y^5},
\end{aligned}
$$

and

$$
\begin{aligned}
&w_1 = z^1-2e^{z^5},\ w_2 = (z^2-\frac{1}{6}\,z^3+\frac{1}{30}\,z^4)(z^4)^{-\frac{1}{6}},\\
&w_3 = (z^3-\frac{1}{4}\,z^4)(z^4)^{-\frac{2}{3}},\ w_4 = (z^4)^{\frac{1}{6}},\ w_5 = z^5.
\end{aligned}
$$

Then we have the expression of the flat coordinates

$$
t_1 = w_1,\ t_2 = w_2-\frac{1}{12}w_3^2\,w_4,\ t_3 = w_3w_4,\ t_4 = w_4,\ t_5 = w_5.
$$

The potential F is given by

$$\begin{aligned}F = &\frac{1}{2}\,{t_1}^2t_5 + \frac{1}{2}\,t_1t_2t_4 - \frac{1}{6912}\,{t_3}^4 + \frac{1}{17280}\,{t_3}^3{t_4}^3\\
&-\frac{1}{288}\,t_2t_4{t_3}^2 - \frac{1}{34560}\,{t_4}^6{t_3}^2 + \frac{1}{24}\,t_1{t_3}^2 + \frac{1}{1440}\,t_3{t_4}^4t_2\\
&-\frac{1}{48}\,{t_2}^2{t_4}^2 - \frac{1}{60480}\,{t_4}^7t_2 + \frac{1}{345600}\,{t_4}^9t_3 - \frac{1}{7603200}\,{t_4}^{12}\\
&+\frac{1}{12}\,e^{t_5}{t_3}^2 + \frac{1}{6}\,e^{t_5}t_3{t_4}^3 + \frac{1}{120}\,e^{t_5}{t_4}^6 + t_2t_4e^{t_5} + \frac{1}{2}\,e^{2t_5}\\
&+\frac{1}{24}\,\frac{t_3{t_2}^2}{t_4} - \frac{1}{216}\,\frac{t_2{t_3}^3}{{t_4}^2} + \frac{1}{4320}\,\frac{{t_3}^5}{{t_4}^3}\end{aligned}$$

with the Euler vector field

$$E = t_1\partial_1 + \frac{5}{6}t_2\partial_2 + \frac{1}{2}t_3\partial_3 + \frac{1}{6}t_4\partial_4 + \partial_5.$$

Example 3.20. $[C_4, k = 2, m = 0]$ *Let R be the root system of type C_4, take $k = 2$, then $d_1 = 1, d_2 = d_3 = d_4 = 2$, and*

$$\begin{aligned}y^1 &= e^{2i\pi x_5}\left(\xi_1 + \xi_2 + \xi_3 + \xi_4\right),\\
y^2 &= e^{4i\pi x_5}\sum_{1\le a<b\le 4}\xi_a\xi_b,\\
y^3 &= e^{4i\pi x_5}\sum_{1\le a<b<c\le 4}\xi_a\xi_b\xi_c,\\
y^4 &= e^{4i\pi x_5}\xi_1\xi_2\xi_3\xi_4,\\
y^5 &= 2i\pi x_5,\end{aligned}$$

where ξ_j are defined as in the last example. The metric $(\ ,\ )^\sim$ has the form

$$((dx_i, dx_j)^\sim) = \frac{1}{4\pi^2}\begin{pmatrix} 1 & 1 & 1 & 1 & 0\\ 1 & 2 & 2 & 2 & 0\\ 1 & 2 & 3 & 3 & 0\\ 1 & 2 & 3 & 4 & 0\\ 0 & 0 & 0 & 0 & -\frac{1}{2}\end{pmatrix}.$$

Introduce the following variables

$$z^1 = y^1 + 8\,e^{y^5},\ \ z^5 = y^5,$$
$$z^2 = y^2 + 6\,y^1 e^{y^5} + 24\,e^{2y^5},$$
$$z^3 = y^3 + 4\,y^2 + 12\,y^1 e^{y^5} + 32\,e^{2y^5},$$
$$z^4 = y^4 + 2\,y^3 + 4\,y^2 + 8\,y^1 e^{y^5} + 16\,e^{2y_5}.$$

Then the flat coordinates are given by

$$t_1 = z^1 - 4e^{z^5},\ \ t_2 = z^2 - 2z^1 e^{z^5} + 6e^{2z^5},$$
$$t_3 = (z^3 - \frac{1}{6}\,z^4)(z^4)^{-\frac{1}{4}},\ \ t_4 = (z^4)^{\frac{1}{4}},\ \ t_5 = z^5.$$

The Euler vector field and the potential are given respectively by

$$E = \frac{1}{2} t_1\partial_1 + t_2\partial_2 + \frac{3}{4} t_3\partial_3 + \frac{1}{4} t_4\partial_4 + \frac{1}{2}\partial_5.$$
$$F = \frac{1}{2}\,{t_2}^2 t_5 + \frac{1}{4}\,{t_1}^2 t_2 + \frac{1}{2}\,t_4 t_3 t_2 + \frac{1}{1440}\,{t_4}^5 t_3 - \frac{1}{48}\,{t_4}^2 {t_3}^2 - \frac{1}{36288}\,{t_4}^8 - \frac{1}{96}\,{t_1}^4 + \frac{1}{2}\,e^{2\,t_5} {t_1}^2 + \frac{1}{6}\,e^{t_5} t_1 {t_4}^4 + \frac{2}{3}\,{t_4}^4 e^{2\,t_5} + e^{t_5} t_1 t_3 t_4 + t_3 t_4 e^{2\,t_5} + \frac{1}{4}\,e^{4\,t_5} + \frac{1}{48}\,\frac{{t_3}^3}{t_4}.$$

In the following, we present more examples and omit all computations and only list the potentials and the Euler vector fields.

Example 3.21. $[C_5, k = 1, m = 2]$ *Let* R *be the root system of type* C_5, *take* $k = 1, m = 2$, *then*

$$F = \frac{1}{2}\,t_6 {t_1}^2 + \frac{1}{2}\,t_1 t_2 t_3 + \frac{1}{2}\,t_1 t_4 t_5 - \frac{1}{72}\,{t_3}^4 {t_5}^4 - \frac{1}{8}\,t_2 t_3 t_4 t_5 - \frac{1}{2268}\,{t_5}^8 - \frac{1}{36288}\,{t_3}^8 - \frac{1}{48}\,{t_3}^2 {t_2}^2 - \frac{1}{48}\,{t_4}^2 {t_5}^2 + \frac{1}{24}\,{t_5}^4 t_2 t_3 + \frac{1}{96}\,{t_3}^4 t_4 t_5 + \frac{1}{1440}\,{t_3}^5 t_2 + \frac{1}{360}\,t_4 {t_5}^5 + t_2 t_3 e^{t_6} - t_4 t_5 e^{t_6} - \frac{2}{3}\,{t_5}^4 e^{t_6} + \frac{1}{6}\,{t_3}^4 e^{t_6} + \frac{1}{2}\,e^{2t_6} + \frac{1}{48}\,\frac{{t_2}^3}{t_3} + \frac{1}{192}\,\frac{{t_4}^3}{t_5}.$$

The Euler vector field is given by

$$E = t_1\partial_1 + \frac{3}{4} t_2\partial_2 + \frac{1}{4} t_3\partial_3 + \frac{3}{4} t_4\partial_4 + \frac{1}{4} t_5\partial_5 + \partial_6.$$

Example 3.22. $[C_6, k=1, m=2]$ *Let R be the root system of type C_6, take $k=1$, then*

$$\begin{aligned}
F = {} & \frac{1}{2}\,{t_1}^2 t_7 + \frac{1}{24}\,t_1 {t_3}^2 + \frac{1}{2}\,t_1 t_2 t_4 + \frac{1}{2}\,t_1 t_5 t_6 - \frac{1}{48}\,{t_2}^2 {t_4}^2 \\
& + \frac{1}{17280}\,{t_4}^3 {t_3}^3 - \frac{1}{48}\,{t_5}^2 {t_6}^2 + \frac{1}{360}\,t_5 {t_6}^5 + \frac{1}{288}\,{t_3}^2 {t_6}^4 \\
& + \frac{17}{5760}\,{t_6}^4 {t_4}^6 - \frac{1}{60480}\,{t_4}^7 t_2 - \frac{1}{72}\,{t_6}^4 {t_4}^3 t_3 - \frac{1}{288}\,t_2 {t_3}^2 t_4 \\
& + \frac{1}{1440}\,t_2 t_3 {t_4}^4 - \frac{1}{96}\,{t_3}^2 t_5 t_6 - \frac{1}{2268}\,{t_6}^8 - \frac{1}{34560}\,{t_4}^6 {t_3}^2 \\
& - \frac{1}{6912}\,{t_3}^4 - \frac{1}{7603200}\,{t_4}^{12} + \frac{1}{24}\,{t_6}^4 t_2 t_4 - \frac{1}{960}\,t_6 {t_4}^6 t_5 \\
& + \frac{1}{345600}\,{t_4}^9 t_3 - \frac{1}{8}\,t_6 t_2 t_4 t_5 + \frac{1}{96}\,t_6 {t_4}^3 t_3 t_5 + \frac{1}{6}\,{t_4}^3 t_3 e^{t_7} \\
& + \frac{1}{120}\,{t_4}^6 e^{t_7} + t_2 t_4 e^{t_7} - t_5 t_6 e^{t_7} + \frac{1}{12}\,{t_3}^2 e^{t_7} - \frac{2}{3}\,{t_6}^4 e^{t_7} \\
& + \frac{1}{2}\,e^{2\,t_7} + \frac{1}{24}\,\frac{{t_2}^2 t_3}{t_4} - \frac{1}{216}\,\frac{t_2 {t_3}^3}{{t_4}^2} + \frac{1}{4320}\,\frac{{t_3}^5}{{t_4}^3} + \frac{1}{192}\,\frac{{t_5}^3}{t_6},
\end{aligned}$$

and the Euler vector field is given by

$$E = t_1\partial_1 + \frac{5}{6}t_2\partial_2 + \frac{1}{2}t_3\partial_3 + \frac{1}{6}t_4\partial_4 + \frac{3}{4}t_5\partial_5 + \frac{1}{4}t_6\partial_6 + \partial_7.$$

4. On the Frobenius manifold structures related to the root system of type B_l and D_l

For the root system R of type B_l, we also define an indefinite metric $(\ ,\ )^{\sim}$ on $\widetilde{V}_{\mathbb{C}} = \widetilde{V} \otimes_{\mathbb{R}} \mathbb{C}$ where $\widetilde{V}$ is the orthogonal direct sum of V and $\mathbb{R}$. Here V is endowed with the W-invariant Euclidean metric

$$(dx_s, dx_n)^{\sim} = \frac{1}{4\pi^2}[(1-\frac{1}{2}\delta_{n,l})s - \frac{l}{4}\,\delta_{n,l}\delta_{s,l}], \quad 1 \le s \le n \le l \tag{4.1}$$

and $\mathbb{R}$ is endowed with the metric

$$(dx_{l+1}, dx_{l+1})^{\sim} = -\frac{1}{4\pi^2 d_k}. \tag{4.2}$$

Here the numbers d_k are defined in (2.1) and (2.2). The basis of the W_a-invariant Fourier polynomials $y_1(\mathbf{x}), \dots, y_{l-1}(\mathbf{x})$, $y_l(\mathbf{x})$ are defined in (2.3)–(2.5). The generators of the ring $\widetilde{W}^{(k)}(B_l)$ have the same form as that of (1.9) and (1.10). It is easy to see that the components of the resulting metric $(g^{ij}(y))$ coincide with those corresponding to the root system of type C_l if we perform the change of coordinates

$$\begin{aligned} & y^j \mapsto \bar{y}^j = y^j,\ y^{l+1} \mapsto \bar{y}^{l+1} = y^{l+1},\ j = 1, \ldots, l-1, \\ & y^l \mapsto \bar{y}^l = (y^l)^2 - \sum_{j=0}^{l-1} 2^{l-s} y^s e^{(k-d_s)y^{l+1}} \end{aligned} \tag{4.3}$$

for $1 \leq k \leq l-1$ and

$$\begin{aligned} & y^j \mapsto \bar{y}^j = y^j,\ y^{l+1} \mapsto \bar{y}^{l+1} = \frac{1}{2} y^{l+1},\ j = 1, \ldots, l-1, \\ & y^l \mapsto \bar{y}^l = (y^l)^2 - \sum_{j=0}^{l-1} 2^{l-s} y^s e^{\frac{1}{2}(l-s)y^{l+1}} \end{aligned} \tag{4.4}$$

for the case when $k = l$. Thus, the Frobenius manifold structure that we obtain in this way from B_l, by fixing the k-th vertex of the corresponding Dynkin diagram, is isomorphic to the one that we obtain from C_l by choosing the k-th vertex of the Dynkin diagram of C_l.

For the root system R of type D_l, the indefinite metric $(\ ,\ )^{\sim}$ on $\widetilde{V} = V \oplus \mathbb{R}$ is defined through the W-invariant Euclidean metric

$$\begin{aligned} & (dx_s, dx_n)_1^{\sim} = \frac{s}{4\pi^2}, \quad 1 \leq s \leq n \leq l-2, \\ & (dx_s, dx_n)^{\sim} = \frac{s}{8\pi^2}, \quad 1 \leq s \leq l-2, n = l-1, l-2, \\ & (dx_{l-1}, dx_{l-1})^{\sim} = (dx_l, dx_l)^{\sim} = \frac{l}{16\pi^2}, \quad (dx_{l-1}, dx_l)^{\sim} = \frac{l-2}{16\pi^2}, \end{aligned} \tag{4.5}$$

and

$$(dx_{l+1}, dx_{l+1})^{\sim} = -\frac{1}{4\pi^2 d_k}. \tag{4.6}$$

Here the numbers d_k are defined in (2.9). The set of generators for the ring $\mathcal{A} = \mathcal{A}^{(k)}(D_l)$ have the same form as that of (1.9) and (1.10), where $y_j(\mathbf{x})$ are defined in (2.13) and (2.14). It can be verified that the components of the resulting metric $(g^{ij}(y))$ coincide with those corresponding to the root system of type C_l if we perform the change of coordinates

$$
\begin{aligned}
&y^j \mapsto \bar{y}^j = y^j,\ j = 1, \cdots, l-2,\ l+1,\\
&y^{l-1} \mapsto \bar{y}^{l-1} = y^{l-1} y^l - \frac{1}{4} \sum_{s=0}^{l-2} [2^{l-s} - (-2)^{l-s}]\ y^s e^{(k-d_s) y^{l+1}}, \qquad (4.7)\\
&y^l \mapsto \bar{y}^l \mapsto \bar{y}^l = (y^l)^2 + (y^{l-1})^2 - \frac{1}{2} \sum_{s=0}^{l-2} [2^{l-s} + (-2)^{l-s}]\ y^s e^{(k-d_s) y^{l+1}}
\end{aligned}
$$

for $1 \leq k \leq l-2$ and

$$
\begin{aligned}
&y^j \mapsto \bar{y}^j = y^j,\ j = 1, \cdots, l-2; \quad y^{l+1} \mapsto \bar{y}^{l+1} = 2y^{l+1},\\
&y^{l-1} \mapsto \bar{y}^{l-1} = y^{l-1} y^l - \frac{1}{4} \sum_{s=0}^{l-2} [2^{l-s} - (-2)^{l-s}]\ y^s e^{\frac{1}{2}(l-s-1) y^{l+1}}, \qquad (4.8)\\
&y^l \mapsto \bar{y}^l = (y^l)^2 e^{\frac{1}{2} y^{l+1}} + (y^{l-1})^2 e^{-\frac{1}{2} y^{l+1}} - \frac{1}{2} \sum_{s=0}^{l-2} [2^{l-s} + (-2)^{l-s}]\ y^s e^{\frac{1}{2}(l-s-1) y^{l+1}}
\end{aligned}
$$

for the case $k = l-1$ and

$$
\begin{aligned}
&y^j \mapsto \bar{y}^j = y^j,\ j = 1, \cdots, l-2; \quad y^{l+1} \mapsto \bar{y}^{l+1} = 2y^{l+1},\\
&y^{l-1} \mapsto \bar{y}^{l-1} = y^{l-1} y^l - \frac{1}{4} \sum_{s=0}^{l-2} [2^{l-s} - (-2)^{l-s}]\ y^s e^{\frac{1}{2}(l-s-1) y^{l+1}}, \qquad (4.9)\\
&y^l \mapsto \bar{y}^l = (y^l)^2 + (y^{l-1})^2 e^{y^{l+1}} - \frac{1}{2} \sum_{s=0}^{l-2} [2^{l-s} + (-2)^{l-s}]\ y^s e^{\frac{1}{2}(l-s) y^{l+1}}
\end{aligned}
$$

for the case $k = l$. Thus, the Frobenius manifold structure that we obtain in this way from D_l, by fixing the k-th vertex of the corresponding Dynkin diagram, is isomorphic to the one that we obtain from C_l by choosing the k-th vertex of the Dynkin diagram of C_l.

5. LG superpotentials for the Frobenius manifolds of $\mathcal{M}_{k,m}(C_l)$-type

We consider a particular class of LG superpotentials consisting of cosine-Laurent series of one variable with tri-degree $(2k, 2m, 2n)$, these being functions of the form[5]

$$\lambda(\varphi) = \big(\cos^2(\varphi) - 1\big)^{-m} \sum_{j=0}^{k+m+n} a_j \cos^{2(k+m-j)}(\varphi), \tag{5.1}$$

where all $a_j \in \mathbb{C}$, $m, n \in \mathbb{Z}_{\geq 0}$ and $k \in \mathbb{N}$. The cosine is considered as an analytic function on the cylinder $\varphi \simeq \varphi + 2\pi$. We denote by $\mathfrak{M}_{k,m,n}$ the space of this kind of cosine Laurent series with the following conditions:

$$a_0 a_{k+m+n} \neq 0, \quad \text{when } m = 0; \tag{5.2}$$

$$a_0 \neq 0, \ \sum_{j=0}^{k+m+n} a_j \neq 0, \quad \text{when } n = 0; \tag{5.3}$$

$$a_0 a_{k+m+n} \neq 0, \ \sum_{j=0}^{k+m+n} a_j \neq 0, \quad \text{when } mn \neq 0. \tag{5.4}$$

By analogy with the construction in [6, 1, 21], the space $\mathfrak{M}_{k,m,n}$ carries a natural structure of Frobenius manifold. The invariant inner product η and the intersection form g of two vectors ∂', ∂'' tangent to $\mathfrak{M}_{k,m,n}$ at a point $\lambda(\varphi)$ can be defined by the following formulae

$$\eta(\partial', \partial'') = (-1)^{k+1} \sum_{|\lambda|<\infty} \operatorname*{res}_{d\lambda=0} \frac{\partial'(\lambda(\varphi)d\varphi)\partial''(\lambda(\varphi)d\varphi)}{d\lambda(\varphi)}, \tag{5.5}$$

$$g(\partial', \partial'') = -\sum_{|\lambda|<\infty} \operatorname*{res}_{d\lambda=0} \frac{\partial'(\log\lambda(\varphi)d\varphi)\partial''(\log\lambda(\varphi)d\varphi)}{d\log\lambda(\varphi)}. \tag{5.6}$$

In these formulae, the derivatives $\partial'(\lambda(\varphi)d\varphi)$ *etc.* are to be calculated keeping φ fixed. The formulae (5.5) and (5.6) uniquely determine multiplication of tangent vectors on $\mathfrak{M}_{k,m,n}$ assuming that the Euler vector field E has the form

$$E = \sum_{j=0}^{k+m+n} a_j \frac{\partial}{\partial a_j}. \tag{5.7}$$

[5]When $k = 1$ and $m = n = 0$, this reduces to $\lambda(\varphi) = a_1 + a_0 \cos^2(\varphi)$. If we set

$$\cos^2(\varphi) = \frac{1 + \cos(2\varphi)}{2}, \quad a_0 = -4e^{\frac{t_2}{2}}, \quad a_1 = t_1 + 2e^{\frac{t_2}{2}}, \quad p = 2\varphi,$$

then the LG superpotential is rewritten as

$$\lambda(p) = t_1 - 2e^{\frac{t_2}{2}} \cos(p),$$

which is exactly the LG superpotential of the $\mathbb{CP}^1$-model obtained in Example I.1 [6].

For tangent vectors ∂', ∂'' and ∂''' to $\mathfrak{M}_{k,m,n}$, one has

$$c(\partial', \partial'', \partial''') = -\sum_{|\lambda|<\infty} \operatorname*{res}_{d\lambda=0} \frac{\partial'(\lambda(\varphi)d\varphi)\partial''(\lambda(\varphi)d\varphi)\partial'''(\lambda(\varphi)d\varphi)}{d\lambda(\varphi)d\varphi}. \tag{5.8}$$

The canonical coordinates $u_1, \cdots, u_{k+m+n+1}$ for this multiplication are the critical values of $\lambda(\varphi)$ and

$$\partial_{u_\alpha} \cdot \partial_{u_\beta} = \delta_{\alpha\beta}\partial_{u_\alpha}, \quad \text{where} \quad \partial_{u_\alpha} = \frac{\partial}{\partial u_\alpha} \tag{5.9}$$

(these are defined only on the semi-simple locus of the manifold).

For clarity, we use the notation

$$\lambda(P) = (P^2-1)^{-m}\sum_{j=0}^{l} a_j P^{2(k+m-j)}, \quad l = k+m+n,$$

$$\dot\lambda(P) = \frac{d\lambda(P)}{dP}, \quad P = \cos(\varphi), \quad P'(\varphi) = \frac{dP}{d\varphi} = -\sin(\varphi) \tag{5.10}$$

and

$$\lambda(\varphi) = a_0(P^2-1)^{-m}P^{-2n}\prod_{j=1}^{l}(P^2-p_j^2), \quad a_0 = e^{2ki\varphi_{l+1}}, \quad p_j = P(\varphi_j). \tag{5.11}$$

Here $P(\varphi)$ has no relation with the function $P(u)$ used in (3.13). Without confusion, we always use $\lambda(P)$ instead of $\lambda(\varphi)$. On comparing coefficients in the two expansions (5.1) and (5.11) of the superpotential we obtain expressions for the a_i in terms of a_0 and the p_j .

Before proceeding to the main result, we give some useful identities.

Lemma 5.1.

$$\lambda'(\varphi_j) = \left.\frac{2PP'(\varphi)\lambda(\varphi)}{P^2-p_j^2}\right|_{\varphi=\varphi_j}, \quad j = 1, \cdots, l. \tag{5.12}$$

Proof. This follows from

$$\lambda'(\varphi) = 2PP'(\varphi)\lambda(\varphi)\left(\sum_{j=1}^{l}\frac{1}{P^2-p_j^2} - \frac{m}{P^2-1} - \frac{n}{P^2}\right)$$

and the definition of $\lambda(\varphi)$ in (5.11). ∎

Let us factorize

$$\lambda'(\varphi) = 2ka_0(P^2-1)^{-m-1}P^{-2n-1}P'(\varphi)\prod_{\alpha=1}^{l+1}(P^2-q_\alpha^2), \quad q_\alpha = P(\psi_\alpha), \tag{5.13}$$

where all q_α^2 are distinct. When $m=0$, we choose $P'(\psi_{l+1})=0$, that is to say,

$$\psi_{l+1} = 0, \pi, \quad \text{i.e.,} \quad q_{l+1} = P(\psi_{l+1}) = 1.$$

Lemma 5.2. *For $1 \le \alpha \le l+1$, we have*

$$\lambda''(\psi_\alpha) = \left.\frac{c_{\alpha,m}PP'(\varphi)\lambda'(\varphi)}{P^2-q_\alpha^2}\right|_{\varphi=\psi_\alpha}, \quad c_{\alpha,m} = 2-\delta_{\alpha,l+1}\delta_{m,0}. \tag{5.14}$$

Proof. By definition, we have

$$\begin{aligned}
\lambda''(\varphi) &= 2ka_0\frac{d}{d\varphi}\left((P^2-1)^{-m-1}P^{-2n-1}\right)\prod_{\alpha=1}^{l+1}(P^2-q_\alpha^2)P'(\varphi) \\
&+ 2ka_0\,(P^2-1)^{-m-1}P^{-2n-1}\frac{d}{d\varphi}\left(\prod_{\alpha=1}^{l+1}(P^2-q_\alpha^2)\right)P'(\varphi) \\
&+ 2ka_0(P^2-1)^{-m-1}P^{-2n-1}\prod_{\alpha=1}^{l+1}(P^2-q_\alpha^2)\,\frac{d^2P}{d^2\varphi} \\
&= \sum_{\alpha=1}^{l+1}\frac{2PP'(\varphi)\lambda'(\varphi)}{P^2-q_\alpha^2} - \frac{(2n+1)P'(\varphi)\lambda'(\varphi)}{P} + \frac{(2m+1)P\lambda'(\varphi)}{P'(\varphi)}.
\end{aligned}$$

So, with the use of (5.13), we get

$$\begin{aligned}
\lambda''(\psi_\alpha) &= \left.\left(\sum_{\alpha=1}^{l+1}\frac{2PP'(\varphi)\lambda'(\varphi)}{P^2-q_\alpha^2} + \frac{(2m+1)P\lambda'(\varphi)}{P'(\varphi)}\right)\right|_{\varphi=\psi_\alpha} \\
&= \begin{cases}
\left.\dfrac{2PP'(\varphi)\lambda'(\varphi)}{P^2-q_\alpha^2}\right|_{\varphi=\psi_\alpha}, & \alpha = 1,\cdots,l, \\
-\left.\dfrac{P\,\lambda'(\varphi)}{P'(\varphi)}\right|_{\varphi=\psi_{l+1}}, & \alpha = l+1, \quad m=0, \\
\left.\dfrac{2PP'(\varphi)\lambda'(\varphi)}{P^2-q_\alpha^2}\right|_{\varphi=\psi_{l+1}}, & \alpha = l+1, \quad m\neq 0
\end{cases} \\
&= \left.\frac{c_{\alpha,m}PP'(\varphi)\lambda'(\varphi)}{P^2-q_\alpha^2}\right|_{\varphi=\psi_\alpha}.
\end{aligned}$$

Thus the lemma is proved. □

We define canonical coordinates

$$u_\alpha = \lambda(\psi_\alpha), \quad \alpha = 1, \cdots, l+1,$$

then

$$\partial_{u_\alpha}\lambda(\varphi)|_{\varphi=\psi_\beta} = \delta_{\alpha\beta}. \tag{5.15}$$

Observe that

$$(P^2-1)^m P^{2n}\partial_{u_\alpha}\lambda(P) = (\partial_{u_\alpha}a_0)P^{2l} + \cdots + \partial_{u_\alpha}a_l$$

is a polynomial of P and

$$(P^2-1)^m P^{2n}\partial_{u_\alpha}\lambda(P)|_{P=q_\beta} = (q_\beta^2-1)^{m+1}q_\beta^{2n-1}\delta_{\alpha\beta},$$

we thus obtain, using the Lagrange interpolation formula,

$$\partial_{u_\alpha}\lambda(\varphi) = \frac{c_{\alpha,m}PP'(\varphi)}{P^2-q_\alpha^2}\frac{\lambda'(\varphi)}{\lambda''(\psi_\alpha)}, \quad \alpha = 1, \cdots, l+1. \tag{5.16}$$

Lemma 5.3.

$$\partial_{u_\alpha}\varphi_\beta = \begin{cases} -\dfrac{c_{\alpha,m}p_\beta P'(\varphi_\beta)}{\lambda''(\psi_\alpha)\,(p_\beta^2-q_\alpha^2)}, \beta = 1, \cdots, l, \\ \dfrac{1}{2ki}\left(\dfrac{\delta_{\alpha,l+1}}{\lambda(\psi_\alpha)} + \dfrac{2c_{\alpha,m}}{\lambda''(\psi_\alpha)}\displaystyle\sum_{s=1}^{l}\dfrac{p_s^2P'(\varphi_s)^2}{(q_{l+1}^2-p_s^2)(q_\alpha^2-p_s^2)}\right), \beta = l+1. \end{cases} \tag{5.17}$$

Proof. By the definition of $\lambda(\varphi)$ in (5.11) and using (5.16), we get

$$\frac{c_{\alpha,m}PP'(\varphi)}{P^2-q_\alpha^2}\frac{\lambda'(\varphi)}{\lambda''(\psi_\alpha)} = \partial_{u_\alpha}\lambda(\varphi) = 2ki\lambda(\varphi)\partial_{u_\alpha}\varphi_{l+1} - \sum_{s=1}^{l}\frac{2p_sP'(\varphi_s)\lambda(\varphi)}{P^2-p_s^2}\partial_{u_\alpha}\varphi_s. \tag{5.18}$$

Putting $\varphi = \varphi_\beta$ for $\beta = 1, \cdots, l$ into (5.18) and using (5.12), we obtain

$$\partial_{u_\alpha}\varphi_\beta = -\frac{c_{\alpha,m}p_\beta P'(\varphi_\beta)}{\lambda''(\psi_\alpha)\,(p_\beta^2-q_\alpha^2)}, \quad \beta = 1, \cdots, l$$

and furthermore,

$$\begin{aligned} \frac{\partial_{u_\alpha}\lambda(\varphi)}{\lambda(\varphi)} &= 2ki\partial_{u_\alpha}\varphi_{l+1} - \sum_{s=1}^{l}\frac{2p_sP'(\varphi_s)}{P^2-p_s^2}\partial_{u_\alpha}\varphi_s \\ &= 2ki\partial_{u_\alpha}\varphi_{l+1} - \frac{2c_{\alpha,m}}{\lambda''(\psi_\alpha)}\sum_{s=1}^{l}\frac{p_s^2P'(\varphi_s)^2}{(P^2-p_s^2)(q_\alpha^2-p_s^2)}. \end{aligned} \tag{5.19}$$

Putting $\varphi=\psi_\beta$ into (5.19), then

$$\frac{\delta_{\alpha\beta}}{u_\beta}=2ki\partial_{u_\alpha}\varphi_{l+1}-\frac{2c_{\alpha,m}}{\lambda''(\psi_\alpha)}\sum_{s=1}^{l}\frac{p_s^2P'(\varphi_s)^2}{(q_\beta^2-p_s^2)(q_\alpha^2-p_s^2)}.$$

Especially, taking $\varphi=\psi_{l+1}$, we obtain the desired formula of $\partial_{u_\alpha}\varphi_{l+1}$. □

Lemma 5.4. *For $\beta,\gamma=1,\cdots,l$, we have*

$$S_{\beta,\gamma}:=\sum_{\alpha=1}^{l+1}\frac{c_{\alpha,m}u_\alpha}{\lambda''(\psi_\alpha)\,(p_\beta^2-q_\alpha^2)(p_\gamma^2-q_\alpha^2)}=\frac{\delta_{\beta\gamma}}{2\,p_\beta^2\,(p_\beta^2-1)}. \tag{5.20}$$

Proof. Letting

$$\lambda(z)=(z-1)^{-m}(a_0z^{k+m}+\cdots+a_lz^{-n})=a_0\,(z-1)^{-m}\,z^{-n}\prod_{j=1}^{l}(z-p_j^2).$$

So, $\lambda(\varphi)=\lambda(z)|_{z=P^2}$ and

$$\frac{d\lambda(z)}{dz}=ka_0(z-1)^{-m-1}z^{-n-1}\prod_{\alpha=1}^{l+1}(z-q_\alpha^2),\qquad \frac{\lambda(z)}{z(z-1)\frac{d\lambda(z)}{dz}}=\frac{\prod_{j=1}^{l}(z-p_j^2)}{\prod_{\alpha=1}^{l+1}(z-q_\alpha^2)},$$

which yields that if $q_\alpha^2\neq 0$ (or 1) for all $\alpha=1,\cdots,l+1$, then $z=0$ (or 1) is not a pole of the function $\dfrac{\lambda(z)}{z(z-1)\frac{d\lambda(z)}{dz}}$.

With the use of (5.14) and $P'(\varphi)^2=1-P^2$, we rewrite $S_{\beta,\gamma}$ as

$$\begin{aligned}
S_{\beta,\gamma} &= \sum_{\alpha=1}^{l+1}\left.\frac{\lambda(\varphi)(P^2-q_\alpha^2)}{P\lambda'(\varphi)P'(\varphi)(P^2-p_\beta^2)(P^2-p_\gamma^2)}\right|_{\varphi=\psi_\alpha}\\
&= -\frac{1}{2}\sum_{\alpha=1}^{l+1}\left.\frac{\lambda(z)(z-q_\alpha^2)}{z(z-1)\frac{d\lambda(z)}{dz}(z-p_\beta^2)(z-p_\gamma^2)}\right|_{z=q_\alpha^2}\\
&= -\frac{1}{2}\sum_{\alpha=1}^{l+1}\left.\operatorname*{res}_{z=q_\alpha^2}\frac{\lambda(z)}{z(z-1)\frac{d\lambda(z)}{dz}(z-p_\beta^2)(z-p_\gamma^2)}\right|_{z=q_\alpha^2}\\
&= \frac{1}{2}\big(\operatorname*{res}_{z=\infty}+\operatorname*{res}_{z=p_\beta^2}+\operatorname*{res}_{z=p_\gamma^2}\big)\frac{\lambda(z)}{\frac{d\lambda(z)}{dz}z(z-1)(z-p_\beta^2)(z-p_\gamma^2)}dz\\
&= \frac{\delta_{\beta\gamma}}{2}\operatorname*{res}_{z=p_\beta^2}\frac{\lambda(z)}{\frac{d\lambda(z)}{dz}z(z-1)(z-p_\beta^2)^2}dz=\frac{\delta_{\beta\gamma}}{2p_\beta^2(p_\beta^2-1)}.
\end{aligned}$$

We thus prove the identity (5.20). □

Lemma 5.5.

$$\frac{\lambda''(\psi_{l+1})}{\lambda(\psi_{l+1})} = -2\left(k + \sum_{s=1}^{l} \frac{p_s^2\, P'(\varphi_s)^2}{(q_{l+1}^2 - p_s^2)^2}\right). \tag{5.21}$$

Proof. Observe that

$$\lambda'(\varphi) = 2PP'(\varphi)\lambda(\varphi)\left(\sum_{s=1}^{l} \frac{1}{P^2 - p_s^2} - \frac{m}{P^2-1} - \frac{n}{P^2}\right), \tag{5.22}$$

which yields

$$P'(\varphi)\left(\sum_{s=1}^{l} \frac{1}{P^2 - p_s^2} - \frac{m}{P^2-1} - \frac{n}{P^2}\right)\Bigg|_{\varphi=\psi_{l+1}} = 0. \tag{5.23}$$

Case 1. $m = 0$. In this case, $P'(\psi_{l+1}) = 0$. Using (5.22) and (5.23), we have

$$\frac{\lambda''(\varphi)}{\lambda(\varphi)}\Bigg|_{\varphi=\psi_{l+1}} = 2(l-k) - \sum_{s=1}^{l} \frac{2q_{l+1}^2}{q_{l+1}^2 - p_s^2} = -2k - \sum_{s=1}^{l} \frac{2\,p_s^2}{q_{l+1}^2 - p_s^2},$$

which is exactly the formula (5.21) because of $q_{l+1} = 1$ and $P'(\varphi_s)^2 = 1 - q_s^2$.

Case 2. $m \neq 0$. In this case, $P'(\psi_{l+1}) \neq 0$. By using (5.23),

$$\sum_{s=1}^{l} \frac{1}{q_{l+1}^2 - p_s^2} = \frac{m}{q_{l+1}^2 - 1} + \frac{n}{q_{l+1}^2}. \tag{5.24}$$

So, using (5.22) and (5.24), we get

$$\begin{aligned}\frac{\lambda''(\varphi)}{\lambda(\varphi)}\Bigg|_{\varphi=\psi_{l+1}} &= 2PP'(\varphi)\frac{d}{d\varphi}\left(\sum_{s=1}^{l} \frac{1}{P^2 - p_j^2} - \frac{m}{p^2-1} - \frac{n}{p^2}\right)\Bigg|_{\varphi=\psi_{l+1}} \\ &= -2\left(k + \sum_{s=1}^{l} \frac{p_s^2 P'(\varphi_s)^2}{(q_{l+1}^2 - p_s^2)^2}\right).\end{aligned}$$

The lemma is proved. □

We are now in a position to state our main theorem in this section.

Theorem 5.6. *Let $\mathfrak{h} : \mathcal{M}_{k,m}(C_l) \to \mathfrak{M}_{k,m,n}$ be induced by the map*

$$(x_1, \cdots, x_{l+1}) \mapsto (\varphi_1, \cdots, \varphi_{l+1}) \tag{5.25}$$

with

$$\varphi_1 = \pi x_1, \quad \varphi_j = \pi(x_j - x_{j-1}), \quad \varphi_{l+1} = \pi x_{l+1}, \quad j = 2, \cdots, l.$$

Then $\mathfrak{h}$ is a k-fold covering map, which is also a local isomorphism between the Frobenius manifolds $\mathcal{M}_{k,m}(C_l)$ and $\mathfrak{M}_{k,m,n}$.

Proof. Let us first prove that the $\mathfrak{h}$ is a k-fold covering map. In fact, by using the formulae (5.10) and (5.11) one has

$$a_0 P^{2l} + \sum_{j=1}^{l} a_j P^{2(l-j)} = a_0 \prod_{j=1}^{l} (P^2 - p_j^2), \quad a_0 = e^{2ki\,\varphi_{l+1}} \tag{5.26}$$

and

$$a_j = (-1)^j a_0\, \sigma_j(p_1^2, \cdots, p_l^2), \quad j = 1, \cdots, l.$$

Observe that

$$p_j^2 = \cos^2 \varphi_j = \frac{\cos(2\varphi_j) + 1}{2}, \quad \cos(2\varphi_j) = \frac{e^{2i\varphi_j} + e^{-2i\varphi_j}}{2} = \frac{\xi_j}{2},$$

and with these one obtains

$$a_0 = e^{ky^{l+1}}, \quad a_j = (-\frac{1}{4})^j \sigma_j(\xi_1 + 2, \cdots, \xi_l + 2) e^{ky^{l+1}}, \quad j = 1, \cdots, l.$$

So the map $\mathfrak{h} : (\tilde{y}_1, \ldots, \tilde{y}_{l+1}) \mapsto (a_0, a_1, \ldots, a_l)$ is given by

$$a_0 = \tilde{y}_{l+1}^k, \tag{5.27}$$

$$a_j = (-\frac{1}{4})^j \left(\sum_{s=1}^{j} 2^{j-s} \binom{l-s}{j-s} \tilde{y}_{l+1}^{k-d_s} \tilde{y}_s + 2^j \binom{l}{j} \tilde{y}_{l+1}^k \right) \tag{5.28}$$

for $j = 1, \ldots, l$, here $d_s = s$ for $s = 1, \cdots, k$ and $d_s = k$ for $s = k+1, \cdots, l$, and the Jacobian of $\mathfrak{h}$ is proportional to $\tilde{y}_{l+1}^{k(k+1)/2-1}$. From the above representation of the map $\mathfrak{h}$ we also have

$$a_l = (-1)^l 4^{-l} \tau^l, \quad \text{for } m = 0.$$

$$\sum_{j=0}^{l} a_j = -(-4)^{-k-1} \tau^l, \quad \text{for } m = l - k > 0;$$

$$a_l = (-1)^l 4^{m-l} \tau^{l-m}, \quad \sum_{j=0}^{l} a_j = -(-4)^{-k-1} \tau^l, \quad \text{for } m \neq 0, m \neq l - k.$$

Note that the condition $m \neq l - k$ is equivalent to $n \neq 0$. So from the definition of $\mathcal{M}_{k,m}(C_l)$ and $\mathfrak{M}_{k,m,n}$ given by Theorem 1 and (5.1)–(5.4) that $\mathfrak{h}$ is a k-fold covering map.

Now let us proceed to prove that $\mathfrak{h}$ is a local isomorphism between the two Frobenius manifold. It is not difficult to check that the Euler vector fields (5.7) and (1.21) coincide. So it suffices to prove that the intersection form (5.6) coincides with the intersection form of the orbit space, and the metric (5.5) coincides with the metric (3.18).

By definition of η in (5.5) and using (5.16), we get

$$\begin{aligned}\eta_{\alpha\beta}(u) &:= \eta(\partial_{u_\alpha},\partial_{u_\beta}) = (-1)^{k+1}\sum_{|\lambda|<\infty}\operatorname*{res}_{d\lambda=0}\frac{\partial_{u_\alpha}(\lambda(\varphi)d\varphi)\partial_{u_\beta}(\lambda(\varphi)d\varphi)}{d\lambda(\varphi)}\\ &= (-1)^{k+1}\sum_{\gamma=1}^{l+1}\operatorname*{res}_{\varphi=[\psi_\gamma]}\frac{c_{\alpha,m}c_{\beta,m}P^2P'(\varphi)^2}{(P^2-q_\alpha^2)(P^2-q_\beta^2)}\frac{\lambda'(\varphi)}{\lambda''(\psi_\alpha)\lambda''(\psi_\beta)}d\varphi.\end{aligned}$$

We remark that $[\psi_\gamma]$ represents four different points $\pm\psi_\gamma$ and $\pm\psi_\gamma+\pi$ satisfying $q_\gamma^2=(e^{i[\psi_\gamma]}+e^{-i[\psi_\gamma]})^2$. Obviously, when $\alpha\neq\beta$, $\eta_{\alpha\beta}(u)=0$. So,

$$\begin{aligned}\eta_{\alpha\alpha}(u) &= (-1)^{k+1}\operatorname*{res}_{\varphi=[\psi_\alpha]}\frac{c_{\alpha,m}^2P^2P'(\varphi)^2}{(q_\alpha^2-P^2)^2}\frac{\lambda'(\varphi)}{\lambda''(\psi_\alpha)^2}d\varphi\\ &= (-1)^k\frac{2c_{\alpha,m}^2}{\lambda''(\psi_\alpha)^2}\operatorname*{res}_{P=\pm q_\alpha}\frac{P^2}{P^2-q_\alpha^2}\frac{\dot\lambda(P)(P^2-1)}{P^2-q_\alpha^2}dP\\ &= (-1)^k\frac{2\,c_{\alpha,m}^2}{\lambda''(\psi_\alpha)^2}\operatorname*{res}_{P=\pm q_\alpha}\frac{P^2}{P^2-q_\alpha^2}\frac{\dot\lambda(P)(P^2-1)}{P^2-q_\alpha^2}dP\\ &= (-1)^{k+1}\frac{2\,c_{\alpha,m}}{\lambda''(\psi_\alpha)}.\end{aligned}$$

We thus obtain

$$\eta_{\alpha\beta}(u)=(-1)^{k+1}\frac{2\,c_{\alpha,m}\delta_{\alpha\beta}}{\lambda''(\psi_\alpha)}.$$

Similarly, we can obtain the formula of $g_{\alpha\beta}(u):=g(\partial_{u_\alpha},\partial_{u_\beta})$ as

$$g_{\alpha\beta}(u)=-\frac{2\,c_{\alpha,m}\delta_{\alpha\beta}}{u_\alpha\lambda''(\psi_\alpha)}.$$

Observe that the vector field $e = \sum_{j=k}^{l} c_j \frac{\partial}{\partial y^j}$ in (3.20) in the coordinates $a_0, \cdots, a_l$ coincides with $e = (-1)^k \sum_{s=0}^{m} (-1)^{m-s} \binom{m}{s} \frac{\partial}{\partial a_{k+m-s}}$. The shifting

$$a_{k+m-s} \longmapsto a_{k+m-s} + c\,(-1)^{m-s} \binom{m}{s}, \quad s = 0, \cdots, m,$$

produces the corresponding shift

$$u_\alpha \longmapsto u_\alpha + c, \quad \alpha = 1, \cdots, l+1$$

of the critical values. This shift does not change the critical points ψ_α neither the values of the second derivative $\lambda''(\psi_\alpha)$. So

$$\mathcal{L}_e g^{\alpha\beta} = \mathcal{L}_e(-\frac{u_\alpha \lambda''(\psi_\alpha)}{2\,c_{\alpha,m}\delta_{\alpha\beta}}) = (-1)^{k+1} \frac{\lambda''(\psi_\alpha)}{2\,c_{\alpha,m}\delta_{\alpha\beta}} = \eta^{\alpha\beta}. \tag{5.29}$$

Finally, we compute the metric $g^{\beta\gamma}(\varphi)$ given by

$$g^{\beta\gamma}(\varphi) : \;=\; (d\varphi_\beta, d\varphi_\gamma) = \sum_{\alpha,\kappa=1}^{l+1} \frac{1}{g_{\alpha\kappa}(u)} \frac{\partial \varphi_\beta}{\partial u_\alpha} \frac{\partial \varphi_\gamma}{\partial u_\kappa} = \sum_{\alpha=1}^{l+1} \frac{1}{g_{\alpha\alpha}(u)} \partial_{u_\alpha}\varphi_\beta\, \partial_{u_\alpha}\varphi_\gamma.$$

Using (5.17), (5.20) and (5.21), we have

Case 1. $1 \le \beta, \gamma \le l$.

$$g^{\beta\gamma}(\varphi) = -\frac{p_\beta\, p_\gamma\, P'(\varphi_\beta)\, P'(\varphi_\gamma)}{2} \sum_{\alpha=1}^{l+1} \frac{c_{\alpha,m}\, u_\alpha}{\lambda''(\psi_\alpha)\,(p_\beta^2 - q_\alpha^2)(p_\gamma^2 - q_\alpha^2)} = \frac{1}{4}\delta_{\beta\gamma}.$$

Case 2. $1 \le \beta \le l$ **and** $\gamma = l+1$.

$$\begin{aligned} g^{\beta,l+1}(\varphi) &= \frac{p_\beta\, P'(\varphi_\beta)}{4ki} \sum_{\alpha=1}^{l+1} \frac{1}{p_\beta^2 - q_\alpha^2} \left(\delta_{\alpha,l+1} + \frac{2\,c_{\alpha,m}\, u_\alpha}{\lambda''(\psi_\alpha)} \sum_{s=1}^{l} \frac{p_s^2\, P'(\varphi_s)^2}{(q_{l+1}^2 - p_s^2)(q_\alpha^2 - p_s^2)} \right) \\ &= \frac{p_\beta\, P'(\varphi_\beta)}{4ki} \left(\frac{1}{p_\beta^2 - q_{l+1}^2} - \sum_{s=1}^{l} \frac{p_s^2\, P'(\varphi_s)^2}{q_{l+1}^2 - p_s^2} \sum_{\alpha=1}^{l+1} \frac{2\,c_{\alpha,m}\, u_\alpha}{\lambda''(\psi_\alpha)(q_\alpha^2 - p_s^2)(q_\alpha^2 - p_\beta^2)} \right) \\ &= 0. \end{aligned}$$

Case 3. $\beta = \gamma = l+1$.

$$\begin{aligned}
g^{l+1,l+1}(\varphi) &= \sum_{\alpha=1}^{l+1} \frac{u_\alpha\, \lambda''(\psi_\alpha)}{8k^2\, c_{\alpha,m}} \left(\frac{\delta_{\alpha,l+1}}{\lambda(\psi_\alpha)} + \frac{2\, c_{\alpha,m}}{\lambda''(\psi_\alpha)} \sum_{s=1}^{l} \frac{p_s^2\, P'(\varphi_s)^2}{(q_{l+1}^2 - p_s^2)(q_\alpha^2 - p_s^2)} \right)^2 \\
&= \frac{1}{8k^2} \frac{\lambda''(\psi_{l+1})}{\lambda(\psi_{l+1})} + \frac{1}{2k^2} \sum_{s=1}^{l} \frac{p_s^2\, P'(\varphi_s)^2}{(q_{l+1}^2 - p_s^2)^2} \\
&+ \frac{1}{2k^2} \sum_{s,j=1}^{l} \frac{p_s^2\, p_j^2\, P'(\varphi_s)^2\, P'(\varphi_j)^2}{(q_{l+1}^2 - p_s^2)\,(q_{l+1}^2 - p_j^2)} \sum_{\alpha=1}^{l+1} \frac{c_{\alpha,m}\, u_\alpha}{\lambda''(\psi_\alpha)(q_\alpha^2 - p_s^2)(q_\alpha^2 - p_j^2)} \\
&= -\frac{1}{4k^2} \left(k + \sum_{s=1}^{l} \frac{p_s^2\, P'(\varphi_s)^2}{(q_{l+1}^2 - p_s^2)^2} \right) + \frac{1}{2k^2} \sum_{s=1}^{l} \frac{p_s^2\, P'(\varphi_s)^2}{(q_{l+1}^2 - p_s^2)^2} \\
&+ \frac{1}{4k^2} \sum_{s,j=1}^{l} \frac{p_s^2\, p_j^2\, P'(\varphi_s)^2\, P'(\varphi_j)^2}{(q_{l+1}^2 - p_s^2)\,(q_{l+1}^2 - p_j^2)} \frac{\delta_{sj}}{p_j^2\,(p_j^2 - q_{l+1}^2)} \\
&= -\frac{1}{4k}.
\end{aligned}$$

Using (5.25), it is easy to know that the intersection form $g^{\alpha\beta}(\varphi)$ coincides with $(\ ,\ )^{\sim}$ defined in (3.1) and (3.2). The coincidence of the metric (5.5) with the metric (3.18) follows (5.29). We thus complete the proof of the theorem. □

Remark 5.7. *On the orbit space of the extended affined Weyl group $\widetilde{W}^{(k)}(D_{k+2})$, Dubrovin and Zhang constructed a quasi-homogenous polynomial Frobenius structure, denoted by $\mathcal{M}_{\mathrm{DZ}}^{(k)}(D_{k+2})$ which is isomorphic to $\mathfrak{M}_{k,1,1}$. Actually, in this case, there is a tri-polynomial description introduced in* [17, 19]*, also used in* [10].

6. Concluding Remarks

For the root systems of type B_l, C_l and D_l, we have constructed families of Frobenius manifold structures on the orbit spaces of the extended affine Weyl groups $\widetilde{W}^{(k)}(R)$ with respect to the choice of an arbitrary vertex on the Dynkin diagram, as was suggested in [18], motivated by the results of [20, 13, 14]. In our construction for the root system C_l, we perform the following two steps:

i) We fix the k-th vertex of the Dynkin diagram and define an extension of the affine Weyl group, and construct a symmetric bilinear form (g^{ij}) on the cotangent space of the orbit space of the extended affine Weyl group.

ii) We find a unity vector field e which is labeled by an integer $0 \le m \le l-k$, and construct a Frobenius manifold structure on the orbit space.

We may ask the question whether one can perform the same construction for the root system of type A_l? Namely, we can perform the first step as it is done in [7] for any $1 \le k \le l$. For the second step, only one choice of the unity vector field e is given in [7] to construct a Frobenius manifold structure on the orbit space. Then is there other choices of the unity vector field? The answer is no, i.e. we can not find a different unity vector field satisfying the conditions of Lemma 3.3. It remains a challenging problem to understand whether the constructions of the present paper can be generalized to the root systems of the types E_6, E_7, E_8, F_4, G_2.

Another open problem is to obtain an explicit realization of the integrable hierarchies associated with the Frobenius manifolds of the type $\widetilde{W}^{(k)}(R)$. So far this problem was solved only for $R = A_l$, see [3, 4, 8, 9, 15, 16] for details. We plan to study these problems in subsequent publications.

Observe that the potential of the semisimple Frobenius manifold structures constructed above from the root systems of type $(C_l, k, m = 0)$ has the form

$$F = \frac{1}{2}\,(t^k)^2 t^{l+1} + \frac{1}{2} t^k \sum_{\alpha,\beta\neq k} \eta_{\alpha\beta} t^\alpha t^\beta + \sum_{j=0}^{n} f_j(t^2, t^3, \dots, t^l, \frac{1}{t^l})\, e^{j\, t^{l+1}},$$

where $f_j(t^2, t^3, \dots, t^l, \frac{1}{t^l}), j = 0, \dots, n$ are some polynomials of their independent variables. The Euler vector field has the form

$$E = \sum_{j=1}^{l} d_j \frac{\partial}{\partial t^j} + r \frac{\partial}{\partial t^{l+1}}.$$

Here $0 < d_j < 1, r > 0$, and they also satisfy the duality relation given in (3.69), (3.70) for the case $m = 0$. We expect that these potentials of semisimple Frobenius manifolds, together with the ones that are constructed in [7], exhaust all solutions of the above form, and we have verified this for the cases when $l = 1, 2, 3$ and $n \le 6$.

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