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Diagrammatic insights into next-to-soft corrections

C.D. White

SUPA, School of Physics and Astronomy, University of Glasgow, Glasgow G12 8QQ, Scotland, UK

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ABSTRACT

We confirm recently proposed theorems for the structure of next-to-soft corrections in gauge and gravity theories using diagrammatic techniques, first developed for use in QCD phenomenology. Our aim is to provide a useful alternative insight into the next-to-soft theorems, including tools that may be useful for further study. We also shed light on a recently observed double copy relation between next-to-soft corrections in the gauge and gravity cases.

1. Introduction

It is well-known that scattering amplitudes in gauge and gravity theories contain infrared divergences. These arise from the emission of *soft* gluons or gravitons, whose 4-momentum tends to zero. The remaining *hard* particles in the amplitude are then said to obey the *eikonal approximation*, and it can be shown that amplitudes factorise in this limit. At tree level, for example, the amplitude for the emission of n hard gluons (momenta $\{p_i\}$) and one soft gluon (momentum k) can be written as

$$\mathcal{A}_{n+1}(\{p_i\},k)=\mathcal{S}_n^{(0)}\mathcal{A}_n(\{p_i\}); \qquad \mathcal{S}_n^{(0)}=\sum_{i=1}^{n}\frac{\epsilon_\mu(k)p_i^\mu}{p_i\cdot k}, \tag{1}$$

where we neglect the coupling constant and colour factors of the soft emission for brevity. Here $\mathcal{A}_n$ is the amplitude for the n hard particles with no additional emission, and $\epsilon_\mu(k)$ is the polarisation vector of the soft gluon. The gravity equivalent of this is known as Weinberg's soft theorem [1], and takes the form

$$\mathcal{M}_{n+1}(\{p_i\},k)=\mathcal{S}_{n,\text{grav.}}^{(0)}\mathcal{M}_n(\{p_i\});$$
$$\mathcal{S}_{n,\text{grav.}}^{(0)}=\sum_{i=1}^{n}\frac{\epsilon_{\mu\nu}(k)p_i^\mu p_i^\nu}{p_i\cdot k}. \tag{2}$$

Until recently, much less has been known about the corrections to Eqs. (1), (2), upon performing a systematic expansion in the momentum of the soft gauge boson. Such corrections are known as next-to-soft, and the hard emitting particles then obey the *next-to-eikonal* approximation. The phenomenological impact of such corrections has been studied in QCD [2–6], and a systematic attempt to classify them has been made in [7–9]. The gravitational consequences of next-to-soft radiation have been explored in [10].

An orthogonal recent body of work has explored such contributions from a more formal point of view. Based on the observation that Weinberg's soft theorem can be interpreted as a Ward identity associated with BMS transformations at past and future null infinity [11,12], Ref. [13] conjectured a tree-level next-to-soft generalisation of Eq. (2), where the subleading soft factor is given by

$$\mathcal{S}_{n,\text{grav.}}^{(1)}=\sum_{i=1}^{n}\frac{\epsilon_{\mu\nu}(k)p_i^\mu k_\rho J^{(i)\rho\nu}}{p_i\cdot k}. \tag{3}$$

Here $J^{\rho\nu}$ is the total angular momentum associated with the hard external leg i, and Ref. [14] gave an analogous result for gauge theory:

$$\mathcal{S}_n^{(1)}=\sum_{i=1}^{n}\frac{\epsilon_\mu(k)k_\rho J^{(i)\mu\rho}}{p_i\cdot k}. \tag{4}$$

These results were subsequently understood from the point of view of the *scattering equations* of [15,16] in Ref. [17], using further symmetry arguments in [18,19], and string theoretic ideas in Refs. [20–22]. Higher dimensions were considered in Ref. [23], and a holographic description of the 4-dimensional gravitational theory pursued in [24]. Possible loop-level corrections to Eqs. (3), (4) have been examined in Refs. [25–27].

The aim of this paper is to explore the above results using Feynman diagrammatic methods previously developed in Refs. [7–9] (which are themselves related to the earlier results of Refs. [28,29]). Our main motivation is to clarify how those results are consistent with the recently proposed theorems. We stress that this analysis is new: whilst Refs. [7–9] and the much earlier work of Refs. [28,29] derive partial results regarding next-to-soft

E-mail address: Christopher.White@glasgow.ac.uk.

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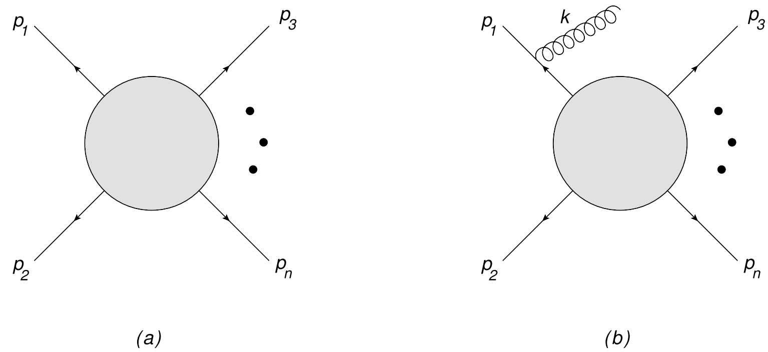

Fig. 1. (a) A hard interaction which produces n particles; (b) Emission of an eikonal gluon from an external leg.

corrections, they do not fully reproduce the results of Eqs. (3), (4). Secondly, it is nearly always useful to have multiple, equivalent ways of thinking about a given piece of physics, and we believe that our point of view may be useful in further studies of the next-to-soft theorems. Finally, connecting the recent results of Refs. [11–14,17–27] with Refs. [7–9] may aid the ongoing effort to use next-to-eikonal effects to increase the accuracy of collider predictions.

The structure of the paper is as follows. In Section 2 we briefly review the content of Refs. [7–9], addressing next-to-eikonal effects using effective Feynman rules. In Section 3, we show how these results reproduce the soft theorem of Eq. (4) for the case of scalar and fermionic emitting particles. The above references did not consider external gluons, and we perform this analysis in Section 4. We will confirm the tree-level results of Eq. (4) for external scalars, fermions and gluons. Finally, in Section 5 we discuss our results and conclude. Some technical details are presented in Appendix A.

2. Review of necessary concepts

In this section, we review the results of Refs. [7–9] and related papers, whose aim is to systematically classify next-to-eikonal contributions to scattering amplitudes in gauge and gravity theories. The starting point is to factorise the amplitude into a *hard function*, which is infrared finite, and a soft function, which collects all soft singularities.[1] Such a factorisation is well-known (see e.g. Ref. [30] for a review in QCD, and Refs. [31,32] for gravity). However, Refs. [7,9] generalised the soft function to include next-to-soft radiative corrections. The method proceeded by writing the propagators for the external particles in a background soft gauge field as first-quantised path integrals [33,34], which can be evaluated perturbatively. The leading term in this expansion is the eikonal approximation, in which external particles do not recoil, and change only by a (Wilson-line) phase [35]. The first subleading term describes the emission of next-to-soft gauge bosons, which are completely external to the hard interaction. Ref. [8] rederived the same results via a systematic expansion of Feynman diagrams to all orders in perturbation theory, and also checked the resulting formalism by reproducing known next-to-eikonal logarithms in Drell–Yan production. These are not the only sources of next-to-soft correction. As Refs. [7–9] explain in detail, one must also worry about soft gluon emissions which originate from inside the hard interaction.

[1] One must also include *jet functions* to keep track of collinear singularities. For the purposes of the present paper, however, we may implicitly absorb the jets into the hard function, as in Refs. [7–9].

Let us illustrate how the results apply to the present context, namely that of dressing an amplitude for the emission of n hard particles by an additional (next-to-)soft emission. One starts with a hard interaction such as that shown in Fig. 1(a). The leading soft singularities come from dressing all external legs (momenta $\{p_i\}$) with a soft gauge boson (momentum k), whose emission is described by an eikonal Feynman rule. This is shown in Fig. 1(b), and the kinematic parts of the eikonal Feynman rules for Yang–Mills theory and gravity are

$$\frac{p_i^\mu}{p_i\cdot k} \quad \text{and} \quad \frac{p_i^\mu p_i^\nu}{p\cdot k} \tag{5}$$

respectively. This clearly leads to the soft factors of Eqs. (3), (4), and at leading soft level one need only worry about the external emission of soft gluons. In Feynman diagram language, this can be understood by the fact that a soft gluon landing inside the hard interaction squares an offshell propagator, which dampens the infrared singular behaviour. In more physical terms, a soft gluon has an infinite Compton wavelength, and thus cannot resolve the substructure of the hard interaction. For the same reason, the above eikonal Feynman rules are independent of the spin of the hard emitting particles.

At next-to-soft level, there are two types of contribution. Firstly, there are next-to-soft gluon emissions external to the hard interaction, as shown in Fig. 2(a). These emissions are described by next-to-eikonal (NE) Feynman rules which, unlike the purely soft limit, depend on the spin of the emitting particles. External fermions and scalars were considered in Yang–Mills theory in Refs. [7,8]; scalars only were considered in the gravity study of Ref. [9]. In Yang–Mills theory, the NE Feynman rules for emission of a (potentially off-shell) gluon from a scalar and fermion are

$$V^\mu_{\text{scal.}} = \frac{k^\mu}{2p_i\cdot k} - \frac{k^2 p^\mu}{2(p_i\cdot k)^2}; \qquad V^\mu_{\text{ferm.}} = V^\mu_{\text{scal.}} - \frac{ik_\nu \Sigma^{\mu\nu}}{p_i\cdot k}, \tag{6}$$

where

$$\Sigma^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu,\gamma^\nu\right] \tag{7}$$

is the generator of Lorentz transformations. There are also NE Feynman rules describing the correlated emission of a pair of soft gluons. The result for emission from an external scalar, for example, is

$$R^{\mu\nu} = \frac{p\cdot k_1 k_2^\mu p^\nu + p\cdot k_2 k_1^\nu p^\mu - p^\mu p^\nu k_1\cdot k_2 - \eta^{\mu\nu} p\cdot k_1 p\cdot k_2}{p\cdot k_1 p\cdot k_2 p\cdot (k_1+k_2)}, \tag{8}$$

where k_1^ν and k_2^μ are the soft gluon 4-momenta. An additional contribution arises for external fermions, again involving the generator

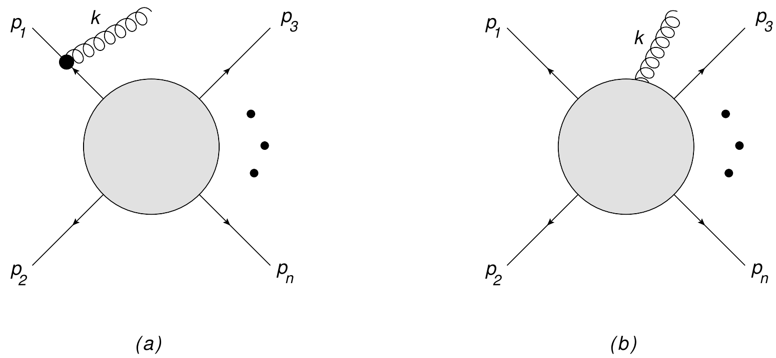

Fig. 2. (a) External emission of a next-to-soft gluon; (b) Internal emission of a soft gluon.

of Lorentz transformations [7,8]. Similar Feynman rules have been obtained for scalar external legs emitting next-to-soft gravitons in Ref. [9], where again there are one and two-graviton vertices.

The second type of contribution at next-to-eikonal level arises from the *internal emission* of a soft boson from inside the hard interaction. Such contributions violate the factorisation of an amplitude into hard and soft parts; physically, this corresponds to the emitted soft boson being able to resolve the finite size of the hard interaction. However, gauge invariance fixes these contributions in terms of derivatives with respect to the external momenta, acting on the hard interaction with no additional emission. This result was first derived by Low [28] for scalar particles, and generalised to fermions by Burnett and Kroll [29]. A further important generalisation, to massless external particles, was carried out by Del Duca [36]. The internal emission contributions decompose into a part which is independent of the spin of the external legs, and an additional spin-dependent piece. The latter is associated only with the hard collinear region, and need not concern us here. The former can be written, in the present notation, as

$$\mathcal{A}^{\text{int.}}_{n+1} = \sum_i \left(\frac{p_{i\mu}}{p_i \cdot k} k_\nu \frac{\partial}{\partial p_i^\nu} - \frac{\partial}{\partial p_i^\mu} \right) \mathcal{A}_n\big(\{p_i\}\big). \tag{9}$$

In the path integral approach of Ref. [7], the first term arises from the non-zero initial position of each hard external line, thus making clear that this contribution arises due to the non-trivial spatial extent of the hard interaction. The analogous result for gravity reads [9]

$$\mathcal{M}^{\text{int.}}_{n+1} = \sum_j \left(\frac{p_{j\mu} p_{j\nu}}{p_j \cdot k} k^\sigma \frac{\partial}{\partial p_j^\sigma} - p_{j\mu} \frac{\partial}{\partial p_j^\nu} \right) \mathcal{M}_n\big(\{p_n\}\big). \tag{10}$$

To summarise, next-to-soft contributions at a given order can be calculated by combining external and internal emission contributions. The former are described using effective NE Feynman rules, whereas the latter obey the iterative formulae of Eqs. (9), (10).

3. The next-to-soft theorem for scalars and fermions

In this section, we demonstrate how the tree-level soft theorems of Eqs. (3), (4) are reproduced by the results of Refs. [7–9], beginning with scalar emitting particles in Yang–Mills theory. As described above, one must combine external and internal emission contributions. The former can be obtained from the NE Feynman rules of Eqs. (6), (8). The two-gluon vertex does not contribute, owing to the fact that we are taking only one gluon soft, and remain at tree level. Furthermore, both terms in the 1-gluon vertex of Eq. (6) vanish: the first due to contraction with a physical polarisation tensor obeying $k^\mu \epsilon_\mu(k) = 0$, and the second due to the onshellness of the emitted gluon ($k^2 = 0$). Thus, only the internal emission contributions are necessary, which may be rewritten as

$$\begin{aligned}\mathcal{A}^{\text{int.}}_{n+1} &= \frac{k^\nu}{p_i \cdot k} \left(p_{i\mu} \frac{\partial}{\partial p_i^\nu} - p_{i\nu} \frac{\partial}{\partial p_i^\mu} \right) \mathcal{A}_n \\ &= -\frac{i k^\nu L^{(i)}_{\mu\nu}}{p_j \cdot k} \mathcal{A}_n,\end{aligned} \tag{11}$$

where we have introduced the orbital angular momentum tensor for the ith particle:

$$L^{(i)}_{\mu\nu} = x_{i\mu} p_{i\nu} - x_{i\nu} p_{i\mu} = i \left(p_{i\mu} \frac{\partial}{\partial p_i^\nu} - p_{i\nu} \frac{\partial}{\partial p_i^\mu} \right). \tag{12}$$

For a scalar particle, the orbital angular momentum is equal to the total angular momentum, $L^{(i)}_{\mu\nu} = J^{(i)}_{\mu\nu}$, and thus we have indeed reproduced the next-to-soft theorem of Eq. (4).

Considering now fermionic emitting particles, the internal emission contribution will be the same as for the scalar case. However, there is now a non-zero external emission contribution, due to the magnetic moment term in Eq. (6). The total next-to-soft contribution is then

$$\begin{aligned}\mathcal{A}_{n+1} &= \frac{k^\nu}{p_i \cdot k} \left(p_{i\mu} \frac{\partial}{\partial p_i^\nu} - p_{i\nu} \frac{\partial}{\partial p_i^\mu} - i\Sigma_{\mu\nu} \right) \mathcal{A}_n \\ &= -\frac{i k^\nu (L^{(i)}_{\mu\nu} + \Sigma_{\mu\nu})}{p_j \cdot k} \mathcal{A}_n,\end{aligned} \tag{13}$$

One may recognise the bracketed factor in the numerator as the sum of the orbital and spin angular momentum of the ith particle, and thus one finds

$$\mathcal{A}_{n+1} = -\frac{i k^\nu J^{(i)}_{\mu\nu}}{p_j \cdot k} \mathcal{A}_n \tag{14}$$

as before.

We may also examine the case of gravity, and the effective Feynman rules for scalar emitting particles were first derived in Ref. [9]. However, that paper defined the graviton in terms of the metric $g_{\mu\nu}$ and its determinant g via

$$\sqrt{-g} g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \tag{15}$$

rather than the more conventional (in high energy physics) choice

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}. \tag{16}$$

Next-to-eikonal Feynman rules for the definition of Eq. (16) are derived here in Appendix A, where we also derive the rules for graviton emission from fermions. The resulting one-graviton vertices for the emission of a graviton of momentum k from a hard line of momentum p are

$$V^{\mu\nu}_{\text{scal.}}=-\frac{p^\mu p^\nu k^2}{2(p\cdot k)^2}+\frac{k^{(\nu}p^{\mu)}}{2p\cdot k}-\frac{\eta^{\mu\nu}}{2};$$
$$V^{\mu\nu}_{\text{ferm.}}=V^{\mu\nu}_{\text{scal.}}-\frac{ik_\rho\Sigma^{\rho(\mu}p^{\nu)}}{2p\cdot k}. \tag{17}$$

As in the Yang–Mills case, all scalar-like external emission contributions vanish. This is due to onshellness of the emitted graviton, and contraction with a physical, traceless polarisation tensor for the graviton:

$$\epsilon_{\mu\nu}(k)k^\mu=\epsilon_{\mu\nu}(k)k^\nu=\epsilon_{\mu\nu}(k)\eta^{\mu\nu}=0. \tag{18}$$

For the scalar case, then, the only next-to-soft contributions arise from internal emissions, which are given by [9][2]

$$\begin{aligned}\mathcal{M}^{\text{int.}}_{n+1}&=\sum_j\left(\frac{p_{j\mu}p_{j\nu}}{p_j\cdot k}k^\rho\frac{\partial}{\partial p_j^\rho}-p_{j\mu}\frac{\partial}{\partial p_j^\nu}\right)\mathcal{M}_n(\{p_i\})\\&=\sum_j\frac{p_{j\mu}k^\rho}{p_j\cdot k}\left(p_{j\nu}\frac{\partial}{\partial p_j^\rho}-p_{j\rho}\frac{\partial}{\partial p_j^\nu}\right)\mathcal{M}_n(\{p_i\})\\&=\sum_j\frac{-ip_{j\mu}k^\rho L^{(j)}_{\rho\nu}}{p_j\cdot k}\mathcal{M}_n(\{p_n\}).\end{aligned} \tag{19}$$

Given that the orbital angular momentum is the total angular momentum in this case, this is the next-to-soft theorem of Eq. (3). For fermionic emitting particles, one must add the additional external emission contribution from Eq. (17), which gives a total next-to-soft amplitude

$$\begin{aligned}\mathcal{M}_{n+1}&=\sum_j\frac{-ip_{j\mu}k^\rho(L^{(j)}_{\rho\nu}+\Sigma_{\rho\nu})}{p_j\cdot k}\mathcal{M}_n(\{p_n\})\\&=\sum_j\frac{-ip_{j\mu}k^\rho J^{(j)}_{\rho\nu}}{p_j\cdot k}\mathcal{M}_n(\{p_n\}),\end{aligned} \tag{20}$$

where we have again recognised the total angular momentum as the sum of the orbital and spin contributions. This is indeed the theorem of Eq. (3).

Before moving on, it is interesting to note that if one neglects the term in $\eta^{\mu\nu}$ in Eq. (17) (which in any case gives zero when contracted with a physical polarisation tensor), each term in the NE one-graviton vertex has the form of a gauge-theory eikonal Feynman rule multiplying a gauge-theory NE rule. This is strongly reminiscent of the double copy between gravity and gauge theory of Refs. [37,38], and indeed it has already been noted in Ref. [26] (see after Eq. (2.15) of that paper) that the next-to-soft theorems in gauge theory and gravity theories appear to be related by the double copy. The similarity between the NE Feynman rules observed here is itself a generalisation of the fact that the eikonal Feynman rules in gauge and gravity theories are related by the double copy. This was explored in detail in Ref. [39], where it was shown that matching up the infrared singularities of Yang–Mills theory and gravity provides all-order evidence for the double copy conjecture.

[2] Note that Ref. [9] contains a number of typos, which have been fixed in Eq. (19).

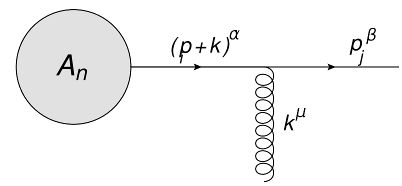

Fig. 3. Emission of a single gluon from the jth external leg of the hard amplitude $\mathcal{A}_n$.

4. The next-to-soft theorem for gluons

In the previous section, we saw how to derive the next-to-soft theorems for external scalars and fermions from a Feynman diagrammatic treatment. In this section, we consider the Yang–Mills result for the case of external gluons. These were not considered in Refs. [7,8], and so we must derive the relevant NE Feynman rules. Our derivation will be analogous to that carried out in Appendix A, although we will consider the sum of internal and emission contributions at the outset.

Let us start with the n-point hard amplitude of Fig. 1(a), and consider all places in which one may add an additional gluon. As described already above, we may emit this either from an external line, or from inside the hard interaction. Using the usual Feynman rules of QCD, one then finds that the $(n+1)$ amplitude is given by

$$\begin{aligned}&\mathcal{A}_{n+1}(\{p_i\},k)\\&=\Bigg(\mathcal{A}^{\mu,\text{int.}}_{n+1}-\sum_{j=1}^n\mathbf{T}_j\mathcal{A}^\alpha_n(p_j+k)\\&\quad\times\frac{[\eta^{\alpha\beta}(-k-2p_j)^\mu+\eta^{\alpha\mu}(2k+p_j)^\beta+\eta^{\beta\mu}(p_j-k)^\alpha]\epsilon_\beta(p)}{(p_j+k)^2}\Bigg)\\&\quad\times\epsilon_\mu(k).\end{aligned} \tag{21}$$

Here the first term collects the internal emission contributions, and $\mathbf{T}_j$ is a colour generator associated with an external emission on line j, which we keep track of for reasons that will become clear. We have also used the notation $A^\alpha_n(p_j+k)$ to denote the n-particle amplitude, but where the hard momentum p_j has been replaced by p_j+k, and where α is the Lorentz index of the jth external gluon line. The emission from this line is then as shown in Fig. 3. One may expand Eq. (21) up to first subleading order in the momentum k, and simplify the result using the transverse nature of the polarisation vector

$$\epsilon_\beta(p_j)p_j^\beta=0 \tag{22}$$

as well as the Ward identity

$$\mathcal{A}^\alpha_n(p_j)p_{j\alpha}=0. \tag{23}$$

The result is

$$\begin{aligned}&\mathcal{A}_{n+1}(\{p_i\},k)\\&=\Bigg\{\mathcal{A}^{\mu,\text{int.}}_{n+1}+\sum_{j=1}^n\mathbf{T}_j\Bigg[\Bigg[\eta^{\alpha\beta}\left(\frac{p_j^\mu}{p_j\cdot k}\right.\\&\quad\left.+\frac{k^\mu}{2p_j\cdot k}\right)+\frac{k^\alpha\eta^{\mu\beta}}{2p_j\cdot k}-\frac{k^\beta\eta^{\alpha\mu}}{p_j\cdot k}\Bigg]\mathcal{A}_{n\alpha}(p_j)\\&\quad+\left(\frac{p_j^\mu}{p_j\cdot k}\eta^{\alpha\beta}k^\sigma-\frac{\eta^{\mu\beta}p_j^\alpha k^\sigma}{2p_j\cdot k}\right)\frac{\partial\mathcal{A}_{n\alpha}(p_j)}{\partial p_j^\sigma}\Bigg]\Bigg\}.\end{aligned} \tag{24}$$

One may simplify this result still further by noting that Eq. (23) implies

$$\frac{\partial}{\partial p^\sigma}\left[p^\alpha \mathcal{A}_{n\alpha(p)}\right]=0 \quad\Rightarrow\quad p^\alpha\frac{\partial\mathcal{A}_{n\alpha}(p)}{\partial p^\sigma}=-\delta^\alpha_\sigma\mathcal{A}_{n\alpha}(p). \tag{25}$$

One then finds

$$\begin{aligned}
&\mathcal{A}_{n+1}\big(\{p_i\},k\big)\\
&=\Bigg\{\mathcal{A}^{\mu,\text{int.}}_{n+1}+\sum_{j=1}^n \mathbf{T}_j\bigg[\eta^{\alpha\beta}\bigg(\frac{p_j^\mu}{p_j\cdot k}+\frac{k^\mu}{2p_j\cdot k}\bigg)\mathcal{A}_{n\alpha}(p_j)\\
&\quad+\eta^{\alpha\beta}\frac{p_j^\mu k^\sigma}{p_j\cdot k}\frac{\partial\mathcal{A}_{n\alpha}(p_j)}{\partial p_j^\sigma}\\
&\quad-\mathcal{A}_{n\alpha}\frac{k_\sigma}{p_j\cdot k}\big(\eta^{\sigma\beta}\eta^{\alpha\mu}-\eta^{\sigma\alpha}\eta^{\beta\mu}\big)\bigg]\epsilon_\beta(p_j)\Bigg\}\epsilon_\mu(k).
\end{aligned} \tag{26}$$

Let us now interpret this result. Firstly, the first and third terms in the square bracket can be written in the form

$$\sum_{j=1}\mathbf{T}_i\mathcal{A}_{n\alpha}\left[V^\mu_{\text{vec.}}\right]_{\alpha\beta}\epsilon_\beta(p), \tag{27}$$

where the effective Feynman rule for the external emission of a gluon up to next-to-soft order is

$$V^\mu_{\text{vec.}}=\frac{p_j^\mu}{p_j\cdot k}+\frac{k^\mu}{2p_j\cdot k}-i\frac{k_\sigma M^{\sigma\mu}}{p_j\cdot k}, \tag{28}$$

and we have recognised the generator of the Lorentz group which acts on vector fields

$$M^{\mu\nu}_{\alpha\beta}=i\left[\delta^\mu_\alpha\delta^\nu_\beta-\delta^\nu_\alpha\delta^\mu_\beta\right]. \tag{29}$$

The first term in Eq. (28) is the eikonal Feynman rule for the emission of a gluon from an external line; the remaining terms produce the next-to-soft external emission contributions. As in the fermion case, there is a spin-independent contribution, and a part involving the spin angular momentum of each external gluon.

The remaining term in the square bracket in Eq. (26) can be related to the internal emission contribution. One may see this by writing

$$\mathcal{A}_{n+1}\big(\{p_i\},k\big)=\mathcal{A}^\mu_{n+1}\epsilon_\mu(k) \tag{30}$$

on the left-hand side, and applying the Ward identity

$$k_\mu\mathcal{A}^\mu_{n+1}=0 \tag{31}$$

which according to the right-hand side of Eq. (26) implies

$$k_\mu\mathcal{A}^{\mu,\text{int.}}_{n+1}+\sum_{j=1}^n\mathbf{T}_j\bigg[\mathcal{A}_{n\alpha}(p_j)+k_\mu\frac{\partial\mathcal{A}_{n\alpha}(p_j)}{\partial p_j^\mu}\bigg]\epsilon_\alpha(p_j)=0. \tag{32}$$

The first term in the square brackets cancels by colour conservation

$$\sum_{j=1}^n\mathbf{T}_j=0, \tag{33}$$

and one thus obtains

$$\mathcal{A}^{\mu,\text{int.}}_{n+1}=-\sum_{i=1}^n\mathbf{T}_j\frac{\partial\mathcal{A}_{n\alpha}}{\partial p_j^\mu}\epsilon_\alpha(p_j). \tag{34}$$

This is in fact a rederivation of part of the internal emission contribution that we have already quoted for the scalar case in Eq. (11), and is similar to the original analysis by Low [28]. The result is incomplete for massless external particles, however, as explained in detail by Del Duca in Ref. [36]. Here, the loophole in the above derivation is that we have not carefully separated out collinear singularities, implicitly absorbing jet functions into the hard function.[3] A more careful analysis leads to the full result of Eq. (11), which is independent of the spin of the emitting particles (additional spin-dependent contributions are associated with the hard collinear region, and thus absent in the soft expansion [36]).

One now obtains the sum of all next-to-soft contributions at tree-level by combining the external and internal emission contributions, as for the scalar and fermion cases. The only surviving NE contribution from Eq. (28) is the spin-dependent piece, and one finds[4]

$$\begin{aligned}
\mathcal{A}_{n+1}&=-\frac{ik^\nu(L^{(i)}_{\mu\nu}+M_{\mu\nu})}{p_j\cdot k}\mathcal{A}_n\\
&=-\frac{ik^\nu J^{(i)}_{\mu\nu}}{p_j\cdot k}\mathcal{A}_n,
\end{aligned} \tag{35}$$

thus reproducing the next-to-soft theorem of Eq. (4).

In this section, we have seen how similar methods to those used in Refs. [7,8] can be used to derive the next-to-soft theorem of Eq. (4) at tree-level. A similar analysis could be carried out for gravity. This would be much more cumbersome, however, due to the lengthy form of the expression for the three-graviton vertex when written in a helicity-independent form.

5. Conclusion

There has recently been a flurry of attention [11–14,17–27] focusing on the behaviour of scattering amplitudes in the next-to-soft approximation, in which a single external particle is taken to have a small, but non-zero, momentum. The structure of such contributions is formally interesting in its own right, but also has distinct practical applications in improving collider physics predictions [2–6]. In this paper, we have examined the recently conjectured next-to-soft theorems of Eqs. (3), (4) using diagrammatic methods developed in Refs. [7–9]. There are a number of motivations for this. Firstly, our approach provides a useful alternative view on how the next-to-soft theorems arise, especially given that it is manifestly independent of the helicities of the emitting particles, and also the space-time dimension. It is interesting, for example, to see how the orbital and spin angular momentum contributions combine to create the coupling to the total angular momentum of each external leg. We also saw that the NE Feynman rules for gravity and gauge theory have a (partial) double-copy structure, which almost certainly underlies the observation made in Ref. [26] that the next-to-soft factors in Eqs. (3), (4) have this property (analogous to the strictly eikonal analysis of Ref. [39]).

We hope that our study clarifies the relationship between the recent studies on next-to-soft theorems, and previous work in the literature. Moreover, we believe that the diagrammatic techniques discussed herein may prove very useful in further examination of e.g. loop corrections. In particular, at loop level one has to worry about the two-gluon (or graviton) vertex (Eq. (8) and its generalisations), such that one gluon is real and the other virtual [40]. Our techniques may also prove useful for investigating the phenomenological consequences of next-to-soft behaviour. Work in this regard is ongoing.

[3] It is for this reason that the original theorems by Low, Burnett and Kroll [28,29] do not fully reproduce the next-to-soft theorems of Eqs. (3), (4), as alluded to in the introduction.

[4] Note that we have now left colour matrices implicit, consistent with the notation throughout the rest of the paper.

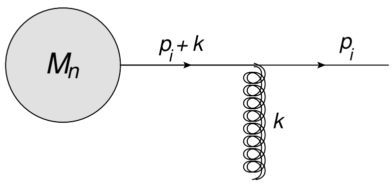

Fig. 4. Emission of a single graviton from the ith external leg of the hard amplitude $\mathcal{M}_n$.

6. Addendum

In the final stages of this paper, the author became aware of the recent Refs. [41,42], which also address how to systematically classify next-to-soft corrections.

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Appendix A. NE Feynman rules for graviton emission

In this appendix, we calculate the effective Feyman rules up to next-to-eikonal order for the emission of gravitons from external scalars or fermions. We begin with the diagram of Fig. 4, which shows a single leg of the hard amplitude (momentum p_i) emitting a graviton (momentum k). Examining first the case of a scalar emitter, one may combine the scalar propagator and graviton-scalar vertex (see e.g. [43]) to get

$$\mathcal{M}_n\left[\frac{p_i^{(\mu}(p_i+k)^{\nu)}-\eta^{\mu\nu}p_i\cdot(p_i+k)}{(p_i+k)^2}\right], \tag{36}$$

where we have used the notation

$$a^{(\mu}b^{\nu)}=a^\mu b^\nu+a^\nu b^\mu$$

and neglected a factor of the gravitational coupling $\kappa/2$ for brevity. Expanding the expression (36) up to first subleading order in the emitted graviton momentum gives

$$\mathcal{M}_n\left[\frac{p_i^\mu p_i^\nu}{p_i\cdot k}-\frac{p_i^\mu p_i^\nu k^2}{2(p_i\cdot k)^2}+\frac{k^{(\nu}p^{\mu)}}{2p\cdot k}-\frac{\eta^{\mu\nu}}{2}\right]. \tag{37}$$

The first term is the eikonal Feynman rule of Eq. (5), and the remainder is then the NE Feynman rule of Eq. (17).

Now consider that the external line in Fig. 4 is a fermion. Again neglecting an overall factor of $\kappa/2$, combining the propagator and vertex gives

$$\mathcal{M}_n\frac{(\not{p}_i+\not{k})}{4}\frac{[\gamma^{(\mu}(2p_i+k)^{\nu)}-2\eta^{\mu\nu}\gamma^\alpha(2p_i+k)_\alpha]}{(p_i+k)^2}u(p_i), \tag{38}$$

where we have explicitly included the spinor associated with the line i. The reason for doing this is that after expanding Eq. (38) to first subleading order in the soft gluon momentum k, we may simplify the result by anticommuting factors of $\not{p}$ and using the Dirac equation

$$\not{p}_i u(p_i)=0. \tag{39}$$

The result is

$$\mathcal{M}_n\left[\frac{p_i^\mu p_i^\nu}{p_i\cdot k}-\frac{p_i^\mu p_i^\nu k^2}{2(p_i\cdot k)^2}+\frac{p_i^{(\mu}k^{\nu)}}{4p_i\cdot k}+\frac{\not{k}\gamma^{(\mu}p_i^{\nu)}}{4p_i\cdot k}-\frac{\eta^{\mu\nu}}{2}\right]u(p_i). \tag{40}$$

The first term is the (spin-independent) eikonal Feynman rule of Eq. (5). For the remaining terms, we may rewrite the combination

$$\frac{\not{k}\gamma^{(\mu}p_i^{\nu)}}{4p_i\cdot k}=\frac{k^{(\mu}p_i^{\nu)}}{4p_i\cdot k}-\frac{ik_\rho\Sigma^{\rho(\mu}p_i^{\nu)}}{8p_i\cdot k}, \tag{41}$$

where we have introduced the spin tensor of Eq. (7). One then finds the NE 1-graviton vertex of Eq. (17).

Note that in this appendix we have not contracted the amplitude with the polarization tensor for the external graviton. This means that the one-graviton vertices we have obtained are also valid for off-shell gravitons.

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