Algorithm for the solution of elastoplastic half-space impact: Force-Indentation Linearisation Method

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Abstract

The governing equation of a half-space impact is generally nonlinear and it is normally solved using numerical techniques that are mostly conditionally stable and require many iteration steps for convergence of the solution. In this paper, we present the force-indentation linearisation method (FILM), an approximate technique that produces closed-form solutions of piecewise linearisation of the governing nonlinear differential equation and is capable of producing accurate impact response for an elastoplastic half-space impact. In contrast to the existing numerical techniques, which discretise the impact force or variable of interest in the time-domain, the present technique discretises the impact force with respect to the indentation using successive piecewise linear approximations. Generalised closed-form solutions were derived for each piecewise approximation, and this was used to develop an iterative algorithm for updating the solutions from one piecewise approximation to the next. The results of the present technique matched with results obtained by direct numerical integration of the governing nonlinear differential equation for

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a half-space impact, and the FILM was found to converge to the results of the numerical solution after a few iterations; typically between five and ten iterations. The FILM is simple, inherently stable, converges quickly, gives accurate results, and it can be implemented manually; these features makes it potentially more attractive than the comparable numerical methods.

Keywords

Half-space, impact, elastoplastic, contact law, force-indentation linearisation method.

LIST OF SYMBOLS

m	mass of the projectile
n	number of discretisations
q	number defining the power law relationship between the contact force and indentation
$r; s$	numbers representing the start and end points of each discretisation
E	effective modulus
F	impact force
F_m	maximum contact force
F_{rs}	generalised linearised contact force for each discretisation
KE	kinetic energy of projectile
K_c	contact stiffness
K_h	hertz contact stiffness
K_p	contact stiffness at the transition point in the elastoplastic loading stage
K_{rs}	generalised linearised contact stiffness for each discretisation
R	effective radius
V_0	initial velocity of projectile
W_e	work done during elastic loading
W_{ep}	work done during elastoplastic loading
δ	indentation depth
$\dot{\delta}$	velocity of projectile or indentation rate
δ_f	fixed or permanent indentation at the end of impact
δ_m	maximum indentation
δ_p	indentation at the transition point in the elastoplastic loading stage

1. Introduction

Projectile impact on structures involves local as well as global deformation response. The local response gives rise to local indentation of the structure at the region of impact whereas the global response gives rise to vibrations in the structure. Adequate modelling of structural impact with significant local indentation requires a combination of an appropriate contact model with the equations of motion of the impact system. In several previous experimental investigations and for many practical impact problems, the conditions are such that the size of the impacted structure (target) in the direction of impact is much larger than the size of the projectile, and the projectile is often stiff enough to be considered as a rigid body. This kind of impact can be modelled as a *half-space impact* i.e. impact on a target of semi-infinite thickness, and it is depicted in Figure 1 for the case of an elastoplastic impact. Half-space impact analysis is very useful for validating experimentally derived contact models. It is characterised by local indentation of the target and the vibrations of the target and projectile are negligible. The impact response of a half-space target is modelled by a single degree-of-freedom motion as shown in equation (1).

$$m\ddot{\delta} + F = 0 \tag{1}$$

where m is the mass of the projectile; δ is the indentation depth, which is equal to the displacement of the projectile, and the initial conditions are $\delta(0) = 0$, $\dot{\delta}(0) = V_0$; F is the impact force and it is subject to the restriction in equation (2). The impact force, F, is related to the indentation and can be determined from an appropriate contact model.

$$F = \begin{cases} F(\delta) & if\ F > 0 \\ 0 & if\ F \leq 0 \end{cases} \tag{2}$$

During low to moderate velocity impact, quasi-static assumptions are applicable and static contact models can be used to estimate the impact force [1]. The relationship between the force and indentation under static conditions is generally expressed according to Meyer's law [2] i.e.

$$F = K_c \delta^q \tag{3}$$

where K_c is the contact stiffness and $q \geq 1$ is a number defining the power law relationship between the contact force and indentation. During elastic impact on a half-space, the Hertz contact model applies where K_c is the Hertzian contact stiffness as derived in [2] and $q = 3/2$. However, real impact events usually result in permanent indentation even at very low impact speeds [1, 2]. Hence, the Hertz contact model is inadequate for accurate estimation of the impact histories and the maximum and end conditions for real impact events. Tabor [3] stated that a typical impact event involves four main stages namely: elastic loading, elastoplastic loading, fully plastic loading, and elastic unloading. Force-indentation models accounting for these four stages of impact have been developed under static conditions [1, 4, 5], but studies on the impact response of post-elastic impact is relatively scarce regardless of the fact that most real impact events are post-elastic. A possible explanation for the scarce literature is the difficulty involved with solving the resulting nonlinear differential or integral equations when the four stages are involved [3]. Although fully plastic loading is not always achieved during impact, elastoplastic loading is achieved most times. The force-indentation relationships for the elastoplastic stage as developed by Johnson [1] and Stronge [4] are logarithmic equations, whereas the one developed in [5] is a cubic equation. The elastoplastic models in [1], [4], and [5] cannot be expressed in terms of Meyer's law (see Eq. 1), and they introduce more computational difficulties when solved numerically. A simpler contact model for analysis of elastoplastic impact has been developed in [6]. All the stages considered in this model are expressed in terms of Meyer's law.

For a typical impact event $q > 1$ in at least one of the stages, and therefore, equation (1) becomes a nonlinear differential equation that can be solved numerically. Goldsmith [2] used a small-increment numerical integration scheme, which is based on discretisation of the nonlinear force in the time-domain, to obtain the impact histories of an elastic impact from the nonlinear integral form of equation (1). The labour of computation required by this scheme is a function of the nature of the nonlinearity in the differential equation and the time-increment used. Abrate [7] used the Newmark time integration scheme for the solution of the impact response of an elastic half-space. The Newmark scheme is a direct numerical integration scheme that discretises the displacement, velocity, and acceleration in the time-domain. In order to ensure convergence of the solution, the Newmark scheme normally requires the application of a fine-tuned time increment and involves many iteration steps. Majeed et al [6] used a nonlinear Finite Element Method (FEM) to obtain the force-time history of an elastoplastic half-space impact, and the FEM results were found to be in good agreement with the results obtained from the simulation of equation (1) using their contact model. Also, numerical integration solvers in computational software packages such as NDSolve integrator in *Mathematica*™ [8] can be used to solve equation (1) when F is a nonlinear function of δ as demonstrated subsequently in this paper. The NDSolve function solves differential equations numerically using an optimised routine that automatically selects an integration method from a list of embedded conventional methods such that convergence and computational efficiency are ensured. Given that the NDSolve function uses conventional numerical methods [8] e.g. explicit Runge-Kutta, implicit Runge-

Kutta, predictor-corrector Adams method, etc, the issues of stability and convergence inherent in these methods spring-up sometimes in the output of the function. Consequently, the NDSolve function may not produce convergent solutions in some cases.

The problem with the FEM and existing numerical integration approaches is that they are conditionally stable, often complicated, and usually require numerous iterations to obtain accurate results and therefore difficult to verify by means of hand calculation. This paper presents a new iterative solution algorithm called the Force-Indentation Linearisation Method (FILM) that uses closed-form solutions of piecewise linearisation to obtain the impact histories of a half-space impact. In contrast to the conventional numerical schemes in which key variables are discretised in the time-domain, the FILM discretises the impact force with respect to indentation depth. The limitations associated with the numerical methods and FEMs as stated above are completely eliminated by the FILM.

2. Concept of the FILM

Closed-form solutions of a half-space impact response can be obtained when a linearised force-indentation relationship is used. Hence, if the nonlinear force-indentation curve between the initial and end states of an impact stage is approximated using several line segments joined end to start, then each line segment representing a linear force-indentation relationship gives a closed-form solution that is only applicable to the range of indentation it covers. The slopes of the line segments represent linearised contact stiffnesses, and the slopes increase gradually between the initial and end states of the impact stage considered. This changing slope results in a changing force-indentation relationship that can be used to update the solution from one line segment to the next.

Figures 2a-c shows a typical nonlinear force-indentation relationship with various approximations. Obviously, Figure 2c with three line segments will give a better description of the impact histories compared to Figures 2a and 2b. The implication is that increasing the number of line segments, produces a better the approximation of the nonlinear force-indentation curve and gives a more accurate estimation of the impact history. Next, a linearisation technique for determining the slope of each piecewise linearisation is presented.

3. Solution of elastic half-space impact using the FILM

3.1. Linearised force-indentation relationship for each discretisation

To demonstrate the technique used in the development of the FILM, the case of an elastic impact is considered. For elastic impact, the force-indentation relationship is depicted in Figure 3 and is given by the Hertz law as shown in equation (4).

$$F = K_h \delta^{3/2} \qquad\qquad 0 \le \delta \le \delta_m \tag{4}$$

where $K_h = (4/3)ER^{1/2}$; E and R are the effective modulus and radius respectively given by: $E = [(1-\upsilon_i^2)/E_i + (1-\upsilon_t^2)/E_t]^{-1}$ and $R = [1/R_i + 1/R_t]^{-1}$. The subscripts i and t stand for impactor (projectile) and target respectively, and υ is the Poisson's ratio. Substituting equation (4) in equation (1), the governing nonlinear differential equation for the impact response of an elastic half-space is given as:

$$m\ddot{\delta} + K_h \delta^{3/2} = 0 \tag{5}$$

To show the development of the FILM for elastic impact, the Herztian contact curve is first approximated by three lines as shown in Figure 3 and then the technique is extended to approximation with n-lines. The linearised contact stiffness between point O and point 1 on the curve is given by:

$$K_{01} = \frac{\overline{1E}}{\overline{EO}} = \frac{F_1}{\delta_1} = \frac{K_h \delta_1^{3/2}}{\delta_1} = K_h \delta_1^{1/2} \tag{6a}$$

Therefore,

$$F_{01} = K_{01}\delta = K_h \delta_1^{1/2}\,\delta \tag{6b}$$

F_{01} is the linearised contact force from point O to point 1 i.e. for $\delta_0 \leq \delta \leq \delta_1$ with $\delta_0 = 0$.

Between points 1 and 2, the linearised contact stiffness is given by the slope of line $\overline{12}$ as:

$$K_{12} = \frac{\overline{2C}}{\overline{C1}} = \frac{F_2 - F_1}{\delta_2 - \delta_1} = K_h \left(\frac{\delta_2^{3/2} - \delta_1^{3/2}}{\delta_2 - \delta_1} \right) \tag{7a}$$

The equation of the line $\overline{12}$ is given as:

$$F_{12} = K_{12}(\delta - \delta_1) + K_h \delta_1^{3/2} \tag{7b}$$

Similarly,

$$K_{23} = \frac{\overline{3A}}{\overline{A2}} = K_h \left(\frac{\delta_3^{3/2} - \delta_2^{3/2}}{\delta_3 - \delta_2} \right) \tag{8a}$$

$$F_{23} = K_{23}(\delta - \delta_2) + K_h \delta_2^{3/2} \tag{8b}$$

Generally,

$$K_{rs} = K_h \left(\frac{\delta_s^{3/2} - \delta_r^{3/2}}{\delta_s - \delta_r} \right); s = r + 1 \tag{9a}$$

$$F_{rs} = K_{rs}(\delta_{rs} - \delta_r) + K_h \delta_r^{3/2} \tag{9b}$$

Assuming the Herztian contact curve was discretised into *n* segments, then the linearised contact force for each segment can be determined using equations (9a) and (9b). If the maximum indentation δ_m is divided into *n* equal segments, then the indentation at the limits of each segment can be written as $\delta_r = r\delta_m/n$ where $r = 0, 1, 2, \dots, n$ i.e. $\delta_0 = 0$, $\delta_1 = \delta_m/n$, $\delta_2 = 2\delta_m/n$, and so on. Substituting these expressions for δ_1 and δ_2 into equation (6a),

$$K_{12} = K_h\left(\frac{(2\delta_m/n)^{3/2} - (\delta_m/n)^{3/2}}{(2\delta_m/n) - (\delta_m/n)}\right) = nK_h\delta_m^{1/2}\left[\left(\frac{2}{n}\right)^{3/2} - \left(\frac{1}{n}\right)^{3/2}\right] \tag{10}$$

Similarly,

$$K_{23} = K_h\left(\frac{(3\delta_m/n)^{3/2} - (2\delta_m/n)^{3/2}}{(3\delta_m/n) - (2\delta_m/n)}\right) = nK_h\delta_m^{1/2}\left[\left(\frac{3}{n}\right)^{3/2} - \left(\frac{2}{n}\right)^{3/2}\right] \tag{11}$$

and generally,

$$K_{rs} = nK_h\delta_m^{1/2}\left[\left(\frac{s}{n}\right)^{3/2} - \left(\frac{r}{n}\right)^{3/2}\right] \tag{12}$$

Hence, for *n* equal divisions of the maximum indentation, the linear contact law for each discretisation is given by:

$$F_{rs} = K_{rs}(\delta_{rs} - \delta_r) + K_h\delta_r^{3/2} = nK_h\delta_m^{1/2}\left[\left(\frac{s}{n}\right)^{3/2} - \left(\frac{r}{n}\right)^{3/2}\right](\delta_{rs} - \delta_r) + K_h\delta_r^{3/2} \tag{13}$$

where $r = 0, 1, 2, \dots, n-1$ represents the initial state of each linear approximation of the $(r+1)$th discretisation, and $s = r + 1$ represents the end state.

It has been shown that during low to moderate velocity impact on a half-space the energy absorbed in the form of elastic waves can be neglected for both elastic [9] and elastoplastic

[10] impact. Also, frictional losses and rate effects are negligible because impacts with moderate velocities can be safely dealt with using quasi-static assumptions [1]. The kinetic energy of the projectile is essentially used for deformation work of the half-space target. Therefore, the maximum indentation, δ_m, of a low to moderate velocity half-space impact can be determined from the energy-balance between the initial kinetic energy of the projectile and the local deformation work in the target. For elastic impact on a half-space by a projectile of mass m and having an initial velocity of V_0, the energy-balance is given as:

$$\frac{1}{2} m V_0^2 = \int_0^{\delta_m} K_h \delta^{3/2} \, d\delta \tag{14}$$

Evaluating the integral in equation (14) and solving for δ_m gives:

$$\delta_m = \left(\frac{5 m V_0^2}{4 K_h} \right)^{2/5} \tag{15}$$

3.2. Application of the linearised force-indentation relationship to the solution of elastic half-space impact

Here the generalised linear contact law obtained using the FILM is applied in deriving closed-form solutions for each piecewise linearisation of the elastic half-space impact. Substituting equation (13) in equation (1), a linear differential equation is obtained as shown:

$$m \ddot{\delta}_{rs} + K_{rs} \delta_{rs} = K_{rs} \delta_r - K_h \delta_r^{3/2} \tag{16}$$

Equation (16) is a non-homogeneous linear differential equation and the complete solution can be readily obtained as:

$$\delta_{rs} = R_{rs}\,\mathrm{Sin}(\omega_{rs}t + \varphi_{rs}) + C_{rs} \tag{17}$$

where $R_{rs} = (A_{rs}^2 + B_{rs}^2)^{1/2}$; $\omega_{rs} = \sqrt{K_{rs}/m}$; $\varphi_{rs} = \mathrm{Tan}^{-1}(B_{rs}/A_{rs})$; and $C_{rs} = \left(K_{rs}\delta_r - K_h\delta_r^{3/2}\right)/K_{rs}$ is the particular solution. As usual, the constants A_{rs} and B_{rs} are determined from the initial conditions $\delta(t_r) = \delta_r$ and $\dot{\delta}(t_r) = \dot{\delta}_r$. Differentiating equation (17) with respect to time, the velocity is obtained as:

$$\dot{\delta}_{rs} = \omega_{rs}R_{rs}\,\mathrm{Cos}(\omega_{rs}t + \varphi_{rs}) \tag{18}$$

During elastic impact the same contact law is applicable to the loading and restitution stages and therefore, equations (17), (18), and (13) are applicable to the restitution stage as well.

3.3. Determination of the initial conditions for each linearisation

When $r = 0$, the linearised contact law starts from the origin and the initial conditions are $\delta(t_o) = \delta_o = 0$ and $\dot{\delta}(t_0) = \dot{\delta}_0 = V_0$. Since the start-point of F_{12} is the same as the endpoint of F_{01}, then the initial conditions of F_{12} are the same as the end conditions of F_{01}. In general, the initial conditions of F_{rs} for any $r \geq 1$ are the same as the end conditions of the preceding discretisation. Since the displacements at the boundaries of each discretisation are known i.e. $\delta_r = r\delta_m/n$, then one initial condition is already available. To get the second initial condition, the known displacements are substituted into equation (17) to get the time at the initial condition as:

$$t_r = (\mathrm{ArcSin}[(\delta_r - C_{rs})/R_{rs}] - \varphi_{rs})/\omega_{rs} \tag{19}$$

From trigonometry, $\mathrm{ArcSin}(\theta) + \mathrm{ArcCos}(\theta) = \pi/2$. Therefore,

$$t_r = \left(\frac{\pi}{2} - \mathrm{ArcCos}[(\delta_r - C_{rs})/R_{rs}] - \varphi_{rs}\right)/\omega_{rs} \tag{20}$$

The second initial condition is then determined by substituting equation (20) in equation (18). More generally, the time at the boundaries of each discretisation can be expressed as:

$$t_r = \left(\frac{\pi}{2} \pm \mathrm{ArcCos}[(\delta_r - C_{rs})/R_{rs}] - \varphi_{rs}\right)/\omega_{rs} \tag{21}$$

In equation (21), the second term in the outer bracket of the numerator is negative during the loading stage (as in equation (20)) and positive during the restitution stage. The change in the sign of this term during the restitution stage is due to the fact that the contact force is reversed. At maximum conditions, this term vanishes so that $t_m = (\pi/2 - \varphi_{rs})/\omega_{rs}$. In the restitution stage r represents the end state of each contact force while s represents the initial state, and $r = n-1, n-2, n-3, \ldots, 1, 0$ while $s = r+1$. For each discretisation, equation (21) is used to determine the time boundaries for the linearised contact force, and then, equations (17), (18), and (13) are used to extract data points for the indentation, velocity, and contact force histories within the time boundaries.

As a result of how the FILM is formulated it produces closed-form solutions for each piecewise linearisation of equation (5). The closed-form solutions for each piecewise linearisation do not oscillate or diverge and this makes the FILM inherently stable unlike other commonly used numerical methods [2, 7] and the FEM. The above procedure of the FILM was developed into a customised program in order to simulate the impact histories of an elastic impact. The input parameter used for the simulation are: $m_i = 1\,kg$, $V_0 = 0.1\,m/s$, $K_h = 12.05\,kN/mm^{1.5}$. To ensure that the impact simulated was an elastic impact, the initial velocity of the projectile was chosen so that the maximum indentation is less than the indentation at yield i.e. $0.28\,mm$. The NDSolve integrator was also used to simulate the elastic impact response for this example. In its basic form the NDSolve function automatically selects the integration method to use in solving a differential equation from a list of embedded conventional methods. Although it is unclear how the integration method used is selected, it is known that the function does so through an optimised algorithm that tries to ensure convergence, computational efficiency, and accuracy in the best possible way [8]. However, the NDSolve function also allows the user to specify the integration method to use from the list of embedded methods, but using this option does not guarantee the best solution. Hence, the NDSolve function has been used in its basic form in the present work. For n = 10 and with five data points extracted within each time interval, the impact histories obtained using the FILM matched with those obtained by direct numerical integration of equation (5) using NDSolve integrator (see Figure 4).

Figure 5 shows the variation of the linearised contact stiffness with each discretisation when n = 30. The plot represents the changing contact stiffness for each piecewise linearisation of the elastic impact force (see equation (4)) when the maximum indentation of the elastic loading is divided into 30 equal segments. The Figure shows an increase in the linearised contact stiffness of each piecewise linearisation from the origin to the maximum indentation.

4. Solution of elastoplastic half-space impact using the FILM

In the previous section, the FILM was used to developed solutions for an elastic half-space impact. Simulation of the FILM was shown to produce accurate results. In this section, the FILM has been used to develop solutions for the impact response of an elastoplastic half-space impact. According to Majeed et al [6], the contact law for an elastoplastic half-space impacted by a spherical projectile can be expressed as:

Stage I: Elastic loading

$$F = K_h \delta^{3/2} \qquad 0 \le \delta \le \delta_p \qquad (22)$$

Stage II: Elastoplastic loading

$$F = K_p(\delta - \delta_p) + K_h \delta_p^{3/2} \qquad \delta_p \le \delta \le \delta_m \qquad (23)$$

Stage III: Restitution (unloading)

$$F = F_m \left(\frac{\delta - \delta_f}{\delta_m - \delta_f}\right)^{3/2} \qquad \delta_f \le \delta \le \delta_m \qquad (24)$$

The expressions for calculating all the constants in equations (22) to (24) can be found in [6], except for F_m and δ_m. The equations for the elastoplastic impact response of a half-space according to this model are given by equations (5), (25) and (26) respectively. The loading period consists of an elastic loading stage, which is modelled according to Hertz's law, and an elastoplastic loading stage that is modelled as a linear force-indentation relationship. The unloading stage is modelled as an elastic unloading with the inclusion of permanent indentation effect. Since the elastoplastic loading stage is modelled with a linear force-indentation relationship, then the FILM can only be applied to the elastic loading and the unloading stages in this case.

$$m\ddot{\delta} + K_p(\delta - \delta_p) + K_h \delta_p^{3/2} = 0 \qquad (25)$$

$$m\ddot{\delta} + F_m \left(\frac{\delta - \delta_f}{\delta_m - \delta_f}\right)^{3/2} = 0 \qquad (26)$$

Application of FILM to elastic loading stage

For the elastic loading, equations (13), (17), (18) and (20) are applied by replacing δ_m with δ_p so that $\delta_r = r\delta_p/n$ and $K_{rs} = nK_h\delta_p^{1/2}\left[(s/n)^{3/2} - (r/n)^{3/2}\right]$.

Solution of the elastoplastic loading stage

The elastoplastic loading stage was modelled by a linear force-indentation relationship. This means that there is no need to apply the FILM to this stage. Equation (25) is a non-homogeneous linear differential equation from which the closed-form solutions are obtained as:

$$\delta = R\,\mathrm{Sin}(\omega t + \varphi) + C \tag{27}$$

$$\dot{\delta} = R\,\mathrm{Cos}(\omega t + \varphi) \tag{28}$$

where $R = (A^2 + B^2)^{1/2}$; $\omega = \sqrt{K_p/m}$; $\varphi = \mathrm{ArcTan}(B/A)$; and $C = \left(K_p\delta_p - K_h\delta_p^{3/2}\right)/K_p$. A and B are determined from the initial conditions of this stage, which coincides with the end conditions of the last discretisation in the elastic loading stage. At the end state of the last discretisation in the elastic loading stage, $s = n$, $\delta(t_n) = \delta_p$, and $\dot{\delta}(t_n)$ is determined by substituting $r = n$ in equation (20), and then substituting the value of t_n in equation (18). The force histories are calculated by substituting equation (27) in equation (23).

The end conditions for the elastoplastic stage are the same as the maximum conditions. As mentioned earlier, the velocity of the projectile at maximum conditions is zero. Using this condition in equation (28), the time taken to reach maximum indentation is derived as:

$$t_m = (\pi/2 - \varphi)/\omega \tag{29}$$

Substituting equation (29) in equation (27), the maximum indentation is derived as:

$$\delta_m = R + C \tag{30}$$

An alternative method based on the energy-balance principle involves equating the kinetic energy of the projectile to the deformation work done on the half-space. Hence, the energy balance gives:

$$KE = W_e + W_{ep} \tag{31}$$

W_e is the work done during the elastic loading from $\delta = 0$ to $\delta = \delta_p$, and W_{ep} is the work done during elastoplastic loading from $\delta = \delta_p$ to $\delta = \delta_m$. Using equations (22) and (23) in equation (31) gives:

$$\frac{1}{2} m V_0^2 = \int_0^{\delta_p} K_h \delta^{3/2}\, d\delta + \int_{\delta_p}^{\delta_m} \left[K_p \delta - K_p \delta_p + K_h \delta_p^{3/2}\right] d\delta \tag{32}$$

Evaluation of the integrals and further simplification leads to a quadratic equation in δ_m.

$$\Omega_1 \delta_m^2 + \Omega_2 \delta_m + \Omega_3 = 0 \tag{33}$$

where $\Omega_1 = K_p$; $\Omega_2 = 2\left(K_h \delta_p^{3/2} - K_p \delta_p\right)$; and $\Omega_3 = K_p \delta_p^2 - (6/5) K_h \delta_p^{5/2} - m V_0^2$. From equation (33), the maximum indentation can be determined as:

$$\delta_m = \left[-\Omega_2 + (\Omega_2^2 - 4\Omega_1\Omega_3)^{1/2}\right] / 2\Omega_1 \tag{34}$$

The time at the maximum indentation can then be obtained from equation (27) as:

$$t_m = [\mathrm{ArcSin}(\delta_m / R - C) + \varphi] / \omega \tag{35}$$

<u>Application of FILM to the unloading stage</u>

In order to illustrate how the FILM can be applied to the unloading stage, which is characterised by a nonlinear elastic force-indentation relationship, the piecewise linearisation technique discussed above is applied to the unloading curve as depicted in Figure 6. Equation (24) can be rewritten as:

$$F = K_u\left(\delta - \delta_f\right)^{3/2} \tag{36}$$

where $K_u = F_m/\left(\delta_m - \delta_f\right)^{3/2}$. Following the same procedure as in section 3, the FILM can be applied to the unloading stage as shown below. For the linearised force,

$$F_{rs} = K_{rs}(\delta - \delta_r) + K_u\left(\delta_r - \delta_f\right)^{3/2} \tag{37}$$

where

$$K_{rs} = nK_u\left(\delta_m - \delta_f\right)^{1/2}\left[\left(\frac{s}{n}\right)^{3/2} - \left(\frac{r}{n}\right)^{3/2}\right]$$

The displacement, velocity, and time boundaries are determined from equations (17), (18), and (21) respectively with the following changes: $C_{rs} = \left(K_{rs}\delta_r - K_u\left(\delta_r - \delta_f\right)^{3/2}\right)/K_{rs}$; $\delta_r = \delta_f + r\left(\delta_m - \delta_f\right)/n$; $r = n-1, n-2, n-3, \ldots, 1, 0$; and $s = r+1$. When $r = n-1$, the initial conditions are equal to those at the maximum conditions i.e. $\delta(t_m) = \delta_m$ and $\dot{\delta}(t_m) = \dot{\delta}_m = 0$. Subsequently, the initial conditions are determined by the known displacements δ_r and the velocities obtained by putting equation (21) into equation (18).

The application of the FILM to the solution of an elastoplastic half-space impact as discussed above was developed into a customized program and the impact problem considered in [6] was solved successfully using the program. The code was run for ten linear approximations (i.e. n = 10) in the elastic loading and the unloading stages, and the impact histories were found to matched with those obtained by numerical integration of equations (5), (25) and (26) using NDSolve integrator (see Figure 7).

5. Conclusions

This paper has focussed on the solution of the nonlinear differential equation for modelling half-space impact. A new solution technique called the Force-Indentation Linearisation Method (FILM) that uses closed-form solutions of piecewise linearisation to determine the impact histories of a half-space impact has been developed and presented. The FILM was used for the solution of elastic and elastoplastic half-space impact, and simulations proved that the FILM produces accurate results. The FILM converged to the required solution after a few iterations, typically between 5 and 10 iterations, making it amenable to manual implementation. Hence, the FILM solution can be checked manually.

With conventional numerical methods including finite element analyses, the labour and complexity of the solution scheme increases with the complexity of the nonlinearity that models the impact force in equation (1). Interestingly, the FILM retains the same simplicity and relative ease in computation irrespective of the kind of nonlinearity that models the impact force. Hence, the FILM can readily be extended to obtain the solutions for a general half-space impact governed by Meyer's law (equation (3)) and other impact models that cannot be expressed according to Meyer's law [1, 4, 5].

Acknowledgement

The first author wishes to acknowledge the Commonwealth Scholarship Commission (CSC), UK for funding his PhD research (CSC award ref: NGCA-2011-60), of which this work is a part.

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LIST OF FIGURES

Figure 1. Elastoplastic half-space impact of a rigid spherical indenter on a compliant flat target (a) before impact (b) during impact (c) after impact.

Figure 2. Linear approximations for discretised nonlinear force-indentation relationship (a) one line (b) two lines (c) three lines.

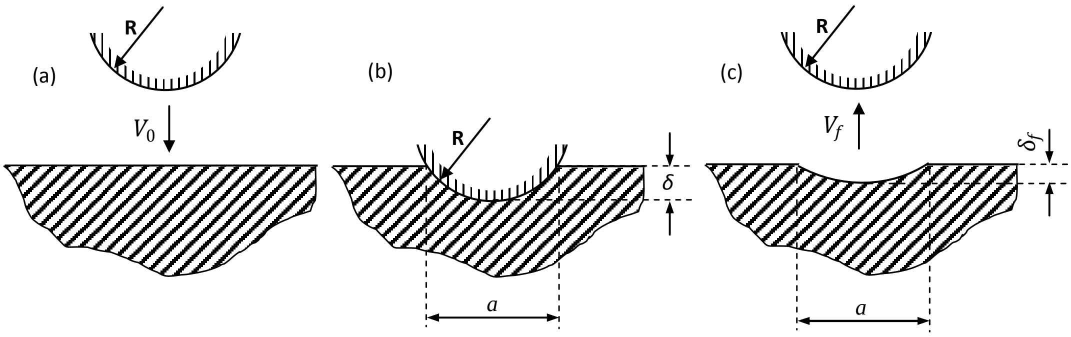
Figure 3. Tri-linear approximation of elastic loading law demonstrating the technique of the FILM.

Figure 4. Comparison of the impact histories of an elastic half-space obtained by FILM and numerical integration in Mathematica.

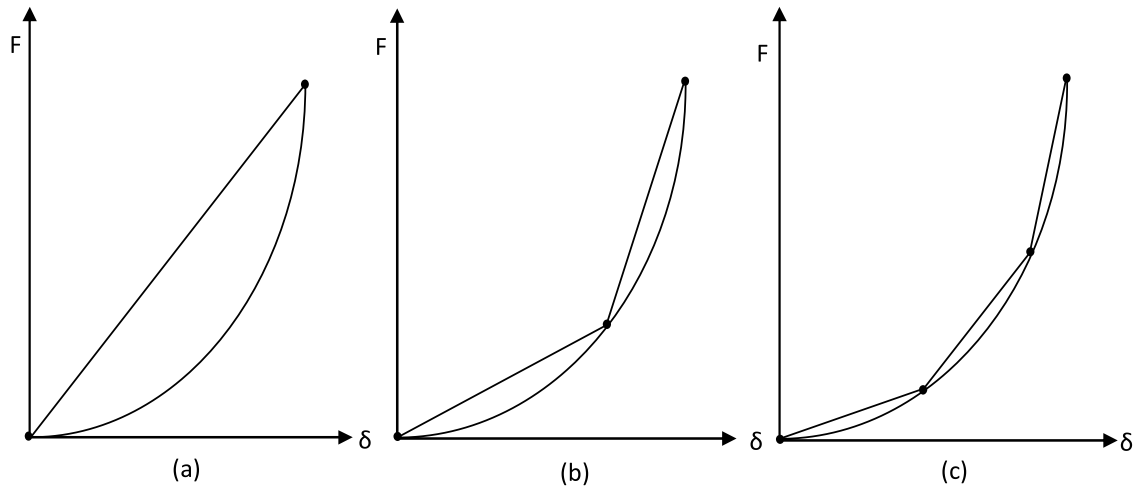
Figure 5. Variation of the linearised contact stiffness for n = 30.

Figure 6. Tri-linear approximation of an elastic unloading law.

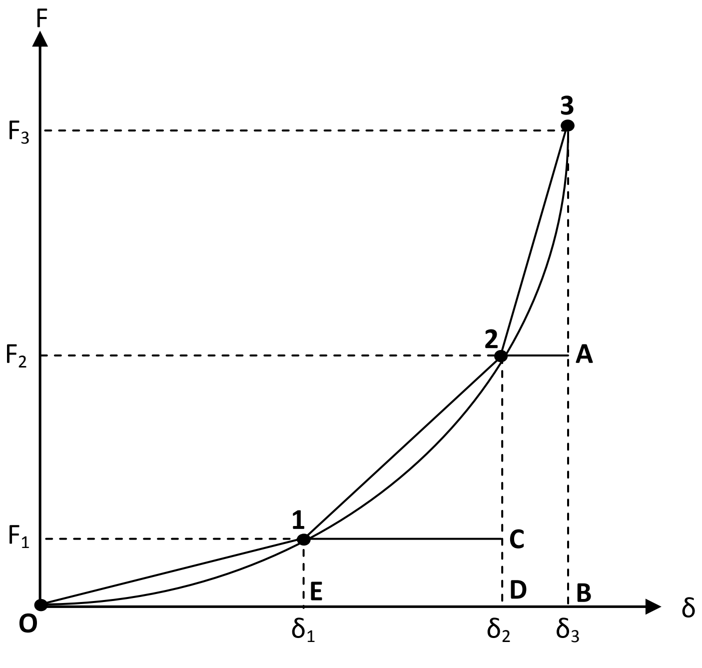
Figure 7. Comparison of elastoplastic half-space impact histories obtained by FILM and by numerical integration.

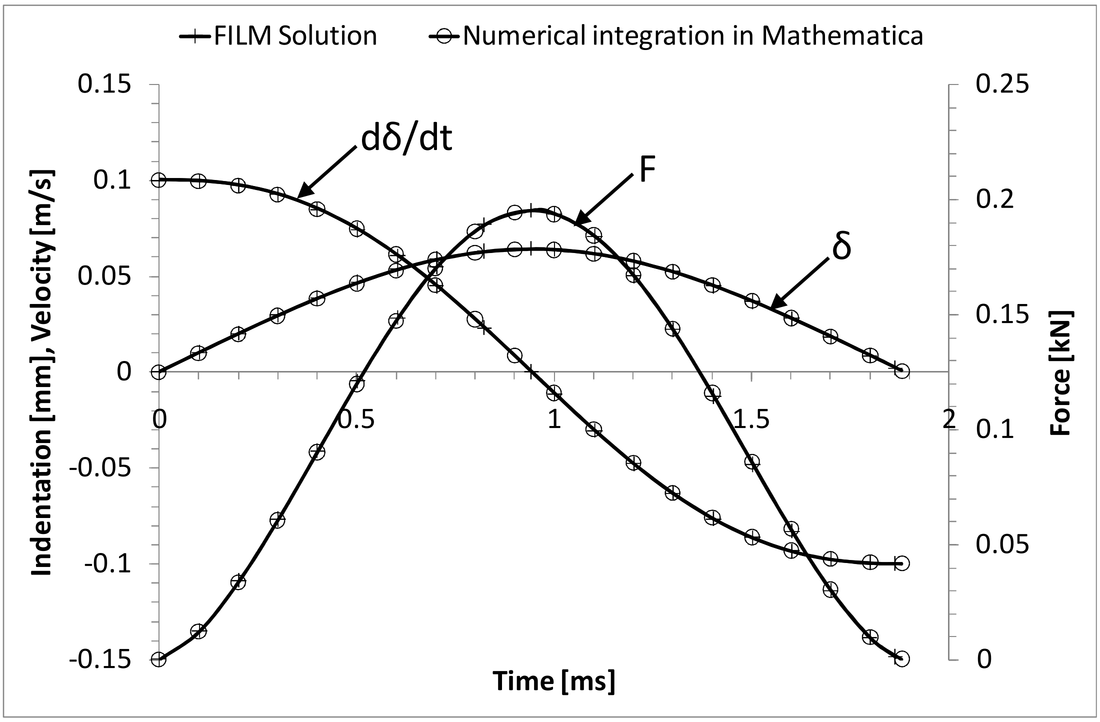


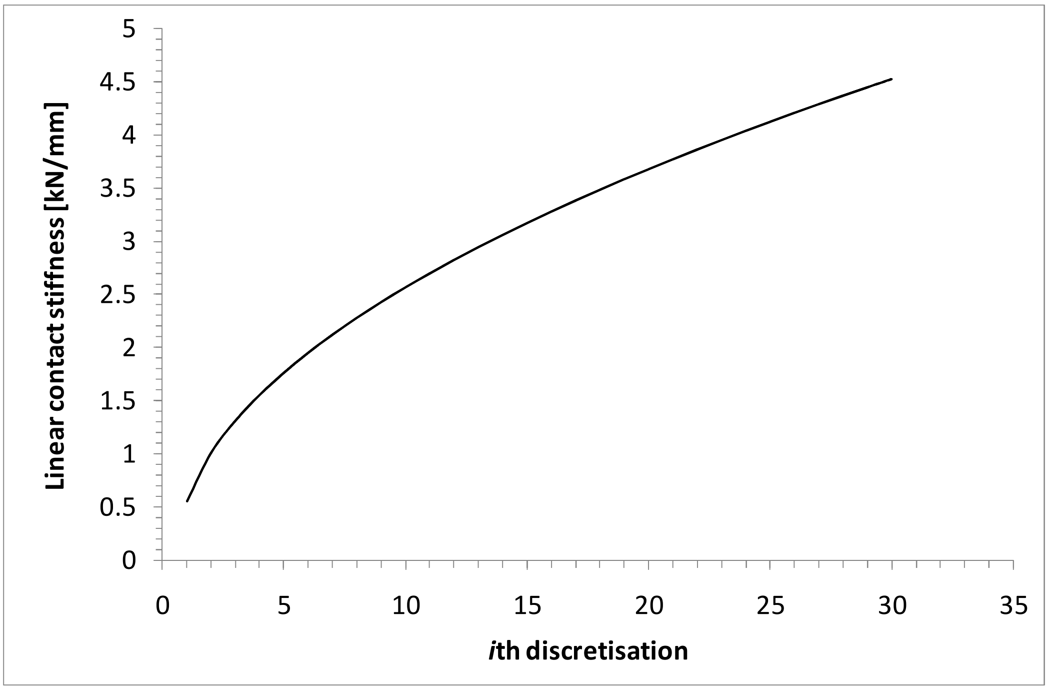
(a)
R
V_0
(b)
R
δ
a
(c)
R
V_f
δ_f
a



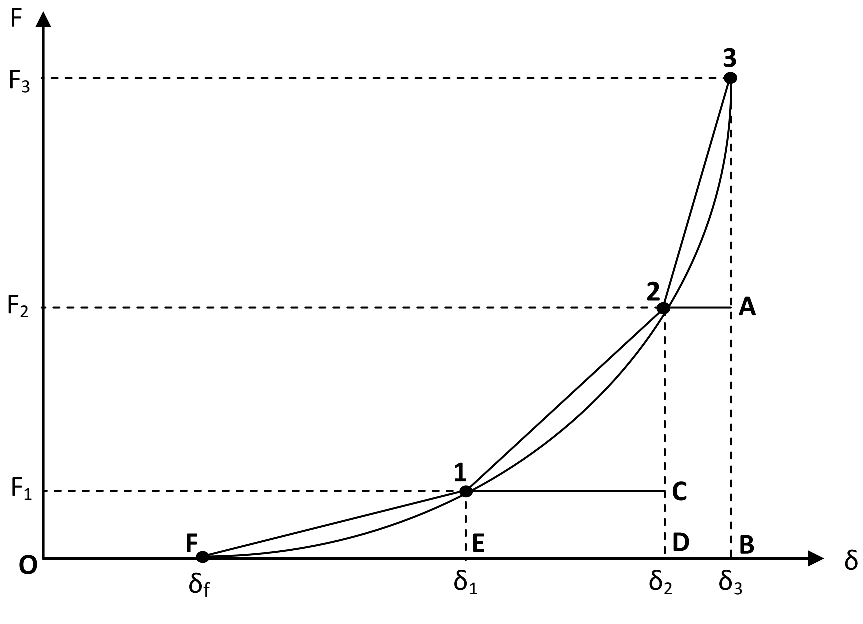
F
δ
(a)
F
δ
(b)
F
δ
(c)


F
F3
F2
F1
O
1
2
3
A
C
E
D
B
δ
δ1
δ2
δ3

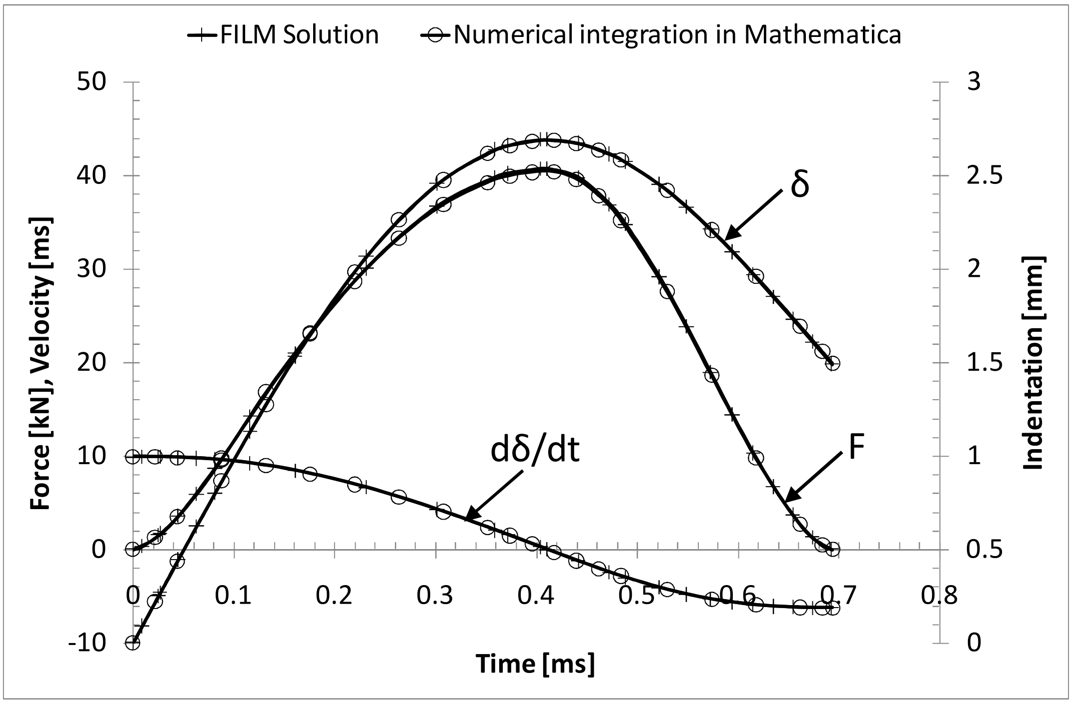

FILM Solution
Numerical integration in Mathematica
dδ/dt
F
δ
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
0
0.5
1
1.5
2
0.25
0.2
0.15
0.1
0.05
0
Indentation [mm], Velocity [m/s]
Force [kN]
Time [ms]



Linear contact stiffness [kN/mm]
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
0
5
10
15
20
25
30
35
*i*th discretisation



F
3
F_3
2
F_2
A
1
F_1
C
F
E
D
B
O
δ
δ_f
δ_1
δ_2
δ_3


FILM Solution
Numerical integration in Mathematica
Force [kN], Velocity [ms]
Indentation [mm]
Time [ms]
δ
F
dδ/dt
50
40
30
20
10
0
-10
3
2.5
2
1.5
1
0.5
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8