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FACTORIZATION IN GENERALIZED CALOGERO-MOSER SPACES

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Abstract. Using a recent construction of Bezrukavnikov and Etingof, [BE], we prove that there is a factorization of the Etingof-Ginzburg sheaf on the generalized Calogero-Moser space associated to a complex reflection group. In the case \( W = S_n \), this confirms a conjecture of Etingof and Ginzburg, [EG].

1. Introduction

In this paper we apply a recent construction of Bezrukavnikov and Etingof, [BE], to the study of the centres of the rational Cherednik algebras at \( t = 0 \). The affine varieties \( X_c(W) \), corresponding to these centres are called the generalized Calogero-Moser spaces and are known to influence the representation theory of the algebras. We show that there exists an isomorphism of schemes

\[
\Phi : \pi^{-1}_W(b) \xrightarrow{\sim} \pi^{-1}_{W_b}(0),
\]

where \( \pi_W : X_c(W) \rightarrow h/W \) and \( W_b \) is a parabolic subgroup of \( W \) associated to the orbit \( b \in h/W \) (for precise definitions see Section 2). In order to relate the representation theory of the algebras \( H_{0,e} \) to the varieties \( X_c \), Etingof and Ginzburg introduced the coherent sheaf \( \mathcal{R}[W] \), defined by \( \Gamma(X_c, \mathcal{R}[W]) = H_{0,e} \).

Our main result describes the pushforward of \( \mathcal{R}[W]_{|\pi^{-1}_W(b)} \) by \( \Phi \).

Theorem. On \( \pi^{-1}_{W_5}(0) \) there is an isomorphism of \( W \)-equivariant sheaves

\[
\Phi_*(\mathcal{R}[W]_{|\pi^{-1}_{W_5}(0)}) \simeq \text{Ind}_{W_b}^{W_5} \mathcal{R}[W_b]_{|\pi^{-1}_{W_b}(0)}
\]

In particular, this theorem proves that the sheaf \( \mathcal{R}[W] \) on the Calogero-Moser space associated to the symmetric group factorizes as conjectured by Etingof and Ginzburg, [EG, page 319].

2. The rational Cherednik algebra

2.1. Definitions and notation. Let \( W \) be a complex reflection group, \( h \) its reflection representation over \( \mathbb{C} \) with rank \( h = n \), and \( S \) the set of all complex reflections in \( W \). The idempotent in \( CW \) corresponding to the trivial representation will be denoted \( e_W \). Let \( (\cdot,\cdot) : h \times h^* \rightarrow \mathbb{C} \) be the natural pairing defined by \( (y,x) = x(y) \). For \( s \in S \), fix \( \alpha_s \in h^* \) to be a basis of the one dimensional space \( \text{Im}(s-1)_{|h^*} \) and \( \alpha_s^\vee \in h \) a basis of the one dimensional space \( \text{Im}(s-1)_{|h} \) such that \( \alpha_s(\alpha_s^\vee) = 2 \). Choose \( c : S \rightarrow \mathbb{C} \) to be a \( W \)-invariant function and \( t \) a complex number. The rational Cherednik algebra, \( H_{t,c}(W) \), as introduced by Etingof and Ginzburg, [EG, page 250], is the quotient of the skew group algebra of the tensor algebra, \( T(h \oplus h^*) \rtimes W \), by the ideal generated by the relations

\[
T(h \oplus h^*) \rtimes W
\]
\[
[x_1, x_2] = 0 \quad [y_1, y_2] = 0 \quad [x_1, y_1] = t(y_1, x_1) - \sum_{s \in S} c(s)(y_1, \alpha_s)(\alpha_s^*, x_1)s
\]

\forall x_1, x_2 \in h^* and y_1, y_2 \in h.

Since there is an isomorphism \( H_{M, A^e}(W) \cong H_{t, e}(W) \) for any \( \lambda \in \mathbb{C}^* \), we can restrict ourselves to considering the cases \( t = 0 \) or \( 1 \).

2.1.1. Parabolic subgroups. For a point \( b \in h \), the stabilizer subgroup of \( W \) with respect to \( b \) will be denoted \( W_b \). By a theorem of Steinberg, [S2, Theorem 1.5], \( W_b \) is itself a complex reflection group. If \((h^*W_b)^\perp\) denotes the vector subspace of \( h \) consisting of all vectors \( y \in h \) such that \( x(y) = 0 \ \forall y \in h^*W_b \), then \( h = h^*W_b \oplus (h^*W_b)^\perp \) is a decomposition of \( h \) as a \( W_b \)-module. Note that \((h^*W_b)^\perp\) is a faithful reflection representation of \( W_b \) of minimal rank.

2.1.2. Centralizer algebras. We recall the centralizer construction described in [BE, 3.2]. Let \( A \) be an \( \mathbb{C} \)-algebra equipped with a homomorphism \( H \rightarrow A^\times \), where \( H \) is a finite group. Let \( G \) be another finite group such that \( H \) is a subgroup of \( G \). The algebra \( C(G, H, A) \) is defined to be the centralizer of \( A \) in the right \( A \)-module \( P := \text{Fun}_H(G, A) \) of \( H \)-invariant, \( A \)-valued functions on \( G \). By making a choice of left coset representatives of \( H \) in \( G \), \( C(G, H, A) \) is realized as the algebra of \(|G/H| \) by \(|G/H| \) matrices over \( A \). For \( w, g \in G \) and \( f \in \text{Fun}_H(G, A) \), \( w \cdot f(g) := f(gw) \) defines, by linearity, an embedding \( \iota : \mathbb{C}G \hookrightarrow C(G, H, A) \).

Let \( e_G \in \mathbb{C}G \) and \( e_H \in \mathbb{C}H \) denote the idempotents corresponding to the trivial representation of \( G \) and \( H \) respectively, where \( \mathbb{C}H \) is considered as a subalgebra of \( A \).

Lemma 2.1. There are isomorphisms of \( \mathbb{C}G \cdot Z(A) \)-bimodules

\[ C(G, H, A) \cdot \iota(e_G) \cong \text{Fun}_H(G, Ae_H) \cong \text{Ind}_H^G A e_H, \]

where \( Z(A) \) denotes the centre of \( A \). Here \( \mathbb{C}G \) acts on \( C(G, H, A) \) by multiplication on the left via \( \iota \) and on the left of \( \text{Fun}_H(G, Ae_H) \) also via \( \iota \).

Proof. The second isomorphism is clear from the definition of \( \text{Fun}_H(G, A) \). Let \( \delta \in \text{Fun}_H(G, A) \) be the function defined by \( \delta(g) = e_H \), for all \( g \in G \). We define a linear map \( \zeta \) from \( C(G, H, A) \cdot \iota(e_G) \) to \( \text{Fun}_H(G, Ae_H) \) and a map \( \eta \) in the opposite direction by

\[
\zeta : M \cdot \iota(e_G) \mapsto M(\delta) \\
\eta : f \mapsto \left(h(-) \mapsto f(-) \sum_{g \in G} h(g)\right),
\]

where \( M \in C(G, H, A) \), \( f \in \text{Fun}_H(G, Ae_H) \) and \( h \in \text{Fun}_H(G, A) \). After fixing left coset representatives of \( H \) in \( G \), a direct calculation shows that \( \eta \) is both a left and right inverse to \( \zeta \). The \( G \)-equivariance of \( \zeta \) is clear since

\[ g \cdot (M \cdot \iota(e_G)) = g \cdot M(\delta) = \iota(g)(M(\delta)) = (\iota(g)M)(\delta) = \zeta(g \cdot M \cdot \iota(e_G)) \]

The \( Z(A) \)-equivariance of \( \zeta \) is similarly clear. \( \Box \)
2.2. Completing the rational Cherednik algebra. For each point \( b \in \mathfrak{h} \), the completion, \( \tilde{H}_{\mathfrak{t},c}(W)_b \), at the orbit \( W \cdot b \in \mathfrak{h}/W \) of \( H_{\mathfrak{t},c}(W) \) is defined in [BE, 2.4]. However, the notion of completion at \( W \cdot b \) works equally well when \( t = 0 \) because \( H_{0,c}(W) \) can be thought of as a sheaf of algebras on the affine variety \( \mathfrak{h}/W \). Therefore, if \( \mathbb{C}[\mathfrak{h}/W]_b \) denotes the completion of \( \mathbb{C}[\mathfrak{h}/W] \) at \( W \cdot b \), we define the completion of \( H_{0,c}(W) \) at \( b \) to be

\[
\tilde{H}_{0,c}(W)_b := \mathbb{C}[\mathfrak{h}/W]_b \otimes_{\mathbb{C}[\mathfrak{h}/W]} H_{0,c}(W)
\]

(3)

Crucially, we note that [BE, Theorem 3.2] is independent of the parameter \( t \) and hence can be applied to the case \( t = 0 \). We state it here for completeness.

**Theorem 2.2** ([BE], Theorem 3.2). Let \( b \in \mathfrak{h} \), and define \( c' \) to be the restriction of \( c \) to the set \( S_b \) of reflections in \( W_b \). Then one has an isomorphism

\[
\theta : \tilde{H}_{\mathfrak{t},c}(W, \mathfrak{h})_b \to C(W, W_b, \tilde{H}_{\mathfrak{t},c}(W_b, \mathfrak{h})_0),
\]

defined by the following formulas. Suppose that \( f \in P = \text{Fun}_{W_b}(W, \tilde{H}_{\mathfrak{t},c}(W_b, \mathfrak{h})_0) \). Then

\[
(\theta(u)f)(w) = f(wu), u \in W;
\]

(4)

for any \( \alpha \in \mathfrak{h}^* \),

\[
(\theta(x_\alpha)f)(w) = (x_\alpha^{(b)} + (w\alpha, b))f(w),
\]

where \( x_\alpha \in \mathfrak{h}^* \subset H_{\mathfrak{t},c}(W, \mathfrak{h}) \), \( x_\alpha^{(b)} \in H_{\mathfrak{t},c}(W_b, \mathfrak{h}) \); and for any \( a \in \mathfrak{h} \),

\[
(\theta(y_a)f)(w) = y_a^{(b)}f(w) + \sum_{s \in S \cdot \mathfrak{g} W_b} \frac{2e_s}{1 - \lambda_s} \left[ x_\alpha^{(b)} + \alpha_s(b) \right] (f(sw) - f(w)).
\]

where \( y_a \in \mathfrak{h} \subset H_{\mathfrak{t},c}(W, \mathfrak{h}) \) and \( y_a^{(b)} \) the same vector considered now as an element of \( H_{\mathfrak{t},c}(W_b, \mathfrak{h}) \).

3. The Etingof-Ginzburg Sheaf

Let \( Z_c(W) \) denote the centre of \( H_{0,c}(W) \) and \( X_c(W) = \text{maxspec}(Z_c(W)) \), the corresponding affine variety. The space \( X_c(W) \) is called the *generalized Calogero-Moser space* associated to the complex reflection group \( W \) at the parameter \( c \). For \( b \in \mathfrak{h} \), the maximal ideal of \( \mathbb{C}[\mathfrak{h}/W] \) corresponding to \( W \cdot b \) will be written \( \mathfrak{m}(b) \) and the two-sided ideal of \( H_{0,c}(W) \) generated by the elements of \( \mathfrak{m}(b) \) will be denoted \( \mathfrak{r}(b) \). Similarly, the maximal ideal of \( \mathbb{C}[\mathfrak{h}/W_b] \) corresponding to \( W_b \cdot p, p \in \mathfrak{h} \), will be written \( \mathfrak{n}(p) \).

Let \( \mathcal{A} \) denote the set of reflecting hyperplanes of \( W \) in \( \mathfrak{h} \) and, for each \( H \in \mathcal{A} \), let \( L_H \in \mathfrak{h}^* \) be a linear functional whose kernel is \( H \) (e.g. \( \alpha_s \in \mathfrak{h}^* \) if \( s \) is a reflection about \( H \)). Choose homogeneous algebraically independent generators \( F_1, \ldots, F_n \) of \( \mathbb{C}[\mathfrak{h}]^W \) and \( P_1, \ldots, P_n \) of \( \mathbb{C}[\mathfrak{h}]^{W_0} \). The following description of the Jacobian is due to Steinberg, [S1, Lemma].

\[
\Pi_W := \det \left( \frac{\partial F_i}{\partial x_j} \right) = k \prod_{H \in \mathcal{A}} L_H^{e_{PH}^{-1}}
\]

(5)
where \( e_H \) is the order of the cyclic group \( W_H \) of elements of \( W \) that fix \( H \) pointwise and \( k \) a non-zero scalar.

**Lemma 3.1.** For each \( b \in \mathfrak{h} \) the map \( \Psi : \mathbb{C}[[\mathfrak{h}/W_b]]_0 \rightarrow \mathbb{C}[[\mathfrak{h}/W_b]]_0 \) defined by
\[
P_i(x) \mapsto F_i(x + b) - F_i(b)
\]
is an automorphism.

**Proof.** Since \( F_i(x + b) - F_i(b) \in \mathfrak{n}(0) \) for all \( i \) there exist polynomials \( Q_1, \ldots, Q_n \) such that \( F_i(x + b) - F_i(b) = Q_i(P_1, \ldots, P_n) \). The chain rule gives
\[
D := \det \left( \frac{\partial F_i(x + b) - F_i(b)}{\partial x_j} \right) = \det \left( \frac{\partial Q_i}{\partial P_k} \right) \det \left( \frac{\partial P_k}{\partial x_j} \right)
\]
However, \( D = \Pi_W (x + b) \) and this gives
\[
\prod_{H \in A} L^{e_H-1}_H (x + b) = \det \left( \frac{\partial Q_i}{\partial P_k} \right) \prod_{H \in A \text{ with } b \in H} L^{e_H-1}_H (x)
\]
Since \( L_H (x + b) = L_H (x) \) if and only if \( b \in H \), we get
\[
\det \left( \frac{\partial Q_i}{\partial P_k} \right) = \prod_{H \in A \text{ with } b \notin H} L^{e_H-1}_H (x + b)
\]
and
\[
\det \left( \frac{\partial Q_i}{\partial P_k} \right)(0) = \prod_{H \in A \text{ with } b \notin H} L^{e_H-1}_H (b) \neq 0.
\]
Hence, by [E, Exercise 7.25], \( \Psi \) is an isomorphism. \( \square \)

As a consequence of Theorem 2.2, we have an isomorphism of quotient algebras.

**Corollary 3.2.** Let \( \tilde{\theta} : \tilde{H}_{0,c}(W, h) \rightarrow C(W, \tilde{H}_{0,c}(W_b, h)_0) \) be the isomorphism (4). Then \( \tilde{\theta} \) descends to an isomorphism
\[
\theta : H_{0,c}(W, h) \mathrel{\langle\langle} m(b) \mathrel{\rangle\rangle} \rightarrow C \left( W, \tilde{H}_{c}(W_b, h) \mathrel{\langle\langle} m(0) \mathrel{\rangle\rangle} \right).
\]

**Proof.** For \( a \in \mathfrak{h}, \alpha \in \mathfrak{h}^* \) and \( w \in W, (x_{w, \alpha} + (w\alpha, b))(a) = (w\alpha, a) + (w\alpha, b) = (w \cdot x_\alpha)(a + b) \). Therefore \( \theta(g)(f(w)) = (w \cdot g)(x + b)f(w) = g(x + b)f(w) \) for all \( g \in \mathbb{C}[[h]]^W \subset \mathbb{C}[[h]]_0 \) and \( f \in \text{Fun}_{W_c}(W, \tilde{H}_{c}(W_b, h)_0) \).
Now choose \( u \in W_b \), then
\[
u \cdot g(x + b) = g(u^{-1} \cdot x + b) = g(u^{-1} \cdot (x + b)) = g(x + b)
\]
shows that \( g(x + b) \in \mathbb{C}[h]^W_b \). Hence, if \( g \in m(b) \mathrel{\langle\langle} \mathbb{C}[h]^W_b \mathrel{\rangle\rangle} \), then \( g(x + b) \in \mathfrak{n}(0) \mathrel{\langle\langle} \mathbb{C}[h]^W_b \mathrel{\rangle\rangle} \). This shows that \( \theta(g)(f(w)) \in \mathfrak{n}(0) \tilde{H}_{t,c}(W_b)_0 \) and
\[
\theta \mathrel{\langle\langle} m(b) \tilde{H}_{t,c}(W, h)_0 \mathrel{\rangle\rangle} \subseteq C \left( W, \tilde{H}_{c}(W_b, h) \mathrel{\langle\langle} m(0) \mathrel{\rangle\rangle} \right).
\]
The ideal \( m(b) \) in \( \mathbb{C}[\mathfrak{h}]^W \) is generated by \( F_1(x) - F_1(b), \ldots, F_n(x) - F_n(b) \) and we have \( \theta(F_i(x) - F_i(b))f(w) = (F_i(x+b) - F_i(b))f(w) \). The statement of Lemma 3.1 is equivalent to the fact that
\[
\{ F_1(x+b) - F_1(b), \ldots, F_n(x+b) - F_n(b) \} \cdot \mathbb{C}[[\mathfrak{h}/W_b]]_0 = n(0)\mathbb{C}[[\mathfrak{h}/W_b]]_0.
\]
This, together with (7), implies that
\[
\theta(m(b)\mathcal{H}_{1,c}(W, \mathfrak{h})_b) = C(W, W_b, n(0)\mathcal{H}_{1,c}(W_b, \mathfrak{h})_b).
\]
and the isomorphism follows. \( \square \)

By [EG, Proposition 4.15], \( \mathbb{C}[\mathfrak{h}]^W \) is contained in the centre of \( H_{0,c}(W, \mathfrak{h}) \) and the embedding defines a surjective morphism \( \pi_W: X_c(W) \to \mathfrak{h}/W \). The algebra \( Z_{0,c}(W)/(m(b)) \) is the coordinate ring of the scheme-theoretic pull-back \( \pi_W^{-1}(b) \). Comparing the centres of the algebras in Corollary 3.2 gives an isomorphism of (non-reduced) schemes.

**Corollary 3.3.** For \( b \in \mathfrak{h} \), there is a scheme-theoretic isomorphism
\[
\Phi: \pi_W^{-1}(b) \xrightarrow{\sim} \pi_{W_b}^{-1}(0).
\]

**Proof.** The Satake isomorphism, [EG, Theorem 3.1], is the map \( Z_{0,c}(W) \to \mathfrak{e}_W H_{0,c}(W, \mathfrak{h}) \mathfrak{e}_W \) defined by \( z \mapsto z \mathfrak{e}_W \). Since \( m(b)H_{0,c}(W, \mathfrak{h}) \) is a centrally generated ideal in \( H_{0,c}(W, \mathfrak{h}) \),
\[
m(b)H_{0,c}(W, \mathfrak{h}) \cap \mathfrak{e}_WH_{0,c}(W, \mathfrak{h}) \mathfrak{e}_W = (\mathfrak{e}_W m(b)),
\]
where the right-hand side is considered as an ideal in \( \mathfrak{e}_WH_{0,c}(W, \mathfrak{h}) \mathfrak{e}_W \). Therefore the Satake isomorphism descends to an isomorphism
\[
S_{W,b}: \frac{Z_{0,c}(W)}{m(b)Z_{0,c}(W)} \xrightarrow{\sim} \mathfrak{e}_W \left( \frac{H_{0,c}(W, \mathfrak{h})}{(m(b))} \right) \mathfrak{e}_W.
\]
As noted in [BE, Lemma 3.1 (ii)], the isomorphism (6) restricts to an isomorphism of subalgebras
\[
\theta: \mathfrak{e}_W \left( \frac{H_{0,c}(W, \mathfrak{h})}{(m(b))} \right) \mathfrak{e}_W \xrightarrow{\sim} \mathfrak{e}_{W_b} \left( \frac{H_{0,c}(W_b, \mathfrak{h})}{(n(0))} \right) \mathfrak{e}_{W_b},
\]
where \( \theta(\mathfrak{e}_W) = \mathfrak{e}_{W_b} \). Here we have implicitly identified the spherical algebra on the right-hand side with a subalgebra of \( C \left( W, W_b, \frac{H_{0,c}(W_b, \mathfrak{h})}{(n(0))} \right) \). It is possible, though uninformative, to give an explicit description of this identification. Combining the isomorphisms of (9) and (10) produces the comorphism
\[
(\Phi^*)^{-1} = S_{W,0}^{-1} \circ \theta \circ S_{W,b}: \frac{Z_{0,c}(W)}{m(b)Z_{0,c}(W)} \xrightarrow{\sim} \frac{Z_{0,c}(W_b)}{n(0)Z_{0,c}(W_b)}
\]
corresponding to \( \Phi \). \( \square \)

Etingof and Ginzburg, [EG, page 247], introduced an important coherent sheaf on \( X_c(W) \), which we now recall.

**Definition 3.4.** The **Etingof-Ginzburg sheaf** is the coherent sheaf \( \mathcal{R}[W] \) on \( X_c(W) \) corresponding to the finitely generated \( Z_{0,c}(W) \)-module \( H_{0,c}(W) \mathfrak{e}_W \).

The coordinate ring of a Zariski-open subset \( U \subseteq X_c(W) \) will be written \( Z_{0,c}(W)_U \). We now conclude;
Theorem 3.5. Let $\mathcal{R}[W]$ be the Etingof-Ginzburg sheaf on $X_c(W)$ and $\mathcal{R}[W_b]$ the Etingof-Ginzburg sheaf on $X_c(W_b)$. For $b \in \mathfrak{h}/W$ we have an isomorphism of $W$-equivariant sheaves on $\pi^{-1}_W(0)$

\begin{equation}
\Phi_\ast \left( \mathcal{R}[W]_{|\pi^{-1}_W(b)} \right) \simeq \text{Ind}^W_{W_b} \mathcal{R}[W_b]_{|\pi^{-1}_{W_b}(0)}.
\end{equation}

Proof. Since $\pi^{-1}_W(b)$ is an affine scheme, to show that we have an isomorphism of $W$-equivariant sheaves as stated in (12) it suffices to show that the global sections are isomorphic as $(W, Z_{0,c}(W_b) / \langle n(0) \rangle) =: Z$-bimodules. Taking global sections gives

\begin{equation}
\Phi_\ast \left( \mathcal{R}[W]_{|\pi^{-1}_W(b)} \right)(\pi^{-1}_W(0)) = \left( \frac{H_{0,c}(W)}{\langle m(b) \rangle} \right) e_W
\end{equation}

and

\begin{equation}
\text{Ind}^W_{W_b} \mathcal{R}[W_b]_{|\pi^{-1}_{W_b}(0)}(\pi^{-1}_{W_b}(0)) = \text{Ind}^W_{W_b} \left( \frac{H_{0,c}(W_b, h)}{\langle n(0) \rangle} \right) e_{W_b}.
\end{equation}

Thus we must show that

\begin{equation}
\text{He} : = \left( \frac{H_{0,c}(W)}{\langle m(b) \rangle} \right) e_W \simeq \text{Ind}^W_{W_b} \left( \frac{H_{0,c}(W_b, h)}{\langle n(0) \rangle} \right) e_{W_b}
\end{equation}

as $(W, Z)$-bimodules. Applying the isomorphism $\theta$ (of (6)) to $\text{He}$, and noting that the restriction of $\theta$ to $\mathcal{C}W$ is the map $\iota$, gives

\begin{equation}
\theta : \text{He} \simeq C \left( W, W_b, \frac{H_{0,c}(W, h)}{\langle n(0) \rangle} \right) \iota(e_W).
\end{equation}

However, we now have two different actions of $Z$ on $\text{He}$. It acts on $\text{He}$, viewed as global sections, via the map $\Phi^\ast$, but acts on the right of $C \left( W, W_b, \frac{H_{0,c}(W, h)}{\langle n(0) \rangle} \right) \iota(e_W)$ via $\theta^{-1}$. These two actions are the same: as stated in (11),

\begin{equation}
\Phi^\ast = S_{W,b}^{-1} \circ \theta^{-1} \circ S_{W,b,0},
\end{equation}

therefore

\begin{equation}
h e_W \cdot \Phi^\ast(z) = h e_W \cdot S_{W,b}^{-1} \circ \theta \circ S_{W,b,0}(z) = h e_W \cdot e_W \cdot \theta^{-1}(e_{W_b} \cdot z) = h e_W \cdot \theta^{-1}(z),
\end{equation}

where $z \in Z$ and $h e_W \in \text{He}$ (recall that $\theta(e_W) = e_{W_b}$, c.f. (11)). Noting that $Z$ is a subalgebra of the centre of $H_{0,c}(W, h) / \langle n(0) \rangle$, the required bimodule isomorphism is given by Lemma 2.1 where $G = W$, $H = W_b$ and $A = H_{0,c}(W, h) / \langle n(0) \rangle$. \hspace{1cm} \Box

Example 3.6. In the case $W = S_n$, $\mathfrak{h} = \mathbb{C}^n$, the Calogero-Moser space $X_c(S_n)$ has been shown by Etingof and Ginzburg, [EG, Theorem 1.23], to be isomorphic to the classical Calogero-Moser space as introduced by Kazhdan, Kostant and Sternberg and studied by Wilson, [W]. It is known to be smooth for $c \neq 0$ ([EG, Corollary 16.2] or [W, Proposition 1.7]), therefore [EG, Theorem 1.7 (i)] implies that $\mathcal{R}[S_n]$ is a vector bundle of rank $n!$ on $X_c(S_n)$. Identifying $\mathbb{C}^n/S_n$ with $\text{Sym}^n(\mathbb{C})$, a point of $\mathbb{C}^n/S_n$ has the form $n_1 x_1 + \cdots + n_k x_k$, where $n_1 + \cdots + n_k = n$ and $x_1, \ldots, x_k \in \mathbb{C}$ are pairwise distinct. Given $b \in \mathbb{C}^n$ such that $S_n \cdot b = n_1 x_1 + \cdots + n_k x_k$, the stabilizer $\left( S_n \right)_b$ is conjugate to $S_{n_1} \times \cdots \times S_{n_k}$. For $W = S_n$, the isomorphism of Corollary 3.3 induces, after factoring out nilpotent elements, an isomorphism of varieties

\begin{equation}
\pi^{-1}_{S_n}(b) \simeq \pi^{-1}_{S_{n_1}}(0) \times \cdots \times \pi^{-1}_{S_{n_k}}(0).
\end{equation}
In [W, Lemma 7.1], Wilson explicitly constructs an isomorphism between the subvarieties of the classical Calogero-Moser space coinciding with the varieties of (13). Let ⊠ denote the external tensor product of vector bundles, then Theorem 3.5 implies that
\[ Φ^∗ \left( ℜ[S_n]|_{_{S_n^{-1}}(b)} \right) \simeq \text{Ind}^{S_n}_{S_{n_1} \times \cdots \times S_{n_k}} \left( ℜ[S_{n_1}]|_{_{S_{n_1}^{-1}}(0)} \boxtimes \cdots \boxtimes ℜ[S_{n_k}]|_{_{S_{n_k}^{-1}}(0)} \right) \]
as $S_n$-equivariant vector bundles. This confirms the conjectured factorization given in [EG, 11.27].

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References


