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Two-particle multi-mode interference

Gergely Ferenczi, Václav Potoček and Stephen M Barnett

School of Physics and Astronomy, University of Glasgow, Glasgow G12 8QQ, UK

E-mail: g.ferenczi.1@research.gla.ac.uk

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Abstract

The statistics of particles incident on a beam splitter has important applications in quantum protocols such as teleportation or Bell state analysis. We study particle pairs with multiple degrees of freedom in terms of exchange symmetry and show that the particle statistics at a beam splitter can be controlled for suitably chosen states. We propose an experimental test of these ideas using orbital angular momentum entangled photons.

Keywords: Hong–Ou–Mandel effect, quantum optics, exchange statistics, orbital angular momentum

1. Introduction

In a quantum mechanical description identical particles must be treated as indistinguishable. These come in two different families, bosons and fermions, distinguished by their spin or helicity. States describing bosons are symmetric under the exchange of particles and states describing fermions are antisymmetric under the exchange of particles ([1] section 14).

In the quantum mechanical description of any process different alternatives by which the process might happen add at the amplitude level rather than at the probability level if the two alternatives lead to physically indistinguishable outcomes ([2] section 1). It is this that gives rise to interference phenomena. In particular whether identical particles are exchanged or not during a physical process constitute such indistinguishable alternatives. As a result of this bosons and fermions exhibit behaviour quite distinct from one another [3].

This interference due to exchanged alternatives has been demonstrated for photons in the classic experiment of Hong, Ou and Mandel [4] in which one photon is incident on each face of a balanced beam splitter. If the modes occupied by the two particles interfere then the two photons always exit the beam splitter together ([5] section 6). This is in contrast to the behaviour for distinguishable particles that exit together with probability 1/2. As the two photons are made indistinguishable by adjusting their arrival times this results in the characteristic Hong–Ou–Mandel (HOM) dip. This provides a useful tool to demonstrate the interference of two modes [6] also in the case when the distinguishable to indistinguishable transition is achieved by a degree of freedom other than time of arrival [7]. The fermion counterpart to HOM interference has also been demonstrated, by showing that two electrons in the same spin state impinge on a beam splitter and the result is that they never leave in the same arm [8]. This is consistent with the Pauli exclusion principle as both phenomena stem from the same underlying principle that quantum states describing fermions are antisymmetric under exchange of the two particles.

It is possible for the photons to exhibit fermion-like statistics at a beam splitter by leaving through different ports if the transverse spatial profile [9] or polarization state [10] is chosen suitably. In both cases these are antisymmetric states forcing the state describing the path degree of freedom to be antisymmetric.

The key feature resulting in the different behaviour of bosons and fermions at a beam splitter is the different symmetry that states describing them satisfy in the path degree of freedom [11]. In this paper we consider particles with multiple quantum numbers and show that control can be exerted over how the overall exchange symmetry is distributed between the exchange symmetries of the two different quantum numbers. We use path and orbital angular momentum (OAM) [12, 13] as physical examples of these quantum numbers, allowing us to probe the symmetry in the path quantum numbers using only a beam splitter. We show that for both bosons and fermions it is possible to obtain the full range of allowed output statistics exhibited by bosons,

fermions or mixtures of them. This includes the extremal cases of purely bosonic and purely fermionic statistics. It is implicit in the construction of our chosen states that the OAM state can have arbitrarily large dimensionality.

2. States of two indistinguishable particles

By far the most important experimental investigations of two-particle interference have been performed with photons. Indeed the HOM effect is now part of the core of quantum optics [5]. Here, however, we seek to describe the phenomenon in generality and so introduce a description of two-particle states applicable to any identical particles. When writing down a two-particle state of a second quantized system we may consider either $\hat{a}_i^\dagger \hat{a}_j^\dagger\,|\mathrm{vac}\rangle$ or $\hat{a}_j^\dagger \hat{a}_i^\dagger\,|\mathrm{vac}\rangle$, where $|\mathrm{vac}\rangle$ denotes the vacuum state and $\hat{a}_i^\dagger$ and $\hat{a}_j^\dagger$ denote creation operators of two modes i and j of a certain degree of freedom that may or may not be distinct. The two physically equivalent alternatives are related by a phase of 0 or π depending on whether the two particles are bosons or fermions respectively. This can be achieved by setting the commutator or the anticommutator of the creation operators to zero. Using the connection between spin and statistics [14] this symmetry requirement on an otherwise arbitrary state characterized by ψ can be expressed as

$$\hat{\mathrm{X}}^{(\hat{a}^\dagger)}|\psi, s\rangle = (-1)^{2s}|\psi, s\rangle, \tag{1}$$

where s is the spin of the particles constituting the particle pair and $\hat{\mathrm{X}}^{(\hat{a}^\dagger)}$ is the particle exchange operator.

A product of operators can always be written as half the sum of their commutator and anticommutator

$$\hat{A}_{ij}^\dagger = \hat{a}_i^\dagger \hat{a}_j^\dagger = \frac{1}{2}\left\{\hat{a}_i^\dagger, \hat{a}_j^\dagger\right\} + \frac{1}{2}\left[\hat{a}_i^\dagger, \hat{a}_j^\dagger\right] = \hat{B}_{ij}^\dagger + \hat{F}_{ij}^\dagger. \tag{2}$$

The significance of this is that even without requiring either of the terms in it to vanish, the states produced by their action on $|\mathrm{vac}\rangle$ are endowed with the defining property of bosons and fermions (1) under particle exchange. Particle exchange is performed by the exchange of the properties of the particles thus in this formalism by the exchange of the indices

$$\hat{\mathrm{X}}^{(\hat{a}^\dagger)}\colon \hat{A}_{ij}^\dagger\,|\mathrm{vac}\rangle \mapsto \hat{A}_{ji}^\dagger\,|\mathrm{vac}\rangle. \tag{3}$$

The two terms in (2) satisfy

$$\begin{aligned} \hat{B}_{ji}^\dagger\,|\mathrm{vac}\rangle &= \hat{B}_{ij}^\dagger\,|\mathrm{vac}\rangle, \\ \hat{F}_{ji}^\dagger\,|\mathrm{vac}\rangle &= -\hat{F}_{ij}^\dagger\,|\mathrm{vac}\rangle, \end{aligned} \tag{4}$$

hence we may identify them as boson and fermion pair creation operators respectively. Under any transformation of a particle pair creation operator $\hat{A}^\dagger$ the symmetric part $\hat{B}^\dagger = (\hat{A}^\dagger + \hat{\mathrm{X}}^{(\hat{a}^\dagger)}\hat{A}^\dagger)/2$ keeps track of bosonic behaviour and the antisymmetric part $\hat{F}^\dagger = (\hat{A}^\dagger - \hat{\mathrm{X}}^{(\hat{a}^\dagger)}\hat{A}^\dagger)/2$ keeps track of fermionic behaviour. The restriction to bosons or fermions corresponds to dropping the appropriate half of the expression.

At present i and j denote two modes of a single degree of freedom. To exchange particles there is only one pair of properties to exchange. We now wish to apply the same considerations to a particle pair with two degrees of freedom. In anticipation of the resulting differences we make a conceptual distinction between the exchange of two particles occupying two modes and the exchange of the modes occupied by two particles. In the present situation, with the particles having only one degree of freedom, the two are equivalent.

2.1. Composite modes

In the case of two degrees of freedom, the indices i and j in (3) are replaced by pairs of numbers (i_1, i_2) and (j_1, j_2). The subscripts 1 and 2 distinguish between the two different degrees of freedom and i_k and j_k are two modes of the degree of freedom k. In the particle pair creation operators we group the indices of the same degree of freedom together according to $\hat{a}_{i_1 i_2}^\dagger \hat{a}_{j_1 j_2}^\dagger = \hat{A}_{i_1 j_1 i_2 j_2}^\dagger$ so that the second degree of freedom is seen as an extension to the first in (2). We distinguish the degrees of freedom only by numbers as there is no restriction on what these may be other than that they must be independent degrees of freedom. For example position and spin coordinates are a suitable choice of degrees of freedom for this analysis however position and momentum coordinates are not as these two degrees of freedom act nontrivially on the same state space. A position state can be expressed as a superposition of momentum modes so exchanging the momentum properties of two particles is necessarily accompanied by exchanging the position properties. This is the case for all conjugate degrees of freedom, e.g. angular position and angular momentum, but also more generally for any two noncommutative observables, for example x and y components of spin.

The exchange of the numbers corresponding to only one of the degrees of freedom

$$\hat{\mathrm{X}}^{(1)}\colon \hat{A}_{i_1 j_1 i_2 j_2}^\dagger\,|\mathrm{vac}\rangle \mapsto \hat{A}_{j_1 i_1 i_2 j_2}^\dagger\,|\mathrm{vac}\rangle, \tag{5}$$

or $\hat{\mathrm{X}}^{(2)}$ similarly defined, is no longer sufficient to implement particle exchange. Particle exchange now means the simultaneous exchange of both properties [1, 15]. The equivalence between the exchange of particles and of occupied modes remains valid only by thinking of modes in a slightly more general way. We consider modes to be composite if they are specified by several constituent modes from distinct degrees of freedom. The exchange of the two occupied composite modes is then equivalent to particle exchange in the above setting. If the two particles are degenerate with respect to one of the constituent degrees of freedom we need not consider that degree of freedom as part of the composite mode for the analysis of exchange symmetries as long as this degeneracy remains. In this way degeneracy with respect to a degree of freedom corresponds to the removal of that degree of freedom. Degeneracy with respect to a composite degree of

freedom means degeneracy with respect to all its constituent degrees of freedom.

In the case of more degrees of freedom, we can always partition a composite degree of freedom into two constituent degrees of freedom which themselves may or may not be further decomposable. For this reason it is sufficient to consider only two degrees of freedom to illustrate the principal difference between one and multiple degrees of freedom.

For two degrees of freedom the equivalence between particle exchange and the exchange of modes takes the form ([15] section 58)

$$\hat{\mathsf{X}}^{(1)}\hat{\mathsf{X}}^{(2)} = \hat{\mathsf{X}}^{(2)}\hat{\mathsf{X}}^{(1)} = \hat{\mathsf{X}}^{\left(\hat{a}^{\dagger}\right)}. \tag{6}$$

Symmetry under exchange of particles is distributed between the two constituent degrees of freedom in the sense of the above relation. The exchange operators are Hermitean and square to the identity so they each have possible eigenvalues of 1 or $\dashv$. The combination of (6) and of (1) gives the constraint on the eigenvalues[1] of exchange operators

$$\mathsf{X}^{(1)}\mathsf{X}^{(2)} = (-1)^{2s}. \tag{7}$$

If the symmetries in the two degrees of freedom are well defined, bosons must have the same exchange symmetry in both whereas fermions must have opposite symmetries [10]. When looking at only one of the degrees of freedom it is possible to have a state that does not have a well defined symmetry under the exchange of mode numbers and yet does not violate the requirement (1) that the particle pair are either bosons or fermions and not a superposition of the two

$$|\psi, s\rangle = \alpha \left|\mathsf{X}^{(1)} = 1, s\right\rangle + \beta \left|\mathsf{X}^{(1)} = -1, s\right\rangle. \tag{8}$$

This is not possible if there is only one degree of freedom available or equivalently when degeneracy in all but one degree of freedom is imposed on the particle pair. Unless α or β vanish applying either $\hat{\mathsf{X}}^{(1)}$ or $\hat{\mathsf{X}}^{(2)}$ to this state will produce a state linearly independent of $|\psi, s\rangle$. This is what is meant by the state of the first or second degree of freedom not having a well defined symmetry. However it will always be the case that after applying both exchange operators we obtain $(-1)^{2s}|\psi, s\rangle$. Each of the exchange operators need not correspond to symmetries of the state but their product must.

2.2. Exchanges of constituent modes

The exchange operators may also be symmetries of transformations on particle pairs. The significance of this possibility is that all unitary transformations on two particles that commute with the exchange operators $\hat{\mathsf{X}}^{(1)}$, $\hat{\mathsf{X}}^{(2)}$ preserve the symmetry structure of the state (8) by leaving $|\alpha|$ and $|\beta|$ untouched. If a transformation is defined on a single particle then a sufficient condition for it to commute with the exchange operators in its two-particle extension is to act independently on the two degrees of freedom

$$\hat{a}^{\dagger}_{i_1 i_2} \mapsto \sum_{j_1, j_2} T_{i_1 j_1 i_2 j_2} \hat{a}^{\dagger}_{j_1 j_2} = \sum_{j_1, j_2} T^{(1)}_{i_1 j_1} T^{(2)}_{i_2 j_2} \hat{a}^{\dagger}_{j_1 j_2}. \tag{9}$$

In the rest of this paper we set the two degrees of freedom to be propagation direction and transverse spatial structure. More specifically we will study the interference of two incoming particles (bosonic or fermionic) after a coherent partial transmission and reflection on a particle beam splitter and we specify transverse spatial structure by the OAM of the particles. We show in the following that the beam splitter satisfies the requirement that it conserves the numbers $|\alpha|$ and $|\beta|$ in (8) and transforms the states under this constraint in such a way that it is possible to obtain information about $|\beta|^2$, and thereby about the symmetry structure of the state, from coincidence count rates.

3. Beam splitter

The ideal beam splitter conserves particle number. Moreover, when acting on a pair of particles the beam splitter Hamiltonian and the finite transformation generated by it must commute with the particle exchange operator so as to respect the indistinguishability of identical particles. In particular when the particles have only the port degree of freedom the particle exchange operator can be replaced by the port mode exchange operator. However, as the beam splitter acts only on the port modes[2] independently of any other degree of freedom the particle pair might have, the induced transform must be symmetric with respect to the latter in all cases. Hence

$$\left[\hat{\mathsf{X}}^{\left(\hat{a}^{\dagger}\right)}, \hat{\mathsf{T}}^{(\mathrm{II})}_{\mathrm{BS}}\right] = \left[\hat{\mathsf{X}}^{(\mathrm{p})}, \hat{\mathsf{T}}^{(\mathrm{II})}_{\mathrm{BS}}\right] = 0, \tag{10}$$

where $\hat{\mathsf{X}}^{(\mathrm{p})}$ denotes the port mode exchange operator, $\hat{\mathsf{T}}_{\mathrm{BS}}$ denotes the beam splitter transformation and the superscript (II) denotes the fact that the extension of this action to two-particle state space is being considered.

The transformation introduced by the symmetric beam splitter on the port modes of a single particle can be expressed as [5]

$$\begin{pmatrix} \hat{a}^{\dagger}_1 \\ \hat{a}^{\dagger}_2 \end{pmatrix} \mapsto \begin{pmatrix} t & r \\ r & t \end{pmatrix} \begin{pmatrix} \hat{a}^{\dagger}_1 \\ \hat{a}^{\dagger}_2 \end{pmatrix}. \tag{11}$$

The labeling of the modes is illustrated in figure 1. The transformation (11) is required to be unitary, as imposed by the fact that it must be canonical. For both a bosonic and a fermionic beam splitter the restrictions this places on the transition matrix in (11) are given by [11]

$$r^* t + r t^* = 0, \quad |t|^2 + |r|^2 = 1. \tag{12}$$

In addition we note that we have taken the beam splitter to be symmetric and we choose the convention $t = |t|$, $r = \mathrm{i}|r|$. For

[1] The eigenvalue of an operator is denoted by the same symbol as the operator but without the caret.

[2] Any change on a transverse spatial profile due to reflection may be neglected by a suitable choice of coordinate system for the input and output modes.

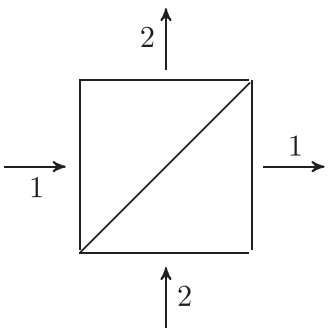

Figure 1. The labels of the two input ports and the two output ports. Note that the labelling is such that in the purely transmissive limit the labelling of a state before its encounter with the beam splitter will remain unaltered afterwards. In the purely reflective limit the labels are switched.

a symmetric beam splitter it is also useful to define an operator that swaps the two port modes

$$\hat{\mathsf{M}}\colon \hat{a}_1^\dagger \leftrightarrow \hat{a}_2^\dagger, \tag{13}$$

as this commutes with $\hat{\mathsf{T}}_{\mathrm{BS}}$,

$$\left[\hat{\mathsf{M}}, \hat{\mathsf{T}}_{\mathrm{BS}}\right] = 0. \tag{14}$$

Note that $\hat{\mathsf{M}}$ is distinct from $\hat{\mathsf{X}}^{(\mathrm{p})}$ in that it is also defined for a single particle whereas exchange is not and in the case of multiple particles, it replaces port mode 1 by port mode 2 and vice versa in each of them regardless of the others.

3.1. Two-particle modes

Although input states of interest will be one particle per port states as there is opportunity to tune their exchange symmetry, in general states with both particles in the same port will be obtained as some of the output. To obtain a simple form of the transition matrix on particle pairs it is useful to express these states as eigenstates of $\hat{\mathsf{M}}$ [16]. These take the form

$$\hat{B}_\pm^\dagger\,|\mathrm{vac}\rangle = \frac{1}{2}\left(\hat{B}_{11}^\dagger \pm \hat{B}_{22}^\dagger\right)|\mathrm{vac}\rangle \tag{15}$$

which, using the notation established in (2), is equivalent to $\frac{1}{2}(\hat{a}_1^\dagger\hat{a}_1^\dagger \pm \hat{a}_2^\dagger\hat{a}_2^\dagger)|\mathrm{vac}\rangle$, and picks up a + or − sign under (13). If the sole degree of freedom is the port mode number then such states can be realized by bosons exclusively. The beam splitter transformation (11) generalizes to

$$\begin{pmatrix} \hat{B}_+^\dagger \\ \hat{B}_{12}^\dagger \\ \hat{B}_-^\dagger \\ \hat{F}_{12}^\dagger \end{pmatrix} \mapsto \begin{pmatrix} r^2+t^2 & 2rt & 0 & 0 \\ 2rt & r^2+t^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{B}_+^\dagger \\ \hat{B}_{12}^\dagger \\ \hat{B}_-^\dagger \\ \hat{F}_{12}^\dagger \end{pmatrix}. \tag{16}$$

The requirement of symmetry under the action of $\hat{\mathsf{M}}$ restricts the form of the matrix in (16) to be block-diagonal, with the only non-zero elements appearing in two 2 × 2 submatrices (here in the top left and bottom right parts of the matrix). The symmetry associated with $\hat{\mathsf{X}}^{(\mathrm{p})}$, however, requires the transformation matrix to be composed of a 3 × 3 and a 1 × 1 submatrix, each on the diagonal. The operator $\hat{F}_{12}^\dagger$ being prevented from mixing with the port exchange symmetric particle pair creation operators. The combination of the two conditions means that only those states may mix that are in the same eigenspace of each of the two operators, leading to the restricted structure of the above transition matrix. There is an inequivalence introduced between the set of states symmetric in the port mode and the set of states antisymmetric in the port mode (that is, with respect to $\hat{\mathsf{M}}$) due to exchange. While it is possible to have states antisymmetric in the port numbers without exchange antisymmetry (for example $\hat{B}_-^\dagger\,|\mathrm{vac}\rangle$ defined above), exchange antisymmetry necessarily means antisymmetry in the port numbers.

3.2. Second degree of freedom

If the particle pair is appended with a second degree of freedom, in this example OAM (the OAM modes are labeled by indices l, j and the port modes are labeled by the indices p and q), then all four modes in (16) can be bosonic or fermionic with a suitable choice of exchange symmetry in the OAM degree of freedom. Using the definition (2) for particles with two degrees of freedom gives

$$\hat{B}_{pqlj}^\dagger = \frac{1}{2}\left\{\hat{a}_{pl}^\dagger, \hat{a}_{qj}^\dagger\right\} = \hat{B}_{\{pq\}\{lj\}}^\dagger + \hat{B}_{[pq][lj]}^\dagger \tag{17}$$

$$\hat{F}_{pqlj}^\dagger = \frac{1}{2}\left[\hat{a}_{pl}^\dagger, \hat{a}_{qj}^\dagger\right] = \hat{F}_{\{pq\}[lj]}^\dagger + \hat{F}_{[pq]\{lj\}}^\dagger. \tag{18}$$

Here { } and [] around index pairs denote symmetrization and antisymmetrization respectively in the index pairs, for example[3]

$$\hat{F}_{\{pq\}[lj]}^\dagger = \frac{1}{2}\left(\hat{F}_{pq[lj]}^\dagger + \hat{F}_{qp[lj]}^\dagger\right), \tag{19}$$

$$\hat{F}_{pq[lj]}^\dagger = \frac{1}{2}\left(\hat{F}_{pqlj}^\dagger - \hat{F}_{pqjl}^\dagger\right). \tag{20}$$

The transition matrix (16) remains the same upon including a second degree of freedom when the $\hat{B}_\pm^\dagger$, $\hat{B}_{12}^\dagger$, $\hat{F}_{12}^\dagger$ entries in both the left- and right-hand side are substituted as

$$\begin{aligned} \hat{B}_\pm^\dagger &\mapsto \hat{B}_{\pm\,lj}^\dagger = \frac{1}{2}\left(\hat{B}_{11lj}^\dagger \pm \hat{B}_{22lj}^\dagger\right), \\ \hat{B}_{12}^\dagger &\mapsto \hat{B}_{\{12\}lj}^\dagger, \\ \hat{F}_{12}^\dagger &\mapsto \hat{B}_{[12]lj}^\dagger. \end{aligned} \tag{21}$$

However we may equally well use $\hat{F}^\dagger$ instead of $\hat{B}^\dagger$ for all four substitutions. This is made possible by the OAM exchange symmetry, only implicit in this notation, conforming accordingly.

It is the port mode symmetry and exchange symmetry that is of significance in (16), not the bosonic or fermionic

[3] Here we illustrate the two-step expansion of the expression $\hat{F}_{\{pq\}[lj]}^\dagger$. Analogously { } and [] would be expanded in all other occurrences. The order in which multiple expansions are done does not matter.

nature of the particle pair. As far as the port modes are concerned the same range of behaviour is realizable for boson pairs or fermion pairs. The phenomenon of HOM interference generalizes to converting a one particle per port state to one with both particles in the same port with unit probability in the 50 : 50 beam splitter ($r = i\ t = i/\sqrt{2}$) case given that the input state is symmetric in the port degree of freedom: according to (16), an input state of the form $\hat{B}_{12}^{\dagger}\,|\mathrm{vac}\rangle$ is then converted to $\hat{B}_{+}^{\dagger}\,|\mathrm{vac}\rangle$ and vice versa. States antisymmetric in the port mode, $\hat{B}_{-}^{\dagger}\,|\mathrm{vac}\rangle$ or $\hat{F}_{12}^{\dagger}\,|\mathrm{vac}\rangle$, remain unaltered. This property of the 50 : 50 beam splitter of being able to remove exchange symmetric states from the realm of one particle per port states makes it useful as a filter for probing the symmetry structure of states.

4. General one particle per port input state

A general input state that contains one particle in each input port with an arbitrary OAM distribution can in general be split into symmetric and antisymmetric components with respect to OAM exchange

$$\begin{aligned}\left|\psi_{\mathrm{in}}\right\rangle &= \sum_{l,j=-L}^{L} c_{lj}\hat{a}_{1\,l}^{\dagger}\hat{a}_{2j}^{\dagger}\,|\mathrm{vac}\rangle \\ &= \sum_{l=-L}^{L} c_{ll}\hat{A}_{12ll}^{\dagger}\,|\mathrm{vac}\rangle \\ &\quad + 2\sum_{\substack{j,l=-L\\ j<l}}^{L}\left(c_{\{lj\}}\hat{A}_{12\{lj\}}^{\dagger} + c_{[lj]}\hat{A}_{12[lj]}^{\dagger}\right)|\mathrm{vac}\rangle \qquad (22)\end{aligned}$$

using the notation for creation operator products introduced in (2). The factor of two in the second term arises from the use of the symmetrization and antisymmetrization notation around the index pairs. The coefficients c_{lj} assign an amplitude to each distinguishable way the angular momenta can be distributed between the two ports. The OAM exchange symmetry properties of a state parametrized in this way may readily be classified by the symmetry properties of the matrix C composed of matrix elements c_{lj}. The most convenient feature of representing the state by the matrix C is that states related by the exchange of OAM numbers correspond to elements of the matrix C that are related by transposition. The diagonal terms correspond to components of the state that are degenerate in angular momentum and the off-diagonal symmetric and antisymmetric components under transposition correspond to OAM exchange symmetric and antisymmetric states respectively. It suffices to classify only the OAM degree of freedom in this way because the properties of the port degree of freedom (relevant to the beam splitter) are then set automatically by (6) as soon as the particle pair is specified to be bosonic or fermionic.

The ability to introduce a phase between the lower and upper triangles of C allows one to tune the exchange symmetry of the OAM state. Note that while we control the OAM exchange symmetry of the state explicitly by the phase the port exchange symmetry is necessarily also controlled by virtue of the restriction (6). Only the port exchange antisymmetric part of the state will lead to coincidence counts after a 50 : 50 symmetric beam splitter. Thus this control over phase between the distinguishable alternatives $\hat{a}_{1\,l}^{\dagger}\hat{a}_{2j}^{\dagger}$ and $\hat{a}_{1j}^{\dagger}\hat{a}_{2\,l}^{\dagger}$ translates directly into variation in coincidence counts. In general the amount of variation introduced into the coincidence counts is dependent on the values of l and j.

5. An example for photons

To be able to clearly demonstrate control over the coincidence counts we seek to be able to introduce a phase between the interfering terms that is sensitive to which arm carries the larger angular momentum but not to their magnitudes

$$c_{jl} = \mathrm{e}^{\mathrm{i}\,\mathrm{sgn}(j-l)\phi_R}c_{lj}. \qquad (23)$$

For a state of this type the phase between interfering terms is the same for all interfering pairs of modes. This is not a requirement but makes the demonstration of the effect clear as the coincidence counts of all OAM components are suppressed by the same amount hence OAM dependent detection is not required to observe the functional dependence of coincidence counts on the relative phase.

A relative phase between c_{lj} and c_{jl} can be introduced by rotating the field in one of the ports thereby introducing a phase that depends linearly on the rotation angle θ with a constant of proportionality that is the OAM of the photon in that port [17]. Let the phase be introduced in port 2. Then a biphoton carrying angular momenta l and j will pick up a phase $\mathrm{e}^{\mathrm{i}l\theta}$ if the l units of angular momenta are carried in port 2 and it will pick up a phase $\mathrm{e}^{\mathrm{i}j\theta}$ if the j units of angular momenta are carried in port 2. The difference in phase between the interfering alternatives now depends on the difference in angular momenta between the two particles $l - j$. Thus to obtain the desired phase (23) by this method what is required of the initially prepared state is to have a fixed difference between the angular momenta of the photon pair. In the matrix picture a state of this type has non-zero elements only on two shifted diagonals that are related to each other by transposition of C. Further, if any of the elements in one of the shifted diagonals is zero then the corresponding element obtained by transposition must also be zero.

Spontaneous parametric downconversion is an angular momentum conserving process so a pump beam of well defined OAM λ creates downconverted states for which the sum of angular momenta of the two photons sum to λ [18, 19]. By reversing the direction of OAM in one of the ports, which may be achieved using a Dove prism, the OAM difference in the two ports becomes constant λ. The resulting state is not yet suitable for an interference experiment as no term in the state has an interfering partner (assuming $\lambda \neq 0$). In other words, the two down-converted photons would behave as distinguishable particles on the beam splitter. In the matrix picture this can be seen quite clearly as the non-zero coefficients lie on a single shifted diagonal with the elements

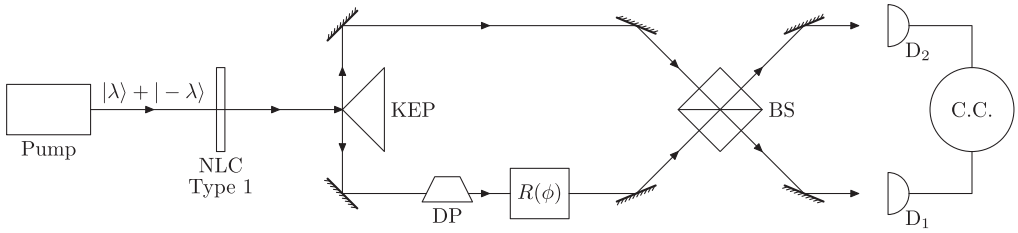

Figure 2. The $\frac{1}{\sqrt{2}}(|\lambda\rangle + |-\lambda\rangle)$ pump beam illuminates a nonlinear crystal (NLC) that performs type I downconversion. The resulting state is split into two arms by a knife edge prism (KEP). In the bottom arm the OAM is flipped using a Dove prism (DP) and a rotation is performed using a system of two more Dove prisms (note that a single Dove prism rotated appropriately about the beam axis can perform the composite reflection and rotation). This setup is also suitable for use with an $|\lambda = 0\rangle$ pump to illustrate that an OAM degenerate state cannot be affected by a rotation in arm 1.

of the shifted diagonal obtained by transposition being zero. However a pump beam with opposite angular momentum $-\lambda$ will produce exactly the missing terms required for interference

$$|\lambda\rangle \mapsto \sum_{l=-L+\lambda}^{L} c_{l,l-\lambda}\, |l, l-\lambda\rangle, \tag{24}$$

$$|-\lambda\rangle \mapsto \sum_{l=-L+\lambda}^{L} c_{l-\lambda,l}\, |l-\lambda, l\rangle, \tag{25}$$

where in Dirac notation, the first label denotes the OAM in port 1 and the second label denotes OAM in port 2, $|l, j\rangle = \hat{a}_{1l}^{\dagger}\hat{a}_{2j}^{\dagger}\,|\mathrm{vac}\rangle$. The arrow $\mapsto$ denotes both downconversion and the subsequent reflection of the beam in port 2 about a plane containing its optical axis. The property that the interfering terms are present with equal amplitudes is ensured by the fact that the amplitudes do not depend on which[4] photon has which angular momentum ($c_{lj} = c_{jl}$) [20]. Hence a pump beam of the superposition $(|\lambda\rangle + |-\lambda\rangle)/\sqrt{2}$ produces the state

$$\sum_{l=-L+\lambda}^{L} \frac{c_{l,l-\lambda}}{\sqrt{2}}(|l, l-\lambda\rangle + |l-\lambda, l\rangle). \tag{26}$$

This state contains only OAM exchange symmetric terms to begin with but upon introducing a phase by rotation of port 2 before the beam splitter, as illustrated in figure 2, the exchange antisymmetric part is introduced

$$\sum_{l=-L+\lambda}^{L} \frac{c_{l,l-\lambda}}{\sqrt{2}} \mathrm{e}^{\mathrm{i}\left(l-\frac{\lambda}{2}\right)\theta}\left[\cos\left(\frac{\lambda}{2}\theta\right)(|l, l-\lambda\rangle + |l-\lambda, l\rangle) - \mathrm{i}\sin\left(\frac{\lambda}{2}\theta\right)(|l, l-\lambda\rangle - |l-\lambda, l\rangle)\right]. \tag{27}$$

In the above only the OAM exchange symmetry is given explicitly, the port exchange symmetries are left implicit.

If the pump beam were to be in the superposition $(|\lambda\rangle - |-\lambda\rangle)/\sqrt{2}$ then the downconverted state would have only OAM exchange antisymmetric terms to begin with. These two possible pump beams for $\lambda = 1$ correspond to first order Hermite–Gaussian modes oriented at right angles to each other. These were used to demonstrate the existence of both peaks and dips in multimode HOM interference [9].

As photons are bosons the OAM and the port exchange symmetries must be the same. Further we know from section 3 that only the port antisymmetric states give rise to coincidence counts after passing through a symmetric 50 : 50 beam splitter. Thus the variation in coincidence counts is expected to vary as

$$\sin^2\left(\frac{\lambda}{2}\theta\right). \tag{28}$$

If the same experiment is run with a Gaussian pump (zero OAM) then after a reflection in one of the arms we have the state

$$\sum_{l=-L}^{L} c_{ll}\, |l, l\rangle. \tag{29}$$

While the rotation of one of the arms introduces a phase that is not global it does not lead to a variation of coincidence counts as there are no two two-particle terms in this sum that are interfering. The only effect to be observed is HOM interference irrespective of the introduced phase. This could in principle be used for calibration or as a control run to make sure that no variation is attributable to some other feature of the experiment.

6. Conclusion

We have shown that the statistics of particles incident on a beam splitter can be tuned if they possess an additional degree of freedom. The range of statistics attainable includes that of identical bosons, identical fermions and those of distinguishable particles. The entire range of behaviour is attainable in implementations whether the underlying particles are bosons or fermions. This could be perceived as to shed new light on the role of the particle type in interference experiments and means of overriding the natural statistics induced by spin, or as a convenient way of studying bosonic and fermionic as well as intermediate statistics in a single experiment. Alternatively, one could use the theory presented in the main text to inquire on the symmetry properties of a given two-particle state.

[4] The photons are to be distinguished in this context by which port they are in.

We proposed an experimental set-up by which the above concepts can be tested using OAM-entangled photons generated by spontaneous parametric down conversion. In this scheme the change attained in the output statistics depends on the angular momentum of the pump beam and a rotation angle.

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References

[1] Pauli W 1980 *General Principles of Quantum Mechanics* (Berlin: Springer)
[2] Feynman R P, Leighton R B and Sands M 1965 The Feynman Lecures On Physics vol 3 (Reading, MA: Addison-Wesley)
[3] Tichy M C 2014 Interference of identical particles from entanglement to boson-sampling *J. Phys.* B **47** 103001
[4] Hong C K, Ou Z-Y and Mandel L 1987 Measurement of subpicosecond time intervals between two photons by interference *Phys. Rev. Lett.* **59** 2044
[5] Loudon R 2000 *The Quantum Theory of Light* 3rd edn (Oxford: Oxford University Press)
[6] Peruzzo A, Laing A, Politi A, Rudolph T and O'Brien J L 2010 Multimode quantum interference of photons in multiport integrated devices *Nat. Commun.* **2** 224
[7] Karimi E, Giovannini D, Bolduc E, Bent N, Miatto F M, Padgett M J and Boyd R W 2014 Exploring the quantum nature of the radial degree of freedom of a photon via Hong–Ou–Mandel interference *Phys. Rev.* A **89** 013829
[8] Liu R C, Odom B, Yamamoto Y and Tarucha S 1998 Quantum interference in electron collision *Nature* **391** 263
[9] Walborn S F, de Oliveira A N, Pádua S and Monken C H 2003 Multimode Hong–Ou–Mandel interference *Phys. Rev. Lett.* **90** 143601
[10] Zeilinger A 1998 Quantum entanglement: a fundamental concept finding its applications *Phys. Scr.* **T76** 203
[11] Loudon R 1998 Fermion and boson beam-splitter statistics *Phys. Rev.* A **58** 4904
[12] Allen L, Barnett S M and Padgett M J 2003 *Optical Angular Momentum* (Bristol: Institute of Physics Publishing)
[13] Yao A M and Padgett M 2011 Orbital angular momentum: origins, behaviour and aplications *Adv. Opt. Photonics* **3** 161
[14] Pauli W 1950 On the connection between spin and statistics *Prog. Theor. Phys.* **5** 526
[15] Dirac P A M 1958 *The Principles of Quantum Mechanics* 4th edn (Oxford: Oxford University Press)
[16] Jeffers J 2000 Interference and the lossless lossy beam splitter *J. Mod. Opt.* **47** 1819
[17] Leach J, Padgett M J, Barnett S M, Franke-Arnold S and Courtial J 2002 Measuring the orbital angular momentum of a single photon *Phys. Rev. Lett.* **88** 257901
[18] Mair A, Vaziri A, Weihs G and Zeilinger A 2001 Entanglement of the orbital angular momentum states of photons *Nature* **412** 313
[19] Franke-Arnold S, Barnett S M, Padgett M J and Allen L 2002 Two-photon entanglement of orbital angular momentum states *Phys. Rev.* A **65** 033823
[20] Yao A M 2011 Angular momentum decomposition of entangled photons with an arbitrary pump *New J. Phys.* **13** 053048