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Spacecraft formation flying has shown to be promising approach to enhance mission capabilities. Nevertheless, formation flying presents several control challenges which escalate as the numbers of elements in the formation is increased. The objective of this paper is to develop decentralised control algorithms to regulate the station-keeping, reconfiguration and collision avoidance of spacecraft in formation around eccentric reference orbits using the combination of a Linear Quadratic Regulator (LQR) and an Artificial Potential Function (APF). Within this control scheme, the LQR will provide station-keeping and reconfiguration capabilities toward desired positions, while optimizing fuel consumption and the APF will ensure collision free manoeuvres between the elements of the formation during manoeuvres. The controller is designed under the assumption of continuous thrust as a standard LQR problem using the Pontryagin minimum principle, an APF based in normalized Gaussian functions and the Tschauner and Hempel (TH) equations as the relative dynamics model.

I. INTRODUCTION

Spacecraft formation flying (SFF) has shown to be a promising approach to enhance mission capabilities\(^1\). This paradigm exhibits technological and economic benefits as well as enhancing mission reliability by distributing major tasks among the elements of the formation in orbit. The size reduction tendency in satellites – as in the case of microspacecraft – is turning this characteristic into an enabling component for future SFF missions with a large number of elements (e.g. lowering manufacture costs). Several new missions based in CubeSats are currently under development or launched. ArduSat-1 and ArduSat-X\(^2\) is a mission of two CubeSats deployed at the beginning of August 2013, running under an open source platform\(^3\), designed to provide access real-time experiments in space for general users. The United Kingdom will launch by the end of 2013 the mission UKube-1\(^4\), a miniature satellite designed to test cutting edge communication technology in space. Nevertheless, SFF presents several control challenges which escalate as the numbers of elements in the formation are increased and autonomy requirements are contemplated. Having a large numbers of elements in formation with automatic controllers requires executing complex coordinated and collective response manoeuvres, fast algorithm computations, optimal management of consumable resources and reliable collision avoidance systems. To meet all these challenges, the design of controllers based on the combination of linear quadratic regulators (LQR) and artificial potential fields (APF) is proposed in this work. Within this control scheme, the LQR will provide station-keeping and reconfiguration capabilities toward desired positions, while optimizing fuel consumption and the APF will ensure collision free manoeuvres between the elements of the formation during manoeuvres. Early research regarding the use of APF to design controllers in SFF has been developed in work like\(^5, 6, 7\) displaying interesting dynamical properties, pattern formations and stability conditions for spacecraft in formation. Also, work regarding bifurcating potential fields for swarms has been developed in\(^8\) however optimality and eccentric reference orbits are left aside. Digital optimal control schemes for elliptical reference orbits have been developed\(^9, 10, 11, 12\) for the NASA benchmark tetrahedron constellation providing fast computational performance algorithms for online applications nevertheless, the results presented do not contemplate formations with large number of satellites and collision avoidance is not considered. Moreover, a previous investigation explored the advantages of using LQR and APF applied to SFF\(^13\) although these results focused on formations with reduced number of elements, close proximity manoeuvres and circular reference orbits. The objective of this paper is to develop reliable decentralised optimal control algorithms to
regulate the station-keeping, reconfiguration and collision avoidance of large spacecraft formations around eccentric reference orbits using an LQR-APF control scheme. For this purpose, a continuous thrust optimal control scheme will be designed using the Pontryagin minimum principle, the Tshauner and Hempel equations obtained from the procedure developed in Ref. 13 and an APF based on a Gaussian function. In general, the methodology used in this work involves the solution of a tracking optimal control problem by means of the solution of the Riccati Matrix Differential Equation (RMDE) with time-varying coefficients and using these results to obtain the optimal control and state. The algorithm will be simulated in two different cases: manoeuvring of a single spacecraft into a desired position while collision is avoided and orbit transfer of multiple spacecraft.

II. RELATIVE DYNAMICS MODEL

This section presents the development of the linearized relative dynamics equations for eccentric orbits used throughout this paper. The following development\textsuperscript{13} is not exhaustive and the reader can access the full details in Ref. 14. Vector quantities are denoted by lower-case bold letters, matrices by upper-case bold letters and scalar quantities by lower-case plain letters. The position of a spacecraft in a formation in an Earth-centred inertial frame (ECI) is denoted by the equation:

\[
\mathbf{r} = \mathbf{r}_c + \mathbf{\rho}
\]  

[1]

where \( \mathbf{r}_c \) denotes the position of the centre of the formation (either a spacecraft or a reference point) and \( \mathbf{\rho} \) corresponds to the position of the relative spacecraft. With the assumption that \( |\mathbf{\rho}| \ll |\mathbf{r}_c| \) under the gravitational attraction of a main body, the equation of motion can be linearized around the formation centre to yield:

\[
\dot{\mathbf{\rho}} = -\frac{\mu}{|\mathbf{r}_c|^3} \left( \mathbf{\rho} - \frac{3 \mathbf{r}_c \cdot \mathbf{\rho}}{|\mathbf{r}_c|^2} \mathbf{r}_c \right)
\]  

[2]

The relative dynamics between spacecraft is better described in a Local-Vertical-Local-Horizontal (LVLH) reference frame denoted by \( \mathbf{L} \) - also known as the Euler-Hill Frame - with origin at the leader satellite, angular velocity \( \mathbf{\dot{\theta}} \) normal to the orbital plane and coordinates \( x, y, z \). Using kinematics, the relative acceleration in the ECI frame can be also measured in the LVLH:

\[
\ddot{\mathbf{\rho}} = [\dot{\mathbf{L}}] \mathbf{\rho} + 2 \mathbf{\dot{\mathbf{\theta}}} \times \mathbf{\rho} + \mathbf{\theta} \times (\mathbf{\dot{\mathbf{\theta}}} \times \mathbf{\rho})
\]  

[3]

It is possible to express \( \mathbf{r}_c, \mathbf{\rho} \) and \( \mathbf{\dot{\theta}} \) in LVLH coordinates (as observed in Fig. 1):

\[
\mathbf{\rho} = xi + yj + zk
\]  

[4]

\[
\mathbf{R} = \Re i
\]  

[5]

\[
\dot{\mathbf{\hat{\theta}}} = \dot{\theta}k
\]  

[6]

Fig. 1: Reference frames used in the development of the relative dynamics equations.

Combining Eqs. [2] and [3] the linearized relative dynamics in eccentric reference orbits can be written as a combination of the states \( x, y, z, R \) and \( \theta \). In addition, using the fundamental equations of planetary motion for \( R \) and \( \dot{\theta} \) the relative motion equations are:

\[
R = \frac{a(1 - e^2)}{1 + ec\cos\hat{\theta}}
\]  

[7]

\[
\dot{\hat{\theta}} = \frac{n(1 + ec\cos\hat{\theta})^2}{(1 - e^2)^{3/2}}
\]  

[8]

\[
\begin{pmatrix}
\dot{x} \\
\dot{y} \\
\dot{z}
\end{pmatrix} = -2
\begin{bmatrix}
0 & -\hat{\theta} & 0 \\
\hat{\theta} & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
- \hat{\theta}^2
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
+ n^2 \frac{(1 + ec\cos\hat{\theta})^2}{(1 - e^2)^3/2}
\begin{pmatrix}
2x \\
y \\
-z
\end{pmatrix}
+ \begin{pmatrix}
u_x \\
u_y \\
u_z
\end{pmatrix}
+ \begin{pmatrix}
f_x \\
f_y \\
f_z
\end{pmatrix}
\]  

[9]

where:

\[
n = \sqrt{\frac{\mu}{a^3}}
\]  

[10]

is the natural frequency of the orbit and the terms \( u_x, u_y, u_z, f_x, f_y, f_z \).
and $f_x$, $f_y$, $f_z$ are accounting for control inputs and disturbances respectively. The terms on the right hand side of Eq. [9] corresponds to Coriolis acceleration, centripetal acceleration, accelerating rotation of the reference frame and the virtual gravity gradient terms with respect to the formation reference. Eq. [9] can be represented as a linear time-varying (LTV) system of equations in state-space format in compact form:

$$\dot{x}(t) = A(t)x(t) + Bu(t) + Wf(t) \quad [11]$$

with:

$$x^T = [x \dot{x} y \dot{y} z \dot{z}]$$

where:

$$A(t) = \begin{bmatrix} 0 & 2 \dot{\theta} & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 + \frac{2}{1 + e \cos \theta} \\ 1 + \frac{2}{1 + e \cos \theta} \\ 1 + \frac{2}{1 + e \cos \theta} \\ 1 + \frac{2}{1 + e \cos \theta} \end{bmatrix}$$

and:

$$B(\theta) = W(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Q(\theta) = \begin{bmatrix} \frac{1}{1 + e \cos \theta} & \theta^T \\ \theta & 1 + \frac{2}{1 + e \cos \theta} \end{bmatrix}$$

III. THE OPTIMAL RENDEZVOUS PROBLEM

The problem of optimal rendezvous has been studied since many years ago, especially from the development of Lawden’s primer vector theory \cite{15}. The concept of rendezvous is also applicable to formation reconfiguration since this is simply a case of rendezvous where the final state is not the origin of the Hill frame. The optimal rendezvous using models of relative dynamics for eccentric orbits was first investigated in detail in the work done in \cite{16} and analytical solutions were proposed in \cite{17}. In this section, the mathematical development of the standard linear optimal control approach (without constraints) based in Pontryagin’s minimum principle \cite{18,19,20} i is presented. The tracking optimal control problem, a generalization of the regulator system, consists in minimizing the cost function:

$$J = \frac{1}{2} \left[ (z-x)^T Q(\theta) [z-x] + u^T R(\theta) u \right] d\theta \quad [14]$$

where $z(\theta)$ is the reference state and $Q(\theta) \geq 0$ and $R(\theta) > 0$ are weighting matrices defined as in \cite{16}:

$$Q(\theta) = \frac{1}{1 + e \cos \theta} Q \quad [15]$$

$$R(\theta) = \left( \frac{1}{1 + e \cos \theta} \right)^2 R \quad [16]$$

and:
subjected to the state system defined by Eq. [13], which for convenience is re-expressed as:

$$\dot{x} = A(\theta)x + B(\theta)u + W(\theta)f$$  \[17\]

The procedure, called the sweep method in \textsuperscript{19}, requires the backwards solution of the Riccati Matrix Differential Equation \(S\) and the Adjoint Riccati Equation \(g\), both with time-varying matrices:

$$\dot{S} = -SA(\theta) - A(\theta)^T S + SB(\theta) R(\theta)^{-1} B(\theta) S - Q(\theta)$$  \[18\]

$$\dot{g} = [SB(\theta) R(\theta)^{-1} B(\theta) - A(\theta)^T] g - Q(\theta)x$$  \[19\]

These equations are solved iteratively through numerical algorithms such as Runge-Kutta or Newton methods implemented in MATLAB\textsuperscript{\textregistered} leading to the closed-loop optimal control law:

$$u(\theta) = -R^{-1}(\theta)B(\theta)^T [S(\theta)x - g(\theta)]$$  \[20\]

\section*{IV. ARTIFICIAL POTENTIAL FIELD}

During manoeuvres, as the number of spacecraft in a formation is increased the risk of collision between the elements is also increased. APF can be used to add a collision avoidance scheme into the dynamical models of spacecraft formation flying (SFF). Important work have been presented regarding APF applied to SFF (as mentioned in Section 1) and one of the main advantages of its use is the insensitivity to the dynamical behaviour of the reference point (e.g. true anomaly). In the present work it is assumed that each spacecraft does not have previous knowledge of the states of the rest of the elements in the formation and this information would be available only through the sensor devices included in each spacecraft leading to real-time motion planning. When using APF for collision avoidance the obstacles to be avoided are treated as forbidden regions and therefore as spacecraft approaches to one of these obstacles an increasing repulsive force is generated upon them.

Ideally, this APF would adapt itself according to shape and size of the obstacle as well as their state causing the approaching spacecraft to reduce its velocity during the avoiding manoeuvre. The type of function to be used in this work is a Gaussian function due to the relatively simple procedure to adjust the parameters such as magnitude and region of influence. The normalized Gaussian function used in this research is:

$$V(x_{so}) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left( -\frac{x_{so}^2}{2\sigma^2} \right)$$  \[21\]

where \(x_{so} = x_i - x_o\) is the distance from the spacecraft to the object to be avoided and \(\sigma\) is a parameter to regulate the region of influence. A sample plot of the potential can be observed in Fig. 2 where the amplitude and the region of influence has been adjusted to 40 units and 20 distance units respectively and the object to be avoided is located at the point (0,0,0). The known acceleration term \(f(x)\) corresponding to the APF to be included in Eq. [17] is Eq. [21] multiplied by the component of the velocity of the spacecraft in the direction of the object to be avoided\textsuperscript{20} and divided by the mean motion \(\dot{n}\), that is:

$$v_{so} = \begin{pmatrix} x_{so} \cdot \nu \end{pmatrix} x_{so} \frac{\nu_{so}}{\|x_{so}\|}$$  \[22\]

where \(\nu\) corresponds to the velocity of the spacecraft and \(\|x_{so}\|\) is the 2-norm of the position of the spacecraft with respect to the object to avoid. Therefore:

$$f(x) = \frac{1}{\sigma \sqrt{2\pi}} v_{so} \begin{pmatrix} \exp \left( -\frac{x_{so}^2}{2\sigma^2} \right) \\ \exp \left( -\frac{y_{so}^2}{2\sigma^2} \right) \\ \exp \left( -\frac{z_{so}^2}{2\sigma^2} \right) \end{pmatrix}$$  \[23\]

Fig. 2: Plot of the Gaussian function.
V. SOLUTION OF THE LQR/APF CONTROL SCHEME

The general methodology proposed in this paper to solve the optimal rendezvous problem with an LQR/APF control scheme is summarized in the next steps. Additionally, the control scheme can also be studied in the diagram located in Fig. 3.

1) Evaluate the Riccati Matrix Differential Equation offline and backwards in time with previously selected final condition (e.g. \( s_N = 0 \) for this work).

2) For every spacecraft in the formation:
   a) Evaluate the Adjoint Riccati Equation offline and backwards in time with previously selected final condition (e.g. \( g_N = 0 \) for this work) and the reference trajectory.
   b) Solve the optimal control using the results obtained in the previous step and the initial conditions of the state.
   c) Solve the state equation including the term of the APF using the results of the previous step and the initial conditions of the state.

The dynamical model in Eq. [17] is a linear time-varying system of differential equations in continuous-time format. In order to address this system a practical procedure is to approximate it to a digital version assuming A, B, R and Q are constant during every true anomaly step but varying as it is proceeded from one step to another (zero-order-hold). MATLAB® is chosen to perform the simulations presented next.

\[
\begin{align*}
\mathbf{x}(k+1) &= 
\begin{bmatrix}
\mathbf{A}(k) & \mathbf{B}(k) \\
\mathbf{W}(k) & \mathbf{P}(k) & \mathbf{Q}(k)
\end{bmatrix} \\
\mathbf{u}(k) &= \mathbf{x}(k+1) - \mathbf{x}^*(k)
\end{align*}
\]

Fig 3: Diagram of the procedure to solve the LQR/APF control scheme.

VI. RESULTS

VI.I Case One

The present case makes reference to the orbital parameters used in the work done in \(^9\) where the several phases of the NASA benchmark tetrahedron constellation mission where studied in detail. This paper will focus on phase I of this mission for which the orbital elements of the reference orbit are summarized in Table 1. The purpose is to correct the drift of a spacecraft from some initial state to a desired state while collision with a static object is avoided. The task done by the LQR is to drive the separation distance from the initial to state to the desired state to zero while the control input is optimized. The mass of the spacecraft is 90 kg \(^9\) and the weights on the diagonal of the \( Q \) and \( R \) matrices are selected to be \([100 \quad 100 \quad 100 \quad 30 \quad 30 \quad 30] \) and \([10 \quad 10 \quad 10]\) respectively determined through a trial-and-error procedure. Fig. 4 shows the state response generated during the correction of the drift and the optimal control for which the maximum acceleration needed to perform the control is \(9e^{-5} km/s^2\) and the desired state is reached in approximately 0.5 days. This period of time is larger than those obtained in \(^9\) due to the deviation from the original trajectory in order to avoid collision. In Fig. 5, in the lower right corner, it is also observed the collision avoidance movement in green, the original trajectory without avoidance scheme in blue and the obstacle with an asterisk starting from left to right. The initial state (in km and km/s) is set to:

\[
x_0 = \begin{bmatrix} 2.5e-4 & -1.18 & -1.4e-5 & 0.302 & -9e-6 & 0.012 \end{bmatrix}^T
\]

Table 1: Orbital Parameters for Phase I

<table>
<thead>
<tr>
<th>Orbital Parameter</th>
<th>Phase I</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radius of the periogee (ER)</td>
<td>1.2</td>
</tr>
<tr>
<td>Radius of the apogee (ER)</td>
<td>12</td>
</tr>
<tr>
<td>Semimajor axis (km)</td>
<td>42,095</td>
</tr>
<tr>
<td>Eccentricity</td>
<td>0.818</td>
</tr>
<tr>
<td>Period (days)</td>
<td>1</td>
</tr>
<tr>
<td>ER = Earth Radius</td>
<td></td>
</tr>
</tbody>
</table>

Table 1: Orbital Parameters for Phase I

Fig. 4: LQR/APF Response. Position and Control
Input.

### Case Two

In the next case an orbital manoeuvre involving three spacecraft is presented. Three spacecraft are located at initial positions distributed along a reference circle of 350 meters diameter. The objective is to transfer the formation to a desired position along a smaller reference circle of 250 meters diameter while the spacecraft avoid collision with a static obstacle, as observed in the schematic representation in Fig. 6. The notation SPX PX is used to denote the number of spacecraft and the position (e.g. SP2 P1 for spacecraft two, position one). As in the previous case, the orbital parameters for the reference orbit are presented in Table 1.

The mass of the spacecraft is selected to be 1 kg and the weights on the diagonal of the $Q$ and $R$ matrices are $[100 \ 100 \ 100 \ 20 \ 20 \ 20]$ and $[1 \ 1 \ 1]$ respectively. Fig. 7 shows the movement of the three spacecraft in the in-track and radial plane. In order to have a more appropriate view of this response the movement of spacecraft two is presented at the bottom of Fig. 7 where the optimal trajectory with collision avoidance can be observed. The optimal position and control of spacecraft two can be observed in Fig. 8 where the time to reach the desired state is approximately 0.15 days and the maximum optimal control required during the manoeuvre is $1 \times 10^{-4} \ km/s^2$.

![Fig. 5: LQR/APF Response. Velocity and Collision Avoidance Movement.](image)

![Fig. 6: Schematic representation for case two.](image)

![Fig. 7: LQR/APF response for case two.](image)
VII. CONCLUSIONS

The proposed algorithm establishes a basis for the future work of the authors in control algorithms which not only contemplate fuel optimization and large formations, but also includes collision avoidance schemes for dynamical models that allows elliptical reference orbits. In this work a linear optimal control scheme with collision avoidance for elliptic reference orbits was presented and tested. The algorithm relies in Pontryagin’s minimum principle to provide results involving the solution to the Matrix Riccati Differential Equation with time-varying parameters. In this paper, solving the standard LQR problem provided an effective starting point for the actual problem. The procedure is computationally light and manageable and the results obtained provide evidence that the proposed control scheme is feasible. The use of APF helped to prevent collision in the test cases and the control scheme has the potential to consider a larger number of spacecraft.

In this paper, the standard LQR problem without constraints in combination with APF was solved to provide results to the problem of spacecraft formation flying manoeuvring with collision avoidance. However, practical systems closer to reality contemplate restrictions in the control input, the state (or both) and multi-stage subproblems. Therefore, the first future consideration is the use of constrained models.

Additionally, further work will be performed to the APF in order to provide it with the ability of adaptation according to the dynamical requirements of the problem improving the collision avoidance system effectiveness. Velocity decrease when approaching to the object to avoid and smooth avoidance trajectories are the second future consideration for the present work.

Fig. 8: Position and optimal control for spacecraft two.
IX. REFERENCES


14. TILLERSON, M. J. Coordination and Control of Multiple Spacecraft using Convex Optimization Techniques. 2000. (PhD). University of Texas at Austin, Texas, United States.


