BP: CLOSE ENCOUNTERS OF THE E_∞ KIND

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Abstract. Inspired by Stewart Priddy's cellular model for the p-local Brown-Peterson spectrum BP, we give a construction of a p-local E_∞ ring spectrum R which is a close approximation to BP. Indeed we can show that if BP admits an E_∞ structure then these are weakly equivalent as E_∞ ring spectra. Our inductive cellular construction makes use of power operations on homotopy groups to define homotopy classes which are then killed by attaching E_∞ cells.

Introduction

The notion of an E_∞ *ring spectrum* arose in the 1970s, and was studied in depth by Peter May *et al* in [8], then later reinterpreted in the framework of [10] as equivalent to that of a *commutative S-algebra*. A great deal of work on the existence of E_∞ structures using various obstruction theories has led to a considerable enlargement of our range of known examples. A useful recent discussion of relationships between various aspects of these topics can be found in [26].

However, despite this, there are some gaps in our knowledge. The question that is a major motivation of this paper is

- *Does the p-local Brown-Peterson spectrum BP for a prime p admit an E_∞ ring structure?*

This has been flagged up as an outstanding problem for almost four decades, despite various attempts to answer it.

Around 1980, Stewart Priddy [31] showed how to build an efficient cellular model for the spectrum BP. This stimulated the later work of [12] (where the basic method was analysed and extended to E_∞ ring spectra), then [5] (where outstanding issues about the spectrum case were addressed) and [3] (where the analogous multiplicative theory was described using topological André-Quillen homology in place of ordinary homology). However, none of this answers the above question!

Some other recent results also add to the uncertainty. Niles Johnson and Justin Noel [14] have shown that for some small primes at least, the natural orientation map of ring spectra $MU \longrightarrow BP$ cannot be E_∞ (or even H_∞). On the other hand, Mike Hill, Tyler Lawson and Niko Naumann [11, 19] have shown that for the primes 2 and 3, $BP\langle 2\rangle$ admits an E_∞ ring structure. Finally, partial results on higher coherence of the multiplication on BP have been

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proved by Birgit Richter [32], and Maria Basterra and Mike Mandell [6] (the latter uses ideas pioneered in an influential but unpublished preprint of Igor Kříž [16]).

Our main purpose in this paper is to give a prescription for constructing a close approach to BP at a prime p. We will show that there is a connective finite type p-local E_∞ ring spectrum R such that the following hold.

- The homotopy $\pi_* R$ is torsion-free.
- There is a morphism of ring spectra $BP \longrightarrow R$ which is a rational weak equivalence.
- If BP admits an E_∞ ring structure then there is a weak equivalence of E_∞ ring spectra $R \xrightarrow{\sim} BP$.

Our construction proceeds in two main stages, the first of which yields a morphism of p-local E_∞ ring spectra $R_\infty \longrightarrow MU_{(p)}$ so that the composition

$$R_\infty \longrightarrow MU_{(p)} \xrightarrow{\varepsilon} BP$$

with the Quillen projection ε is a morphism of ring spectra which induces an epimorphism on $\pi_*(-)$ and is a rational equivalence. The second stage gives a morphism of E_∞ ring spectra $R_\infty \longrightarrow R$ which is a rational equivalence and where $\pi_*(R)$ is torsion free. One source of difficulty with our construction is that if R is an E_∞ realisation of BP, then there can be no E_∞ morphism $R \longrightarrow MU_{(p)}$ by [12, theorem 2.11]. If we could produce any map of spectra $R \longrightarrow BP$ which is an equivalence on the bottom cell then the composition $BP \longrightarrow R \longrightarrow BP$ would be a weak equivalence and so would each of the maps $BP \longrightarrow R$ and $R \longrightarrow BP$.

1. Attaching E_∞ cells to commutative S-algebras

We recall the idea of attaching E_∞ cells to a commutative S-algebra. Details can be found in [10], and it was exploited in [3] to describe topological André-Quillen homology of CW commutative S-algebras. We will make use of various obstructions involving free commutative S-algebras. Recall from [10] that if X is an S-module then the free commutative S-algebra on X is

$$\mathbb{P}X = \mathbb{P}_S X = \bigvee_{r \geqslant 0} X^{(r)}/\Sigma_r.$$

When X is cofibrant, for each $r \geqslant 1$ the natural projection provides a weak equivalence

$$E\Sigma_r \ltimes_{\Sigma_r} X^{(r)} \xrightarrow{\sim} X^{(r)}/\Sigma_r. \tag{1.1}$$

Let E be a commutative S-algebra and let $f\colon \bigvee_i S^n \longrightarrow E$ be a map from a finite wedge of n-spheres. Then there is a unique extension of f to a morphism of commutative S-algebras $\widetilde{f}\colon \mathbb{P}(\bigvee_i S^n) \longrightarrow E$ from the free commutative S-algebra on $\bigvee_i S^n$. Then the pushout diagram of commutative S-algebras

$$\begin{tikzcd}
\mathbb{P}(\bigvee_i S^n) \arrow[r, "\widetilde{f}"] \arrow[d, "\mathbb{P}(\mathrm{inc})"'] \arrow[dr, phantom, "\ulcorner"] & E \arrow[d] \\
\mathbb{P}(\bigvee_i D^{n+1}) \arrow[r] & E//f
\end{tikzcd}$$

defines $E//f$ which we can regard as obtained from E by attaching E_∞ cells. In fact, we can take

$$E//f = \mathbb{P}(\bigvee_i D^{n+1}) \wedge_{\mathbb{P}(\bigvee_i S^n)} E$$

where $\mathbb{P}(\bigvee_i D^{n+1})$ and E are $\mathbb{P}(\bigvee_i S^n)$-algebras in the evident way.

The homology of extended powers has been well studied and we can deduce the following.

Proposition 1.1. *For $n \in \mathbb{N}$, we have*

$$H_*(\mathbb{P}S^{2n-1};\mathbb{Q}) = \Lambda_{\mathbb{Q}}(x_{2n-1}), \quad H_*(\mathbb{P}S^{2n};\mathbb{Q}) = \mathbb{Q}[x_{2n}],$$

where $x_m \in H_m(\mathbb{P}S^m;\mathbb{Q})$ is the image of the homology generator of $H_m(S^m;\mathbb{Q})$.

Proof. The weak equivalences of (1.1) combine to give a weak equivalence

$$\bigvee_{r\geqslant 0} E\Sigma_r \ltimes_{\Sigma_r} (S^{2n-1})^{(r)} \xrightarrow{\sim} \bigvee_{r\geqslant 0} (S^{2n-1})^{(r)}/\Sigma_r = \mathbb{P}S^{2n-1}.$$

By [20, chapter VIII], for $r \geqslant 2$ we have

$$H_*(E\Sigma_r \ltimes_{\Sigma_r} (S^{2n-1})^{(r)};\mathbb{Q}) = H_*((S^{2n-1})^{(r)};\mathbb{Q})_{\Sigma_r} = 0,$$

since the permutation action of Σ_r on the factors is equivalent to the sign representation,

$$H_*((S^{2n-1})^{(r)};\mathbb{Q}) \cong \mathbb{Q}^-,$$

which is a summand of the regular representation $\mathbb{Q}[\Sigma_r]$, hence it has trivial cohomology, and in particular trivial coinvariants. Thus we have

$$H_*(\mathbb{P}S^{2n-1};\mathbb{Q}) = \Lambda_{\mathbb{Q}}(x_{2n-1}),$$

where

$$x_{2n-1} \in H_{2n-1}(D_1S^{2n-1};\mathbb{Q}) \cong H_{2n-1}(E\Sigma_r \ltimes_{\Sigma_r} (S^{2n-1})^{(r)};\mathbb{Q}).$$

Similarly,

$$\mathbb{P}S^{2n} = \bigvee_{r\geqslant 0} D_rS^{2n} \sim \bigvee_{r\geqslant 0} E\Sigma_r \ltimes_{\Sigma_r} (S^{2n})^{(r)},$$

but this time the Σ_r action on the factors is trivial giving

$$H_*((S^{2n-1})^{(r)};\mathbb{Q}) \cong \mathbb{Q},$$

hence

$$H_*(E\Sigma_r \ltimes_{\Sigma_r} (S^{2n})^{(r)};\mathbb{Q}) = H_*((S^{2n})^{(r)};\mathbb{Q})_{\Sigma_r} = \mathbb{Q}$$

concentrated in degree $2nr$. It follows easily that $H_*(\mathbb{P}S^{2n};\mathbb{Q})$ is polynomial on the stated generator. □

The next result is fundamental, see [8, 17, 18, 24, 25]. Here we use the convention that the excess of the empty exponent sequence is $\mathrm{exc}(\varnothing) = \infty$.

Theorem 1.2. *If X is connective then for a prime p, $H_*(\mathbb{P}X;\mathbb{F}_p)$ is the free commutative graded $\mathbb{F}_p$-algebra generated by elements $Q^I x_j$, where x_j for $j \in J$ gives a basis for $H_*(X;\mathbb{F}_p)$, and $I = (\varepsilon_1, i_1, \varepsilon_2, \dots, \varepsilon_\ell, i_\ell)$ is admissible with $\mathrm{exc}(I)+\varepsilon_1 > |x_j|$ when p is odd, while $I = (i_1, \dots, i_\ell)$ is admissible with $\mathrm{exc}(I) > |x_j|$ when $p = 2$.*

Using the notation $R\langle G\rangle$ for the free commutative graded algebra over R on a collection of homogeneous generators G, this gives the following formulae. Thus for p odd,

$$H_*(\mathbb{P}X;\mathbb{F}_p) = \mathbb{F}_p\left\langle Q^I x_j : j \in J,\ \mathrm{exc}(I) + \varepsilon_1 > |x_j|\ \right\rangle$$

is polynomial on the stated generators with $|Q^I x_j|$ even and exterior on those generators with $|Q^I x_j|$ odd, while for $p = 2$,

$$\begin{aligned} H_*(\mathbb{P}X;\mathbb{F}_2) &= \mathbb{F}_2\left\langle Q^I x_j : j \in J,\ \mathrm{exc}(I) > |x_j|\ \right\rangle \\ &= \mathbb{F}_2[Q^I x_j : j \in J,\ \mathrm{exc}(I) > |x_j|\ ]. \end{aligned}$$

Remark 1.3. If E is a commutative S-algebra and that $f\colon X \longrightarrow E$ is a map of spectra for which the induced homomorphism

$$f_*\colon H_*(X;\mathbb{F}_p) \longrightarrow H_*(E;\mathbb{F}_p)$$

is trivial. Then by Theorem 1.2, the induced ring homomorphism

$$\widetilde{f}_*\colon H_*(\mathbb{P}X;\mathbb{F}_p) \longrightarrow H_*(E;\mathbb{F}_p)$$

is also trivial since it is a homomorphism of algebras over the Dyer-Lashof algebra.

We record some results on the attaching of E_∞ cones to commutative S-algebras and its effect on ordinary homology. We will make repeated use of the Künneth spectral sequence of [10]. By [4] this is multiplicative, and for a prime p an extension of the work of [6, 13] shows that it has Dyer-Lashof operations.

First we give some easy observations on rational homology.

Proposition 1.4. *Suppose that E is a connective commutative S-algebra and let $n \in \mathbb{N}$.*
(a) *If $f\colon S^{2n-1} \longrightarrow E$ is a map for which the induced homomorphism $f_*\colon H_*(S^{2n-1};\mathbb{Q}) \longrightarrow H_*(E;\mathbb{Q})$ is trivial, then*

$$H_*(E//f;\mathbb{Q}) = H_*(E;\mathbb{Q})[w],$$

where $w \in H_{2n}(E//f;\mathbb{Q})$.
(b) *If $f\colon S^{2n} \longrightarrow E$ is a map for which the induced homomorphism $f_*\colon H_*(S^{2n};\mathbb{Q}) \longrightarrow H_*(E;\mathbb{Q})$ is trivial, then*

$$H_*(E//f;\mathbb{Q}) = \Lambda_{H_*(E;\mathbb{Q})}[z],$$

where $z \in H_{2n+1}(E//f;\mathbb{Q})$.

Proof. Recall Proposition 1.1.
(a) There is a multiplicative Künneth spectral sequence [4, 10] of form

$$\mathrm{E}^2_{s,t} = \mathrm{Tor}^{H_*(\mathbb{P}S^{2n-1};\mathbb{Q})}_{s,t}(\mathbb{Q}, H_*(E;\mathbb{Q})) \Longrightarrow H_{s+t}(E//f;\mathbb{Q}),$$

where we have

$$\mathrm{E}^2_{*,*} = \Gamma_{H_*(E;\mathbb{Q})}(w) = H_*(E;\mathbb{Q})[w],$$

with $w \in \mathrm{E}^2_{1,2n-1}$. As w is an infinite cycle for degree reasons, the result follows.
(b) Here the relevant Künneth spectral sequence

$$\mathrm{E}^2_{s,t} = \mathrm{Tor}^{H_*(\mathbb{P}S^{2n};\mathbb{Q})}_{s,t}(\mathbb{Q}, H_*(E;\mathbb{Q})) \Longrightarrow H_{s+t}(E//f;\mathbb{Q}),$$

has

$$\mathrm{E}^2_{*,*} = \Lambda_{H_*(E;\mathbb{Q})}(z)$$

with $z \in \mathrm{E}^2_{1,2n}$ which is an infinite cycle for degree reasons. □

Of course we can replace a single sphere by a wedge of spheres in this result.

Now we will describe analogous results in positive characteristic. When the context makes this unambiguous, we will often write $H_*(-)$ for $H_*(-;\mathbb{F}_p)$ and $\otimes$ for $\otimes_{\mathbb{F}_p}$. In the following, a *p-truncated algebra* will mean a quotient $\mathbb{F}_p$-algebra of the form

$$\mathrm{TP}_p(x) = \mathbb{F}_p[x]/(x^p).$$

and we will denote this by $\mathrm{TP}(x)$ when the prime p is clear. It is standard that a divided power algebra on an element x,

$$\Gamma_{\mathbb{F}_p}(x) = \mathbb{F}_p\{1, \gamma_1(x), \gamma_2(x), \ldots\},$$

is a tensor product of p-truncated algebras:

$$\Gamma_{\mathbb{F}_p}(x) = \bigotimes_{r\geqslant 0} \mathrm{TP}(\gamma_{p^r}(x)).$$

Here the product is given by

$$\gamma_r(x)\gamma_s(x) = \binom{r+s}{r}\gamma_{r+s}(x)$$

and so for every r,

$$\gamma_r(x)^p = 0.$$

Furthermore, if r has p-adic expension

$$r = r_0 + r_1 p + \cdots + r_\ell p^\ell,$$

where $r_i = 0, 1, \ldots, p-1$, then there is a non-zero element $c_r \in \mathbb{F}_p$ for which

$$\gamma_r(x) = c_r \gamma_1(x)^{r_0}\gamma_p(x)^{r_1}\cdots\gamma_{p^\ell}(x)^{r_\ell}.$$

To prove the odd primary case in our next result, we make use of work of Hunter [13].

Proposition 1.5. *Let p be an odd prime and let E be a connective commutative S-algebra. Suppose that $n \in \mathbb{N}$ and $f\colon S^{2n-1} \longrightarrow E$ is a map for which the induced homomorphism $f_*\colon H_*(S^{2n-1};\mathbb{F}_p) \longrightarrow H_*(E;\mathbb{F}_p)$ is trivial. Then the Künneth spectral sequence*

$$\mathrm{E}^2_{s,t} = \mathrm{Tor}^{H_*(\mathbb{P}S^{2n-1};\mathbb{F}_p)}_{s,t}(\mathbb{F}_p, H_*(E;\mathbb{F}_p)) \Longrightarrow H_{s+t}(E//f;\mathbb{F}_p)$$

has the following properties.

(a) *The homology of $\mathbb{P}S^{2n-1}$ is the free commutative graded algebra*

$$H_*(\mathbb{P}S^{2n-1};\mathbb{F}_p) = \mathbb{F}_p\left\langle Q^I x_{2n-1} : \mathrm{exc}(I) + \varepsilon_1 > 2n-1 \right\rangle. \tag{1.2}$$

(b) *The E^2-term is a tensor product*

$$\mathrm{E}^2_{*,*} = H_*(E;\mathbb{F}_p) \otimes \mathcal{D} \otimes \mathcal{E},$$

of subalgebras, where $\mathcal{D}$ and $\mathcal{E}$ have the following descriptions:

- *$\mathcal{D}$ is a tensor product of infinitely many divided power algebras each having the form $\Gamma_{\mathbb{F}_p}([Q^I x_{2n-1}])$ with a generator $[Q^I x_{2n-1}] \in \mathrm{E}^2_{1,|Q^I x_{2n-1}|}$ for each odd degree exterior generator occurring in (1.2);*
- *$\mathcal{E}$ is an exterior algebra with a generator $[Q^I x_{2n-1}] \in \mathrm{E}^2_{1,|Q^I x_{2n-1}|}$ for each even degree polynomial generator $Q^I x_{2n-1}$ occurring in (1.2).*

(c) *In the above spectral sequence,*

$$\mathrm{E}^2_{*,*} = \cdots = \mathrm{E}^{p-1}_{*,*}, \quad \mathrm{E}^p_{*,*} = \mathrm{E}^\infty_{*,*},$$

where the differential d^{p-1} *acts on the divided power generators of* $\mathcal{D}$ *by*

$$d^{p-1}\gamma_{p^r}([Q^I x_{2n-1}]) \doteq \begin{cases} [\beta Q^{(|Q^I x_{2n-1}|+1)/2} Q^I x_{2n-1}]\gamma_{p^r-p}([Q^I x_{2n-1}]) & \text{if } r \geqslant 1, \\ 0 & \text{if } r = 0, \end{cases}$$

where $\doteq$ *means 'equal up to multiplication by a unit in* $\mathbb{F}_p$*'.*

Proof. Using a standard Koszul resolution over the free algebra $H_*(\mathbb{P}S^{2n-1})$ we obtain the stated form for the E^2-term. The statement about the differentials involves a suitable reinterpretation of [13, proposition 11] together with the multiplicative structure of the spectral sequence. □

The situation for $p = 2$ is simpler to describe and we state it in greater generality than we actually need for the present work.

Proposition 1.6. *Let* $p = 2$ *and let* E *be a connective commutative* S*-algebra. Suppose that* $0 \leqslant n \in \mathbb{Z}$ *and* $f\colon S^n \longrightarrow E$ *is a map whose induced homomorphism* $f_*\colon H_*(S^n;\mathbb{F}_2) \longrightarrow H_*(E;\mathbb{F}_2)$ *is trivial. Then the Künneth spectral sequence*

$$\mathrm{E}^2_{s,t} = \mathrm{Tor}^{H_*(\mathbb{P}S^n;\mathbb{F}_2)}_{s,t}(\mathbb{F}_2, H_*(E;\mathbb{F}_2)) \Longrightarrow H_{s+t}(E//f;\mathbb{F}_2)$$

has the following properties.
(a) *The homology of* $\mathbb{P}S^n$ *is the graded polynomial algebra*

$$H_*(\mathbb{P}S^n;\mathbb{F}_2) = \mathbb{F}_2[Q^I x_n : \mathrm{exc}(I) > n\ ].$$

(b) *The* E^2*-term is the exterior algebra*

$$\mathrm{E}^2_{*,*} = \Lambda_{H_*(E;\mathbb{F}_2)}([Q^I x_n] : \mathrm{exc}(I) > n),$$

with a generator $[Q^I x_n] \in \mathrm{E}^2_{1,|Q^I x_n|}$ *for each polynomial generator* $Q^I x_n$ *in* (a).
(c) *This spectral sequence has trivial differentials from* E^2 *onwards.*

Proof. Using a standard Koszul resolution over the free algebra $H_*(\mathbb{P}S^n)$ we obtain the stated form for the E^2-term. The exterior generators all lie in $\mathrm{E}^2_{1,*}$ and so must be infinite cycles. The multiplicative structure of the spectral sequence shows that all differentials are trivial. □

Theorem 1.7. *Let* p *be a prime. Suppose that* E *is a connective commutative* S*-algebra and that* $f\colon S^{2n-1} \longrightarrow E$ *is a map for which the induced homomorphism* $f_*\colon H_*(S^{2n-1};\mathbb{F}_p) \longrightarrow H_*(E;\mathbb{F}_p)$ *is trivial. Then there is an element* $u \in H_{2n}(E//f;\mathbb{F}_p)$ *such that if* p *is odd,*

$$H_*(E//f;\mathbb{F}_p) = H_*(E;\mathbb{F}_p)\left\langle Q^I u : \mathrm{exc}(I) + \varepsilon_1 > 2n, \text{ with } i_1 = 0 \text{ if } \varepsilon_1 = 1 \text{ and } |Q^I u| \text{ odd}\right\rangle,$$

while if $p = 2$*,*

$$H_*(E//f;\mathbb{F}_2) = H_*(E;\mathbb{F}_2)[Q^I u : \mathrm{exc}(I) > 2n].$$

Proof. Taking into account the results of Propositions 1.5 and 1.6, we find that the E^∞-term is a tensor product of algebras

$$\mathrm{E}^\infty_{*,*} = H_*(E) \otimes \mathcal{D}' \otimes \mathcal{E}',$$

where

- $\mathcal{D}'$ is a tensor product of infinitely many p-truncated algebras each having the form $\mathrm{TP}([Q^I x_{2n-1}])$ with a generator $[Q^I x_{2n-1}] \in \mathrm{E}^{\infty}_{1,|Q^I x_{2n-1}|}$ for each exterior generator in (1.2),
- $\mathcal{E}'$ is an exterior algebra with a generator $[Q^I x_{2n-1}] \in \mathrm{E}^{\infty}_{1,|Q^I x_{2n-1}|}$ for each polynomial generator $Q^I x_{2n-1}$ listed in (1.2).

When p is odd, each exterior generator $Q^I x_{2n-1}$ is of odd degree so it gives rise to a p-truncated algebra $\mathrm{TP}([Q^I x_{2n-1}])$ concentrated in even degrees. When $p = 2$, $\mathcal{D}'$ is trivial and $\mathcal{E}'$ is generated by elements $[Q^I x_{2n-1}]$ satisfying $[Q^I x_{2n-1}]^2 = 0$. In each case we need to show these generators represent elements which are not nilpotent in $H_*(E//\gamma)$. We do this using Dyer-Lashof operations, using a well known argument, see for example [21].

If p is an odd prime, set

$$k = \frac{|Q^I x_{2n-1}| + 1}{2},$$

and if $p = 2$, take

$$k = |Q^I x_{2n-1}| + 1.$$

Then in the E^2-term we have

$$Q^k[Q^I x_{2n-1}] = [Q^k Q^I x_{2n-1}] \neq 0$$

since $Q^k Q^I$ is admissible. This shows that in $H_*(E//\gamma)$ $[Q^I x_{2n-1}]$ represents an element whose p-th power is represented by $[Q^k Q^I x_{2n-1}]$, thus resolving the multiplicative extensions in the filtration. □

Remark 1.8. When $p = 2$ this results applies to all the cases of Proposition 1.6. Thus for $f\colon S^n \longrightarrow E$ with $n \geqslant 0$ we have

$$H_*(E//f; \mathbb{F}_2) = H_*(E; \mathbb{F}_2)[Q^I u : \mathrm{exc}(I) > n + 1].$$

2. Power operations for E_∞ ring spectra

We refer to [7] for work on power operations, in particular Bruner's chapters IV and V. Our main use of this is in connection with applying 'the first operation above the p-th power' $\beta\mathcal{P}^{k+1}$ to give a homotopy element of degree $2k$. Here are the results we will use.

At the prime 2, we have

Theorem 2.1. *Suppose that E is a connective 2-local E_∞ ring spectrum for which $0 = \eta 1 \in \pi_1 E$. Then for $r \geqslant 1$, the operation $\mathcal{P}^{2^{r+1}-1}$ is defined on $\pi_{2^{r+1}-2}E$, giving a map*

$$\mathcal{P}^{2^{r+1}-1}\colon \pi_{2^{r+1}-2}E \longrightarrow \pi_{2^{r+2}-3}E.$$

Moreover, the indeterminacy is trivial and the operation $2\mathcal{P}^{2^{r+1}-1}$ is trivial.

Proof. We will write $n = 2^{r+1} - 2$.

Applying [7, proposition V.1.5] to the skeleton $D^1_2 S^{2n}$, we have

$$i = 1, \quad j = n + 1, \quad \varphi(i) = 1,$$

and so

$$n \equiv -2 \mod (2),$$

hence the operation $\mathcal{P}^{2^{r+1}-1}$ is defined on $\pi_{2^{r+1}-2}E$. Also, by [7, theorem V.1.8] we have with $j = a = b = 0$ and $w \in \pi_{2^{r+1}-2}E$,

$$2\mathcal{P}^{2^{r+1}-1}w = 0$$

since by assumption the natural map $\pi_1 S \longrightarrow \pi_1 E$ is trivial. Similarly, since $n \equiv 2 \bmod (4)$, the indeterminacy is trivial by [7, table V.1.3]. □

For odd primes we have

Theorem 2.2. *Let p be an odd prime. Suppose that E is a connective p-local E_∞ ring spectrum for which $0 = \alpha_1 1 \in \pi_{2p-3}E$.*
Then for $r \geqslant 1$, the operation $\beta\mathcal{P}^{p^r}$ is defined on $\pi_{2(p^r-1)}E$ giving a map

$$\beta\mathcal{P}^{p^r} : \pi_{2(p^r-1)}E \longrightarrow \pi_{2(p^{r+1}-1)-1}E.$$

Moreover the indeterminacy is trivial and the operation $p\beta\mathcal{P}^{p^r}$ is trivial.

Proof. We will assume that all spectra are localised at p. Recall that $\alpha_1 \in \pi_{2p-3}S$ is a non-zero p-primary stable homotopy element of lowest positive degree.

Using the results and notation of [7, proposition V.1.5], the fact that this operation is defined on $\pi_{2(p^r-1)}E$ this follows since

$$\psi\Big(2p^r(p-1) - 1 - 2(p^r-1)(p-1)\Big) = \psi(2(p-1)-1) = \left\lfloor \frac{2(p-1)-1}{2(p-1)} \right\rfloor = 0.$$

For triviality of the indeterminacy, see [7, table V.1.1].

By [7, theorem V.1.8], for each $y \in \pi_{2(p^r-1)}E$ there is an element $\alpha \in \pi_{2(p-1)-2}S$ for which

$$p(\beta\mathcal{P}^{p^r}y) = \alpha y^p.$$

But $\alpha \in \pi_{2(p-2)}S = 0$, hence $p(\beta\mathcal{P}^{p^r})$ is indeed trivial. □

The next result tells us how this works in the Adams spectral sequence in good situations.

Lemma 2.3. *Let p be a prime*
(i) *If $p = 2$, then under the assumptions of* Theorem 2.1, *if $w \in \pi_{2^{r+1}-2}E$ is detected in the 1-line of the Adams spectral sequence by $W \in \mathrm{Ext}^{1,2^{r+1}-1}_{\mathcal{A}(2)_*}(\mathbb{F}_2, H_*E)$, then $\mathcal{P}^{2^{r+1}-1}w$ is detected in the 1-line by*

$$\mathcal{P}^{2^{r+1}-1}W \in \mathrm{Ext}^{1,2^{r+2}-2}_{\mathcal{A}(2)_*}(\mathbb{F}_2, H_*E),$$

where $\mathcal{P}^{2^{r+1}-1}$ is the algebraic Steenrod operation of [7, 23, 29]. *This can be calculated by applying the Dyer-Lashof operation $Q^{2^{r+1}-1}$ to the element of $\mathcal{A}(2)_* \otimes H_*E = H_*(H \wedge E)$ representing W.*
(ii) *If p is odd, then the assumptions of* Theorem 2.2, *if $w \in \pi_{2(p^r-1)}E$ is detected in the 1-line of the Adams spectral sequence by $W \in \mathrm{Ext}^{1,2p^r-1}_{\mathcal{A}(p)_*}(\mathbb{F}_p, H_*E)$, then $\beta\mathcal{P}^{p^r}w$ is detected in the 1-line by*

$$\beta\mathcal{P}^{p^r}W \in \mathrm{Ext}^{1,2(p^{r+2}-1)}_{\mathcal{A}(p)_*}(\mathbb{F}_p, H_*E),$$

where $\beta\mathcal{P}^{p^r}$ is the algebraic Steenrod operation of [7, 23, 29]. *This can be calculated by applying the Dyer-Lashof operation βQ^{p^r} to the element of $\mathcal{A}(p)_* \otimes H_*E = H_*(H \wedge E)$ representing W.*

Proof. This follows from work of Milgram and Bruner [7, 29]. Note that for $p = 2$, [7, theorem 2.5(i)] should read

$$\beta\mathcal{P}^i \colon \mathrm{Ext}^{s,t} \longrightarrow \mathrm{Ext}^{s+t-i,2t}.$$ □

3. Outline of a construction

In this section and later ones, we will always be working with (connective) p-local spectra for some prime p. When referring to cells, finite type conditions, etc, we will always mean in that context.

Starting with the p-local sphere S, we will construct a sequence of commutative S-algebras

$$S = R_0 \longrightarrow R_1 \longrightarrow \cdots \longrightarrow R_{n-1} \longrightarrow R_n \longrightarrow \cdots, \tag{3.1}$$

where R_n is obtained from R_{n-1} by attaching a single E_∞ cell of dimension $2(p^n-1)$. The rational homotopy of the colimit $R_\infty = \operatorname{colim}_n R_n$ is

$$\mathbb{Q} \otimes \pi_* R_\infty = \mathbb{Q}[u_n : n \geqslant 1],$$

where $u_n \in \pi_{2(p^n-1)} R_\infty$ arises in $\pi_{2(p^n-1)} R_n$. Next we could inductively kill the torsion part of the homotopy of R by non-trivally attaching E_∞ cones on Moore spectra, thus we do not change the rational homotopy. Then we obtain a commutative R_∞-algebra R for which

$$\mathbb{Q} \otimes \pi_* R = \mathbb{Q} \otimes \pi_* R = \mathbb{Q}[u_n : n \geqslant 1].$$

4. Construction of the R_n

We begin with the construction of the sequence (3.1). We will use the notation $u_0 = p$.

Let $n \geqslant 1$. Suppose that a sequence of cofibrations of commutative S-algebras

$$S = R_0 \longrightarrow R_1 \longrightarrow \cdots \longrightarrow R_{n-1}$$

exists in which there are compatible homotopy elements $u_r \in \pi_{2(p^r-1)} R_k$ for $0 \leqslant r \geqslant k$, satisfying

$$\mathbb{Q} \otimes \pi_* R_k = \mathbb{Q} \otimes \pi_* R_{k-1}[u_k].$$

Then by Theorem 2.2, assuming that it is not trivial, the element $\beta \mathcal{P}^{p^{n-1}} u_{n-1}$ is of order p; we let $f_n \colon S^{2p^n-3} \longrightarrow R_{n-1}$ be a representative of this homotopy class. Thus as in [3, 10] we can form the pushout diagram of commutative S-algebras

$$\begin{tikzcd} \mathbb{P}S^{2p^n-3} \arrow[r, "\widetilde{f}_n"] \arrow[d] & R_{n-1} \arrow[d] \\ \mathbb{P}D^{2p^n-2} \arrow[r] & R_n \end{tikzcd}$$

in which $\widetilde{f}_n$ is the extension of f_n to a map from the free commutative S-algebra $\mathbb{P}S^{2p^n-3}$. We remark that we can work equally well with commutative R_{n-1}-algebras and define R_n using the pushout diagram

$$\begin{tikzcd} \mathbb{P}_{R_{n-1}} S^{2p^n-3} \arrow[r, "\widetilde{f}_n"] \arrow[d] & R_{n-1} \arrow[d] \\ \mathbb{P}_{R_{n-1}} D^{2p^n-2} \arrow[r] & R_n \end{tikzcd}$$

and we will make use of both viewpoints. We also have

$$R_n \cong R_{n-1} \wedge_{\mathbb{P}S^{2p^n-3}} \mathbb{P}D^{2p^n-2} \cong R_{n-1} \wedge_{\mathbb{P}_{R_{n-1}} S^{2p^n-3}} \mathbb{P}_{R_{n-1}} D^{2p^n-2}$$

Since f_n has order p, there is a commutative diagram of R_{n-1}-modules

$$\begin{tikzcd}
 & & S^{2p^n-2} \arrow[d, dotted] \arrow[dr, "p"] \arrow[ddl, dashed] & \\
S^{2p^n-3} \arrow[r, "f_n"] & R_{n-1} \arrow[r] & C_{f_n} \arrow[r] \arrow[d] & S^{2p^n-2} \\
 & & R_n &
\end{tikzcd}$$

in which the dashed arrow provides a homotopy class $u_n \in \pi_{2p^n-2}R_n$.

There is a Kunneth spectral sequence [10] of the form

$$\mathrm{E}^2_{r,s} = \mathrm{Tor}_{s,t}^{\mathbb{Q}\otimes\pi_*R_{n-1}[w_{2p^n-3}]}(\mathbb{Q}\otimes\pi_*R_{n-1}, \mathbb{Q}\otimes\pi_*R_{n-1}) \Longrightarrow \mathbb{Q}\otimes\pi_{s+t}R_n,$$

where

$$\mathbb{Q}\otimes\pi_*R_{n-1}[w_{2p^n-3}] = \mathbb{Q}\otimes\pi_*\mathbb{P}_{R_{n-1}}S^{2p^n-3}$$

is an exterior algebra, so

$$\mathrm{E}^2_{r,s} = \mathbb{Q}\otimes\pi_*R_{n-1}[U_n]$$

with generator U_n of bidegree $(1, 2p^n-3)$. Thus the spectral sequence collapses and we easily obtain

$$\mathbb{Q}\otimes\pi_*R_n = \mathbb{Q}\otimes\pi_*R_{n-1}[u_n].$$

We still need to verify the following key result.

Lemma 4.1. *The element $\beta\mathcal{P}^{p^{n-1}}u_{n-1} \in \pi_{2p^n-3}R_{n-1}$ is non-zero and has order p. Furthermore, the mod p Hurewicz image of u_n is trivial.*

Passing to the limit, we see that since each morphism $R_{n-1} \longrightarrow R_n$ is a cofibration,

$$R_\infty = \operatorname{hocolim}_n R_n$$

and

$$\pi_*R_\infty = \operatorname{colim}_n \pi_*R_n.$$

Working rationally this gives

$$\mathbb{Q}\otimes\pi_*R_\infty = \mathbb{Q}\otimes\pi_*S[u_n : n \geqslant 1] = \mathbb{Q}[u_n : n \geqslant 1].$$

5. Killing the torsion

The homotopy of the commutative S-algebra R_∞ has finite type and R is a CW commutative S-algebra with one E_∞ cell in each degree of the form $2(p^n-1)$ with $n \geqslant 1$.

Now we proceed to kill the torsion in π_*R_∞ by induction on degree. Let $R^0 = R_\infty$. Suppose that we have constructed $R^0 \longrightarrow R^{m-1}$ so that π_kR^{m-1} is torsion free for $k \leqslant m-2$ and the natural map induces an isomorphism

$$\mathbb{Q}\otimes\pi_*R^0 \cong \mathbb{Q}\otimes\pi_*R^{m-1}.$$

Now following [3, 5] we attach m-cells minimally to kill the torsion of $\pi_{m-1}R^{m-1}$. In fact, following a suggestion of Tyler Lawson, we can do slightly more: factoring the attaching maps through Moore spectra of the form $S^{m-1} \cup_{p^r} D^m$, we can define R^m using the pushout diagram

$$\begin{array}{ccc} \mathbb{P}(\bigvee_i S^{m-1} \cup_{p^{r_i}} D^m) & \longrightarrow & R^{m-1} \\ \downarrow & & \downarrow \\ \mathbb{P}(\bigvee_i C(S^{m-1} \cup_{p^{r_i}} D^m)) & \longrightarrow & R^m \end{array}$$

and so we have

$$\mathbb{Q} \otimes \pi_* R^0 \cong \mathbb{Q} \otimes \pi_* R^m.$$

Continuing in this way, we obtain a sequence of cofibrations

$$R_\infty = R^0 \longrightarrow R^1 \longrightarrow \cdots \longrightarrow R^{m-1} \longrightarrow R^m \longrightarrow \cdots$$

whose limit is

$$R = \operatorname*{colim}_m R^m = \operatorname*{hocolim}_m R^m.$$

Furthermore, the natural map $R_\infty \longrightarrow R$ induces an epimorphism

$$\pi_* R_\infty \longrightarrow \pi_* R$$

and a rational isomorphism

$$\mathbb{Q} \otimes \pi_* R \longrightarrow \mathbb{Q} \otimes \pi_* R_\infty = \mathbb{Q}[u_n : n \geqslant 1].$$

6. Some recursive formulae

We give the odd primary case first, the 2-primary case is similar.

The case $p > 2$. Let p be an odd prime and assume that all spectra are p-local. Starting with $R_0 = S$, the p-local sphere, we will inductively assume that there is a sequence of E_∞ ring spectra

$$S = R_0 \longrightarrow R_1 \longrightarrow \cdots \longrightarrow R_{n-1}$$

so that the following hold:
(A) for $1 \leqslant r \leqslant n$ there are homotopy classes $\alpha_{[r]} \in \pi_{2(p^r-1)-1}R_{r-1}$ of order p, and homology classes $z_r \in H_{2(p^r-1)}(R_r; \mathbb{F}_p)$;
(B) the $\mathcal{A}(p)_*$-coaction is given by

$$\psi(z_r) = 1 \otimes z_r + \zeta_1 \otimes z_{r-1}^p + \zeta_2 \otimes z_{r-2}^{p^2} + \cdots + \zeta_{r-1} \otimes z_1^{p^{r-1}} + \zeta_r \otimes 1, \tag{6.1}$$

where we identify $z_i \in H_*(R_r; \mathbb{F}_p)$ with the image of $z_i \in H_*(R_i; \mathbb{F}_p)$ under the induced homomorphism $H_*(R_i; \mathbb{F}_p) \longrightarrow H_*(R_r; \mathbb{F}_p)$ whenever $i < r$;
(C) $\alpha_{[r]}$ is detected in filtration 1 the Adams spectral sequence by the class with cobar representative

$$\zeta_1 \otimes z_{r-1}^p + \zeta_2 \otimes z_{r-2}^{p^2} + \cdots + \zeta_{r-1} \otimes z_1^{p^{r-1}} + \zeta_r \otimes 1, \tag{6.2}$$

and $\alpha_{[1]} = \alpha_1 \pi_{2p-3} S$.

Given this data, we construct the morphism of E_∞ ring spectra $R_{n-1} \longrightarrow R_n$ as follows.

Choose a representative $f_n\colon S^{2p^n-3} \longrightarrow R_{n-1}$ for $\alpha_{[n]}$. Attach an E_∞ cone to R_{n-1} by forming the pushout $R_{n-1}//\alpha_{[n]}$ in the diagram

$$\begin{tikzcd}
\mathbb{P}S^{2p^n-3} \arrow[r, "\widetilde{f}_n"] \arrow[d] & R_{n-1} \arrow[d] \\
\mathbb{P}D^{2p^n-2} \arrow[r] & R_{n-1}//\alpha_{[n]}
\end{tikzcd}$$

and set $R_n = R_{n-1}//\alpha_{[n]}$. Since $\alpha_{[n]}$ has order p, there is a commutative diagram of S-modules

$$\begin{tikzcd}
 & & S^{2p^n-2} \arrow[dl, dashed] \arrow[d, dotted] \arrow[dr, "p"] & \\
S^{2p^n-3} \arrow[r, "f_n"] & R_{n-1} \arrow[r] \arrow[dr, dashed] & C_{f_n} \arrow[r] \arrow[d] & S^{2p^n-2} \\
 & & R_n &
\end{tikzcd}$$

in which the dashed arrow provides a homotopy class $u_n \in \pi_{2p^n-2}R_n$. The homology class z_n is represented by the image of the ordinary cell attached to form the mapping cone C_{f_n}.

Lemma 6.1. *The homotopy class u_n lies in the Toda bracket $\langle p, \alpha_{[n]}, 1\rangle \subseteq \pi_{2(p^n-1)}R_n$, and in the Adams spectral sequence it has filtration 1 and cobar representative*

$$\bar{\tau}_0 \otimes z_n + \bar{\tau}_1 \otimes z_{n-1}^p + \bar{\tau}_2 \otimes z_{n-2}^{p^2} + \cdots + \bar{\tau}_{n-1} \otimes z_1^{p^{n-1}},$$

where $\bar{\tau}_j$ denotes the conjugate of the exterior generator $\tau_j \in \mathcal{A}(p)_{2p^j-1}$.

Proof. This Toda bracket should be interpreted in the sense of modules over R_{n-1}. Thus the first two variables are in π_*R_{n-1} while the last is in π_*R_n viewed as a module over π_*R_{n-1}.

Now in the Adams E_2-term, we have the relation

$$h_0[\zeta_1 \otimes z_{n-1}^p + \zeta_2 \otimes z_{n-2}^{p^2} + \cdots + \zeta_{n-1} \otimes z_1^{p^{n-1}} + \zeta_n \otimes 1] = 0$$

since $p\alpha_{[n]} = 0$, and using (6.1) we obtain

$$\begin{aligned}
d_1\Big(\sum_{1\leqslant s\leqslant n} \bar\tau_s \otimes z_{n-s}^{p^s}\Big) &= \sum_{1\leqslant s\leqslant n} 1\otimes \bar\tau_s \otimes z_{n-s}^{p^s} \\
&\qquad - \Big(\sum_{\substack{1\leqslant s\leqslant n\\ 0\leqslant j\leqslant s}} \bar\tau_j \otimes \zeta_{s-j}^{p^j} \otimes z_{n-s}^{p^s} + \sum_{1\leqslant s\leqslant n} 1\otimes \bar\tau_s \otimes z_{n-s}^{p^s}\Big) \\
&\qquad\qquad + \Big(\sum_{1\leqslant s\leqslant n} \bar\tau_s \otimes 1 \otimes z_{n-s}^{p^s} + \sum_{\substack{1\leqslant s\leqslant n\\ 1\leqslant k\leqslant n-s}} \bar\tau_s \otimes \zeta_k^{p^s} \otimes z_{n-s-k}^{p^{s+k}}\Big) \\
&= - \sum_{\substack{1\leqslant s\leqslant n\\ 0\leqslant j\leqslant s-1}} \bar\tau_j \otimes \zeta_{s-j}^{p^j} \otimes z_{n-s}^{p^s} + \sum_{\substack{1\leqslant s\leqslant n\\ 1\leqslant k\leqslant n-s}} \bar\tau_s \otimes \zeta_k^{p^s} \otimes z_{n-s-k}^{p^{s+k}} \\
&= - \sum_{1\leqslant s\leqslant n} \bar\tau_0 \otimes \zeta_s \otimes z_{n-s}^{p^s} \\
&\qquad\qquad - \Big(\sum_{\substack{1\leqslant s\leqslant n\\ 1\leqslant j\leqslant s-1}} \bar\tau_j \otimes \zeta_{s-j}^{p^j} \otimes z_{n-s}^{p^s} - \sum_{\substack{1\leqslant s\leqslant n\\ 1\leqslant k\leqslant n-s}} \bar\tau_s \otimes \zeta_k^{p^s} \otimes z_{n-s-k}^{p^{s+k}}\Big) \\
&= \sum_{1\leqslant s\leqslant n} -\bar\tau_0 \otimes \zeta_s \otimes z_{n-s}^{p^s}.
\end{aligned}$$

We also have

$$\begin{aligned}
d_1(-z_n) &= -1\otimes z_n + \Big(\sum_{1\leqslant s\leqslant n} \zeta_t \otimes z_{n-t}^{p^t} + 1\otimes z_n\Big) \\
&= \sum_{1\leqslant s\leqslant n} \zeta_s \otimes z_{n-s}^{p^s}.
\end{aligned}$$

Therefore we have

$$\Big[\sum_{0\leqslant s\leqslant n} \bar\tau_s \otimes z_{n-s}^{p^s}\Big] = \Big[\bar\tau_0 \otimes z_n + \sum_{1\leqslant s\leqslant n} \bar\tau_s \otimes z_{n-s}^{p^s}\Big] \in \Big\langle h_0, \Big[\sum_{1\leqslant s\leqslant n} \bar\tau_s \otimes z_{n-s}^{p^s}\Big], 1\Big\rangle.$$

So modulo higher Adams filtration, the Toda bracket $\langle p, \alpha_{[n]}, 1\rangle$ is represented in the Adams spectral sequence by

$$\Big\langle h_0, \Big[\sum_{1\leqslant s\leqslant n} \bar\tau_s \otimes z_{n-s}^{p^s}\Big], 1\Big\rangle. \qquad \square$$

Note that Lemma 2.3(ii) gives

$$\begin{aligned}
\beta\mathcal{P}^{p^n}\Big[\sum_{0\leqslant s\leqslant n} \bar\tau_s \otimes z_{n-s}^{p^s}\Big] &= \Big[\beta Q^{p^n} \sum_{0\leqslant s\leqslant n} \bar\tau_s \otimes z_{n-s}^{p^s}\Big] \\
&= \Big[\sum_{0\leqslant s\leqslant n} \beta Q^{p^s}(\bar\tau_s) \otimes Q^{p^n-p^s}(z_{n-s}^{p^s})\Big] \\
&= \Big[\sum_{0\leqslant s\leqslant n} \beta\bar\tau_{s+1} \otimes (Q^{p^{n-s}-1} z_{n-s})^{p^s}\Big]
\end{aligned}$$

(by [7, theorem III.2.3])

$$= \left[\sum_{0 \leqslant s \leqslant n} \beta \bar{\tau}_{s+1} \otimes (z_{n-s}^{p})^{p^s} \right]$$

$$= \left[\sum_{0 \leqslant s \leqslant n} \zeta_{s+1} \otimes z_{n-s}^{p^{s+1}} \right]$$

(by [7, theorem III.2.3] again)

$$= \left[\sum_{1 \leqslant s \leqslant n} \zeta_s \otimes z_{n+1-s}^{p^s} \right]$$

$$= \alpha_{[n+1]}.$$

The case $p = 2$. With similar notation to that for odd primes, we have

Lemma 6.2. *The element u_n lies in the Toda bracket $\langle 2, w_{n-1}, 1 \rangle \subseteq \pi_{2^{n+1}-2} R(n)$, and in the Adams spectral sequence it has filtration 1 with cobar representative*

$$\zeta_1 \otimes s_n + \zeta_2 \otimes s_{n-1}^2 + \zeta_3 \otimes s_{n-2}^{2^2} + \cdots + \zeta_n \otimes s_1^{2^{n-1}} + \zeta_{n+1} \otimes 1,$$

where ζ_j denotes the conjugate of the Milnor generator generator $\xi_j \in \mathcal{A}(2)_{2^j-1}$.

7. The map to $H\mathbb{F}_p$

There is a morphism of commutative S-algebras $R_\infty \longrightarrow MU_{(p)}$, and composing this with the Quilen morphism of ring spectra $\varepsilon \colon MU_{(p)} \longrightarrow BP$ we obtain morphisms of ring spectra

$$\begin{tikzcd} R_\infty \arrow[r] \arrow[rr, bend left] & BP \arrow[r] & H\mathbb{F}_p \end{tikzcd}$$

and we would like to understand their induced maps in homotopy and homology.

Lemma 7.1. *Let p be a prime.*
(i) *If p is odd, suppose that $s_1, s_2, s_2, \ldots$ is a sequence of elements $s_n \in \mathcal{A}(p)_{2(p^n-1)}$ with coproducts*

$$\psi(s_n) = \zeta_n \otimes 1 + \zeta_{n-1} \otimes s_1^{p^{n-1}} + \cdots + \zeta_1 \otimes s_{n-1}^p + 1 \otimes s_n.$$

Then $s_n = \zeta_n$.
(ii) *If $p = 2$, suppose that $s_1, s_2, s_2, \ldots$ is a sequence of elements $s_n \in \mathcal{A}(p)_{2^n-1}$ with coproducts*

$$\psi(s_n) = \zeta_n \otimes 1 + \zeta_{n-1} \otimes s_1^{2^{n-1}} + \cdots + \zeta_1 \otimes s_{n-1}^2 + 1 \otimes s_n.$$

Then $s_n = \zeta_n$.

Proof. We recall that there are no non-trivial coaction primitives in positive degrees, i.e., viewing $\mathcal{A}(p)_*$ as a left $\mathcal{A}(p)_*$-comodule, a standard change of rings isomorphism gives

$$\mathrm{Ext}^{0,*}_{\mathcal{A}(p)_*}(\mathbb{F}_p, \mathcal{A}(p)_*) \cong \mathrm{Ext}^{0,*}_{\mathbb{F}_p}(\mathbb{F}_p, \mathbb{F}_p) = \mathbb{F}_p.$$

(i) For $n = 1$, we have

$$\begin{aligned} \psi(s_1 - \zeta_1) &= (\zeta_1 \otimes 1 + 1 \otimes s_1) - (\zeta_1 \otimes 1 + 1 \otimes \zeta_1) \\ &= 1 \otimes (s_1 - \zeta_1). \end{aligned}$$

So $s_1 = \zeta_1$.

Now suppose that for $k < n$, $s_k = \zeta_k$. Then

$$\begin{aligned}\psi(s_n - \zeta_n) &= \sum_{0\leqslant j\leqslant n} \zeta_j \otimes s_{n-j}^{p^j} - \sum_{0\leqslant j\leqslant n} \zeta_j \otimes \zeta_{n-j}^{p^j} \\ &= 1 \otimes (s_n - \zeta_n),\end{aligned}$$

so we have $s_n = \zeta_n$. By induction this holds for all n.

The proof of (ii) is similar. □

Remark 7.2. Since $H_*(BP;\mathbb{F}_p)$ can be identified with a subalgebra of $\mathcal{A}(p)_*$, we can also characterize a family of polynomial generators $t_n \in H_{2(p^n-1)}(BP;\mathbb{F}_p)$ by the coaction formulae

$$\psi(t_n) = \begin{cases} \sum_{0\leqslant j\leqslant n} \zeta_j \otimes t_{n-j}^{p^j} & \text{if } p \text{ is odd,} \\ \sum_{0\leqslant j\leqslant n} \zeta_j^2 \otimes t_{n-j}^{2^j} & \text{if } p = 2. \end{cases}$$

Theorem 7.3. *The morphism of ring spectra $R_\infty \longrightarrow BP$ induces epimorphisms in $\pi_*(-)$, $H_*(-;\mathbb{Z}_{(p)})$ and $H_*(-;\mathbb{F}_p)$.*

Proof. We indicate two rather different proofs.

First proof: The morphism of ring spectra $R_\infty \longrightarrow BP \longrightarrow H\mathbb{F}_p$ induces a homorphism in homology sending the elements z_n to elements $s_n \in \mathcal{A}(p)_*$ for which Lemma 7.1 applies. By Remark 7.2, this means that $z_n \mapsto \zeta_n$ if p is odd, and $z_n \mapsto \zeta_n^2$ if $p = 2$.

Second proof: First assume that p is odd. Consider the morphism of ring spectra $R_1 \longrightarrow R_\infty \longrightarrow BP$. The $z_1 \in H_{2(p-1)}(R_1;\mathbb{F}_p)$ maps to an element $t \in H_{2(p-1)}(BP;\mathbb{F}_p)$ with $\mathcal{A}_*$-coaction

$$\psi(t) = \zeta_1 \otimes 1 + 1 \otimes t.$$

The only such element is t_1.

The homomorphism $H_*(BP;\mathbb{F}_p) \longrightarrow H_*(H\mathbb{F}_p;\mathbb{F}_p) = \mathcal{A}(p)_*$. Also $MU \longrightarrow H\mathbb{F}_p$ is a morphism of commutative S-algebras whose image is $H_*(BP;\mathbb{F}_p) \subseteq H_*(H\mathbb{F}_p;\mathbb{F}_p)$. Therefore the action of the Dyer-Lashof operations on $H_*(H\mathbb{F}_p;\mathbb{F}_p)$ restricts to $H_*(BP;\mathbb{F}_p)$. Now t_r maps to ζ_r, so we can determine the Dyer-Lashof action using [7, theorem III.2.3]. Then

$$Q^{p^{s-1}} \cdots Q^{p^2} Q^p \zeta_1 = \zeta_s,$$

hence

$$Q^{p^{s-1}} \cdots Q^{p^2} Q^p t_1 = t_s.$$

Thus the element $Q^{p^{s-1}} \cdots Q^{p^2} Q^p z_1 \in H_*(R_1;\mathbb{F}_p)$ maps to $t_s \in H_*(BP;\mathbb{F}_p)$. Since

$$H_*(BP;\mathbb{F}_p) = \mathbb{F}_p[t_s : s \geqslant 1],$$

we see that $H_*(R_1;\mathbb{F}_p) \longrightarrow H_*(BP;\mathbb{F}_p)$ is epic, hence so is $H_*(R_\infty) \longrightarrow H_*(BP;\mathbb{F}_p)$.

In fact the z_s all lift to elements of $H_*(R_\infty;\mathbb{Z}_{(p)})$ and it easily follows that $H_*(R_\infty;\mathbb{Z}_{(p)}) \longrightarrow H_*(BP;\mathbb{Z}_{(p)})$ is epic.

For $p = 2$, the arguments are similar, but with ζ_s^2 in place of ζ_s, and $Q^{2^{r+1}}$ in place of Q^{p^r} throughout.

To show that the induced homomorphism $\pi_* R_\infty \longrightarrow \pi_* BP$ in homotopy is epic, we need to verify that a family of polynomial generators for $\pi_* BP$ is in the image. When p is odd,

Lemma 6.1 together with the above discussion, shows that in the Adams spectral sequence for $\pi_* BP$, u_n maps to an element represented by

$$\left[\bar{\tau}_0 \otimes t_n + \bar{\tau}_1 \otimes t_{n-1}^p + \bar{\tau}_2 \otimes t_{n-2}^{p^2} + \cdots + \bar{\tau}_{n-1} \otimes t_1^{p^{n-1}}\right] \in \mathrm{Ext}^{1,2p^n-1}_{\mathcal{A}(p)_*}(\mathbb{F}_p, H_*(BP;\mathbb{F}_p))$$

which correspond to a homotopy element with Hurewicz image in $H_*(BP;\mathbb{Z}_{(p)})$ of the form

$$pt_n \quad (\mathrm{mod}\ p, \mathrm{decomposables}).$$

By Milnor's criterion, this is a polynomial generator.

The argument for $p = 2$ is similar, with u_n mapping to an element having cobar representative

$$[\zeta_1 \otimes t_n + \zeta_2 \otimes t_{n-1}^2 + \zeta_3 \otimes t_{n-2}^{2^2} + \cdots + \zeta_{n+1} \otimes 1] \in \mathrm{Ext}^{1,2^n-1}_{\mathcal{A}(2)_*}(\mathbb{F}_2, H_*(BP;\mathbb{F}_2)). \qquad \square$$

8. Relationship to BP

We start with an easy lemma. For an abelian group G, we write $\mathrm{tors}\, G$ for the torsion subgroup.

Lemma 8.1. *Let $Y \xrightarrow{g} Z$ be a fibration of p-local spectra and let*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Y \arrow[d, "g"] \\ * \arrow[r] & Z \end{tikzcd}$$

be a pullback square. Assume that the following hold:

- *$f_*\colon \pi_*(X) \longrightarrow \pi_*(Y)$ is monic;*
- *$\mathrm{tors}\, \pi_*(Z) = 0$;*
- *$\mathrm{tors}\, \pi_*(X) = \pi_*(X)$.*

Suppose that $\alpha \in \mathrm{tors}\, \pi_m(Y)$ is non-zero and has order p^e. Then there is a map

$$u\colon S^m \cup_{p^e} D^{m+1} \longrightarrow X$$

for which the composition

$$\begin{tikzcd} S^m \arrow[r, "\mathrm{inc}"] \arrow[rrr, bend right] & S^m \cup_{p^e} D^{m+1} \arrow[r, "u"] & X \arrow[r, "f"] & Y \end{tikzcd}$$

represents α.

Proof. By assumption, f_* induces an isomorphism

$$f_*\colon \pi_*(X) \xrightarrow{\cong} \mathrm{tors}\, \pi_*(Y).$$

hence there is a unique element $\alpha' \in \pi_m(X)$ for which $f_*(\alpha') = \alpha$ and the order of α' is also p^e. A representative of α' must factor through $S^m \cup_{p^e} D^{m+1}$,

$$\begin{tikzcd} S^m \arrow[r] \arrow[rr, bend left] \arrow[drr, dashed] & S^m \cup_{p^e} D^{m+1} \arrow[r, "u"'] & X \arrow[d, "f"] \\ & & Y \end{tikzcd}$$

showing that the desired u exists, and the dashed arrow represents α. $\square$

Corollary 8.2. *The map g factors through the mapping cone of fu.*

$$\begin{tikzcd} S^m \cup_{p^e} D^{m+1} \arrow[r, "fu"] & Y \arrow[r] \arrow[d, "g"'] & \mathrm{C}_{fu} \arrow[dl, dashed] \\ & Z & \end{tikzcd}$$

Proof. This follows from the commutative diagram

$$\begin{tikzcd} S^m \cup_{p^e} D^{m+1} \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r] \arrow[d, "f"'] & \mathrm{C}_u \arrow[d] \\ S^m \cup_{p^e} D^{m+1} \arrow[r, "fu"] & Y \arrow[r] \arrow[d, "g"'] & \mathrm{C}_{fu} \arrow[dl, dashed] \\ & Z & \end{tikzcd}$$

in which gf is the trivial map and the dashed arrow is obtained by mapping the cone trivially. □

Theorem 8.3. *Let p be a prime. If BP admits an E_∞ structure then there is a weak equivalence of commutative S-algebras $R \xrightarrow{\sim} BP$.*

Proof. Since $\pi_* BP$ is torsion-free, the inductive construction of R_n from R_{n-1} gives morphisms of commutative S-algebras

$$\begin{tikzcd} R_{n-1} \arrow[rr] \arrow[dr] & & R_n \arrow[dl, dotted] \\ & BP & \end{tikzcd}$$

and passing to the colimit we obtain a morphism $R_\infty \longrightarrow BP$. By Theorem 7.3, $\pi_* R_\infty \longrightarrow \pi_* BP$ is an epimorphism and on tensoring with $\mathbb{Q}$ it becomes an isomorphism.

On replacing $R_\infty \longrightarrow BP$ with a fibration of commutative S-algebras $T^0 \longrightarrow BP$ with fibre J_0, we are in the situation of Lemma 8.1. Now we can inductively adjoin cones on wedges of Moore spectra $S^m \cup_{p^e} D^{m+1}$ where $m \geqslant 1$ to form morphisms of E_∞ ring spectra $T^{m-1} \longrightarrow T^m$. At each stage Corollary 8.2 shows that we can extend to a diagram of morphisms

$$\begin{tikzcd} T^0 \arrow[r] \arrow[drrr] & T^1 \arrow[r] \arrow[drr] & \cdots\cdots \arrow[r] & T^{m-1} \arrow[r] \arrow[d] & T^m \arrow[r] \arrow[dl] & \cdots \\ & & & BP & & \end{tikzcd}$$

and the homotopy colimit $\operatorname{hocolim}_m T^m$ is easily seen to admit a weak equivalence to

$$\operatorname*{hocolim}_m T^m \xrightarrow{\sim} BP.$$

As $R \sim \operatorname{hocolim}_m T^m$, this shows that $R \sim BP$. □

As defined, it is not clear if R is a minimal atomic commutative S-algebra; however, by construction, R_∞ is nuclear and hence is minimal atomic according to results of [3]. We can produce a core $R^c \longrightarrow R$, i.e., a morphism of commutative S-algebras with R^c nuclear and which induces a monomorphism on $\pi_*(-)$. In particular, $\pi_*(R^c)$ is torsion-free.

Lemma 8.4. *Let A be a connective p-local commutative S-algebra for which $\pi_*(A)$ is torsion-free. Then there is a morphism of commutative S-algebras $R_\infty \longrightarrow A$. In particular, the natural morphism $R_\infty \longrightarrow R$ admits a factorisation through any core $R^c \longrightarrow R$ for R.*

$$\begin{tikzcd} R_\infty \arrow[r] \arrow[rr, bend left] & R^c \arrow[r] & R \end{tikzcd}$$

Proof. Since our cellular construction of R_∞ involves attaching E_∞ cells to kill torsion elements in homotopy, it is straightforward to see that at each stage we can extend the unit $S \longrightarrow A$, in the limit this gives a morphism $R_\infty \longrightarrow A$. □

As R and more generally any core R^c have torsion-free homotopy concentrated in even degrees, standard arguments of [1] show that there are morphisms of ring spectra $BP \longrightarrow R$ and $BP \longrightarrow R^c$ associated with complex orientations with p-typical formal group laws. Our earlier arguments show that these are rational weak equivalences. Of course we have not shown that $BP \sim R$ even as (ring) spectra. One way to prove this would be to produce any map of spectra $R \longrightarrow BP$ that is an equivalence on the bottom cell, for then the composition $BP \longrightarrow R \longrightarrow BP$ would be a weak equivalence, therefore so would each of the maps $BP \longrightarrow R$ and $R \longrightarrow BP$. It is tempting to conjecture that R (or equivalently R^c) is always weakly equivalent to BP, but we have no hard evidence for this beyond what we have described above.

Appendix A. Toda brackets and Massey products

For the sake of completeness, we describe the kind of Toda brackets and Massey products we use. Details of this material can be developed in the spirit of the exposition of Toda brackets by Whitehead [33].

Toda brackets in the homotopy of R-modules. We will work with (left) S-modules in the sense of [10]. We will usually omit S from notation, for example $\wedge$ will denote $\wedge_S$ and so on.

Let R be a commutative S-algebra and let M be a left R-module. We will require Toda brackets of the following form. Let $\alpha \in \pi_a R$, $\beta \in \pi_b R$ and let $\gamma \in \pi_c M$, and suppose that

$$\alpha\beta = 0 = \beta\gamma.$$

Choosing representatives $f\colon S^a \longrightarrow R$, $g\colon S^b \longrightarrow R$, $h\colon S^c \longrightarrow M$, the maps

$$\begin{tikzcd} S^{a+b} \arrow[r] \arrow[rrr, bend left, "fg"] & S^a \wedge S^b \arrow[r, "f\wedge g"'] & R \wedge R \arrow[r] & R \end{tikzcd} \qquad \begin{tikzcd} S^{b+c} \arrow[r] \arrow[rrr, bend left, "gh"] & S^b \wedge S^c \arrow[r, "g\wedge h"'] & R \wedge M \arrow[r] & M \end{tikzcd}$$

are null homotopic. Now choosing explicit null homotopies

$$k\colon D^{a+b+1} \longrightarrow R, \quad \ell\colon D^{b+c+1} \longrightarrow M,$$

we obtain maps

$$\begin{tikzcd} D^{a+b+1} \wedge S^c \arrow[r, "k\wedge h"'] \arrow[rr, bend left, "kh"] & R \wedge M \arrow[r] & M \end{tikzcd} \qquad \begin{tikzcd} S^a \wedge D^{b+c+1} \arrow[r, "f\wedge \ell"'] \arrow[rr, bend left, "f\ell"] & S^b \wedge S^c \arrow[r] & M \end{tikzcd}$$

which agree on the boundary $S^{a+b+c} \doteq S^{a+b} \wedge S^c \doteq S^a \wedge S^{b+c}$. Therefore we obtain a map $S^{a+b+c+1} \longrightarrow M$ in the usual way representing the bracket $\langle \alpha, \beta, \gamma \rangle$.

Recollections on Massey products. We follow the sign conventions of [15, section 5.4].

Let (A, d) be a dga where $A = A^*$ is $\mathbb{Z}$-graded. If $W \in A^*$ is a homogeneous element, we set

$$\bar{W} = (-1)^{1+\deg W} W.$$

Suppose that $X, Y, Z \in A^*$ are homogeneous elements which are cycles so that the Massey product $\langle [X], [Y], [Z] \rangle$ is defined, i.e., $[X][Y] = 0 = [Y][Z]$. Choose $U, V \in A^*$ so that

$$d(U) = \bar{X}Y, \quad d(V) = \bar{Y}Z.$$

Then $d(\bar{X}V + \bar{U}Z) = 0$ and

$$[\bar{X}V + \bar{U}Z] \in \langle [X], [Y], [Z] \rangle \subseteq H^*(A, d).$$

The indeterminacy is the subset

$$[X] \cdot H^*(A, d) + \cdot [Z] \cdot H^*(A, d) \subseteq H^*(A, d).$$

References

[1] J. F. Adams, *Stable homotopy and generalised homology*, Chicago Lectures in Mathematics, University of Chicago Press, 1995. Reprint of the 1974 original.

[2] J. F. Adams and A. Liulevicius, *The Hurewicz homomorphism for* MU *and* BP, J. London Math. Soc. (2) **5** (1972), 539–545.

[3] A. Baker, H. Gilmour, and P. Reinhard, *Topological André-Quillen homology for cellular commutative S-algebras*, Abh. Math. Semin. Univ. Hambg. **78** (2008), no. 1, 27–50.

[4] A. Baker and A. Lazarev, *On the Adams spectral sequence for R-modules*, Algebr. Geom. Topol. **1** (2001), 173–199.

[5] A. J. Baker and J. P. May, *Minimal atomic complexes*, Topology **43** (2004), no. 2, 645–665.

[6] M. Basterra and M. A. Mandell, *The multiplication on BP*, J. Topology **6** (2013), 285–310.

[7] R. R. Bruner, J. P. May, J. E. McClure, and M. Steinberger, *H_∞ ring spectra and their applications*, Lect. Notes in Math., vol. 1176, 1986.

[8] F. R. Cohen, T. J. Lada, and J. P. May, *The homology of iterated loop spaces*, Lect. Notes in Math., vol. 533, 1976.

[9] W. G. Dwyer, *Strong convergence of the Eilenberg-Moore spectral sequence*, Topology **13** (1974), 255–265.

[10] A. D. Elmendorf, I. Kříž, M. A. Mandell, and J. P. May, *Rings, modules, and algebras in stable homotopy theory*, Mathematical Surveys and Monographs, vol. 47, 1997 (with an appendix by M. Cole).

[11] M. Hill and T. Lawson, *Automorphic forms and cohomology theories on Shimura curves of small discriminant*, Adv. Math. **225** (2010), no. 2, 1013–1045.

[12] P. Hu, I. Kříž, and J. P. May, *Cores of spaces, spectra, and E_∞ ring spectra*, Homology Homotopy Appl. **3** (2001), no. 2, 341–354.

[13] T. Hunter, *On the homology spectral sequence for topological Hochshild homology*, Trans. Amer. Math. Soc. **348** (1996), no. 10, 3941–3953.

[14] N. Johnson and J. Noel, *For complex orientations preserving power operations, p-typicality is atypical*, Topology Appl. **157** (2010), no. 14, 2271–2288.

[15] S. O. Kochman, *Bordism, stable homotopy and Adams spectral sequences*, Fields Institute Monographs, vol. 7, American Mathematical Society, 1996.

[16] I. Kříž, *Towers of E_∞ ring spectra with an application to BP*, unpublished preprint.

[17] N. J. Kuhn, *The transfer and James-Hopf invariants*, Math. Z. **196** (1987), 391-405.

[18] N. J. Kuhn and J. McCarty, *The mod* 2 *homology of infinite loopspaces*, Algebr. Geom. Topol. **13** (2013), 687-745.

[19] T. Lawson and N. Naumann, *Commutativity conditions for truncated Brown-Peterson spectra of height* 2, J. Topol. **5** (2012), no. 1, 137–168.

[20] L. G. Lewis, J. P. May, M. Steinberger, and J. E. McClure, *Equivariant stable homotopy theory*, Lect. Notes in Math., vol. 1213, 1986.

[21] H. Ligaard and I. Madsen, *Homology operations in the Eilenberg-Moore spectral sequence*, Math. Z. **143** (1975), 45–54.

[22] H. Matsumura, *Commutative Ring Theory*, 2nd ed., Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, 1989.

[23] J. P. May, *A general algebraic approach to Steenrod operations*, Lect. Notes in Math. **168** (1970), 153–231.

[24] ———, *Homology operations on infinite loop spaces*, Proc. Sympos. Pure Math. **XXII** (1971), 171–185.

[25] ———, *H_∞ ring spectra and their applications*, Proc. Sympos. Pure Math., XXXII (1978), 229–243.

[26] ———, *What precisely are E_∞ ring spaces and E_∞ ring spectra?* **16** (2009), 215–282.

[27] J. P. May, F. Quinn, N. Ray, and Tornehave. J., *E_∞ ring spaces and E_∞ ring spectra*, Lect. Notes in Math., vol. 577, 1977.

[28] J. McCleary, *A user's guide to spectral sequences*, 2nd ed., Camb. Studies in Adv. Math., vol. 58, Cambridge University Press, 2001.

[29] R. J. Milgram, *Group representations and the Adams spectral sequence*, Pacific J. Math. **41** (1972), 157–182.

[30] J. W. Milnor and J. C. Moore, *On the structure of Hopf algebras*, Ann. of Math. (2) **81** (1965), 211–264.

[31] S. Priddy, *A cellular construction of BP and other irreducible spectra*, Math. Z. **173** (1980), no. 1, 29–34.

[32] B. Richter, *A lower bound for coherences on the Brown-Peterson spectrum*, Algebr. Geom. Topol. **6** (2006), 287–308 (electronic).

[33] G. W. Whitehead, *Recent advances in homotopy theory*, Vol. 5, 1970. Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics.

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