A 3D Beam-Column Element Implemented within a Hybrid Force-Based Method

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Abstract

This paper describes a force-based beam-column element implemented using a hybrid force-based solution strategy. The element can accommodate elastic-plastic strain hardening material behaviour under various loadings including axial, torsion, bending and shear deformation, both in and out of the plane of the element. In order to overcome difficulties associated with conventional displacement-based and force-based methods a hybrid force based-method is proposed. This alternative approach is based on simultaneous use of the principles of minimum total potential energy and minimum complementary potential energy. Here the primary equation is the equilibrium equation rather than the compatibility equation (the latter takes precedence when following a displacement based solution strategy). The predictions of the element using this solution procedure are compared against predictions from Abaqus™, showing excellent agreement.

Keywords: Force Based Method; Hybrid Method; Beam Element

Introduction

Generally, inelastic behaviour in frame structures can be studied by two main approaches (i) the Concentrated Inelastic Approach (CIA) [1] and (ii) the Distributed Inelastic Approach (DIA), which can be further subdivided into techniques using either customised beam elements [2] or, more commonly, using continuum elements. In the beam-based DIA, each structural member is modelled by numerous discrete homogeneous domains, usually of square cross-section, running along the length of the member (similar in structure to a fibrous composite material). This approach permits calculation of the gradual spread of inelastic behaviour over the member cross-section and length as deformation proceeds. Such elements can provide accurate solutions, enabling the tracking of phenomena such as cracking and residual stress. The technique is much less demanding in terms of computational resource than a typical full 3D DIA, based on continuum elements.

The main goal of the current work is to use this beam-based 'fibre' modelling DIA to develop a 3D force-based beam-column element, implemented within a solution strategy involving force rather than displacement as the primary unknown variable. The element is formulated to incorporates arbitrary cross-sections and accommodate axial, torsional, bending and shear deformations, both in and out of the plane of the element and 3D elastic-plastic strain hardening material behaviour [3].

The structure of the rest of the paper is as follows; the first section is a brief explanation of the force-based method as compared with the more usual displacement method and is followed by a brief description of the fundamental equations in the force-displacement domain. Next, force-based solution strategies are described before focusing on the concept and governing element equations used in this approach. In the final section, a numerical example is presented comparing numerical displacement-based predictions generated using Abaqus™ , previously published work [4] and the results of the current investigation. The comparison demonstrates the accuracy of both the element and the solution procedure developed here. Predictions of the various approaches are critically assessed before concluding the paper.

Force Method

The most significant difference between the displacement and force-based methods results from the fact that the deformation field is more complex and discontinuous compared to the corresponding force field, especially in inelastic regions where the deformation field can have steep gradients. Further, while the distribution of the internal force vector components are known throughout all the frame elements, the same cannot be said for the deformation field. These

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factors have caused a significant reprisal in attempts to formulate force-based inelastic fields. In addition, new generation procedures in force-based methods have started to improve the currently unsystematic approach to choosing the system's redundancy. These new procedures are based on either structural topology [5] or the computational method. The latter can be classed in three categories, namely the Integrated Force Method [6], the Large Increment Method [7] and the hybrid force-based method.

Fundamental Equations

The principle of complementary potential energy has been an attractive approach for structural analysis in force-based methods because it provides the exact flexibility matrix, in contrast to the classical displacement-based method, which is based on the principle of total potential energy [8,9]. Using both these principals simultaneously leads to a hybrid method [1, 2]. Let the structure domain, (Ω), be defined by,

$$\Omega = \{(x_k, k = 1,2,3) \in R \,, \partial\Omega_\varepsilon \cap \partial\Omega_\sigma = 0 \,, \partial\Omega_\varepsilon \cup \partial\Omega_\sigma = \partial\Omega \quad ,\}.$$

Minimizing the total, π_{TPE} and complementary, π_{CPE} energy functions (i.e. the principles of virtual displacement and virtual force work),

$$\frac{\partial\pi_{TPE}}{\partial u} = \int_\Omega \delta\, \varepsilon_{ij}{}^T\, \sigma_{ij} d\Omega - \int_{\partial\Omega_\sigma} \delta u_i^T\, t_i(\partial\Omega_\sigma) d(\partial\Omega_\sigma) - \int_\Omega \delta u_i^T\, b_i(\Omega) d\Omega = 0 \tag{1}$$

$$\frac{\partial\pi_{CPE}}{\partial\sigma} = \int_\Omega \delta\sigma_{ij}^T\, E^{-1}\, \sigma_{ij} d\Omega - \int_{\partial\Omega_\sigma} \delta t_i(\partial\Omega_\sigma)^T u_i\ d(\partial\Omega_\sigma) - \int_\Omega \delta b_i(\Omega)^T u_i\ d\Omega = 0 \tag{2}$$

leads to the main governing equations. Where u is the continuous local element displacement field, b_i is the body force vector, t_i is the external traction vector acting on the external surface Ω_σ , σ_{ij} is the Cauchy stress tensor and ε_{ij} is the strain tensor. The equilibrium and compatibility equation in the force-displacement domain based on those minimized potential energy principals can be derived for whole of the structure as,

$$\begin{bmatrix} C_D \\ C_N \end{bmatrix} . \mathrm{F} = \begin{bmatrix} P_D \\ P_N \end{bmatrix},\ P_N = \hat{P}_N \tag{3}$$

$$[C_D^T \quad C_N^T] . \begin{bmatrix} D_D \\ D_N \end{bmatrix} = \delta \,, D_D = \hat{D}_D \tag{4}$$

where F is a vector containing internal forces, moments and torsion, C is the equilibrium matrix which can be decomposed into the Dirichlet (or first or essential) boundary conditions, C_D which corresponds to the displacement at the boundaries and the Neumann boundary conditions (or secondary or Natural boundary condition) C_N which corresponds to the externally applied force. The external force, P and nodal displacement, D components P_D, P_N and D_D, D_N can also be decomposed in a similar way. Based on this decomposition, predefined displacements and forces on nodes can be defined and are denoted as; $\hat{D}_D$ and $\hat{P}_N$ respectively.

The Solution Procedure

In general, for indeterminate structures the matrix, C_N , alone is insufficient information to calculate a unique solution for the internal forces and so compatibility conditions are also required to fully formulate the problem. Methods to include the compatibility equations into the classic (conventional or standard) force-based method usually involve changing the rank of the equilibrium matrix. This change is based on segregating redundant force components and can be achieved using various different techniques such as the Cutter [10,11,12], Eigenvalue [13], Reduced Rank [12], Lower-Upper decomposition [14], Orthogonal-Upper decomposition [15] and General Inverse [7, 16] techniques. The common objective in each of these methods is to produce a special form of the compatibility equation based on a predefined determined structure. Nevertheless, finding an optimum predefined determined structure under static equilibrium is not a straightforward procedure [10, 12]. The general form of the solution for the equilibrium equation is,

$$F = \alpha(C_N)\hat{P}_N + \beta(C_N, C_D, K) \tag{5}$$

where the term, $\beta(C_N, C_D, K)$, is an unbalanced load vector that has to be calculated based on the compatibility equation, and can be defined using various methods. The proposed solution procedure is summarized in Table.1

Table 1. The solution procedure algorithm.

Solution Algorithm		
1	Built the external load vector	P
2	Built the equilibrium matrix,	C
3	Decompose P and C based on boundary condition	P_D, P_N, C_D, C_N
4	Calculate the right-side inverse form,	$\alpha(C_N) = {C_N}^T\left(C_N {C_N}^T\right)^{-1}$
5	Calculate the unbalance load vector matrix	$\beta(C_N, C_D, K, F) = p_r . k_r . \delta$
6	Calculate the elements force vector,	$F = \alpha(C_N)\hat{P}_N + \beta(C_N, C_D, K)$
7	Calculate the elements deformation vector,	$\delta = K^{-1}F$
8	Calculate the nodal displacement vector,	$D = \left(C_N {C_N}^T\right)^{-1} C_N \delta$
9	Control structure compatibility within a predefined tolerance	$Ex = {C_N}^T D - \delta$
10	Calculate the stiffness reduction factor	$\gamma = {C_N}^T\left(C_N {C_N}^T\right)^{-1} C_N$
12	Calculate the precipitator function	$p_r = -\delta^T s(s^T K^{-T} s)$
13	Go to 4, unless accuracy is within the predefined tolerance	

where the K is the segregated stiffness matrix and δ is a vector containing deformation components caused by the force components contained in F.

Element Formulation

A 3D beam element of arbitrary cross section with twelve nodal degrees of freedom, denoted here as, $\bar{f} = \bar{f}_{i=1,..,12}$ (see Figure 1C) has been formulated to include axial force N, torsion T, the bending moments M_2, M_3 and shear deformations V_2, V_3 both in and out of the plane of the element (see Figure 1B) and at both ends of the element (see Figure 1A).

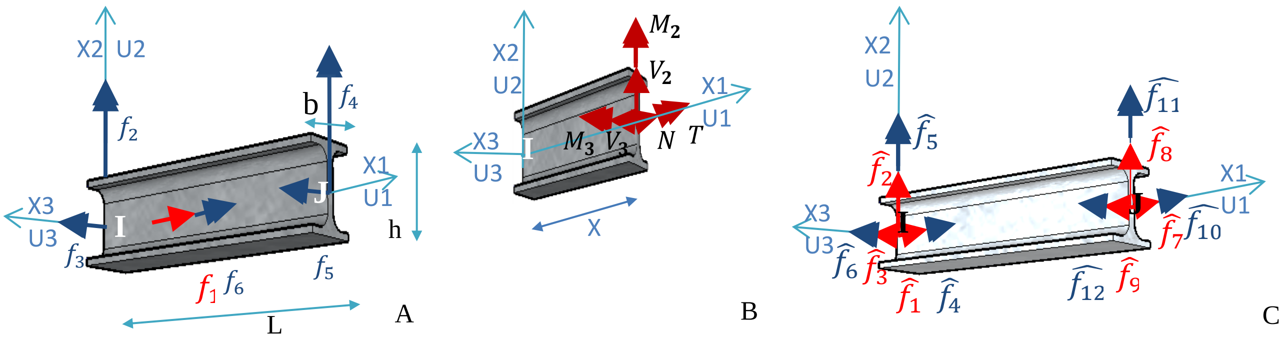

Figure 1.Beam-Column Element Configuration.1A) Element degrees of freedom configuration.1B) Section force vector components definition over the cross section.1C) Nodal degrees of freedom configuration.

This element is implemented in the hybrid force based method by interpolating the force instead of the displacement along the element by defining b(x)

$$\begin{bmatrix} N \\ M_2 \\ M_3 \\ V_2 \\ V_3 \\ T \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1+x/L & 0 & x/L & 0 & 0 \\ 0 & 0 & -1+x/L & 0 & x/L & 0 \\ 0 & 0 & -1/L & 0 & -1/L & 0 \\ 0 & 1/L & 0 & 1/L & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{bmatrix}, \quad F_{Sec} = b(x).f \tag{6}$$

where F_{sec} is the force vector acting on any given section. This definition is based on the elemental degrees of freedom $f_{i=1,..,6}$ (see Figure 1A) as opposed to the nodal degrees of freedom $\hat{f}_i$ which is the common form when using in the displacement-based method). This change from the nodal to the element domain is essential when using force-based solution strategies and eliminates numerical issues resulting from the interdependence between some of the nodal degrees of freedom. Satisfying the total potential energy over the section makes it possible to satisfy equilibrium at each section. This leads to a relationship between force and deformation for each section incorporating axial, rotation, bending and shear effects,

$$\begin{bmatrix} N \\ M_2 \\ M_3 \\ V_2 \\ V_3 \\ T \end{bmatrix} = \begin{bmatrix} \int_A E_T dA & -\int_A x_2 E_T dA & \int_A x_3 E_T dA & 0 & 0 & 0 \\ -\int_A x_2 E_T dA & \int_A x_2^2 E_T dA & -\int_A x_2 x_3 E_T dA & 0 & 0 & 0 \\ \int_A x_3 E_T dA & -\int_A x_2 x_3 E_T dA & \int_A x_3^2 E_T dA & 0 & 0 & 0 \\ 0 & 0 & 0 & \int_{A_{s2}} G_T dA & 0 & -\int_{A_{s2}} x_3 G_T dA \\ 0 & 0 & 0 & 0 & \int_{A_{s3}} G_T dA & -\int_{A_{s3}} x_2 G_T dA \\ 0 & 0 & 0 & -\int_{A_{s2}} x_3 G_T dA & -\int_{A_{s3}} x_2 G_T dA & \left(\int_{A_{s2}} x_2^2 G_T dA + \int_{A_{s3}} x_3^2 G_T dA \right) \end{bmatrix} . \begin{bmatrix} \frac{\partial u}{\partial x_1} \\ \frac{\partial \phi_2}{\partial x_1} \\ \frac{\partial \phi_3}{\partial x_1} \\ \frac{\partial \gamma_2}{\partial x_1} \\ \frac{\partial \gamma_3}{\partial x_1} \\ \frac{\partial \varphi}{\partial x_1} \end{bmatrix}, \tag{7}$$

$$F_{Sec} = {fl_{Sec}}^{-1} . D_{Sec} \tag{8}$$

The end deformation components D_i including the tensile and compressive deformations, the bending deformations with respect to the transverse (2,3) axes and the torsional deformation with respect to the (1) axis are defined by u, ϕ, φ and can be computed by integrating the deformation along the element length $D_i = \int_L b(x)^T . D_{Sec} . dx$. By then substituting Equations (6) and (8) into this definition of D_i , find ,

$$D_i = F_{ij} f_i \tag{9}$$

This defines the main relationship between force and displacement in the local element space where,

$$F_{ij} = \int_L b(x)^T . fl_{Sec} . b(x) \; . dx \tag{10}$$

and

$$D_i = \begin{bmatrix} u_1 & \phi_2^i & \phi_3^i & \phi_2^j & \phi_3^j & \varphi \end{bmatrix}^T \tag{11}$$

where the matrix that transfers the degrees of freedom from the nodal to the element domain, Tr is defined as,

$$Tr = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/L & 0 & 1/L & 0 \\ 0 & -1/L & 0 & -1/L & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1/L & 0 & -1/L & 0 \\ 0 & 1/L & 0 & 1/L & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad , \quad \bar{f} = Tr . f \quad , \quad \bar{D} = Tr^T . D \tag{12}$$

Using this transformation matrix, it is possible to rewrite the main relationship between force and displacement (Equation 9) in nodal space to apply nodal boundary conditions as,

$$\overline{D_i} = (Tr^T . F_{ij} . Tr) . \overline{f_i} \tag{13}$$

This means that $\bar{F} = Tr^T . F_{ij} . Tr$ is the element flexibility matrix in the local element space based on the nodal DOFs, which are the final quantities, calculated by prepared code.

Numerical Examples

The following example is produced using the element and solution procedure described above and is compared against both previously published work [4] and numerical results produced by the commercial implicit finite element code Abaqus^{TM}. The latter provides a reliable benchmark case for comparison. The performance of the code, and in particular, the predictions of the new element implemented within the code, are assessed in terms of both accuracy and efficiency.

The example is designed to test the performance of the new element during the simultaneous application of bending moments and shear deformations. Figure 2A shows a cantilever beam. The beam is subject first to a monotonic loading and second to a cyclic loading P at its free end. The elements have a bilinear material behaviour defined as $(E = 2.85 \times 10^{10}, E_t = 1.9 \times 10^8 \text{ and } \sigma_0 = 1.9 \times 10^9)(N/Cm^2)$ [4]. The equivalent Abaqus^{TM} simulation used 100 beam elements B31 throughout the beam structure. Figure 2B demonstrates the stress distribution results at maximum load across the whole of the structure.

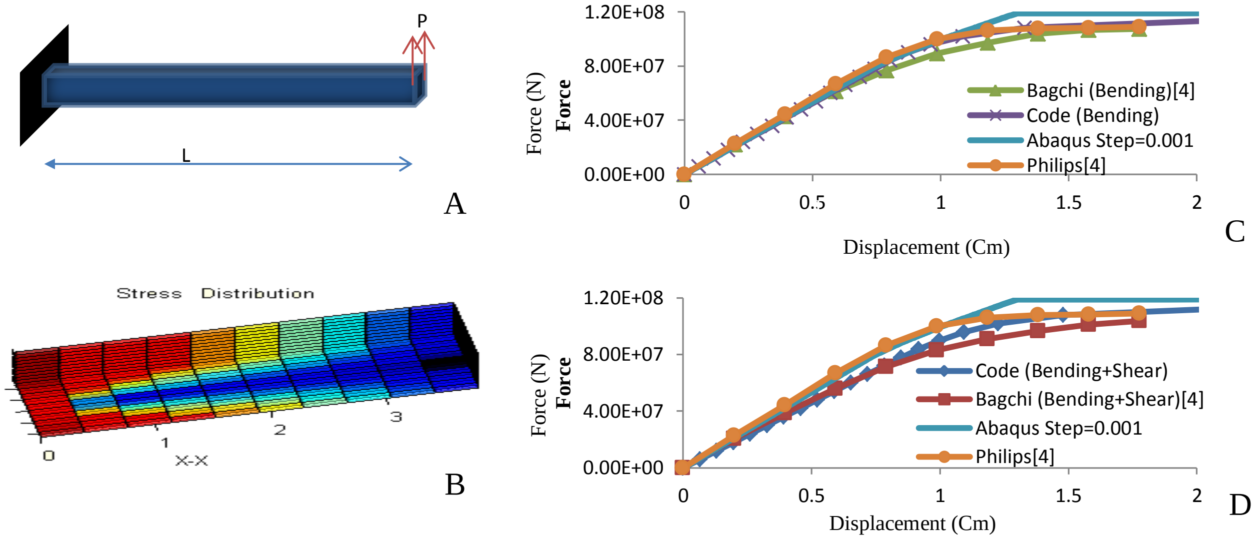

Figure 2. Comparison of code result under bending and shear-bending effect.2A) The element configuration. 2B)The stress distribution in whole of the element. 2C) The load-displacement relation at free end under bending effects. 2D) The load-displacement relation at free end under both bending and shear effect.

Predictions using the new element show close correspondence with the previous research result both in pure bending (see Figure 2C) and in combined bend-shear loading (see Figure 2D) when compared with the Abaqus^{TM} prediction. The detailed behaviour is demonstrated by axial stress distribution in whole of the element (see Figure 2B).

The cyclic predictions are compared against Abaqus^{TM} in Figure 3 using two different methods of satisfying equilibrium over the section: (a) an iteration procedure and (b) a non-iterative procedure [17].

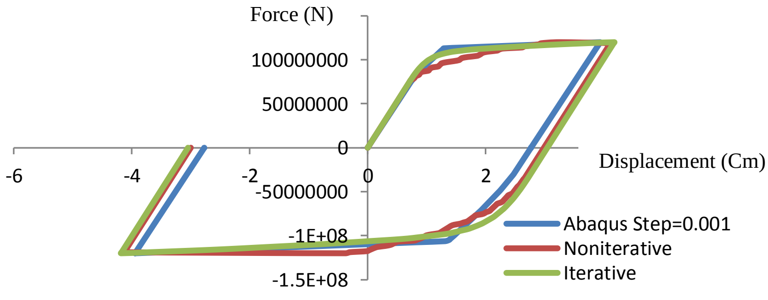

Figure 3. Comparison of code and Abaqus^{TM} result for iterative and non-iterative procedure.

Although the prediction of inelastic behaviour, based on the non-iterative procedure is not smooth in the transfer region, (i.e. the region where the deformation changes from elastic to inelastic behaviour), estimates of the extreme displacements are the same as when using the iterative procedure. The significant advantage of using the non-iterative procedure is the reduction in computational cost. For the same accuracy in the prediction of the maximum load, the computational time is 15 times less than for the iterative procedure.

Conclusions

A force-based beam-column element of general section, able to incorporate 3D elastic-plastic linear strain hardening behaviour has been formulated for the first time within the framework of the hybrid solution scheme. This procedure reduces the computational time and increases the accuracy by changing the computational domain from displacement to force and by simultaneously incorporating both the total and complementary potential energy functions. The numerical example clearly demonstrates the excellent performance of both the new hybrid force-based method and the new element in terms of accuracy, robustness and computational efficiency. The element can be extended for special cases (e.g., residual stress effect and crack propagation) and optimised in terms of the discretisation size.

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