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CHARACTERISTICS FOR \mathcal{E}_∞ RING SPECTRA

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ABSTRACT. We introduce a notion of characteristic for connective p -local \mathcal{E}_∞ ring spectra and study some basic properties. Apart from examples already pointed out by Markus Szymik, we investigate some examples built from Hopf invariant 1 elements in the stable homotopy groups of spheres and make a series of conjectures about spectra for which they may be characteristics; these appear to involve hard questions in stable homotopy theory.

INTRODUCTION

In ordinary ring theory, the characteristic of a unital ring is really part of the structure, although often not introduced in elementary courses except in the context of fields. Less standard is a generalisation of the notion to algebras over a commutative ring and we discuss this in Section 1. The main aim of this note is to introduce an appropriate notion of characteristic for derived commutative rings, at least in the topological context of commutative S -algebras (also known as \mathcal{E}_∞ ring spectra). Our approach could be extended to other versions of derived commutative rings such as simplicial commutative algebras or \mathcal{E}_∞ -algebras over a fixed commutative ring, but we focus on the topological version.

In [17, 18], Markus Szymik introduced a notion of characteristic for a commutative S -algebra. We consider what properties a more general notion of characteristic might be expected to have in this setting, at least for connective algebras localised at a prime. We do not attempt to work in the chromatic setting since we rely on the theory of minimal atomic commutative S -algebras which does not seem to extend to such an intrinsically non-connective context.

As well as setting up a general notion of characteristic, we discuss possible candidates for characteristics of some important standard examples, and state conjectures which appear to involve non-trivial questions in stable homotopy theory. One possible approach to proving these might involve old work of Joel Cohen [12], however, to date we have been unable to carry out such a programme. Further related ideas are explored in [4].

We will assume the reader is familiar with the basic theory of \mathcal{E}_∞ ring spectra in their avatar as commutative S -algebras [13], discussions of cellular aspects can be found in [1, 2, 5].

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1. MOTIVATION: CHARACTERISTICS IN ALGEBRA

If R is a (not necessarily commutative) ring with unity $1 \neq 0$, there is a ring homomorphism $\eta_R: \mathbb{Z} \rightarrow R$ called the *unit* or *characteristic* homomorphism, defined by

$$\eta_R(n) = n1 = \begin{cases} \underbrace{1 + \cdots + 1}_n & \text{if } n > 0, \\ -\underbrace{(1 + \cdots + 1)}_{-n} & \text{if } n < 0, \\ 0 & \text{if } n = 0. \end{cases}$$

Since $1 \in R$ is non-zero, $\ker \eta_R$ is a proper ideal of \mathbb{Z} and there is a quotient monomorphism $\bar{\eta}_R: \mathbb{Z}/\ker \eta_R \rightarrow R$ which allows us to identify the quotient ring $\mathbb{Z}/\ker \eta_R$ with image $\eta_R \mathbb{Z} \subseteq R$, the *characteristic subring* of R . Thus there is a unique non-negative integer $\text{char } R \geq 0$ such that $\ker \eta_R = (\text{char } R) \triangleleft \mathbb{Z}$, and this is called the *characteristic* of R .

We can generalise this to unital \mathbb{k} -algebras over a commutative ring \mathbb{k} . Here \mathbb{k} -algebra is used in its most general sense: a \mathbb{k} -algebra A is a unital ring equipped with a unital homomorphism $\eta_A: \mathbb{k} \rightarrow A$ whose image is central. The ideal $\ker \eta_A \triangleleft \mathbb{k}$ is not necessarily principal, and the quotient homomorphism $\bar{\eta}_A: \mathbb{k}/\ker \eta_A \rightarrow A$ defines what might reasonably be called the *characteristic subalgebra* of A . This construction is functorial with respect to \mathbb{k} -algebra homomorphisms, i.e., given an algebra homomorphism $\varphi: A \rightarrow B$, there is a commutative diagram

$$\begin{array}{ccc} \mathbb{k}/\ker \eta_A & \xrightarrow{\varphi_0} & \mathbb{k}/\ker \eta_B \\ \bar{\eta}_A \downarrow & & \downarrow \bar{\eta}_B \\ A & \xrightarrow{\varphi} & B \end{array}$$

and in particular, if φ is an isomorphism, so is φ_0 .

The latter generalisation seems natural and we use it as motivation for our discussion of the analogue for commutative S -algebras. In that context there do not appear to be obvious notions of ideals or quotient objects (see [14] for recent work on related questions), so care is needed in making suitable definitions. There are some features of a notion of characteristic in that setting which seem desirable, in particular some kind of functoriality and homotopy invariance. We tacitly assume that a characteristic of a commutative S -algebra R should be a factorisation of

its unit $\iota_R: S \rightarrow R$ of the form

$$\begin{array}{ccccc}
 & & \iota_R & & \\
 & \curvearrowright & & \curvearrowleft & \\
 S & \xrightarrow{\iota_{R_0}} & R_0 & \xrightarrow{\bar{\iota}_R} & R
 \end{array}$$

where $\bar{\iota}_R$ is a morphism of commutative S -algebras. Motivated by obvious functoriality properties of characteristics for \mathbb{k} -algebras, the following basic properties appear to be reasonable requirements.

Functoriality: If $f: R \rightarrow R'$ is a morphism of commutative S -algebras, then there is a diagram of commutative S -algebras

$$\begin{array}{ccc}
 R_0 & \xrightarrow{f_0} & R'_0 \\
 \bar{\iota}_R \downarrow & & \downarrow \bar{\iota}_{R'} \\
 R & \xrightarrow{f} & R'
 \end{array}$$

which is homotopy commutative.

Homotopy invariance: If $g: R \xrightarrow{\sim} R''$ is a weak equivalence of commutative S -algebras then there is a diagram of commutative S -algebras

$$\begin{array}{ccc}
 R_0 & \xrightarrow{g_0} & R''_0 \\
 \bar{\iota}_R \downarrow & & \downarrow \bar{\iota}_{R''} \\
 R & \xrightarrow{g} & R''
 \end{array}$$

which is homotopy commutative.

Functoriality implies that characteristics of homotopy equivalent commutative S -algebras are homotopy equivalent.

We will show that our definition of characteristic does possess homotopy invariance, but does not appear to satisfy functoriality in this sense, but nevertheless it does satisfy a weaker version of this property.

2. BACKGROUND MATERIAL ON S -MODULES AND COMMUTATIVE S -ALGEBRAS

We will assume the reader is familiar the framework provided by [13], in particular we will work with the simplicial monoidal model category of S -modules \mathcal{M}_S and the associated simplicial model category of commutative S -algebras \mathcal{C}_S . Actually we will work with the p -local versions of these for some prime p , and later S will denote the p -local sphere spectrum but no essential differences occur in that setting. We will write $\iota_A: S \rightarrow A$ for the unit of a commutative S -algebra, which is taken to be part of its structure.

We choose a cofibrant replacement $S^0 \xrightarrow{\sim} S$ for S in model category \mathcal{M}_S of S -modules of [13] (for example we could use the functorial cofibrant replacement). We may consider the slice category S^0/\mathcal{M}_S of S -modules under S^0 . Every commutative S -algebra A admits a canonical morphism of S -modules

$$\begin{array}{ccccc}
 & & & & \\
 & \curvearrowright & & \curvearrowleft & \\
 S^0 & \xrightarrow{\sim} & S & \xrightarrow{\iota_A} & A
 \end{array}$$

making it an object of S^0/\mathcal{M}_S . This gives rise to a functor

$$\tilde{\mathcal{U}}: \mathcal{C}_S \rightarrow S^0/\mathcal{M}_S$$

which has a left adjoint

$$\tilde{\mathbb{P}}: S^0/\mathcal{M}_S \rightarrow \mathcal{C}_S,$$

the reduced free functor [1, section 5] which gives a Quillen adjunction

$$\mathcal{C}_S \begin{array}{c} \xleftarrow{\tilde{\mathbb{P}}} \\ \xrightarrow{\tilde{\mathbb{U}}} \end{array} S^0/\mathcal{M}_S$$

In S^0/\mathcal{M}_S , coproducts are defined using pushouts in \mathcal{M}_S , and we use the symbol \bigvee^{S^0} to indicate such coproducts. So for $X, Y \in S^0/\mathcal{M}_S$, $X \bigvee^{S^0} Y$ is the pushout of $X \leftarrow S^0 \rightarrow Y$ in \mathcal{M}_S , and since the reduced free algebra functor $\tilde{\mathbb{P}}: \mathcal{M}_S \rightarrow \mathcal{C}_S$ is a left adjoint it preserves coproducts, hence

$$\tilde{\mathbb{P}}(X \bigvee^{S^0} Y) \cong \tilde{\mathbb{P}}X \wedge \tilde{\mathbb{P}}Y.$$

In the p -local connective setting, we will use ideas on minimal atomic S -modules and commutative S -algebras which may be found in [5, 6, 15]. In particular, the notions of nuclear CW S -modules and commutative S -algebras will play a central rôle in our work. This depends in turn on the theory of cellular and CW objects [13] in \mathcal{M}_S and \mathcal{C}_S . When discussing cellular constructions we will refer to multiplicatively defined cell objects built in \mathcal{C}_S using the phrase \mathcal{E}_∞ cell, and reserve cell for objects built in \mathcal{M}_S . Given a cell object X in \mathcal{M}_S we will write $X^{[n]}$ for the n -skeleton, while for a cell object Y in \mathcal{C}_S we will write $Y^{(n)}$. Finally, given a morphism f out of a cell object we will write $f^{[n]}$ or $f^{(n)}$ for the restriction to the n -skeleton.

Following the Referee's suggestion, we summarise briefly some of the key ideas about minimal atomic S -modules and commutative S -algebras developed in [5, 6]. In the following we switch viewpoints and regard spectra as S -modules.

First we recall from [6, section 1] that p -local connective S -module X with $\pi_0(X)$ non-trivial and cyclic is *minimal atomic* if any map $f: Y \rightarrow X$ for which $\pi_0(f) \otimes \mathbb{F}_p$ and $\pi_n(f)$ ($n \geq 0$) are monomorphisms is a weak equivalence. Then X is minimal atomic if and only if the Hurewicz homomorphism $h: \pi_*(X) \rightarrow H_*(X; \mathbb{F}_p)$ is trivial in all positive degrees. Notice that this characterisation does not depend on a choice of CW structure on X ; another useful way to characterise minimal atomic spectra is in terms of a special kind of CW structure. Following [6, section 2], a CW S -module Z is *nuclear* if $Z^{[0]} = S^0$ and for each $n \geq 1$, the attaching map of the $(n+1)$ -cells

$$f_n: \bigvee_i S^n \rightarrow Z^{[n]}$$

satisfies

$$\ker \pi_n(f_n) \subseteq p \pi_n \left(\bigvee_i S^n \right).$$

Then X is minimal atomic if and only if there is a weak equivalence $f: Z \rightarrow X$ where Z is a nuclear CW S -module.

Following [5, section 3], a p -local connective commutative S -algebra A with $\pi_0(A)$ non-trivial and cyclic is *minimal atomic* if given any morphism of commutative S -algebras $g: B \rightarrow A$ for which $\pi_0(g) \otimes \mathbb{F}_p$ and $\pi_n(g)$ ($n \geq 0$) are monomorphisms is a weak equivalence. Then A is minimal atomic if and only if the TAQ-Hurewicz homomorphism (induced by the universal derivation $A \rightarrow \Omega_S(A)$) $\text{taq}: \pi_*(X) \rightarrow \text{TAQ}_*(A, S; \mathbb{F}_p)$ is trivial in all positive degrees. There is also a notion of nuclear commutative S -algebra defined in terms of attaching \mathcal{E}_∞ cells. A CW

commutative S -algebra C is *nuclear* if $C^{(0)} = S$ and for each $n \geq 1$, the attaching map of its $(n + 1)$ -dimensional \mathcal{E}_∞ cells is induced from a map

$$g_n: \bigvee_i S^n \rightarrow C^{(n)}$$

by passing to the induced morphism of commutative S -algebras $\mathbb{P}\left(\bigvee_i S^n\right) \rightarrow C^{(n)}$ where

$$\ker \pi_n(g_n) \subseteq p \pi_n\left(\bigvee_i S^n\right).$$

Then A is minimal atomic if and only if there is a weak equivalence $b: C \rightarrow X$ where C is a nuclear CW commutative S -algebra.

3. CHARACTERISTICS OF CONNECTIVE p -LOCAL COMMUTATIVE S -ALGEBRAS

Let $\iota_R: S \rightarrow R$ be such a connective commutative S -algebra. We remark that if we started with R being non-connective then we could replace it with its connective cover in our discussion below, so we do not lose anything by assuming connectivity. The induced ring homomorphism $(\iota_R)_*: \pi_0(S) \rightarrow \pi_0(R)$ could have a non-trivial kernel, and also might not be surjective. If we focus on $\pi_0(-)$ it might seem reasonable to define the characteristic of R to be this kernel. However, this neglects the kernel of $(\iota_R)_*: \pi_*(S) \rightarrow \pi_*(R)$ in positive degrees. So another definition might be the (graded) kernel of $(\iota_R)_*$. These definitions are closely wedded to the algebra, and instead we propose a different approach which makes the characteristic a commutative S -algebra equipped with a morphism into R .

Conventions: For ease of notation and other simplifications, from now on we work with connective p -local commutative S -algebras A for some fixed rational prime $p > 0$. Thus S is to be interpreted as the p -local sphere spectrum, and we will assume that $\pi_0(A)$ is a cyclic $\mathbb{Z}_{(p)}$ -module. The use of *finite-type* is always in the p -local context of p -local cells or $\mathbb{Z}_{(p)}$ -modules.

Let R be a connective p -local commutative S -algebra.

Definition 3.1. A *characteristic morphism* of R is a morphism of commutative S -algebras $j: T \rightarrow R$ where T is a finite-type CW commutative S -algebra, where the \mathcal{E}_∞ skeleta of T are defined inductively using maps of the form

$$\begin{array}{ccc} \bigvee_i S^n & \xrightarrow{\quad f^n \quad} & T^{(n)} \\ & \searrow \quad \quad \quad \nearrow & \\ & S & \xrightarrow{\quad \iota_{T^{(n)}} \quad} & T^{(n)} \end{array}$$

factoring through the unit of $T^{(n)}$, and which satisfy the conditions

$$(3.1a) \quad \ker[j_*^n: \pi_n(\bigvee_i S^n) \rightarrow \pi_n(T^{(n)})] \subseteq p \pi_n(\bigvee_i S^n),$$

$$(3.1b) \quad \text{im } f_*^n = \text{im}[(\iota_{T^{(n)}})_*: \pi_n(S) \rightarrow \pi_n(T^{(n)})] \cap \ker[j_*^{(n)}: \pi_n(T^{(n)}) \rightarrow \pi_n(R)].$$

The domain of any characteristic morphism is called a *characteristic* for R .

Note: Condition (3.1a) says that the CW structure on T is *nuclear*, hence T is also *minimal atomic*. Further properties of such commutative S -algebras are discussed in [5, section 3].

Of course this definition begs the question of whether such characteristics exist and also whether or not they are in any sense unique. Notice also that the attaching maps of \mathcal{E}_∞ cells all originate as maps into the sphere spectrum S .

Lemma 3.2. *Let R be a connective p -local commutative S -algebra.*

- (a) *Characteristics for R exist.*
- (b) *Suppose that $f: R \rightarrow R'$ is a morphism of commutative S -algebras and that $j: T \rightarrow R$ and $j': T' \rightarrow R$ are characteristic morphisms. Then there is a morphism of commutative S -algebras $T \rightarrow T'$.*
- (c) *Suppose that T_1 and T_2 are two characteristics for R . Then there is a homotopy equivalence of commutative S -algebras $T_1 \xrightarrow{\simeq} T_2$. Therefore characteristics are unique up to homotopy equivalence.*

Proof. We will make use of the notation in Definition 3.1.

(a) We can build the skeleta of a nuclear CW commutative S -algebra inductively making sure that conditions of (3.1) are satisfied. In detail, assuming the n -skeleton $T^{(n)}$ as been constructed, consider the epimorphism

$$\begin{array}{c} \iota_{T^{(n)}}^{-1} \left(\text{im}[(\iota_{T^{(n)}})_* : \pi_n(S) \rightarrow \pi_n(T^{(n)})] \cap \ker[j_*^{(n)} : \pi_n(T^{(n)}) \rightarrow \pi_n(R)] \right) \subseteq \pi_n(S) \\ \downarrow \\ \text{im}[(\iota_{T^{(n)}})_* : \pi_n(S) \rightarrow \pi_n(T^{(n)})] \cap \ker[j_*^{(n)} : \pi_n(T^{(n)}) \rightarrow \pi_n(R)] \subseteq \pi_n(T^{(n)}) \end{array}$$

and after choosing a minimal set of generators for the codomain, lift them to elements of the domain. These can be used to form a suitable composition

$$f^n: \bigvee_i S^n \longrightarrow S \xrightarrow{\iota_{T^{(n)}}} T^{(n)}$$

satisfying (3.1). The $(n+1)$ -skeleton $T^{(n+1)}$ is defined by the following pushout diagram in \mathcal{C}_S ,

$$\begin{array}{ccc} \mathbb{P}(\bigvee_i S^n) & \longrightarrow & \mathbb{P}(\bigvee_i D^{n+1}) \\ \widetilde{f}^n \downarrow & \lrcorner & \downarrow \\ T^{(n)} & \longrightarrow & T^{(n+1)} \end{array}$$

where \widetilde{f}^n is induced using the freeness of the functor $\mathbb{P} = \mathbb{P}_S: \mathcal{M}_S \rightarrow \mathcal{C}_S$. The existing morphism $j^{(n)}: T^{(n)} \rightarrow R$ extends to a morphism $j^{(n+1)}: T^{(n+1)} \rightarrow R$.

(b) We will inductively build compatible morphisms of commutative S -algebras $g^n: T^{(n)} \rightarrow T'$.

Assume that for some $n \geq 0$, we have a morphism of commutative S -algebras $g^n: T^{\langle n \rangle} \rightarrow T'$ making the following diagram of solid arrows commute, where ι always denotes a suitable unit.

$$(3.2) \quad \begin{array}{ccccc} & & & T^{\langle n \rangle} & \xrightarrow{j^{\langle n \rangle}} & R \\ & \dashrightarrow & f^n \dashrightarrow & \nearrow \iota & & \downarrow f \\ \bigvee_i S^n & \dashrightarrow & S & \xrightarrow{g^n} & T' & \xrightarrow{j'} & R' \\ & & \searrow \iota & \downarrow & & & \\ & & & T' & \xrightarrow{j'} & R' \end{array}$$

Note that we are not asserting the the right hand square with dotted edge commutes, however the adjacent triangle does. The dashed arrows represent the factorisation of the attaching map f^n of the \mathcal{E}_∞ $(n + 1)$ -cells of T and the diagram of solid and dashed arrows commutes. By construction, $j^{\langle n \rangle} \circ f^n$ is null homotopic, hence so is the lower composition

$$\bigvee_i S^n \longrightarrow S \xrightarrow{\iota} T' \xrightarrow{j'} R'.$$

Therefore the image of the induced group homomorphism

$$\pi_n(\bigvee_i S^n) \rightarrow \pi_n(T')$$

is contained in $\ker[(j')_*: \pi_n(T') \rightarrow \pi_n(R)]$. It follows that there is an extension of g^n to a morphism of commutative S -algebras $g^{n+1}: T^{\langle n+1 \rangle} \rightarrow T'$. By induction on n and passing to the colimit, we obtain a morphism $g: T \rightarrow T'$.

(c) We make use of the fact that nuclear complexes are (minimal) atomic; see [6, proposition 2.3] and [5, theorem 3.4] for the multiplicative case. Using this, it is enough to construct morphisms of commutative S -algebras

$$\begin{array}{ccc} & g_1 & \\ T_1 & \xrightarrow{\quad} & T_2 \\ & g_2 & \end{array}$$

by applying (b) to the identity morphism $R \rightarrow R$. Since the compositions $g_2 \circ g_1$ and $g_1 \circ g_2$ are weak equivalences and therefore homotopy equivalences by Whitehead's Theorem, therefore so is each g_r . \square

Lemma 3.3. *Let $g: R \rightarrow R'$ be a weak equivalence of connective p -local commutative S -algebras and let $k: T \rightarrow R'$ be a characteristic morphism. Then there is a morphism of commutative S -algebras $j: T \rightarrow R$ such that $g \circ j \simeq k$ and j is a characteristic morphism for R .*

Proof. We can make use of the simplicial structure of \mathcal{C}_S to form cylinder objects; for an algebra $A \in \mathcal{C}_S$ the cylinder $A \otimes I$ is the domain for homotopies between morphisms. We also require

a version of Peter May's HELP in \mathcal{C}_S , see [13]: Given a commutative diagram of solid arrows

$$\begin{array}{ccccc}
C^{(n)} & \xrightarrow{i_1} & C^{(n)} \otimes I & \xleftarrow{i_0} & C^{(n)} \\
\downarrow & \searrow & \downarrow & \searrow & \downarrow \\
& & A & \xrightarrow{\sim} & B \\
& \swarrow & \downarrow & \swarrow & \downarrow \\
C^{(n+1)} & \xrightarrow{i_1} & C^{(n+1)} \otimes I & \xleftarrow{i_0} & C^{(n+1)}
\end{array}$$

there is an extension to the larger commutative diagram with dotted arrows. Here C is a connective CW algebra and $i_0, i_1: X \rightarrow X \otimes I$ are the two morphisms corresponding to the ends of the cylinder on X ; the vertical morphisms involve inclusions of skeleta.

We apply HELP to give the inductive step in constructing a morphism $T \rightarrow R$. Assume that we have a suitable morphism $j^{(n)}: T^{(n)} \rightarrow R$ so that $g \circ j^{(n)} \simeq k^{(n)}$. Given a homotopy $h_n: T^{(n)} \otimes I \rightarrow R'$ with $h_n \circ i_0 = k^{(n)}$ and $h_n \circ i_1 = g \circ j^{(n)}$, there is a commutative diagram of solid arrows

$$\begin{array}{ccccc}
T^{(n)} & \xrightarrow{i_1} & T^{(n)} \otimes I & \xleftarrow{i_0} & T^{(n)} \\
\downarrow & \searrow^{j^{(n)}} & \downarrow & \searrow^{h_n} & \downarrow \\
& & R & \xrightarrow{g} & R' \\
& \swarrow^{j^{(n+1)}} & \downarrow & \swarrow^{h_{n+1}} & \downarrow \\
T^{(n+1)} & \xrightarrow{i_1} & T^{(n+1)} \otimes I & \xleftarrow{i_0} & T^{(n+1)}
\end{array}$$

and by HELP an extension to a larger diagram exists. The induction is grounded in the case $n = 0$ where $T^{(0)} = S$. \square

Here is a summary of our results which provide substitutes for the functoriality and homotopy invariance conditions of Section 1.

Theorem 3.4. *Let \mathcal{C}_S be the category of p -local commutative S -algebras.*

- (a) *Every connective object $R \in \mathcal{C}_S$ has a characteristic $\text{char } R$ which is well-defined up to homotopy equivalence in \mathcal{C}_S .*
- (b) *Given a morphism of commutative S -algebras $R \rightarrow R'$, there is a morphism $\text{char } R \rightarrow \text{char } R'$.*
- (c) *Given a weak equivalence $g: R \rightarrow R'$, there are characteristic morphisms $j: T \rightarrow R$ and $k: T \rightarrow R'$ which fit into the following homotopy commutative diagram.*

$$\begin{array}{ccc}
& T & \\
j \swarrow & & \searrow k \\
R & \xrightarrow{g} & R'
\end{array}$$

The next result shows that our notion of characteristic really only depends on the kernel of the induced algebraic unit; in particular all commutative S -algebras with torsion free homotopy have the same characteristics.

Proposition 3.5. *Suppose that $\iota_R: S \rightarrow R$ and $\iota_{R'}: S \rightarrow R'$ are two commutative S -algebras so that $\ker(\iota_R)_* \subseteq \ker(\iota_{R'})_* \subseteq \pi_*(S)$. Then there is a morphism of commutative S -algebras $\text{char } R \rightarrow \text{char } R'$.*

Proof. It is straightforward to show that for $T = \text{char } R$ and $T' = \text{char } R'$, given a morphism $T^{(n)} \rightarrow T'$ on the n -skeleton, there is an extension to the $(n+1)$ -skeleton, giving a commutative diagram

$$\begin{array}{ccc}
 & S & \\
 \swarrow & & \searrow \\
 T^{(n)} & \xrightarrow{\quad} & T^{(n+1)} \\
 \searrow & & \swarrow \\
 & T' &
 \end{array}$$

and by induction on n , it follows that a morphism $T \rightarrow T'$ exists. \square

Corollary 3.6. *Suppose that $\iota_R: S \rightarrow R$ and $\iota_{R'}: S \rightarrow R'$ are two commutative S -algebras for which $\ker(\iota_R)_* = \ker(\iota_{R'})_* \subseteq \pi_*(S)$. Then there is a homotopy equivalence of commutative S -algebras $\text{char } R \xrightarrow{\cong} \text{char } R'$.*

Proof. By the Proposition there are morphisms $\text{char } R \rightarrow \text{char } R' \rightarrow \text{char } R$ whose compositions are weak equivalences since $\text{char } R$ and $\text{char } R'$ are both (minimal) atomic. \square

We end this discussion with an observation that relates the notion of a characteristic to that of a nuclear S -module. Let R be a connective p -local commutative S -algebra. Then as described in [6], beginning with the 0-cell S^0 we can construct a nuclear CW complex X by attaching cells only to the bottom cell, and a map $X \rightarrow R$ which induces a monomorphism on $\pi_*(-)$ and $\mathbb{F}_p \otimes \pi_0(-)$ (i.e., a core for R). There is a unique extension to a morphism of commutative S -algebras $\tilde{\mathbb{P}}X \rightarrow R$.

Proposition 3.7. *If $\tilde{\mathbb{P}}X$ is a minimal atomic commutative S -algebra then the morphism $\tilde{\mathbb{P}}X \rightarrow R$ described above is a characteristic for R .*

Proof. Suppose that $R_0 \rightarrow R$ is a characteristic morphism for R . Then it is straightforward to construct a map $X \rightarrow R_0$ under S^0 , and this has a unique extension to a morphism of commutative S -algebras $\tilde{\mathbb{P}}X \rightarrow R_0$. Another cellular argument constructs a morphism of commutative S -algebras $R_0 \rightarrow \tilde{\mathbb{P}}X$. If $\tilde{\mathbb{P}}X$ is minimal atomic then the two composite endomorphisms are weak equivalences and so each morphism is also a weak equivalence. \square

The reason for the conditional statement here is that in general for a CW complex $S^0 \rightarrow Y$, while $\tilde{\mathbb{P}}Y$ being minimal atomic implies Y is minimal atomic, the converse need not be true. An example for $p = 2$ is provided by $Y = \Sigma^{-2}\Sigma^\infty\mathbb{C}P^\infty$, see [9]. In the examples we will consider later, this condition will in fact be satisfied.

4. EXAMPLES

4.1. Prime power characteristics. We begin with a generalisation of examples discussed in [17]. For a prime p and $r \geq 1$, we may form $S//p^r$ as the pushout of the solid square in

$$\begin{array}{ccc}
 \mathbb{P}S^0 & \xrightarrow{\quad} & \mathbb{P}D^1 \\
 \swarrow \tilde{p}^r & \downarrow & \downarrow \\
 S & \xrightarrow{\quad} & S//p^r \\
 \nwarrow \sim & \uparrow & \\
 \tilde{S} & \xrightarrow{\quad} & S//p^r
 \end{array}$$

where \tilde{p}^r is the multiplicative extension of a map $S^0 \rightarrow S$ of degree p^r and the left-hand triangle is the functorial cofibration/acyclic fibration factorisation of \tilde{p}^r . There is also a homotopy equivalence of commutative S -algebras

$$\tilde{\mathbb{F}}C_{p^r} \xrightarrow{\simeq} S//p^r$$

where C_{p^r} is the mapping cone of a map $S^0 \rightarrow S^0$ of degree p^r viewed as an object of S^0/\mathcal{M}_S . Notice that $\pi_0(S//p^r) = \mathbb{Z}/p^r$, so this ring has characteristic p^r . Furthermore, there is a (non-unique) morphism of commutative S -algebras $S//p^r \rightarrow H\mathbb{Z}/p^r$.

When $r = 1$, Steinberger's splitting result [11, theorem III.4.1] (see also [3, section 10] for a more recent approach) implies that the spectrum $S//p^r$ splits as a wedge of suspensions of the Eilenberg-Mac Lane spectrum $H\mathbb{F}_p = H\mathbb{Z}/p$. This means that the unit induces a ring homomorphism $\pi_*(S) \rightarrow \pi_*(S//p)$ whose kernel contains p and all positive degree elements of the domain. This shows that $S//p$ is a characteristic of $H\mathbb{F}_p$. More generally, we have the following.

Lemma 4.1. *Let R be a p -local commutative S -algebra for which the ring $\pi_0(R)$ has characteristic p . Then $S//p$ is a characteristic of R and R is a wedge of suspensions of $H\mathbb{F}_p$.*

Proof. The unit map $S \rightarrow R$ factors through the mapping cone C_p so there is an associated morphism of commutative S -algebras $S//p \rightarrow R$ which also factors the unit. By the above discussion, the kernel of $\pi_*(S//p) \rightarrow \pi_*(R)$ contains p and all positive degree elements. Furthermore, $\pi_*(R)$ is a graded \mathbb{F}_p -vector space and a standard argument shows that given a basis b_i ($i \in I$) there is a weak equivalence

$$R \sim \bigvee_{i \in I} \Sigma^{|b_i|} H\mathbb{F}_p. \quad \square$$

When $r > 1$, we still have $\pi_0(S//p^r) = \mathbb{Z}/p^r$, but the splitting result of [11, theorem III.4.2] requires further conditions and does not imply a splitting of $S//p^r$. When p is odd, the element $\alpha_1 \in \pi_{2p-3}(S)$ survives in $\pi_{2p-3}(S//p^r)$, and we can attach an \mathcal{E}_∞ cell to kill its image, giving a commutative S -algebra $S//(p^r, \alpha_1)$ for which

$$\pi_{2p-2}(S//(p^r, \alpha_1)) = \mathbb{Z}_{(p)}u'_1,$$

where the new cell gives an integral homology class $x'_{2p-2} \in H_{2p-2}(S//(p^r, \alpha_1))$ so that the Hurewicz image of u'_1 is px'_{2p-2} . In order to obtain a commutative S -algebra satisfying the condition that 1 is in the image of $\beta\mathcal{P}_*^1$ acting on $H_{2p-1}(-)$, we need to attach an \mathcal{E}_∞ cell of dimension $2p - 1$ to kill u'_1 , giving $S//(p^r, \alpha_1, u'_1)$ which does split as a wedge of suspensions of the Eilenberg-Mac Lane spectra $H\mathbb{Z}/p^s$ for $0 \leq s \leq r$. It is tempting to state the following.

Conjecture 4.2. For an odd prime p and $r > 1$, $S//(p^r, \alpha_1)$ is a characteristic for $H\mathbb{Z}/p^r$.

When $p = 2$, we have a similar situation with η in place of α_1 . Then $1 \in H_0(S//(2^r, \eta, u'_1))$ is in the image of Sq^3 acting on $H_3(S//(2^r, \eta, u'_1))$, so $S//(2^r, \eta, u'_1)$ splits as a wedge of suspensions of the Eilenberg-Mac Lane spectra $H\mathbb{Z}/2^s$ for $0 \leq s \leq r$. Again we are led to make a conjecture.

Conjecture 4.3. For $r > 1$, $S//(2^r, \eta)$ is a characteristic for $H\mathbb{Z}/2^r$.

The reader is warned that we have no direct evidence for this and the discussion of Section 4.3 suggests that it may be far too optimistic.

4.2. Odd-primary examples associated with Hopf invariant one elements. Let p be an odd prime. Then we can kill the element α_1 to form $S//\alpha_1$ which has an \mathcal{E}_∞ morphism $S//\alpha_1 \rightarrow \ell$ to the connective cover of the Adams summand of $KU_{(p)}$ which is known to possess an essentially unique \mathcal{E}_∞ structure by results of [7, 8]. Then

$$\begin{aligned}\pi_{2p-2}(S//\alpha_1) &= \mathbb{Z}_{(p)}u_1, \\ H_{2p-2}(S//\alpha_1; \mathbb{Z}_{(p)}) &= \mathbb{Z}_{(p)}x_{2p-2},\end{aligned}$$

where the Hurewicz image of u_1 is px_{2p-2} .

As in the discussion for $S//p^r$, α_1 , u'_1 , we can apply Steiberger's splitting results to show that $S//\alpha_1, u_1$ splits as a wedge of $H\mathbb{Z}_{(p)}$ and suspensions of Eilenberg-Mac Lane spectra $H\mathbb{Z}_{p^r}$ for various $r \geq 1$.

Conjecture 4.4. For an odd prime p and $r > 1$, $S//\alpha_1$ is a characteristic for $H\mathbb{Z}_{(p)}$.

4.3. 2-primary examples associated with Hopf invariant one elements. In this section we take $p = 2$ and consider the four elements $2 \in \pi_0(S)$, $\eta \in \pi_1(S)$, $\nu \in \pi_3(S)$ and $\sigma \in \pi_3(S)$ of Hopf invariant 1. We also set $H = H\mathbb{F}_2$ and denote the mod 2 dual Steenrod algebra by $\mathcal{A}_* = \mathcal{A}(2)_*$ and the Steenrod algebra by $\mathcal{A}^* = \mathcal{A}(2)^*$. By results of [2], the mod 2 homology of $S//\eta$, $S//\nu$ and $S//\sigma$ are all polynomial on admissible Dyer-Lashof monomials on generators $x_1 \in H_1(S//2)$, $x_2 \in H_2(S//\eta)$, $x_4 \in H_4(S//\nu)$ and $x_8 \in H_4(S//\sigma)$:

$$\begin{aligned}H_*(S//2) &= \mathbb{F}_2[Q^I x_1 : \text{exc}(I) > 1], & H_*(S//\eta) &= \mathbb{F}_2[Q^I x_2 : \text{exc}(I) > 2], \\ H_*(S//\nu) &= \mathbb{F}_2[Q^I x_4 : \text{exc}(I) > 4], & H_*(S//\sigma) &= \mathbb{F}_2[Q^I x_8 : \text{exc}(I) > 8].\end{aligned}$$

The \mathcal{A}_* -coaction on the generator x_{2^d} is given by

$$\psi(x_{2^d}) = \zeta_1^{2^d} \otimes 1 + 1 \otimes x_{2^d},$$

and the coaction on a generator $Q^I x_2$ can be found using formulae in [3], at least in principal. Dually, the Steenrod action satisfies

$$\text{Sq}_*^{2^d}(x_{2^d}) = 1,$$

and we also have

$$\text{Sq}_*^{2^{d+k}}(x_{2^d}^{2^k}) = 1.$$

This suggests that ν and σ might map to zero in $\pi_*(S//\eta)$, at least modulo Adams filtration 2. We will examine this in detail later.

There are \mathcal{E}_∞ morphisms

$$\begin{array}{ccccccc} S//2 & & S//\eta & & S//\nu & & S//\sigma \\ & \searrow & \searrow & & \swarrow & & \swarrow \\ & & & H & & & \\ & & & 11 & & & \end{array}$$

which induce ring homomorphisms

$$\begin{array}{ccccccc}
 H_*(S//2) & & H_*(S//\eta) & & H_*(S//\nu) & & H_*(S//\sigma) \\
 & \searrow & \searrow & & \swarrow & \swarrow & \\
 & & & H_*(H) & & & \\
 & & & \parallel & & & \\
 & & & \mathcal{A}_* & & &
 \end{array}$$

which allow us to identify their images as \mathcal{A}_* -subcomodule algebras of the dual Steenrod algebra \mathcal{A}_* .

Lemma 4.5. *The above homomorphisms give epimorphisms of \mathcal{A}_* -comodule algebras*

$$\begin{array}{ll}
 H_*(S//2) \xrightarrow{\cong} \mathbb{F}_2[\zeta_s : s \geq 1] & H_*(S//\eta) \twoheadrightarrow \mathbb{F}_2[\zeta_s^2 : s \geq 1] \\
 H_*(S//\nu) \twoheadrightarrow \mathbb{F}_2[\zeta_s^4 : s \geq 1] & H_*(S//\sigma) \twoheadrightarrow \mathbb{F}_2[\zeta_s^8 : s \geq 1]
 \end{array}$$

onto subalgebras of \mathcal{A}_* .

Proof. It is easy to see that in each case, the generator x_{2^d} maps to $\zeta_1^{2^d}$. It follows that

$$Q^{(i_1, \dots, i_\ell)} x_{2^d} \mapsto \begin{cases} (Q^{(i_1/2^d, \dots, i_\ell/2^d)} \zeta_1)^{2^d} & \text{if every } i_r \text{ is divisible by } 2^d, \\ 0 & \text{otherwise,} \end{cases}$$

hence the image is a subring of $\mathbb{F}_2[\zeta_s^{2^d} : s \geq 1]$. Making use of Steinberger's determination of the Dyer-Lashof action on \mathcal{A}_* we see that the image contains all of the elements $\zeta_s^{2^d}$, therefore the image is exactly this subring of 2^d -th powers. \square

There is a commutative diagram of \mathcal{E}_∞ morphisms as indicated by solid arrows

$$(4.1) \quad \begin{array}{ccccccc}
 S//2 & \longrightarrow & MO & & & & \\
 \uparrow \text{dotted} & & \uparrow & & & & \\
 S//\eta & \longrightarrow & MU & \longrightarrow & kU & & \\
 \uparrow \text{dotted} & & \uparrow & & \uparrow & \searrow & \\
 S//\nu & \longrightarrow & MSp & \longrightarrow & kO & \longrightarrow & H\mathbb{Z} \longrightarrow H\mathbb{F}_2 \\
 \uparrow \text{dotted} & & \uparrow & & \uparrow & \nearrow & \\
 S//\sigma & \longrightarrow & MO\langle 8 \rangle & \longrightarrow & tmf & &
 \end{array}$$

where the left-most horizontal morphisms exist because the bottom cells of the Thom spectra support non-trivial actions of Steenrod operations of the form Sq^{2^d} by the Wu formulae. We will consider the possible existence of suitable morphisms corresponding to the vertical dotted arrows.

Steinberger's work shows that there is a splitting of $S//2$ into a wedge of suspensions of $H = H\mathbb{F}_2$, hence the ring homomorphism $\pi_*(S) \rightarrow \pi_*(S//2)$ induced by the unit is trivial in positive degrees. It follows that there are \mathcal{E}_∞ morphisms to $S//2$ from each of $S//\eta$, $S//\nu$ and $S//\sigma$. In each case, under the induced homomorphism in homology, $x_{2^d} \mapsto \zeta_1^{2^d}$. For the case of $S//\eta$ we can also deduce that such a map exists using the fact that the inclusion of the bottom cell induces an isomorphism

$$\pi_1(S^0) \xrightarrow{\cong} \pi_1(C_2),$$

so in $H_*(S//2)$ the 2-cell is attached to the bottom cell by η since $\text{Sq}^2(x_1^2) = 1$.

Lemma 4.6. *Under the ring homomorphism $\pi_*(S) \rightarrow \pi_*(S//\eta)$ induced by the unit $S \rightarrow S//\eta$, $\nu \mapsto 0$, hence there is an \mathcal{E}_∞ morphism $S//\nu \rightarrow S//\eta$ inducing a ring homomorphism $H_*(S//\nu) \rightarrow H_*(S//\eta)$ under which $x_4 \mapsto x_2^2$.*

Proof. Recall that there is a homotopy equivalence of \mathcal{E}_∞ ring spectra

$$\tilde{\mathbb{P}}C_\eta \simeq S//\eta.$$

The long exact sequence for the homotopy of the mapping cone C_η has an exact portion

$$\begin{array}{ccccccc} \pi_3(S^1) & \xrightarrow{\eta} & \pi_3(S^0) & \longrightarrow & \pi_3(C_\eta) & \longrightarrow & \pi_2(S^1) \xrightarrow{\eta} \pi_2(S^0) \\ \cong \downarrow & & \parallel & & & & \cong \downarrow \parallel \\ \pi_2(S^0) & & \mathbb{Z}/8\nu & & & & \pi_1(S^0) \parallel \\ \parallel & & \parallel & & & & \parallel \\ \mathbb{Z}/2\eta^2 & & \mathbb{Z}/8\nu & & & & \mathbb{Z}/2\eta \parallel \\ & & & & & & \parallel \\ & & & & & & \mathbb{Z}/2\eta^2 \\ \eta^2 \longmapsto & \eta^3 = 4\nu & & & \eta \longmapsto & \eta^2 & \end{array}$$

showing that

$$\pi_3(S^0)/\eta^3 = \pi_3(S^0)/4\nu \xrightarrow{\cong} \pi_3(C_\eta).$$

In $H_*(S//\eta)$ we have $\text{Sq}^4(x_2^2) = 1$. If we realise $S//\eta$ as a minimal CW S -module, its 4-skeleton has one cell in each of the dimensions 0, 2 and 4. The attaching map of the 4-cell to the 2-skeleton is detected by Sq^4 , therefore it must be one of the generators of $\pi_3(C_\eta)$ and so homotopic to a map which factors through S^0 where it agrees with $\pm\nu \bmod 4$. Thus there is a map $C_\nu \rightarrow S//\eta$ which extends to a morphism of \mathcal{E}_∞ ring spectra

$$\tilde{\mathbb{P}}C_\nu \xrightarrow{\cong} S//\nu \rightarrow S//\eta,$$

inducing the stated homomorphism in homology. □

The situation for $S//\nu$ and $S//\sigma$ is more involved.

Lemma 4.7. *There is no morphism of \mathcal{E}_∞ ring spectra $S//\sigma \rightarrow S//\nu$ for which the induced homology homomorphism sends x_8 to x_4^2 .*

Proof. The mapping cone of ν has an associated long exact sequence

$$\begin{array}{ccccccc}
 \pi_7(S^3) & \xrightarrow{\nu} & \pi_7(S^0) & \twoheadrightarrow & \pi_7(C_\nu) & \longrightarrow & \pi_7(S^4) \xrightarrow{\nu} \pi_7(S^1) \\
 \cong \downarrow & & \parallel & & & & \cong \downarrow & \cong \downarrow \\
 \pi_4(S^0) & & \parallel & & & & \pi_3(S^0) & \pi_6(S^0) \\
 \parallel & & \parallel & & & & \parallel & \parallel \\
 0 & & \mathbb{Z}/16\sigma & & & & \mathbb{Z}/8\nu & \mathbb{Z}/2\nu^2
 \end{array}$$

where the element $2\nu \in \pi_3(S^0)$ has Adams filtration 2. It follows that

$$\pi_7(C_\nu) \cong \pi_7(S^0) \oplus 2\pi_7(S^4) \cong \mathbb{Z}/16\sigma \oplus \mathbb{Z}/4\widetilde{2\nu}.$$

In $H_*(S//\nu)$ we have $Sq^8(x_4^2) = 1$, but it does not follow that the attaching map of the 8-cell to the 4-skeleton C_ν can be taken to be σ . In fact, the attaching map can be seen to be $\sigma + \widetilde{2\nu}$ since the natural symmetrisation map

$$C_\nu \wedge C_\nu \rightarrow E\Sigma_2 \times_{\Sigma_2} (C_\nu \wedge C_\nu)$$

induces the fold map on the 4-cells and a careful analysis in integral homology shows that the attaching map is as claimed (I learnt this argument from Peter Eccles; see Figure 4.1). The upshot is that there is no morphism of \mathcal{E}_∞ ring spectra $S//\sigma \rightarrow S//\nu$ since there can be no map $C_\sigma \rightarrow S//\nu$ extending the unit. \square

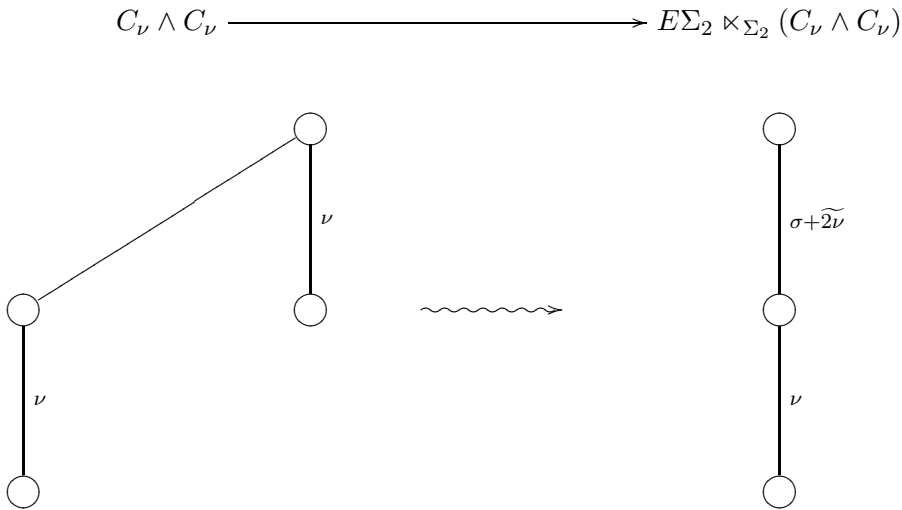


FIGURE 4.1.

It follows that the mapping cone of the composition

$$\begin{array}{ccccc}
 & & \nu + \sigma & & \\
 & & \curvearrowright & & \\
 S^3 \vee S^7 & \xrightarrow{\nu \vee \sigma} & S^0 \vee S^0 & \xrightarrow{\text{fold}} & S^0
 \end{array}$$

gives rise to the minimal atomic commutative S -algebra

$$S//(\nu, \sigma) = \widetilde{\mathbb{P}}C_{\nu+\sigma}.$$

Then there are \mathcal{E}_∞ morphisms

$$S//\nu \rightarrow S//(\nu, \sigma) \rightarrow MSp \rightarrow kO$$

and in fact $S//(\nu, \sigma) \rightarrow MSp$ is an 8-equivalence in the classical sense. By [5, proposition 5.1], all of these are minimal atomic spectra.

For $S//\sigma$, there is a morphism of commutative S -algebras $S//\sigma \rightarrow tmf$ and we expect this to be a characteristic. There is a morphism $S//\sigma \rightarrow H\mathbb{F}_2$ inducing a ring homomorphism

$$H_*(S//\sigma) \rightarrow H_*(H\mathbb{F}_2) = \mathcal{A}_*$$

whose image consists of the eighth powers.

The following result is even more challenging to verify.

Lemma 4.8. *There is no morphism of \mathcal{E}_∞ ring spectra $S//\sigma \rightarrow S//\eta$ for which the induced homology homomorphism sends x_8 to x_2^4 .*

Proof. The point is that the image of σ in $\pi_*(S//\eta)$ is non-zero. This was shown in an unpublished calculation by John Rognes (private communication 2013). This image has order 4 (instead of 16), and in the Adams spectral sequence, it has filtration 2 rather than 1. \square

Conjecture 4.9. Each of the following is a characteristic, where the maps are compositions of those mentioned above with standard ones.

$$\begin{array}{ccccc}
 & & S//(\eta, \sigma) & & \\
 & \swarrow & \downarrow & \searrow & \\
 MU & \longrightarrow & kU & \longrightarrow & H\mathbb{Z}_{(2)} \\
 & & & & \\
 & & S//(\nu, \sigma) & & \\
 & \swarrow & & \searrow & \\
 MSp & \longrightarrow & & \longrightarrow & kO \\
 & & & & \\
 & & S//\sigma \rightarrow tmf & &
 \end{array}$$

Let us consider what the statements in this conjecture really amount to. The first is equivalent to the unit homomorphism $\pi_*(S) \rightarrow \pi_*(S//(\eta, \sigma))$ being trivial in positive degrees. The second statement is equivalent to the equalities

$$\ker[\pi_*(S) \rightarrow \pi_*(S//\nu, \sigma)] = \ker[\pi_*(S) \rightarrow \pi_*(kO)] = \ker[\pi_*(S) \rightarrow \pi_*(MSp)],$$

where the second equality was proved by Stan Kochman [16, part I], but the first would also imply it. The third statement is equivalent to the equality

$$\ker[\pi_*(S) \rightarrow \pi_*(S//\sigma)] = \ker[\pi_*(S) \rightarrow \pi_*(tmf)].$$

To end, we mention some further results on these spectra.

Proposition 4.10. *The 2-local morphisms $S//\eta \rightarrow kU$ and $S//\nu \rightarrow kO$ induce epimorphisms on $\pi_*(-)$.*

Proof. For $S//\eta \rightarrow kU$, it suffices to show that it induces an epimorphism on $\pi_2(-)$. This is clear since the Toda bracket $\langle 2, \eta, 1_{S//\eta} \rangle \subseteq \pi_2(S//\eta)$ is defined, where $S//\eta$ is viewed as an S -module, and the bracket is to be taken as a function on the set

$$\pi_0(S) \times \pi_1(S) \times \pi_1(S//\eta).$$

By naturality, this maps to the Toda bracket $\langle 2, \eta, 1_{kU} \rangle \subseteq \pi_2(kU)$ which contains a generator of $\pi_2(kU)$ and has indeterminacy $2\pi_2(kU)$ as pointed out in [19, page 64].

For $S//\nu \rightarrow kO$, it is sufficient to show that the generators a, b of the groups $\pi_4(kO), \pi_4(kO)$ come from $\pi_4(S//\nu), \pi_8(S//\nu)$. We can appeal to [19, page 64] (see also [6, lemma 7.3]), where it is shown that the Toda brackets $\langle 8, \nu, 1_{kO} \rangle$ and $\langle 8, \nu, a \rangle$ contain a, b respectively. Analogues of these brackets can be defined in $\pi_*(S//\nu)$ and are preimages of the kO versions. Zhouli Xu has pointed out that the Toda brackets $\langle \eta^2, \eta, 2 \rangle$ and $\langle \eta, \eta^2, \eta, \eta^3 \rangle$ in $\pi_*(kO)$ also contain a, b , furthermore they make sense in $\pi_*(S//\nu)$; this time we use the traditional Toda brackets defined in the homotopy of a ring spectrum. \square

There are factorisations of these \mathcal{E}_∞ morphisms

$$S//\eta \rightarrow T_{kU} \rightarrow kU, \quad S//\nu \rightarrow T_{kO} \rightarrow kO,$$

where each second factor is a characteristic morphism. Hence these characteristic morphisms induce epimorphisms on $\pi_*(-)$. Motivated by these examples, we are led to make a conjecture on a characteristic morphism for tmf .

Conjecture 4.11. A 2-local characteristic morphism $T_{tmf} \rightarrow tmf$ induces an epimorphism on $\pi_*(-)$.

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