A comment on pulsatile pipe flow

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This article is concerned with analytic solutions of flows in cylindrical and annular pipes subject to an arbitrary time dependent pressure gradient and arbitrary initial flow.

Consider unsteady flow within a circular pipe driven by a time-dependent pressure gradient. The flow is unidirectional with $u(r,t)$ in the axial direction satisfying

$$\frac{\partial u}{\partial t} = g(t) + \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right), \tag{1}$$

with

$$u(a,t) = 0 \quad ; \quad u(0,t) \quad \text{bounded}, \quad \text{all} \quad t. \tag{2}$$

Here ν, ρ and a are respectively the kinematic viscosity, the density and the pipe radius and $\rho g(t) = -\partial p/\partial z$, with $-\partial p/\partial z$ the pressure gradient (z is the axial direction). This problem has a long history dating back to Sexl (1928), Lambossy (1952), Womersley (1955 a,b) and Uchida (1956). Other authors who have worked on closely related problems including pressure gradients which are impulses, exponentially decreasing functions etc. include Syzmanski (1932), Ito (1953), Lance (1956), Sanyal (1956), Verma (1960) and Smith (1997). Drazin and Riley (2006) provide the solution

$$u(r,t) = \frac{P}{4\mu}\left(a^2 - r^2\right) - \mathrm{Re}\left\{\frac{\mathrm{i}c}{\omega\rho}\left(1 - \frac{\mathrm{J}_0\left(\mathrm{i}^{3/2}kr\right)}{\mathrm{J}_0\left(\mathrm{i}^{3/2}ka\right)}\right)\mathrm{e}^{\mathrm{i}\omega t}\right\}, \quad k^2 = \frac{\omega}{\nu}, \tag{3}$$

where, here,

$$\frac{\partial p}{\partial z} = -P - \mathrm{Re}\left\{c\mathrm{e}^{\mathrm{i}\omega t}\right\}, \quad c = c_1 - \mathrm{i}c_2 \tag{4}$$

which is essentially the form given by Sexl (1928), Uchida (1956) and Womersley (1955a). This form is ubiquitous in the literature and yet it is unsatisfactory as it assumes a very specific initial condition. The problem one wishes to solve is (1) with (2) together with an arbitrary initial condition, let us say

$$u(r,0) = f(r). \tag{5}$$

In fact it is relatively straightforward (see Appendix) to write down a solution to (1) subject to (2) and (5):

$$u(r,t) = 2\sum_{n=1}^{\infty}\frac{\mathrm{J}_0(\alpha_n r)\mathrm{e}^{-\nu\alpha_n^2 t}}{a\mathrm{J}_1(\alpha_n a)}\left\{\frac{\int_0^a rf(r)\mathrm{J}_0(\alpha_n r)dr}{a\mathrm{J}_1(\alpha_n a)} + \frac{\int_0^t \mathrm{e}^{\nu\alpha_n^2 t'}g(t')dt'}{\alpha_n}\right\}, \tag{6}$$

where $\alpha_n, n = 1, 2, ..,$ are the countably infinite roots of $\mathrm{J}_0(\alpha_n a) = 0$.

The volume flow rate is given by

$$Q = 4\pi \sum_{n=1}^{\infty} \frac{e^{-\nu\alpha_n^2 t}}{\alpha_n} \left\{ \frac{\int_0^a rf(r)J_0(\alpha_n r)dr}{aJ_1(\alpha_n a)} + \frac{\int_0^t e^{\nu\alpha_n^2 t'} g(t')dt'}{\alpha_n} \right\}. \tag{7}$$

We observe immediately that we have a solution for any initial flow and time-dependent pressure gradient, with no possible singularities: $J_1(\alpha_n a)$ is never zero since the zeros of $J_0(z)$ and $J_1(z)$ interlace. We note that when the flow is initially quiescent, we obtain the same expression for the flow as Szymanski (1932) and, additionally, if $\partial p/\partial z =$ constant, then as $t \to \infty$ we retrieve Poiseuille flow, since

$$a^2 - r^2 = \frac{8}{a} \sum_{n=1}^{\infty} \frac{J_0(\alpha_n r)}{\alpha_n^3 J_1(\alpha_n a)}. \tag{8}$$

Of course, because of the generality of the above expressions, flows subject to impulsive pressure gradients and gradients subject to exponential decay, for example, are easily derived from (6): indeed, the solution obtained by Ito (1953) for a linear pressure gradient can be shown to be a special case of (6).
We may also consider flow in an annular region subject to an arbitrary time-dependent pressure gradient where the outer radius is a while the inner radius is b ($b < a$). We obtain

$$u(r,t) = \pi \sum_{n=1}^{\infty} \frac{V_0(\alpha_n r)e^{-\nu\alpha_n^2 t}}{1 + X_0(\alpha_n)} \left\{ \frac{\pi\alpha_n^2 \int_b^a rf(r)V_0(\alpha_n r)dr}{2(1 - X_0(\alpha_n))} + \int_0^t e^{\nu\alpha_n^2 t'} g(t')dt' \right\} \tag{9}$$

where

$$V_0(\alpha_n r) = J_0(\alpha_n r)Y_0(\alpha_n b) - J_0(\alpha_n b)Y_0(\alpha_n r),$$

$$X_0(\alpha_n) = J_0(\alpha_n b)/J_0(\alpha_n a), \quad n = 1, 2, ...,$$

where here α_n are the countably infinite zeros of $V_0(\alpha_n a) = 0$. The corresponding volumetric flow rate is

$$Q = 4\pi \sum_{n=1}^{\infty} \frac{e^{-\nu\alpha_n^2 t}(1 - X_0(\alpha_n))}{\alpha_n^2(1 + X_0(\alpha_n))} \left\{ \frac{\pi\alpha_n^2 \int_b^a rf(r)V_0(\alpha_n r)dr}{2(1 - X_0(\alpha_n))} + \int_0^t e^{\nu\alpha_n^2 t'} g(t')dt' \right\}. \tag{10}$$

It is straightforward to show that as $b \to 0$ expressions (9) and (10) reduce to (6) and (7) respectively.

Appendix

The eigenfunctions of the related homogeneous problem satisfy

$$\frac{d}{dr}\left(r\frac{d\phi(r)}{dr} \right) + \alpha^2 r\phi(r) = 0, \tag{A 1}$$

$$\phi(a) = 0 \tag{A 2}$$

and admit the general solution

$$\phi(r) = AJ_0(\alpha r) + BY_0(\alpha r). \tag{A 3}$$

Boundedness and the no slip condition them imply, respectively, that $B = 0$ and

$$J_0(\alpha a) = 0, \tag{A 4}$$

yielding, through superposition, the general solution

$$u(r,t) = \sum_{n=1}^{\infty} a_n(t)\mathrm{J}_0(\alpha_n r). \tag{A 5}$$

With $f(r) = \sum_{n=1}^{\infty} a_n(0)\mathrm{J}_0(\alpha_n r)$ we deduce, using orthogonality, that

$$a_n(0) = \frac{2}{(\mathrm{J}_1(\alpha_n a))^2} \int_0^a r f(r)\mathrm{J}_0(\alpha_n r)dr. \tag{A 6}$$

Substitution of (A 5) into the inhomogeneous problem results in

$$\sum_{n=1}^{\infty} \left\{ \frac{da_n(t)}{dt} + \nu\alpha_n^2 a_n(t) \right\} \mathrm{J}_0(\alpha_n r) = g(t). \tag{A 7}$$

From orthogonality we deduce that

$$a_n(t) = a_n(0)\mathrm{e}^{-\nu\alpha_n^2 t} + \frac{2}{a\alpha_n \mathrm{J}_1(\alpha_n a)} \int_0^t g(t')\mathrm{e}^{-\nu\alpha_n^2(t-t')}dt', \tag{A 8}$$

giving rise to the solution (6) provided in the main text. The derivation of (9) is similar except that the orthogonal functions employed are $\mathrm{V}_0(\alpha_n r)$.

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