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Attitude Dynamics of the Classical Dumbbell

Problem with Solar Radiation Pressure

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Introduction

This work extends the attitude dynamics and stability properties of the classical planar dumbbell problem, namely two masses connected by a rigid massless tether [1–3], by considering the effect of a solar radiation pressure (SRP) gradient between the tip masses. In principle, this SRP gradient can be used as a means of attitude station-keeping of tethered satellite systems, without the need for mechanical systems or thrusters to maintain a fixed attitude. Attitude stabilization of a satellite by SRP was first proposed in 1959 [4], and further in 1965 [5], where the latter considered differential reflectivity on the spacecraft due to local surface irregularities. Since then, the concept of employing differential SRP for semi-passive attitude control and stabilisation of tethered satellites, for example using articulated reflective surfaces, has been investigated in [6].

This work adapts the widely used model of a tethered satellite system [7, 8] by introducing SRP forces to the tip masses, acting in the radial direction from the Sun. Therefore, lightness numbers are assigned to the masses, which is equivalent to assigning a variable surface reflectivity. This can be achieved, for example, using electro-chromic coatings which consist of an electro-active material that changes its surface reflectivity according to an applied electric charge [9, 10]. When a long tether is orbiting a central body, the relative attitude of the system affects the total force acting on

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the dumbbell's centre-of-mass (CoM). This means that the equations of motion (EOM) describing the orbit and attitude of the system are coupled [11].

First, the system's Hamiltonian and the EOM for the coupled orbit/attitude motion are presented. By introducing a large central mass, the satellite is constrained to a circular orbit such that the EOM are decoupled. This is required to enable a stability analysis of the system's equilibrium attitude, showing that the relative equilibria depend on the lightness numbers of the two masses. It is demonstrated that artificial equilibria exist that differ from the well-known solutions of the pure gravity gradient (GG) dumbbell [12]. Clearly, the decoupled EOM are only valid under the assumption that the CoM of the system stays on a circular Keplerian orbit. However, through the introduction of SRP, the Keplerian orbital motion of the system is perturbed. Therefore, it is shown that even without the central mass, the dumbbell can be controlled on a circular non-Keplerian orbit using SRP [13]. Thus, the coupling of the orbit and attitude is reintroduced by deriving appropriate constraints on the lightness numbers. The dynamical behaviour of the system is investigated through iso-energy curves of the Hamiltonian in phase space. Finally, motion between and controllability around equilibria is demonstrated in the phase space of the problem through the use of quasi-heteroclinic connections and by changing the lightness numbers of the tip masses.

Equations of Motion of Dumbbell with Solar Radiation Pressure

The planar motion of the dumbbell system is described with respect to a Sun-centred inertial frame $\mathcal{I}:(X, Y)$ in the case of the coupled orbit/attitude problem and relative to a rotating orbit frame $\mathcal{O}:(r, \nu)$, originating in the CoM of the system, for the decoupled attitude dynamics (Fig. 1). The axes of frame $\mathcal{O}$ are aligned with the local vertical and the local horizontal relative to the Sun.

The system is modelled as a rigid body with a central mass M_B located in the CoM and two tip masses m_1 and m_2 at each end of a massless tether. The dumbbell parameter $\lambda = l/R$ describes the ratio of total tether length l to orbit radius R . The three masses are approximated as point masses, with the mass ratio $\kappa = m_1/(m_1 + m_2)$ and the total mass $M = m_1 + m_2 + M_B$, with M_B located in the CoM. The Sun's gravitational force is extended by introducing SRP forces to the tip masses, assigning arbitrary lightness numbers $\beta_i = [0, 1]$, with $i = 1, 2$, to each of the masses.

Fig. 1 Geometry of dumbbell system in Sun-centred orbit with lightness numbers β_1 and β_2 , representing SRP forces acting on the tip masses.

The lightness number of an object describes the ratio of SRP force and gravity. Bodies with a high surface reflectivity and a high area-to-mass ratio have a high value of β . Compared to the tip masses, the area-to-mass ratio of the central mass is assumed to be small, so the lightness number of M_B can be neglected.

For a rigid body, the position vector $\mathbf{R}$ of the CoM is defined as

$$\mathbf{R} = \frac{1}{M} \sum_{i=1}^2 m_i \mathbf{R}_i \quad (1)$$

With respect to the inertial frame $\mathcal{I}$, the position vectors of the three masses are described as $\mathbf{R} = R(\cos \nu, \sin \nu)^T$, $\mathbf{R}_1 = \mathbf{R} + \mathbf{r}_1$ and $\mathbf{R}_2 = \mathbf{R} + \mathbf{r}_2$. In here, $\mathbf{r}_1 = (\kappa - 1)l(\cos \theta, \sin \theta)^T$ and $\mathbf{r}_2 = \kappa l(\cos \theta, \sin \theta)^T$ are the position vectors of the tip masses with respect to the CoM, using the constraint $l = r_1 + r_2 = \text{const}$. In previous relations, ν denotes the true anomaly and θ the attitude angle relative to the inertial X -axis, according to Fig. 1. The norms of the position vectors are

$$R_1 = R \left[1 - 2\lambda(1 - \kappa) \cos(\theta - \nu) + \lambda^2(1 - \kappa)^2 \right]^{1/2} \quad (2a)$$

$$R_2 = R \left[1 + 2\lambda\kappa \cos(\theta - \nu) + (\lambda\kappa)^2 \right]^{1/2} \quad (2b)$$

Using previous definitions, the planar EOM for the tethered satellite system including SRP can

now be formulated using a Hamiltonian approach [14].

Coupled orbit/attitude equations of motion

The SRP forces can be included into the potential energy function V , because they act in radial direction and thus originate from a conservative force field

$$\mathbf{g} = -\frac{\mu M_B}{R^2} \hat{\mathbf{R}} - \sum_{i=1}^2 \frac{\tilde{\mu}_i m_i}{R_i^2} \hat{\mathbf{R}}_i = -\nabla V \quad (3)$$

where $\mu = 1.3272 \times 10^{11} \text{ km}^3/\text{s}^2$ denotes the Sun's gravitational parameter and $(\hat{\cdot})$ denotes the unit vector. The so-called effective gravitational parameter $\tilde{\mu}_i = \mu(1-\beta_i)$ for each mass represents the reduced effect of the gravitational force due to a radially outward SRP force [13]. The effective potential energy of the system can now be written as

$$V = -\frac{\mu M_B}{R} - \sum_{i=1}^2 \frac{\tilde{\mu}_i m_i}{R_i} = -\frac{\mu M_B}{R} - \frac{\mu m_1(1-\beta_1)}{R_1} - \frac{\mu m_2(1-\beta_2)}{R_2} \quad (4)$$

The kinetic energy is split into a translational part T_{transl} attached to the CoM and a rotational part T_{rot} , representing the contribution of the two rotating masses to the total kinetic energy such that

$$T_{\text{transl}} = \frac{1}{2} M \dot{\mathbf{R}} \cdot \dot{\mathbf{R}} = \frac{1}{2} (m_1 + m_2 + M_B) (\dot{R}^2 + R^2 \dot{\nu}^2) \quad (5a)$$

$$T_{\text{rot}} = \frac{1}{2} \sum_{i=1}^2 m_i \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i = \frac{1}{2} \frac{m_1 m_2 l^2}{m_1 + m_2} \dot{\theta}^2 \quad (5b)$$

Using the Lagrangian $L = T - V$ and introducing three generalized coordinates $q_j = (R, \nu, \theta)$, the coupled Hamiltonian of the dynamical system can be written as

$$H = \sum_{j=1}^3 \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j - L = \dot{R} \frac{\partial L}{\partial \dot{R}} + \dot{\nu} \frac{\partial L}{\partial \dot{\nu}} + \dot{\theta} \frac{\partial L}{\partial \dot{\theta}} - L \quad (6)$$

After calculating the generalized momenta p_j

$$p_1 = \frac{\partial L}{\partial \dot{R}} = (m_1 + m_2 + M_B) \dot{R} \quad (7a)$$

$$p_2 = \frac{\partial L}{\partial \dot{\nu}} = (m_1 + m_2 + M_B) R^2 \dot{\nu} \quad (7b)$$

$$p_3 = \frac{\partial L}{\partial \dot{\theta}} = \frac{m_1 m_2 l^2}{m_1 + m_2} \dot{\theta} \quad (7c)$$

the coupled Hamiltonian of the system is found as

$$H = \frac{1}{2}M(\dot{R}^2 + R^2\dot{\nu}^2) + \frac{1}{2}\frac{m_1m_2l^2}{m_1+m_2}\dot{\theta}^2 - \frac{\mu M_B}{R} - \frac{\mu m_1(1-\beta_1)}{R_1} - \frac{\mu m_2(1-\beta_2)}{R_2} \quad (8)$$

For generalized coordinates $q_j(t)$, $j = 1, \dots, n$, the trajectory of $\mathbf{q}(t) = (q_1(t), \dots, q_n(t))$ through the configuration space satisfies the Euler-Lagrange equations [14]

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = 0 \quad (9)$$

Using the above equation, the coupled EOM of the system including SRP can be formulated in terms of the free parameters κ , λ , β_1 and β_2 as

$$\begin{aligned} \ddot{R} - R\dot{\nu}^2 + \frac{\mu M_B}{MR^2} + \frac{\mu R m_1(1-\beta_1) \left(1 - \lambda(1-\kappa) \cos(\theta-\nu) \right)}{MR_1^3} \\ + \frac{\mu R m_2(1-\beta_2) \left(1 + \lambda\kappa \cos(\theta-\nu) \right)}{MR_2^3} = 0 \end{aligned} \quad (10a)$$

$$\ddot{\nu} + \frac{2\dot{R}\dot{\nu}}{R} + \frac{\mu \kappa m_2 \lambda \sin(\theta-\nu)}{M} \left(\frac{(1-\beta_2)}{R_2^3} - \frac{(1-\beta_1)}{R_1^3} \right) = 0 \quad (10b)$$

$$\ddot{\theta} + \frac{\mu \sin(\theta-\nu)}{\lambda} \left(\frac{(1-\beta_1)}{R_1^3} - \frac{(1-\beta_2)}{R_2^3} \right) = 0 \quad (10c)$$

Decoupled attitude equations of motion

Assuming a central mass $M_B \gg m_i$, the attitude motion of the system decouples from the orbit dynamics, thus Eqs. (10a) and (10b) reduce to the common two-body problem and no longer depend on θ . Introducing the above condition for M_B into Eq. (10) results in the decoupled EOM of the dumbbell including SRP, given by

$$\ddot{R} + \frac{\mu}{R^2} - R\dot{\nu}^2 = 0 \quad (11a)$$

$$\ddot{\nu} + \frac{2\dot{R}\dot{\nu}}{R} = 0 \quad (11b)$$

$$\ddot{\theta} + \frac{\mu \sin(\theta-\nu)}{\lambda} \left(\frac{(1-\beta_1)}{R_1^3} - \frac{(1-\beta_2)}{R_2^3} \right) = 0 \quad (11c)$$

When the CoM of the system initially follows a circular orbit with $\dot{\nu} = \sqrt{\mu/R^3}$ and $\dot{R} = 0$, Eqs. (11a) and (11b) further reduce to $\ddot{R} = 0$ and $\ddot{\nu} = 0$, respectively. When considering again Fig. 1,

γ represents the angle between the dumbbell axis and the local vertical, thus it can be seen from geometry that $\theta - \nu = \gamma$. Since $\ddot{\nu} = 0$, it follows that $\ddot{\theta} = \ddot{\gamma}$. Inserting this into Eq. (11c) together with the norms of the position vectors, Eq. (2), the decoupled attitude EOM can now be written as

$$\ddot{\gamma} + \frac{\mu \sin \gamma}{\lambda} \left[\frac{1 - \beta_1}{R^3 [1 - 2\lambda(1 - \kappa) \cos \gamma + \lambda^2(1 - \kappa)^2]^{3/2}} - \frac{1 - \beta_2}{R^3 [1 + 2\lambda\kappa \cos \gamma + (\lambda\kappa)^2]^{3/2}} \right] = 0 \quad (12)$$

Relative Equilibria and Stability

It is now shown that the relative equilibria γ_{eq} of the system are a function of the free parameters β_1 and β_2 . For both lightness numbers being zero and a mass ratio of $\kappa = 0.5$, the stable equilibria are 0 and ± 180 degrees, while the unstable equilibria are located at ± 90 degrees, corresponding to the classical gravity gradient dumbbell. The dumbbell is in an equilibrium state whenever $\dot{\gamma} = 0$ and the total torque on the system is zero, thus $\ddot{\gamma} = 0$. Solving the decoupled attitude EOM, Eq. (12), for $\ddot{\gamma} = 0$ gives the equilibrium angles as a function of the four parameters κ , λ , β_1 and β_2 . The condition $\ddot{\gamma} = 0$ has two invariant solutions for $\sin(\gamma_{\text{eq}}) = 0$ and two further solutions as

$$\cos(\gamma_{\text{eq}}) = \left[\frac{(1 - \beta_2)^{\frac{2}{3}} (1 + (1 - \kappa)^2 \lambda^2) - (1 - \beta_1)^{\frac{2}{3}} (1 + (\kappa\lambda)^2)}{(1 - \beta_1)^{\frac{2}{3}} (2\kappa\lambda) + (1 - \beta_2)^{\frac{2}{3}} 2(1 - \kappa)\lambda} \right] \quad (13)$$

The stable/unstable character of the new equilibria is evaluated through a stability analysis [15] for which the eigenvalues of the Jacobian of the linearized system are analyzed. Figure 2 shows the stable (grey) and unstable (black) regions of γ_{eq} as a function of β_1 and β_2 for a chosen reference dumbbell with equal masses $m_1 = m_2$ and tether length $l = 100$ km on a circular orbit at $R = 1$ AU (i.e. Earth distance from the Sun), thus $\lambda = 6.685 \times 10^{-7}$. The lightness numbers are scaled in both figures using $\beta_i = \beta_i^* 2 \times 10^{-6}$ with $\beta_i^* = [0, 1]$ to improve readability, since the differential SRP and gravity gradient forces on the dumbbell are very small at 1 AU distance from the Sun. In the figure, the grey plane indicates the stable equilibrium at 0 degrees. The two black curved planes indicate new unstable equilibria created by introducing SRP to the classical dumbbell problem. The system can now obtain an arbitrary equilibrium state in the range of ± 180 degrees for the differential SRP on the two masses in the range of $\Delta\beta = |\beta_2 - \beta_1| = 2 \times 10^{-6}$. Although achieving such a small difference in lightness number may be technically very challenging, the results illustrate

(a)

(b)

Fig. 2 Stable (grey) and unstable (black) equilibria γ_{eq} as function of lightness numbers β_1^* and β_2^* for dumbbell with equal masses ($\kappa = 0.5$) and tether length $l = 100$ km ($\lambda = 6.685 \times 10^{-7}$) in Sun-centred orbit at 1 AU solar distance.

the possibility of controlling a tethered satellite system using the SRP gradient.

When solving Eq. (13) for β_2 , the possible combinations $(\beta_1, \beta_2)_{\gamma_{\text{eq}}}$ for a given equilibrium

Fig. 3 Possible lightness number sets $(\beta_1^*, \beta_2^*)_{\gamma_{eq}}$ to create unstable equilibria γ_{eq} for dumbbell with equal masses ($\kappa = 0.5$) and tether length $l = 100$ km ($\lambda = 6.685 \times 10^{-7}$) in Sun-centred orbit at 1 AU solar distance.

angle, further referred to as ' β -sets', can be obtained as

$$\beta_2(\beta_1, \gamma_{eq}) = 1 - \left[\frac{[1 + (\kappa\lambda)^2](1 - \beta_1)^{\frac{2}{3}} - 2\kappa\lambda \cos \gamma (1 - \beta_1)^{\frac{2}{3}}}{1 - 2(1 - \kappa)\lambda \cos \gamma + ((1 - \kappa)\lambda)^2} \right]^{\frac{3}{2}} \quad (14)$$

For the chosen reference dumbbell and for equilibrium angles in the interval $[0, 180]$ degrees, the possible $(\beta_1, \beta_2)_{\gamma_{eq}}$ sets are shown in Fig. 3, parametrized in steps of $\Delta\gamma = 10$ degrees. Each point on one of the curves can be chosen to create the respective unstable equilibrium attitude γ_{eq} .

Maintaining System on Circular Orbit using Solar Radiation Pressure

The decoupled attitude EOM, Eq. 12, is only valid under the assumption that the CoM stays on a circular orbit with uniform orbital rate $\omega_0 = \dot{\nu}_0$. This condition is satisfied when introducing a large central mass, as shown above, or approximated when assuming a small tether length, thus $\lambda \ll 1$. However, regardless the tether length, the radial outward SRP force is always perturbing the circular Keplerian motion of the dumbbell around the central body. The effect of SRP is thus to decrease the effective solar gravity experienced by the two masses, as shown earlier by

the introduction of the effective gravitational parameter $\tilde{\mu} = \mu(1 - \beta_i)$. The effective orbital rate $\tilde{\omega} = \sqrt{\tilde{\mu}/R_0^3} < \omega_0$ now corresponds to a circular non-Keplerian orbit in the combined gravitational and SRP force field [13]. However, it is shown below that when the system is orbiting with $\tilde{\omega}$, it can still maintain a circular non-Keplerian orbit.

To this aim, constraints for β_1 and β_2 are derived for the two-mass dumbbell without a central mass that allow circular orbital motion for any given attitude γ . Hereby, the coupling of the orbit and attitude dynamics are reintroduced to the system. Considering the coupled EOM, Eq. (10), without the central mass, thus $M_B = 0$, and introducing the conditions $\ddot{R} = \dot{R} = 0$, $\ddot{\nu} = 0$ and $\dot{\nu} = \tilde{\omega}$ results in two constraint equations for β_1 and β_2

$$\beta_1 = 1 - \frac{\tilde{\omega}_{\text{CoM}}^2}{\mu} R_1^3 \left[\frac{1}{\kappa[1 - \lambda(1 - \kappa) \cos \gamma] + (1 - \kappa)[1 + \lambda\kappa \cos \gamma]} \right] = \beta_1(\gamma) \quad (15a)$$

$$\beta_2 = 1 - \frac{\tilde{\omega}_{\text{CoM}}^2}{\mu} R_2^3 \left[\frac{1}{\kappa[1 - \lambda(1 - \kappa) \cos \gamma] + (1 - \kappa)[1 + \lambda\kappa \cos \gamma]} \right] = \beta_2(\gamma) \quad (15b)$$

Note that the above equations are coupled through the angle γ . Figures 4 and 5 show the constrained $(\beta_1, \beta_2)_\gamma$ sets over the range of attitude angles $\gamma \in [-180, 180]$ degrees for the reference dumbbell. The concept is further illustrated by a family of circular non-Keplerian orbits with decreasing orbital rates $\tilde{\omega}_i \leq \omega_0$, starting with the nominal rate $\omega_0 = \sqrt{\mu/R_0^3}$ for the circular Keplerian orbit. The figures show that for ω_0 it is not possible to stay on the circular orbit for any given attitude, since β_i becomes negative for some intervals of gamma, and so it is not a physical solution. When decreasing the orbital rate to $\tilde{\omega}_1 = (1 - 1.5 \times 10^{-7}) \omega_0$, $\tilde{\omega}_2 = (1 - 3.5 \times 10^{-7}) \omega_0$ and $\tilde{\omega}_3 = (1 - 5.0 \times 10^{-7}) \omega_0$, the region of feasible attitudes can be increased gradually, as visible in Figs. 4 and 5. For $\tilde{\omega}_{\text{CoM}} \leq \tilde{\omega}_3$, all equilibrium attitudes are possible.

The introduction of the new β -constraints to the previously defined β -sets that create a respective unstable equilibrium attitude γ_{eq} , as seen before in Fig. 3, is now shown in Fig. 6. While the original $(\beta_1, \beta_2)_{\gamma_{\text{eq}}}$ sets in the decoupled problem were lines for $\beta_i \in [0, 1]$, the constraints that reintroduce the orbit/attitude coupling now restrict the sets to one point for each γ_{eq} , depending on the chosen orbit rate $\tilde{\omega}$ of the non-Keplerian circular orbit.

Fig. 4 Constrained β_1^* for reference dumbbell ($\kappa = 0.5, \lambda = 6.685 \times 10^{-7}$) to maintain a circular non-Keplerian orbit with orbital rates $\tilde{\omega}_i$ for a given dumbbell attitude γ .

Fig. 5 Constrained β_2^* for reference dumbbell ($\kappa = 0.5, \lambda = 6.685 \times 10^{-7}$) to maintain a circular non-Keplerian orbit with orbital rates $\tilde{\omega}_i$ for a given dumbbell attitude γ .

Fig. 6 Possible $(\beta_1, \beta_2)_{\gamma_{\text{eq}}}$ sets for reference dumbbell ($\kappa = 0.5, \lambda = 6.685 \times 10^{-7}$) to create a respective γ_{eq} and superimposed β -constraints for various circular non-Keplerian orbits with reduced orbital rates $\tilde{\omega}_1 = (1 - 1.5 \times 10^{-7}) \omega_0$, $\tilde{\omega}_2 = (1 - 3.5 \times 10^{-7}) \omega_0$ and $\tilde{\omega}_3 = (1 - 5.0 \times 10^{-7}) \omega_0$.

Attitude Dynamics and Control of Dumbbell using Solar Radiation Pressure

The dynamics of the decoupled problem, Eq. 12, are further analyzed using the above lightness-number constraints that allows the CoM of the dumbbell to stay on a circular non-Keplerian orbit with orbital rate $\tilde{\omega} < \omega_0$.

Phase Space of the Problem

The Hamiltonian of the decoupled system is found using one generalized coordinate $q_{1,\text{dec}} = \gamma$ in the rotating orbit frame $\mathcal{O} : (r, \nu)$, according to Fig. 1. After dividing by $\frac{1}{2}(m_1 + m_2)\omega_0^2 l^2$, the non-dimensional Hamiltonian can be written as (without derivation)

$$\hat{H}_{\text{dec}} = \frac{2}{(m_1 + m_2)\omega_0^2 l^2} \left[\frac{1}{2} \frac{m_1 m_2 l^2}{m_1 + m_2} (\dot{\gamma}^2 - \omega_0^2) - \frac{\mu m_1 (1 - \beta_1)}{R_1} - \frac{\mu m_2 (1 - \beta_2)}{R_2} \right] \quad (16)$$

Since the decoupled Hamiltonian is a constant of motion, for each value of $\hat{H}_{\text{dec}}$, the motion of the system is represented by a two-dimensional phase space $(\gamma, \dot{\gamma})$ [14] with free parameters β_1 and β_2 .

Fig. 7 Superimposed phase spaces $(\gamma, \dot{\gamma})$ of pure gravity gradient dumbbell ($\kappa = 0.5, \lambda = 6.685 \times 10^{-7}$) with $(\beta_1, \beta_2)_A = (0, 0)$ (black solid curves) and same system including SRP with lightness numbers $(\beta_1^*, \beta_2^*)_B = (0.86, 0.15)$ (dashed grey curves) on circular non-Keplerian orbit.

Figure 7 shows the iso-energy curves in the phase space for the reference dumbbell. Arrows indicate the direction of motion along a curve. Whenever the curves are closed, they correspond to librations around the equilibrium point, while the open curves correspond to rotations. Two superimposed phase spaces for different β -sets and the respective location of the stable and unstable equilibria are visible. The first set $(\beta_1, \beta_2)_A = (0, 0)$ (black solid curves) corresponds to the pure gravity gradient dumbbell without SRP, showing the unstable equilibria at ± 90 degrees and the stable ones at 0 and ± 180 degrees (black points). The second set B (grey dashed curves) is chosen according to the derived β -constraints. Here, as an example, the chosen lightness numbers shift the unstable equilibria to ± 45 degree (grey crosses). For a chosen orbit of radius $R_0 = 1 \text{ AU}$ and orbital rate $\omega_0 = 0.0172 \text{ rad/day}$, the non-Keplerian orbit in terms of orbital rate $\tilde{\omega}_{\text{CoM}}$ and the corresponding

set $(\beta_1, \beta_2)_B$ that creates the ± 45 degree equilibria can be obtained from Fig. 6, or likewise Eq. (15). As found above, an orbit rate of $\tilde{\omega}_3 = (1 - 5.0 \times 10^{-7}) \omega_0$ allows for positive β -sets in the full range of attitudes between $[-180, 180]$ degree. Accordingly, the resulting β -set is $(\beta_1^*, \beta_2^*)_B = (0.86, 0.15)$, as obtained from Eq. 15.

Motion in Phase Space

The phase space of the system is characteristic for a particular β -set. Switching to another set, the phase space and the respective equilibria change accordingly, as shown above. This property of the system can be exploited to find quasi-heteroclinic connections between equilibria of different phase spaces. By providing a qualitative switching law between different β -sets, the aim is towards arbitrarily changing the attitude of the dumbbell and further controlling it in the vicinity of a desired (unstable) attitude.

When inspecting again Fig. 7, possible controlled sequences in the phase space in order to change the dumbbell attitude can be obtained. Whenever the dumbbell is in a state at (or close to) an unstable equilibrium (saddle), there are two unstable manifolds for the system to move away from the saddle. The other two stable manifolds always lead towards the unstable point, as can be seen in the detail view of Fig. 8. For example, the system can move along the bold dashed path away from the $\pi/2$ saddle, as indicated through the arrows in the figure. Likewise, there are also two stable manifolds leading towards the $\pi/4$ saddle of the second phase space (dashed grey lines). When switching between the previously chosen sets $(\beta_1, \beta_2)_A$ and $(\beta_1, \beta_2)_B$, at the point in phase space indicated with a bold 'S', the dumbbell will consequently change its equilibrium attitude from $\pi/2$ to $\pi/4$, when following the dashed path. This way, intersections between manifolds of different phase spaces can be exploited. In order to further control the dumbbell in the vicinity of an unstable saddle point, a control sequence such as the one indicated with the dark dotted path can be used. When the system initially moves away from the saddle on one of the outgoing manifolds (within phase space B), then switching to set A (black solid curves) at the point marked with a bold '1' will let it move along a closed path around the stable centre of phase space A. When it reaches point '2', which is the crossing with the ingoing manifold, switching again to set B will complete a closed

Fig. 8 Detail view of superimposed phase spaces $(\gamma, \dot{\gamma})$ of pure gravity gradient dumbbell with $(\beta_1, \beta_2)_A = (0, 0)$ (black solid curves) and dumbbell with lightness numbers $(\beta_1^*, \beta_2^*)_B = (0.86, 0.15)$ (dashed grey curves). Two possible sequences are highlighted: attitude change between unstable equilibria (bold dashed path) and control sequence around an unstable equilibrium (bold dotted path).

loop around the desired unstable equilibrium of $\pi/4$.

Conclusions

Introducing solar radiation pressure to the classical planar rigid-body dumbbell problem creates artificial unstable equilibria that are different from those of the pure gravity gradient dumbbell. In particular, by controlling the lightness numbers of the tip masses, equilibrium attitudes at an arbitrary angle, relative to the local vertical, can be created. Possible control of the dumbbell attitude has been demonstrated through changing the lightness numbers, exploiting the quasi-heteroclinic connections in the variable phase space of the problem. The additional solar radiation pressure forces perturb the circular Keplerian motion of the dumbbell around the central body. Accordingly,

coupling of the orbit and attitude dynamics has been reintroduced by deriving constraints for the lightness numbers, showing that the dumbbell can be maintained on a circular non-Keplerian orbit for arbitrary attitudes using solar radiation pressure. This supports the concept of using solar radiation pressure for attitude station-keeping of tethered satellite systems at relatively low cost, since the lightness numbers, or surface reflectivities, respectively, can in principle be changed using electro-chromic coatings. Therefore, no mechanical systems or thrusters are required to maintain a fixed observation attitude.

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