On Love-type waves in a finitely deformed magnetoelastic layered half-space

Prashant Saxena and Ray W. Ogden

Abstract. In this paper the propagation of Love-type waves in a homogeneously and finitely deformed layered half-space of an incompressible non-conducting magnetoelastic material in the presence of an initial uniform magnetic field is analyzed. The equations and boundary conditions governing linearized incremental motions superimposed on an underlying deformation and magnetic field for a magnetoelastic material are summarised and then specialized to a form appropriate for the study of Love-type waves in a layered half-space. The wave propagation problem is then analyzed for different directions of the initial magnetic field for two different magnetoelastic energy functions, which are generalizations of the standard neo-Hookean and Mooney–Rivlin elasticity models. The resulting wave speed characteristics in general depend significantly on the initial magnetic field as well as on the initial finite deformation, and the results are illustrated graphically for different combinations of these parameters. In the absence of a layer shear-horizontal (SH) surface waves do not exist in a purely elastic material, but the presence of a magnetic field normal to the sagittal plane makes such waves possible, these being analogous to Bleustein–Gulyaev waves in piezoelectric materials. Such waves are discussed briefly at the end of the paper.

Keywords. Nonlinear magnetoelasticity; magnetoacoustics; Love waves; finite deformation.

1. Introduction

In recent years many synthetic elastomers have been developed that are capable of significant changes in their mechanical properties on the application of a magnetic field, as highlighted in the work of, for example, Jolly *et al.* [13], Lokander and Stenberg [16], and Varga *et al.* [26]. They have uses in a variety of engineering applications and have commonly become referred to as examples of 'smart materials'. The problem of wave propagation in such materials when subject to finite deformation in the presence of a magnetic field is, in particular, important in a number of applications, including in the experimental determination of the magnetoelastic properties of the materials concerned.

The underlying theory governing electromechanical and magnetomechanical interactions in deformable continua is well established and can be found in texts such as those by Maugin [17] and Eringen and Maugin [8], which have provided a basis for much of the subsequent work on electroelasticity and magnetoelasticity. The analysis of waves in magnetoelastic materials, in particular, has received significant attention. For example, Maugin and Hakmi [19] studied magnetoelastic surface waves with an orthogonal setting of the initial magnetic field, Abd-Alla and Maugin [1] analyzed surface magnetoelastic waves in anisotropic crystals, Lee and Its [14] discussed Rayleigh-type surface waves in a magnetoelastic medium using the theory of linear elasticity and Hefni *et al.* [10, 11, 12] were concerned with surface and bulk waves in a magnetoelastic perfect conductor.

The constitutive formulation of nonlinear magnetoelasticity developed recently by Dorfmann and Ogden [5] which involves a so-called *total* energy density function has played an underpinning role for problems which involve incremental deformations and motions since it leads to a relatively clear structure of the governing equations. It was used, in particular, by Otténio *et al.* [21] in analyzing static incremental disturbances and their effect on the stability of the surface of a pre-strained magnetoelastic half-space. The same formulation also formed the basis for the work of Destrade and Ogden [4], who considered the propagation of small amplitude homogeneous plane waves superimposed on a finite deformation in the presence of a uniform magnetic field and derived a generalization to the magnetoelastic context of the strong ellipticity condition of pure elasticity.

Following this pattern, in a recent communication [24], within the context of the quasi-magnetostatic approximation, the present authors have studied the propagation of Rayleigh-type surface waves on a half-space of an incompressible magnetoelastic material subject to pure homogeneous strain in the presence of a uniform magnetic field. In the present work, we analyze the propagation of Love-type waves in a layered half-space of incompressible non-conducting magnetoelastic material, again in the presence of a uniform magnetic field and finite homogeneous deformation. Using the notation from [24], we summarize, in Section 2, the governing equations and boundary conditions for finite deformation and then the corresponding equations and boundary conditions for incremental disturbances in a magnetoelastic material. In Section 3, the constitutive laws based on the formulation of Dorfmann and Ogden [5] are used to derive the magnetoelastic moduli tensors, and we write the governing (quasi-magnetostatic) incremental equations for an incompressible magnetoelastic solid explicitly.

In Section 4, we consider a half-space of magnetoelastic material with an overlying layer of different magnetoelastic material of uniform thickness bonded to the half-space, each of the layer and half-space being subject to a finite homogeneous pure strain with one principal direction of strain normal to the interface, the x_2 direction in a system of rectangular Cartesian coordinates (x_1, x_2, x_3). We take the (x_1, x_2) plane to be a principal plane and the initial (uniform) magnetic field to be either parallel to this plane or normal to it. The incremental mechanical displacement vector, as for a Love wave, is assumed to have only a component normal to the (x_1, x_2) plane (the sagittal plane), and to be independent of the coordinate x_3. In Sections 4.1 the problem of Love-type wave propagation is analysed in detail for the case in which the initial magnetic field is parallel to the sagittal plane. It is also shown that in the absence of the magnetic field the results reduce to those for the purely elastic case obtained by Dowaikh [7]. The numerical results are facilitated by use of a simple so-called Mooney–Rivlin magnetoelastic material model and illustrate the dependence of the wave speed (as a function of wave number) on the initial magnetic induction and underlying deformation. It is found that in general an initial magnetic induction in the direction of wave propagation or perpendicular to the direction of wave propagation (but in-plane) tends to increase the wave speed.

The corresponding analysis for the situation in which the initial magnetic field is normal to the sagittal plane is detailed in Section 4.2. For the Mooney–Rivlin magnetoelastic model, the equations governing incremental motion and the increments in magnetic field decouple and are related only through the boundary conditions and in this case an initial magnetic field decreases the wave speed. To consider the coupling through the equations we specialize the constitutive law to a version of the neo-Hookean solid. The underlying magnetic field tends to increase the wave speed in this case.

It was shown by Parekh [22, 23] that waves with an out-of-plane displacement component can exist without a layer on top of the half-space in the presence of a magnetic field. This was inspired by similar results in the electroelastic case by Bleustein [3]. Such waves do not have a counterpart in pure elasticity. Further studies include those by Scott and Mills [25] on surfaces waves in ferromagnetic crystals, by Maugin and Hakmi [18] on magnetoacoustic waves in paramagnetic insulators, a review of the theoretical and the experimental results on wave propagation in magnetoelastic materials by Gulyaev *et al.* [9], and by Liu *et al.* [15] who obtained a closed-form solution for the speed of the shear horizontal surface waves in the context of linear elasticity.

We analyze a similar problem for the nonlinear magnetoelastic case in Section 5. It is observed that such waves exist only when the underlying magnetic field has an out-of-plane component. For the neo-Hookean type material, we consider an underlying plane strain deformation and observe that the wave speed increases with an increase in the magnetic field and in the underlying stretch parallel to the direction of propagation. When there is a stretch perpendicular to the direction of wave propagation, we observe a critical value of this stretch for which the wave speed becomes independent of the magnetic field. For the Mooney–Rivlin magnetoelastic material an explicit formula is obtained for the wave speed and it is found that there is an upper bound on the magnitude of the magnetic induction for which a wave of the considered type exists.

2. Equations of motion and boundary conditions

We consider an incompressible magnetoelastic body which, when undeformed and unstressed and in the absence of magnetic fields, occupies the *reference configuration*, denoted $\mathcal{B}_r$, with boundary $\partial\mathcal{B}_r$. It is then subject to a static deformation due to the combined action of a magnetic field and mechanical surface and body forces. The deformed configuration is denoted $\mathcal{B}$, which has boundary $\partial\mathcal{B}$. The two configurations are related by a deformation function $\boldsymbol{\chi}$ which maps a point $\mathbf{X}$ in $\mathcal{B}_r$ to $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$ in $\mathcal{B}$. The deformation gradient tensor $\mathbf{F} = \mathrm{Grad}\,\boldsymbol{\chi}$ satisfies the incompressibility constraint $J \equiv \det \mathbf{F} = 1$, where Grad is the gradient operator with respect to $\mathbf{X}$. We further assume that the material is electrically non-conducting and that there are no electric fields.

Let $\boldsymbol{\tau}$ be the total Cauchy stress tensor (see, for example, Dorfmann and Ogden [5] for its definition), ρ the mass density, $\mathbf{f}$ the mechanical body force per unit mass, $\mathbf{H}$ the magnetic field vector and $\mathbf{B}$ the magnetic induction vector. Then, the equations to be satisfied in $\mathcal{B}$ are

$$\mathrm{div}\,\boldsymbol{\tau} + \rho\mathbf{f} = \mathbf{0}, \quad \boldsymbol{\tau}^{\mathrm{T}} = \boldsymbol{\tau}, \quad \mathrm{curl}\,\mathbf{H} = \mathbf{0}, \quad \mathrm{div}\,\mathbf{B} = 0, \tag{1}$$

the first being the mechanical equilibrium equation, the second the rotational balance equation, the third the specialization of Ampère's law appropriate to the present situation and the fourth the equation that the magnetic induction vector must satisfy. Here and henceforth, grad, div, curl denote the standard differential operators in $\mathcal{B}$ while Grad, Div, Curl denote the corresponding operators in $\mathcal{B}_r$.

Assuming that there are no surface currents, the boundary conditions corresponding to these governing equations are

$$\boldsymbol{\tau}\mathbf{n} = \mathbf{t}_{\mathrm{a}} + \mathbf{t}_{\mathrm{m}}, \quad \mathbf{n} \times [\![\mathbf{H}]\!] = \mathbf{0}, \quad \mathbf{n} \cdot [\![\mathbf{B}]\!] = 0, \tag{2}$$

on $\partial\mathcal{B}$, where $\mathbf{t}_{\mathrm{a}}$ and $\mathbf{t}_{\mathrm{m}}$ are the boundary tractions due to the mechanical and magnetic forces, respectively, $\mathbf{n}$ is the unit outward normal to $\partial\mathcal{B}$, and $[\![\bullet]\!]$ represents the jump in a vector across the boundary in the sense $[\![\mathbf{a}]\!] = \mathbf{a}_{\mathrm{out}} - \mathbf{a}_{\mathrm{in}}$, $\mathbf{a}_{\mathrm{out}}$ and $\mathbf{a}_{\mathrm{in}}$ being the values of $\mathbf{a}$ on the outside and inside of the body, respectively, evaluated on $\mathcal{B}$. In general, the position vector $\mathbf{x}$ may be prescribed on part of $\partial\mathcal{B}$, in which case $\mathbf{t}_{\mathrm{a}}$ in $(2)_1$ should be prescribed only on the complementary part of $\partial\mathcal{B}$.

Outside the material, which here we take to be a vacuum (although a non-magnetizable, non-polarizable medium could also be considered), we use a superscript * to represent the field quantities. The magnetic vectors satisfy the simple constitutive law $\mathbf{B}^* = \mu_0\mathbf{H}^*$, where μ_0 is the magnetic permeability in vacuum. The governing equations to be satisfied are

$$\mathrm{curl}\,\mathbf{H}^* = \mathbf{0}, \quad \mathrm{div}\,\mathbf{B}^* = 0. \tag{3}$$

Within an incompressible material the total nominal stress is defined as as $\mathbf{T} = \mathbf{F}^{-1}\boldsymbol{\tau}$ and the Lagrangian forms of $\mathbf{H}$ and $\mathbf{B}$ are defined by

$$\mathbf{H}_l = \mathbf{F}^{\mathrm{T}}\mathbf{H}, \quad \mathbf{B}_l = \mathbf{F}^{-1}\mathbf{B}. \tag{4}$$

This allows the equations (1) to be written in terms of these Lagrangian variables:

$$\mathrm{Div}\,\mathbf{T} + \rho\mathbf{f} = \mathbf{0}, \quad (\mathbf{FT})^{\mathrm{T}} = \mathbf{FT}, \quad \mathrm{Curl}\,\mathbf{H}_l = \mathbf{0}, \quad \mathrm{Div}\,\mathbf{B}_l = 0. \tag{5}$$

The boundary conditions (2) are transformed into

$$\mathbf{T}^{\mathrm{T}}\mathbf{N} = \mathbf{t}_{\mathrm{A}} + \mathbf{t}_{\mathrm{M}}, \quad \mathbf{N} \times [\![\mathbf{H}_l]\!] = \mathbf{0}, \quad \mathbf{N} \cdot [\![\mathbf{B}_l]\!] = 0, \tag{6}$$

on $\partial\mathcal{B}_r$, where $\mathbf{N}$ is the unit outward normal to the boundary $\partial\mathcal{B}_r$, $\mathbf{t}_\mathrm{A}$ is the mechanical traction per unit area of $\partial\mathcal{B}_r$, $\mathbf{t}_\mathrm{M} = \boldsymbol{\tau}^*\mathbf{F}^{-\mathrm{T}}\mathbf{N}$ is the traction due to the magnetic forces, and $\boldsymbol{\tau}^*$ is the Maxwell stress in vacuum (where $\mathbf{B}^* = \mu_0\mathbf{H}^*$) given by

$$\boldsymbol{\tau}^* = \frac{1}{\mu_0}\left[\mathbf{B}^*\otimes\mathbf{B}^* - \tfrac{1}{2}\left(\mathbf{B}^*\cdot\mathbf{B}^*\right)\mathbf{I}\right], \tag{7}$$

$\mathbf{I}$ being the identity tensor.

Superimposed on the initial configuration, we consider time-dependent infinitesimal increments $\dot{\boldsymbol{\chi}}(\mathbf{X},t)$ in the motion and $\dot{\mathbf{B}}_l(\mathbf{X},t)$ in the Lagrangian magnetic induction, where t is time. Here and subsequently incremented quantities are represented by a superposed dot. The incremental quantities are then updated from $\mathcal{B}_r$ to $\mathcal{B}$ by standard push-forward operations to give, for example,

$$\dot{\mathbf{T}}_0 = \mathbf{F}\dot{\mathbf{T}}, \quad \dot{\mathbf{B}}_{l0} = \mathbf{F}\dot{\mathbf{B}}_l, \quad \dot{\mathbf{H}}_{l0} = \mathbf{F}^{-\mathrm{T}}\dot{\mathbf{H}}_l, \tag{8}$$

where a subscript 0 indicates a pushed-forward quantity.

The pushed-forward forms are then used in the incremented and updated forms of the governing equations to obtain

$$\operatorname{div}\dot{\mathbf{T}}_0 + \rho\dot{\mathbf{f}} = \rho\mathbf{u}_{,tt}, \quad \mathbf{L}\boldsymbol{\tau} + \dot{\mathbf{T}}_0 = \boldsymbol{\tau}\mathbf{L}^\mathrm{T} + \dot{\mathbf{T}}_0^\mathrm{T}, \quad \operatorname{curl}\dot{\mathbf{H}}_{l0} = \mathbf{0}, \quad \operatorname{div}\dot{\mathbf{B}}_{l0} = 0, \tag{9}$$

where $\mathbf{L} = \operatorname{grad}\mathbf{u}$ and $\mathbf{u}(\mathbf{x},t) = \dot{\boldsymbol{\chi}}(\mathbf{X},t)$. We emphasise that here we are considering the quasimagnetostatic approximation. For a derivation of the equations involving time-dependent increments in electro-magneto-elasticity we refer to, for example, Ogden [20].

The updated incremented boundary conditions are

$$\dot{\mathbf{T}}_0^\mathrm{T}\mathbf{n} = \dot{\mathbf{t}}_{\mathrm{A}0} + \dot{\boldsymbol{\tau}}^*\mathbf{n} - \boldsymbol{\tau}^*\mathbf{L}^\mathrm{T}\mathbf{n}, \tag{10}$$

$$(\dot{\mathbf{B}}_{l0} - \dot{\mathbf{B}}^* + \mathbf{L}\mathbf{B}^*)\cdot\mathbf{n} = 0, \tag{11}$$

$$(\dot{\mathbf{H}}_{l0} - \mathbf{L}^\mathrm{T}\mathbf{H}^* - \dot{\mathbf{H}}^*)\times\mathbf{n} = \mathbf{0}, \tag{12}$$

on $\partial\mathcal{B}$, where $\dot{\mathbf{t}}_{\mathrm{A}0}$ is the incremented updated mechanical traction. Here, the increment in the Maxwell stress is given as

$$\dot{\boldsymbol{\tau}}^* = \frac{1}{\mu_0}[\dot{\mathbf{B}}^*\otimes\mathbf{B}^* + \mathbf{B}^*\otimes\dot{\mathbf{B}}^* - (\dot{\mathbf{B}}^*\cdot\mathbf{B}^*)\mathbf{I}]. \tag{13}$$

3. Constitutive relations

Following Dorfmann and Ogden [5], we consider an incompressible magnetoelastic solid characterized in terms of a total energy density function $\Omega(\mathbf{F},\mathbf{B}_l)$ for which the total nominal stress and the Lagrangian magnetic field are given simply by

$$\mathbf{T} = \frac{\partial\Omega}{\partial\mathbf{F}} - p\mathbf{F}^{-1}, \quad \mathbf{H}_l = \frac{\partial\Omega}{\partial\mathbf{B}_l}, \tag{14}$$

where p is a Lagrange multiplier associated with the constraint of incompressibility.

On incrementing and linearizing the constitutive equations, we obtain

$$\dot{\mathbf{T}} = \boldsymbol{\mathcal{A}}\dot{\mathbf{F}} + \boldsymbol{\mathcal{C}}\dot{\mathbf{B}}_l - \dot{p}\mathbf{F}^{-1} + p\mathbf{F}^{-1}\dot{\mathbf{F}}\mathbf{F}^{-1}, \quad \dot{\mathbf{H}}_l = \boldsymbol{\mathcal{C}}^\mathrm{T}\dot{\mathbf{F}} + \mathsf{K}\dot{\mathbf{B}}_l, \tag{15}$$

where the magnetoelastic 'moduli' tensors are defined by

$$\boldsymbol{\mathcal{A}} = \frac{\partial^2\Omega}{\partial\mathbf{F}\partial\mathbf{F}}, \quad \boldsymbol{\mathcal{C}} = \frac{\partial^2\Omega}{\partial\mathbf{F}\partial\mathbf{B}_l}, \quad \boldsymbol{\mathcal{C}}^\mathrm{T} = \frac{\partial^2\Omega}{\partial\mathbf{B}_l\partial\mathbf{F}}, \quad \mathsf{K} = \frac{\partial^2\Omega}{\partial\mathbf{B}_l\partial\mathbf{B}_l}, \tag{16}$$

or index notation, and taking account of the commutativity of the partial derivatives,

$$\mathcal{A}_{\alpha i\beta j} = \mathcal{A}_{\beta j\alpha i} = \frac{\partial^2\Omega}{\partial F_{i\alpha}\partial F_{j\beta}}, \quad \mathcal{C}_{\alpha i|\beta} = \mathcal{C}_{\beta|\alpha i} = \frac{\partial^2\Omega}{\partial F_{i\alpha}\partial B_{l\beta}}, \quad \mathsf{K}_{\alpha\beta} = \mathsf{K}_{\beta\alpha} = \frac{\partial^2\Omega}{\partial B_{l\alpha}\partial B_{l\beta}}. \tag{17}$$

The vertical bar between the indices on $\mathcal{C}$ is used to separate and distinguish the single subscript from the pair of subscripts that always go together.

The incremented constitutive equations (15) are now updated to obtain

$$\dot{\mathbf{T}}_0 = \boldsymbol{\mathcal{A}}_0\mathbf{L} + \boldsymbol{\mathcal{C}}_0\dot{\mathbf{B}}_{l0} - \dot{p}\mathbf{I} + p\mathbf{L}, \quad \dot{\mathbf{H}}_{l0} = \boldsymbol{\mathcal{C}}_0^{\mathrm{T}}\mathbf{L} + \mathsf{K}_0\dot{\mathbf{B}}_{l0}, \tag{18}$$

where $\boldsymbol{\mathcal{A}}_0, \boldsymbol{\mathcal{C}}_0$ and K_0 are defined in component form as

$$\mathcal{A}_{0piqj} = \mathcal{A}_{0qjpi} = F_{p\alpha}F_{q\beta}\mathcal{A}_{\alpha i\beta j}, \tag{19}$$

$$\mathcal{C}_{0pi|q} = \mathcal{C}_{0q|pi} = F_{p\alpha}F^{-1}_{\beta q}\mathcal{C}_{\alpha i|\beta}, \tag{20}$$

$$\mathsf{K}_{0pq} = \mathsf{K}_{0qp} = F^{-1}_{\alpha p}F^{-1}_{\beta q}\mathsf{K}_{\alpha\beta}, \tag{21}$$

where we have adopted the notation $F^{-1}_{\alpha i} = (\mathbf{F}^{-1})_{\alpha i}$.

On substituting the updated constitutive relations (18) into the governing equations (9), we obtain

$$\mathrm{div}\,(\boldsymbol{\mathcal{A}}_0\mathbf{L} + \boldsymbol{\mathcal{C}}_0\dot{\mathbf{B}}_{l0}) - \mathrm{grad}\,\dot{p} + \mathbf{L}^{\mathrm{T}}\mathrm{grad}\,p = \rho\mathbf{u}_{,tt}, \quad \mathrm{curl}\,(\boldsymbol{\mathcal{C}}_0^{\mathrm{T}}\mathbf{L} + \mathsf{K}_0\dot{\mathbf{B}}_{l0}) = \mathbf{0}, \tag{22}$$

along with

$$\boldsymbol{\mathcal{A}}_0\mathbf{L} + p\mathbf{L} + \mathbf{L}\boldsymbol{\tau} = (\boldsymbol{\mathcal{A}}_0\mathbf{L})^{\mathrm{T}} + p\mathbf{L}^{\mathrm{T}} + \boldsymbol{\tau}\mathbf{L}^{\mathrm{T}}, \quad \boldsymbol{\mathcal{C}}_0\dot{\mathbf{B}}_{l0} = (\boldsymbol{\mathcal{C}}_0\dot{\mathbf{B}}_{l0})^{\mathrm{T}}, \tag{23}$$

from which we deduce the additional symmetries

$$\mathcal{A}_{0piqj} + \delta_{pj}(\tau_{qi} + p\delta_{qi}) = \mathcal{A}_{0ipqj} + \delta_{ij}(\tau_{pq} + p\delta_{pq}), \quad \mathcal{C}_{0pi|q} = \mathcal{C}_{0ip|q}. \tag{24}$$

Equations (22) are taken in conjunction with

$$\mathrm{div}\,\dot{\mathbf{B}}_{l0} = 0, \quad \mathrm{div}\,\mathbf{u} = 0, \tag{25}$$

the latter being the incremental incompressibility condition.

For later reference we also mention the generalized strong ellipticity condition given in [4], in which it was assumed that K_0 is invertible. Here we write the strong ellipticity condition in the form

$$\left(\mathcal{A}_{0piqj} - \mathcal{C}_{0ip|m}\mathsf{K}^{-1}_{0mn}\mathcal{C}_{0jq|n}\right) m_i m_j n_p n_q > 0, \tag{26}$$

which holds for all unit vectors $\mathbf{m}$ and $\mathbf{n}$ such that $\mathbf{m}\cdot\mathbf{n} = 0$ (because of incompressibility).

3.1. Isotropy

For an incompressible isotropic magnetoelastic material, Ω can be expressed in terms of five independent scalar invariants of the right Cauchy–Green deformation tensor $\mathbf{c} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$ and the tensor product $\mathbf{B}_l \otimes \mathbf{B}_l$, and specifically we select the invariants

$$I_1 = \mathrm{tr}\,\mathbf{c}, \quad I_2 = \frac{1}{2}[(\mathrm{tr}\,\mathbf{c})^2 - (\mathrm{tr}\,\mathbf{c}^2)], \quad I_4 = \mathbf{B}_l\cdot\mathbf{B}_l, \quad I_5 = (\mathbf{c}\mathbf{B}_l)\cdot\mathbf{B}_l, \quad I_6 = (\mathbf{c}^2\mathbf{B}_l)\cdot\mathbf{B}_l. \tag{27}$$

Using these invariants, the constitutive relations (14) can be expanded in the forms

$$\mathbf{T} = -p\mathbf{F}^{-1} + 2\Omega_1\mathbf{F}^{\mathrm{T}} + 2\Omega_2\left(I_1\mathbf{I} - \mathbf{c}\right)\mathbf{F}^{\mathrm{T}} + 2\Omega_5\mathbf{B}_l\otimes\mathbf{F}\mathbf{B}_l + 2\Omega_6\left(\mathbf{B}_l\otimes\mathbf{F}\mathbf{c}\mathbf{B}_l + \mathbf{c}\mathbf{B}_l\otimes\mathbf{F}\mathbf{B}_l\right), \tag{28}$$

and

$$\mathbf{H}_l = 2\left(\Omega_4\mathbf{B}_l + \Omega_5\mathbf{c}\mathbf{B}_l + \Omega_6\mathbf{c}^2\mathbf{B}_l\right), \tag{29}$$

where $\Omega_k = \partial\Omega/\partial I_k$, $k = 1, 2, 4, 5, 6$.

Corresponding expansions of the magnetoelastic moduli tensors $\boldsymbol{\mathcal{A}}_0, \boldsymbol{\mathcal{C}}_0$ and K_0 are fairly lengthy and are not given in full here. The components required in this paper can be obtained from the formulas listed in Appendix A. For a complete list we refer to Saxena and Ogden [24].

4. Application to Love waves

Let the initial deformation in the material be given by a pure homogeneous strain

$$x_1 = \lambda_1 X_1, \quad x_2 = \lambda_2 X_2, \quad x_3 = \lambda_3 X_3, \tag{30}$$

where the principal stretches $\lambda_1, \lambda_2, \lambda_3$ are uniform. The component matrix of the deformation gradient is then given by $[\mathbf{F}] = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$. The initial uniform magnetic induction vector is taken to have either components $(B_1, B_2, 0)$ in the material with $(B_1^*, B_2^*, 0)$ outside the material, or $(0, 0, B_3)$ in the material with $(0, 0, B_3^*)$ outside the material. Note that the boundary condition $(2)_3$ requires that $B_2^* = B_2$.

For such a configuration, the in-plane displacement components u_1 and u_2 are coupled with each other in the governing equations, and are independent of the out-of-plane component u_3. Surface waves with in-plane displacement components have been studied by the authors in a separate paper [24]. Here we seek solutions depending on the in-plane variables x_1 and x_2 such that $u_1 = u_2 = 0$ and u_3 depends on (x_1, x_2, t). The incremental incompressibility condition $\mathrm{div}\, \mathbf{u} = 0$ is then automatically satisfied, and with all incremental quantities independent of x_3, $\dot{p}_{,3} = 0$ and, from $(25)_1$ we obtain

$$\dot{B}_{l01,1} + \dot{B}_{l02,2} = 0. \tag{31}$$

On expanding the governing equations (22), we obtain

$$\begin{aligned}\mathcal{A}_{01113} u_{3,11} &+ (\mathcal{A}_{01123} + \mathcal{A}_{02113})\, u_{3,12} + \mathcal{A}_{02123} u_{3,22} + \mathcal{C}_{011|1} \dot{B}_{l01,1} + \mathcal{C}_{011|2} \dot{B}_{l02,1} \\ &+ \mathcal{C}_{011|3} \dot{B}_{l03,1} + \mathcal{C}_{021|1} \dot{B}_{l01,2} + \mathcal{C}_{021|2} \dot{B}_{l02,2} + \mathcal{C}_{021|3} \dot{B}_{l03,2} - \dot{p}_{,1} = 0,\end{aligned} \tag{32}$$

$$\begin{aligned}\mathcal{A}_{01213} u_{3,11} &+ (\mathcal{A}_{01223} + \mathcal{A}_{02213})\, u_{3,12} + \mathcal{A}_{02223} u_{3,22} + \mathcal{C}_{012|1} \dot{B}_{l01,1} + \mathcal{C}_{012|2} \dot{B}_{l02,1} \\ &+ \mathcal{C}_{012|3} \dot{B}_{l03,1} + \mathcal{C}_{022|1} \dot{B}_{l01,2} + \mathcal{C}_{022|2} \dot{B}_{l02,2} + \mathcal{C}_{022|3} \dot{B}_{l03,2} - \dot{p}_{,2} = 0,\end{aligned} \tag{33}$$

$$\begin{aligned}\mathcal{A}_{01313} u_{3,11} &+ 2\mathcal{A}_{01323} u_{3,12} + \mathcal{A}_{02323} u_{3,22} + \mathcal{C}_{013|1} \dot{B}_{l01,1} + \mathcal{C}_{013|2} \dot{B}_{l02,1} + \mathcal{C}_{013|3} \dot{B}_{l03,1} \\ &+ \mathcal{C}_{023|1} \dot{B}_{l01,2} + \mathcal{C}_{023|2} \dot{B}_{l02,2} + \mathcal{C}_{023|3} \dot{B}_{l03,2} = \rho u_{3,tt},\end{aligned} \tag{34}$$

$$(\mathcal{C}_{013|3} u_{3,1} + \mathcal{C}_{023|3} u_{3,2} + \mathsf{K}_{013} \dot{B}_{l01} + \mathsf{K}_{023} \dot{B}_{l02} + \mathsf{K}_{033} \dot{B}_{l03})_{,2} = 0, \tag{35}$$

$$(\mathcal{C}_{013|3} u_{3,1} + \mathcal{C}_{023|3} u_{3,2} + \mathsf{K}_{013} \dot{B}_{l01} + \mathsf{K}_{023} \dot{B}_{l02} + \mathsf{K}_{033} \dot{B}_{l03})_{,1} = 0, \tag{36}$$

$$\begin{aligned}\mathcal{C}_{013|2} u_{3,11} &+ \left(\mathcal{C}_{023|2} - \mathcal{C}_{013|1}\right) u_{3,12} - \mathcal{C}_{023|1} u_{3,22} + \mathsf{K}_{012} \dot{B}_{l01,1} + \mathsf{K}_{022} \dot{B}_{l02,1} \\ &+ \mathsf{K}_{023} \dot{B}_{l03,1} - \mathsf{K}_{011} \dot{B}_{l01,2} - \mathsf{K}_{012} \dot{B}_{l02,2} - \mathsf{K}_{013} \dot{B}_{l03,2} = 0.\end{aligned} \tag{37}$$

Associated boundary conditions will be considered in the specializations that follow.

From here on, we consider two separate cases, for which the underlying magnetic field is, first, parallel to the $(1, 2)$ plane and, second, normal to the plane.

We consider a half-space of magnetoelastic material for which $X_2 < 0$ in the undeformed configuration ($x_2 < 0$ in the deformed configuration) and over which is a layer of a different magnetoelastic material of thickness h in the deformed configuration, so that, for the layer, $0 < x_2 < h$. Quantities in the half-space are distinguished by a prime ($'$); those in the layer are unprimed.

4.1. In-plane magnetic field: $\mathbf{B} = (B_1, B_2, 0)$

Here we take $B_3 = B_3^* = 0$ so that the Maxwell stress and its increment are obtained in component form from equations (7) and (13) as

$$[\boldsymbol{\tau}^*] = \frac{1}{\mu_0} \begin{bmatrix} \frac{1}{2}(B_1^{*2} - B_2^{*2}) & B_1^* B_2^* & 0 \\ B_1^* B_2^* & \frac{1}{2}\left(-B_1^{*2} + B_2^{*2}\right) & 0 \\ 0 & 0 & \frac{1}{2}(-B_1^{*2} - B_2^{*2}) \end{bmatrix}, \tag{38}$$

and

$$[\dot{\boldsymbol{\tau}}^*] = \frac{1}{\mu_0}\begin{bmatrix} \dot{B}_1^*B_1^* - \dot{B}_2^*B_2^* & \dot{B}_1^*B_2^* + \dot{B}_2^*B_1^* & \dot{B}_3^*B_1^* \\ \dot{B}_2^*B_1^* + \dot{B}_1^*B_2^* & \dot{B}_2^*B_2^* - \dot{B}_1^*B_1^* & \dot{B}_3^*B_2^* \\ \dot{B}_3^*B_1^* & \dot{B}_3^*B_2^* & -\left(\dot{B}_1^*B_1^* + \dot{B}_2^*B_2^*\right) \end{bmatrix}, \tag{39}$$

respectively.

Equations (32)–(37) simplify to

$$\mathcal{C}_{011|1}\dot{B}_{l01,1} + \mathcal{C}_{021|1}\dot{B}_{l01,2} + \mathcal{C}_{021|2}\dot{B}_{l02,2} + \mathcal{C}_{011|2}\dot{B}_{l02,1} - \dot{p}_{,1} = 0, \tag{40}$$

$$\mathcal{C}_{012|1}\dot{B}_{l01,1} + \mathcal{C}_{012|2}\dot{B}_{l02,1} + \mathcal{C}_{022|2}\dot{B}_{l02,2} + \mathcal{C}_{022|1}\dot{B}_{l01,2} - \dot{p}_{,2} = 0, \tag{41}$$

$$\mathcal{A}_{01313}u_{3,11} + 2\mathcal{A}_{01323}u_{3,12} + \mathcal{A}_{02323}u_{3,22} + \mathcal{C}_{013|3}\dot{B}_{l03,1} + \mathcal{C}_{023|3}\dot{B}_{l03,2} = \rho u_{3,tt}, \tag{42}$$

$$\left(\mathcal{C}_{013|3}u_{3,1} + \mathcal{C}_{023|3}u_{3,2} + \mathsf{K}_{033}\dot{B}_{l03}\right)_{,2} = 0, \tag{43}$$

$$\left(\mathcal{C}_{013|3}u_{3,1} + \mathcal{C}_{023|3}u_{3,2} + \mathsf{K}_{033}\dot{B}_{l03}\right)_{,1} = 0, \tag{44}$$

$$\mathsf{K}_{022}\dot{B}_{l02,1} + \mathsf{K}_{012}\dot{B}_{l01,1} - \mathsf{K}_{012}\dot{B}_{l02,2} - \mathsf{K}_{011}\dot{B}_{l01,2} = 0, \tag{45}$$

and from $(9)_4$ we obtain

$$\dot{B}_{l01,1} + \dot{B}_{l02,2} = 0. \tag{46}$$

Equations (43), (44) and the assumption of independence of x_3 imply that $\dot{H}_{l03}$ depends only on t and hence we take $\dot{H}_{l03} = f(t)$. We also observe that $\dot{B}_{l01}$ and $\dot{B}_{l02}$ are coupled through equations (40), (41), (45) and (46) and are independent of u_3, while u_3 is coupled with $\dot{B}_{l03}$ through equations (42), (43) and (44). Since we are only interested here in u_3 it suffices to take $\dot{B}_{l01} = \dot{B}_{l02} = 0$ in both half-space and layer. Indeed, in general $\dot{B}_{l01}$ and $\dot{B}_{l02}$ are overdetermined by equations (40), (41), (45) and (46). It follows from $(18)_2$ and the components of $\boldsymbol{\mathcal{C}}_0$ and K_0 given in Appendix A that $\dot{H}_{l01} = \dot{H}_{l02} = 0$.

The governing equations now reduce to

$$\mathcal{A}_{01313}u_{3,11} + 2\mathcal{A}_{01323}u_{3,12} + \mathcal{A}_{02323}u_{3,22} + \mathcal{C}_{013|3}\dot{B}_{l03,1} + \mathcal{C}_{023|3}\dot{B}_{l03,2} = \rho u_{3,tt}, \tag{47}$$

$$\mathcal{C}_{013|3}u_{3,1} + \mathcal{C}_{023|3}u_{3,2} + \mathsf{K}_{033}\dot{B}_{l03} = f(t), \tag{48}$$

in the layer, while in the half-space they are

$$\mathcal{A}'_{01313}u'_{3,11} + 2\mathcal{A}'_{01323}u'_{3,12} + \mathcal{A}'_{02323}u'_{3,22} + \mathcal{C}'_{013|3}\dot{B}'_{l03,1} + \mathcal{C}'_{023|3}\dot{B}'_{l03,2} = \rho' u'_{3,tt}, \tag{49}$$

$$\mathcal{C}'_{013|3}u'_{3,1} + \mathcal{C}'_{023|3}u'_{3,2} + \mathsf{K}'_{033}\dot{B}'_{l03} = f'(t), \tag{50}$$

where $f'(t)$ is the counterpart of $f(t)$ for the half-space.

The boundary conditions (11) and (12) reduce to

$$\dot{B}_2^* = \dot{B}_{l02} = 0, \quad \dot{B}_1^* = \mu_0\dot{H}_1^* = \mu_0\dot{H}_{l01} = 0, \quad \dot{H}_3^* = \dot{H}_{l03} = f(t) \quad \text{on } x_2 = 0, \tag{51}$$

and hence we may take $\dot{B}_1^* = \dot{B}_2^* = 0$ outside the material.

From the boundary condition (10) applied at the layer–vacuum boundary the only non-trivial component is $\dot{T}_{023} = \dot{\tau}_{23}^*$, which yields

$$\mathcal{A}_{02313}u_{3,1} + \mathcal{A}_{02323}u_{3,2} + \mathcal{C}_{023|3}\dot{B}_{l03} = 0 \quad \text{on } x_2 = h, \tag{52}$$

and at the layer–half-space interface $\dot{T}_{023} = \dot{T}'_{023}$, which leads to

$$\mathcal{A}_{02313}u_{3,1} + \mathcal{A}_{02323}u_{3,2} + \mathcal{C}_{023|3}\dot{B}_{l03} = \mathcal{A}'_{02313}u'_{3,1} + \mathcal{A}'_{02323}u'_{3,2} + \mathcal{C}'_{023|3}\dot{B}'_{l03} \quad \text{on } x_2 = 0. \tag{53}$$

Additionally, the displacement must be continuous at the interface, i.e.

$$u_3 = u'_3 \quad \text{on } x_2 = 0. \tag{54}$$

The problem is therefore reduced to solving equations (47) and (48) in $0 < x_2 < h$ and equations (49) and (50) in $x_2 < 0$ using the boundary conditions (52), (53) and (54).

4.1.1. Wave propagation. On the basis of the above equations and boundary conditions we now study Love-type waves propagating in the x_1 direction. We consider harmonic solutions of the form

$$u_3 = P\exp[\mathrm{i}\,(skx_2 + kx_1 - \omega t)], \quad 0 < x_2 < h, \tag{55}$$

$$\dot{B}_{l03} = Q\exp[\mathrm{i}\,(skx_2 + kx_1 - \omega t)], \quad 0 < x_2 < h, \tag{56}$$

$$u_3' = P'\exp(s'kx_2 + \mathrm{i}kx_1 - \mathrm{i}\omega t), \quad x_2 < 0, \tag{57}$$

$$\dot{B}_{l03}' = Q'\exp(s'kx_2 + \mathrm{i}kx_1 - \mathrm{i}\omega t), \quad x_2 < 0, \tag{58}$$

with the condition $\mathrm{Re}\,(s') > 0$ for the wave to decay away from the surface of the half-space. Here $\mathrm{i} = \sqrt{-1}$, k is the wave number, and ω is the angular frequency.

Substitution of (55) and (56) into the governing equation (48) yields

$$[\mathrm{i}(\mathcal{C}_{013|3} + s\mathcal{C}_{023|3})Pk + \mathsf{K}_{033}Q]\mathrm{e}^{\mathrm{i}(skx_2+kx_1-\omega t)} = f(t), \tag{59}$$

which is satisfied non-trivially only when $f(t) = 0$. Similarly, we obtain $f'(t) = 0$. Now using equation (47), and defining the wave speed $v = \omega/k$ we have the two equations

$$\left(\rho v^2 - \mathcal{A}_{01313} - 2s\mathcal{A}_{01323} - \mathcal{A}_{02323}s^2\right)Pk + \mathrm{i}\left(\mathcal{C}_{013|3} + s\mathcal{C}_{023|3}\right)Q = 0, \tag{60}$$

$$\mathrm{i}\left(\mathcal{C}_{013|3} + s\mathcal{C}_{023|3}\right)Pk + \mathsf{K}_{033}Q = 0. \tag{61}$$

For non-trivial solutions for P and Q, the determinant of coefficients must vanish, which yields a quadratic equation for s, which we write compactly as

$$As^2 + 2Bs + C - \rho v^2 = 0, \tag{62}$$

where we have introduced the notations

$$A = \mathcal{A}_{02323} - \frac{\mathcal{C}^2_{023|3}}{\mathsf{K}_{033}}, \quad B = \mathcal{A}_{02313} - \frac{\mathcal{C}_{013|3}\mathcal{C}_{023|3}}{\mathsf{K}_{033}}, \quad C = \mathcal{A}_{01313} - \frac{\mathcal{C}^2_{013|3}}{\mathsf{K}_{033}}. \tag{63}$$

Let s_1 and s_2 be the two solutions of this quadratic. Then the general solution of the considered form is

$$u_3 = \left(P_1\mathrm{e}^{\mathrm{i}s_1kx_2} + P_2\mathrm{e}^{\mathrm{i}s_2kx_2}\right)\exp[\mathrm{i}(kx_1 - \omega t)], \tag{64}$$

$$\dot{B}_{l03} = \left(Q_1\mathrm{e}^{\mathrm{i}s_1kx_2} + Q_2\mathrm{e}^{\mathrm{i}s_2kx_2}\right)\exp[\mathrm{i}\,(kx_1 - \omega t)]. \tag{65}$$

The coefficients P_j and Q_j, $j = 1, 2$, are related by either one of the equations (60) or (61) as

$$Q_j = -\frac{\mathrm{i}k(\mathcal{C}_{013|3} + s_j\mathcal{C}_{023|3})}{\mathsf{K}_{033}}P_j, \quad j = 1, 2. \tag{66}$$

Substituting the solutions (57) and (58) into equations (49) and (50), we obtain a similar quadratic for s', namely

$$A's'^2 + 2\mathrm{i}B's' + \rho'v^2 - C' = 0, \tag{67}$$

where the coefficients are defined by

$$A' = \mathcal{A}'_{02323} - \frac{\mathcal{C}'^2_{023|3}}{\mathsf{K}'_{033}}, \quad B' = \mathcal{A}'_{02313} - \frac{\mathcal{C}'_{013|3}\mathcal{C}'_{023|3}}{\mathsf{K}'_{033}}, \quad C' = \mathcal{A}'_{01313} - \frac{\mathcal{C}'^2_{013|3}}{\mathsf{K}'_{033}}. \tag{68}$$

This has at most one solution satisfying the requirement $\mathrm{Re}(s') > 0$. Equation (50) also yields the connection

$$k(\mathrm{i}\mathcal{C}'_{013|3} + s'\mathcal{C}'_{023|3})P' + \mathsf{K}'_{033}Q' = 0. \tag{69}$$

From the generalized strong ellipticity condition (26), we deduce that

$$A > 0, \quad C > 0, \quad A' > 0, \quad C' > 0, \tag{70}$$

and hence that there is a solution for s' with positive real part provided

$$A'(C' - \rho'v^2) - B'^2 > 0. \tag{71}$$

Substituting the solutions (57), (58), (64) and (65) into the boundary conditions (52), (53) and (54), we obtain

$$\begin{aligned} \mathrm{i}k(\mathcal{A}_{02313} + s_1\mathcal{A}_{02323})P_1\mathrm{e}^{\mathrm{i}s_1kh} \;+\; \mathrm{i}k(\mathcal{A}_{02313} + s_2\mathcal{A}_{02323})P_2\mathrm{e}^{\mathrm{i}s_2kh} \\ + \;\mathcal{C}_{023|3}\left(Q_1\mathrm{e}^{\mathrm{i}s_1kh} + Q_2\mathrm{e}^{\mathrm{i}s_2kh}\right) = 0, \end{aligned} \tag{72}$$

$$\begin{aligned} \mathrm{i}k(\mathcal{A}_{02313} + s_1\mathcal{A}_{02323})P_1 \;+\; \mathrm{i}k(\mathcal{A}_{02313} + s_2\mathcal{A}_{02323})P_2 + \mathcal{C}_{023|3}\left(Q_1 + Q_2\right) \\ = k(\mathrm{i}\mathcal{A}'_{02313} + s'\mathcal{A}'_{02323})P' + \mathcal{C}'_{023|3}Q', \end{aligned} \tag{73}$$

$$P_1 + P_2 = P'. \tag{74}$$

We may then use the relations (66) and (69) to eliminate Q_1, Q_2 and Q' to obtain

$$(s_1A + B)\,\mathrm{e}^{\mathrm{i}s_1kh}P_1 + (s_2A + B)\,\mathrm{e}^{\mathrm{i}s_2kh}P_2 = 0, \tag{75}$$

$$(s_1A + B)\,P_1 + (s_2A + B)\,P_2 + (\mathrm{i}s'A' - B')\,P' = 0, \tag{76}$$

$$P_1 + P_2 - P' = 0. \tag{77}$$

The three linear equations for P_1, P_2, P' have non-trivial solutions provided the determinant of their coefficients vanishes. This gives rise to the secular equation

$$\begin{aligned} [(s_1A + B)(s_2A + B) \;+\; (\mathrm{i}s'A' - B')B](\mathrm{e}^{\mathrm{i}s_2kh} - \mathrm{e}^{\mathrm{i}s_1kh}) \\ + \;(\mathrm{i}s'A' - B')A(s_2\mathrm{e}^{\mathrm{i}s_2kh} - s_1\mathrm{e}^{\mathrm{i}s_1kh}) = 0, \end{aligned} \tag{78}$$

where s_1 and s_2 are the solutions of equation (62) and s' is the solution of (67) with positive real part.

4.1.2. Pure elastic case. We now take the magnetic field to vanish in order to reduce our results to the purely elastic case. For this purpose we take $\mathbf{B} = \mathbf{0}, \boldsymbol{\mathcal{C}} = \mathbf{O}$, $Q_1 = Q_2 = Q' = 0$. Under this specialization, the governing equations (47) and (49) reduce to

$$\mathcal{A}_{01313}u_{3,11} + \mathcal{A}_{02323}u_{3,22} = \rho u_{3,tt}, \quad \mathcal{A}'_{01313}u'_{3,11} + \mathcal{A}'_{02323}u'_{3,22} = \rho' u'_{3,tt}, \tag{79}$$

in the layer and half-space, respectively. The relations (62) and (67) become

$$s^2 = \frac{\rho v^2 - \mathcal{A}_{01313}}{\mathcal{A}_{02323}}, \quad s'^2 = \frac{\mathcal{A}'_{01313} - \rho' v^2}{\mathcal{A}'_{02323}}. \tag{80}$$

For these simplifications, the secular equation (78) becomes

$$\tan(skh) = \frac{s'\mathcal{A}'_{02323}}{s\mathcal{A}_{02323}}, \quad \rho v^2 > \mathcal{A}_{01313}, \tag{81}$$

where $s > 0$ and we note that to qualify for a surface wave the inequality $\rho' v^2 < \mathcal{A}'_{01313}$ must be satisfied and that there are no real solutions for the wave speed if $s^2 < 0$. The above equation is equivalent to equation (3.12) obtained by Dowaikh [7]. Note, however, the result (3.17) in [7] corresponding to $s^2 < 0$ is incorrect. Thus,

$$\mathcal{A}_{01313}/\rho < v^2 < \mathcal{A}'_{01313}/\rho'. \tag{82}$$

For the isotropic linear elastic case, $\mathcal{A}_{01313} = \mathcal{A}_{02323} = \mu$, $\mathcal{A}'_{01313} = \mathcal{A}'_{02323} = \mu'$, where μ and μ' are the shear moduli of the layer and the half-space, respectively. If the transverse wave speed is denoted by $v_T = (\mu/\rho)^{1/2}$ in the layer and $v'_T = (\mu'/\rho')^{1/2}$ in the bulk, then the above secular equation reduces to

$$\tan\left[\left(\frac{v^2}{v_T^2} - 1\right)^{\frac{1}{2}} kh\right] = \frac{\mu'}{\mu}\frac{\left[1 - (v/v'_T)^2\right]^{\frac{1}{2}}}{\left[(v/v_T)^2 - 1\right]^{\frac{1}{2}}}, \quad v_T < v < v'_T, \tag{83}$$

thus recovering the well-known dispersion relation for Love waves in linear elasticity (see, for example, [2]).

4.1.3. Application to a Mooney–Rivlin magnetoelastic material. To illustrate the results, we now consider the energy function of Mooney–Rivlin type magnetoelastic material used by Otténio *et al.* [21]. This is given (in different notation from that used in the latter paper) by

$$\Omega = \frac{\mu}{4}\left[(1+n)(I_1-3)+(1-n)(I_2-3)\right]+lI_4+mI_5, \tag{84}$$

where μ is the shear modulus of the material in the absence of a magnetic field, n is a dimensionless parameter in the range $-1 \le n \le 1$, and l and m are magnetoelastic coupling constants such that $l\mu_0$ and $m\mu_0$ are dimensionless, μ_0 being the magnetic permeability in vacuo.

We note that if the underlying magnetic induction is either parallel or perpendicular to the boundary, i.e. either $\mathbf{B} = (B_1, 0, 0)$ or $\mathbf{B} = (0, B_2, 0)$, then $\mathcal{A}_{01323} = 0 = \mathcal{C}_{013|3}\mathcal{C}_{023|3}$ ($\mathcal{C}_{013|3} = 0$ if $B_1 = 0$ and $\mathcal{C}_{023|3} = 0$ if $B_2 = 0$). Hence $B = 0$ and similarly $B' = 0$. Equations (62) and (67) then simplify to

$$s^2 = (\rho v^2 - C)/A, \quad s'^2 = (C' - \rho' v^2)/A'. \tag{85}$$

We require $s'^2 > 0$ for a surface waves to exist. By taking account of the strong ellipticity condition (26) this requirement imposes the conditions

$$C < \rho v^2 < \rho C'/\rho' \tag{86}$$

on the wave speed, and these inequalities also impose certain restrictions on the energy functions used and the deformations in the layer and the half-space for the existence of Love-type waves.

The secular equation (78) reduces to

$$\tan(skh) = \frac{s'A'}{sA}. \tag{87}$$

We now analyze this equation numerically by plotting the non-dimensionalized squared wave speed $\zeta = \rho v^2/\mu$ against the dimensionless wave number kh. We take the following values of the material constants in order to obtain some representative solutions:

$$l\mu_0 = 2, \quad l'\mu_0 = 1.7, \quad m\mu_0 = 2, \quad m'\mu_0 = 0.7, \quad n = 0.3, \quad n' = 0.8,$$
$$\mu_0 = 4\pi \times 10^{-7}\,\mathrm{N/A^2}, \quad \mu = 2.6 \times 10^5\,\mathrm{N/m^2}, \quad \mu'/\mu = 2, \quad \rho'/\rho = 1/3. \tag{88}$$

We assume the initial deformations in the layer and the half-space to be the same, i.e. $\lambda_1 = \lambda_1', \lambda_2 = \lambda_2', \lambda_3 = \lambda_3'$. An infinite number of propagation modes are obtained due to the dispersive nature of equation (87). Multiple modes of wave propagation corresponding to equation (87) are illustrated in Figure 1 for two sets of representative values ($B_1 = 0.1$ T, $\lambda_1 = 0.7 = \lambda_2^{-1}$) and ($B_2 = 0.5$ T, $\lambda_1 = 1.4 = \lambda_2^{-1}$). For other values of the parameters the pattern of the higher-order modes is similar and we therefore show only the lowest mode henceforth from (87) for each of a selection of values of the magnetic induction and deformation.

For the pure elastic problem (no magnetic field) with a finite initial deformation the results are shown in Figure 2 for different values of initial stretch for the first mode. It is noted from Figures 2(a) and 2(b) that for the linear elastic case ($\lambda_1 = \lambda_2 = \lambda_3 = 1$), the curves intersect the ζ axis at $\mu'\rho/\mu\rho'$, which agrees with the classical solution (obtained by taking the limit $kh \to 0$ in Equation (83)) and is equal to 6 for the values adopted here.

The effect of the magnetic field without a finite deformation on the wave propagation characteristics is illustrated in Figure 3. It is noted that as $kh \to 0$, a magnetic (induction) field B_2 perpendicular to the boundary has no effect while that parallel to the boundary (B_1) changes the wave speed significantly. Either B_1 or B_2 tends to increase the wave speed.

The effect of the magnetic field when there is an initial finite deformation is illustrated in Figure 4 for two different values of λ_1: 0.7 and 1.4. The character of the results is similar qualitatively to the situation when there is no initial stretch.

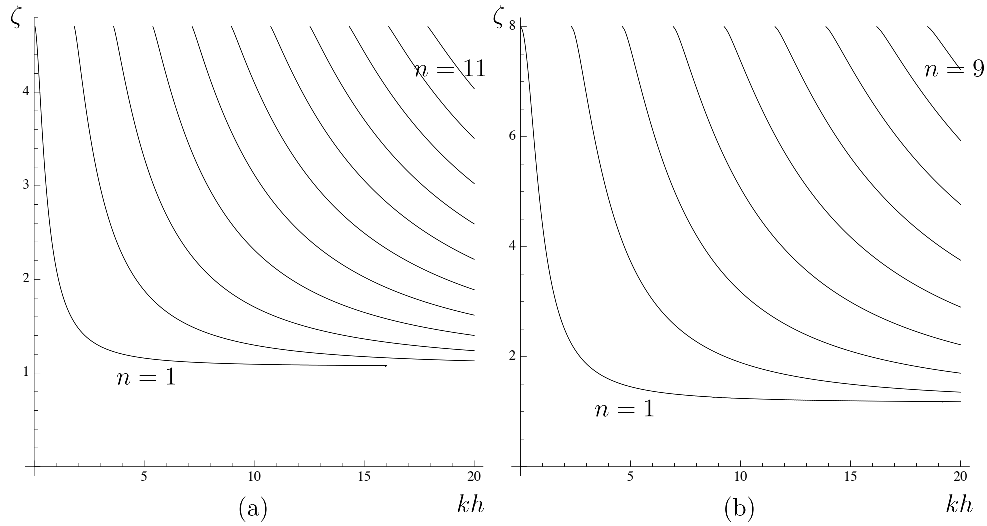

FIGURE 1. Dispersion curves $\zeta = \rho v^2/\mu(0)$ vs. kh for various mode numbers n satisfying equation (87), illustrated for (a) $B_2 = 0 = B_3, B_1 = 0.1$ T, $\lambda_1 = 0.7 = \lambda_2^{-1}, \lambda_3 = 1$, $n = 1$ to $n = 11$; (b) $B_1 = 0 = B_3, B_2 = 0.5$ T, $\lambda_1 = 1.4 = \lambda_2^{-1}, \lambda_3 = 1$, $n = 1$ to $n = 9$.

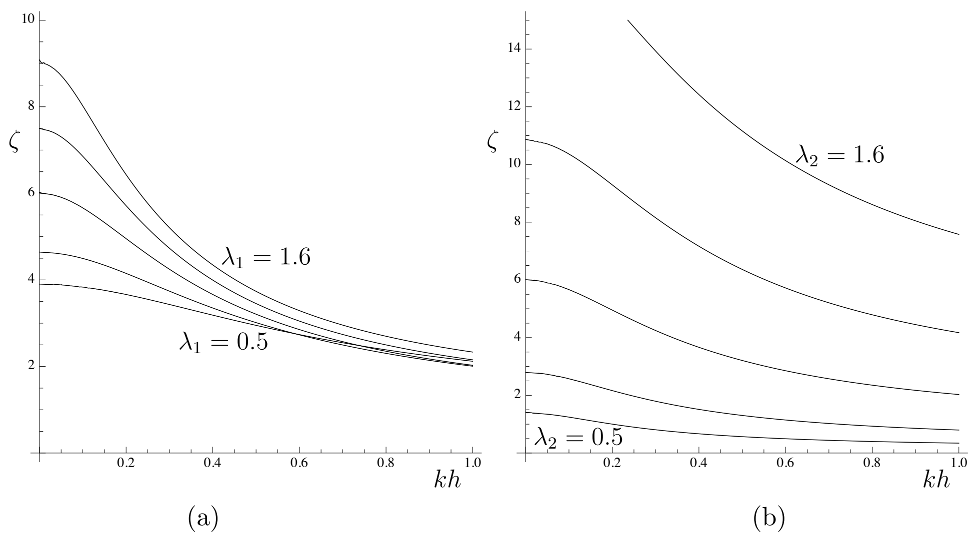

FIGURE 2. First mode dispersion curves $\zeta = \rho v^2/\mu$ vs. kh under finite deformation in the absence of a magnetic field: (a) $\lambda_1 = 0.5, 0.7, 1, 1.3, 1.6$, $\lambda_2 = \lambda_1^{-1}, \lambda_3 = 1$; (b) $\lambda_1 = 1, \lambda_2 = 0.5, 0.7, 1, 1.3, 1.6$, $\lambda_3 = \lambda_2^{-1}$.

4.2. Out-of-plane magnetic field: $\mathbf{B} = (0, 0, B_3)$

We now consider the case when the magnetic field is out of the plane, i.e. in the same direction as the mechanical displacement. The initial and deformed configurations are considered to be the same as in the previous section. For this value of the underlying magnetic induction, using equations (7) and

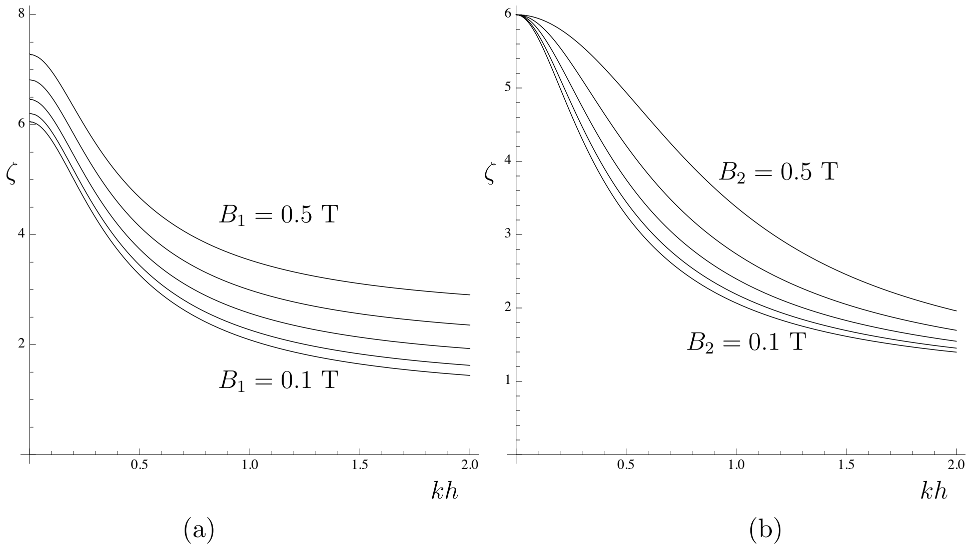

FIGURE 3. First mode dispersion curves $\zeta = \rho v^2/\mu$ vs. kh for the linear elastic case in the presence of a magnetic field: (a) $B_2 = 0$, $B_1 = 0.1, 0.2, 0.3, 0.4, 0.5$ T; (b) $B_1 = 0$, $B_2 = 0.1, 0.2, 0.3, 0.4, 0.5$ T.

(13) the components of the Maxwell stress and its increment are given by

$$[\boldsymbol{\tau}^*] = \frac{B_3^{*2}}{2\mu_0}\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{89}$$

$$[\dot{\boldsymbol{\tau}}^*] = \frac{1}{\mu_0}\begin{bmatrix} -\dot{B}_3^* B_3^* & 0 & \dot{B}_1^* B_3^* \\ 0 & -\dot{B}_3^* B_3^* & \dot{B}_2^* B_3^* \\ \dot{B}_1^* B_3^* & \dot{B}_2^* B_3^* & \dot{B}_3^* B_3^* \end{bmatrix}, \tag{90}$$

respectively.

The governing equations (32)–(37) reduce to $\dot{p}_{,1} = 0$, $\dot{p}_{,2} = 0$, $\dot{B}_{l03,1} = 0$ and $\dot{B}_{l03,2} = 0$ along with (31) and

$$\mathcal{A}_{01313} u_{3,11} + \mathcal{A}_{02323} u_{3,22} + \mathcal{C}_{013|1} \dot{B}_{l01,1} + \mathcal{C}_{023|2} \dot{B}_{l02,2} = \rho u_{3,tt}, \tag{91}$$

$$\left(\mathcal{C}_{023|2} - \mathcal{C}_{013|1}\right) u_{3,12} + \mathsf{K}_{022} \dot{B}_{l02,1} - \mathsf{K}_{011} \dot{B}_{l01,2} = 0 \tag{92}$$

for the layer, and similarly for the half-space. In this case u_3, $\dot{B}_{l01}$ and $\dot{B}_{l02}$ are coupled with each other through equations (91) and (92). Clearly, since there is no dependence on x_3, we may infer that $\dot{B}_{l03}$ is a function of t which may be taken to be zero as for $f(t)$ in Section 4.1.1.

Let $u_3 = \phi$. Since the pairs $\{\dot{B}_{l01}, \dot{B}_{l02}\}$, $\{\dot{B}'_{l01}, \dot{B}'_{l02}\}$ and $\{\dot{B}^*_{l01}, \dot{B}^*_{l02}\}$ satisfy equation (31), we may define potentials ψ, ψ' and ψ^* such that

$$\dot{B}_{l01} = \psi_{,2}, \quad \dot{B}_{l02} = -\psi_{,1}, \quad \dot{B}'_{l01} = \psi'_{,2}, \quad \dot{B}'_{l02} = -\psi'_{,1}, \quad \dot{B}^*_1 = \psi^*_{,2}, \quad \dot{B}^*_2 = -\psi^*_{,1}. \tag{93}$$

Substituting these potentials in the governing equations, we obtain

$$\mathcal{A}_{01313}\phi_{,11} + \mathcal{A}_{02323}\phi_{,22} + \mathcal{C}_{013|1}\psi_{,12} - \mathcal{C}_{023|2}\psi_{,12} = \rho\phi_{,tt}, \tag{94}$$

$$\left(\mathcal{C}_{023|2} - \mathcal{C}_{013|1}\right)\phi_{,12} - \mathsf{K}_{011}\psi_{,22} - \mathsf{K}_{022}\psi_{,11} = 0, \tag{95}$$

in the layer, while for the half-space we obtain

$$\mathcal{A}'_{01313}\phi'_{,11} + \mathcal{A}'_{02323}\phi'_{,22} + \mathcal{C}'_{013|1}\psi'_{,12} - \mathcal{C}'_{023|2}\psi'_{,12} = \rho'\phi'_{,tt}, \tag{96}$$

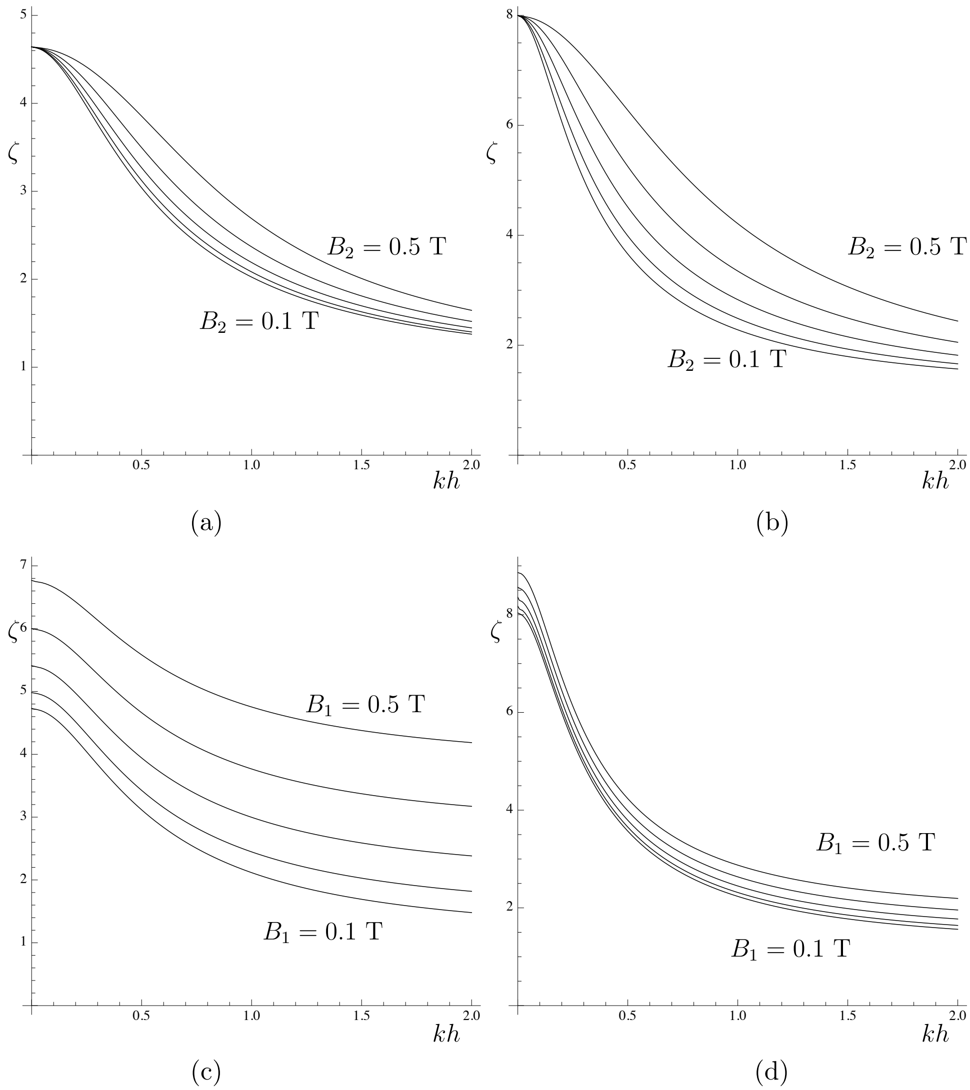

FIGURE 4. First mode dispersion curves $\zeta = \rho v^2/\mu$ vs. kh for the material under finite deformation and magnetic field satisfying equation (87): (a) $\lambda_1 = 0.7, \lambda_2 = \lambda_1^{-1}, \lambda_3 = 1$, $B_1 = 0$, $B_2 = 0.1, 0.2, 0.3, 0.4, 0.5$ T; (b) $\lambda_1 = 1.4, \lambda_2 = \lambda_1^{-1}, \lambda_3 = 1$, $B_1 = 0$, $B_2 = 0.1, 0.2, 0.3, 0.4, 0.5$ T; (c) $\lambda_1 = 0.7, \lambda_2 = \lambda_1^{-1}, \lambda_3 = 1$, $B_2 = 0$, $B_1 = 0.1, 0.2, 0.3, 0.4, 0.5$ T; (d) $\lambda_1 = 1.4, \lambda_2 = \lambda_1^{-1}, \lambda_3 = 1$, $B_2 = 0$, $B_1 = 0.1, 0.2, 0.3, 0.4, 0.5$ T.

$$\left(\mathcal{C}'_{023|2} - \mathcal{C}'_{013|1}\right)\phi'_{,12} - \mathsf{K}'_{011}\psi'_{,22} - \mathsf{K}'_{022}\psi'_{,11} = 0, \tag{97}$$

and outside the material

$$\psi^*_{,11} + \psi^*_{,22} = 0. \tag{98}$$

4.2.1. Incremental boundary conditions. From the boundary conditions (10)–(12) the only non-trivial remaining components are $\dot{T}_{023} = \dot{\tau}^*_{23}$, $\dot{B}_{l02} = \dot{B}^*_2$ and $\dot{H}_{l01} - u_{3,1}H^*_3 - \dot{H}^*_1 = 0$, which, terms of ϕ and the potential functions, yield, at the layer–vacuum interface $x_2 = h$,

$$\mathcal{A}_{02323}\phi_{,2} - \mathcal{C}_{023|2}\psi_{,1} + \psi^*_{,1}H^*_3 = 0, \tag{99}$$

$$\psi_{,1} - \psi^*_{,1} = 0, \tag{100}$$

$$\left(\mathcal{C}_{013|1} - H^*_3\right)\phi_{,1} + \mathsf{K}_{011}\psi_{,2} - \frac{1}{\mu_0}\psi^*_{,2} = 0, \tag{101}$$

and at the layer–half-space interface $x_2 = 0$,

$$\phi = \phi', \tag{102}$$

$$\mathcal{A}_{02323}\phi_{,2} - \mathcal{C}_{023|2}\psi_{,1} = \mathcal{A}'_{02323}\phi'_{,2} - \mathcal{C}'_{023|2}\psi'_{,1}, \tag{103}$$

$$\mathcal{C}_{013|1}\phi_{,1} + \mathsf{K}_{011}\psi_{,2} = \mathcal{C}'_{013|1}\phi'_{,1} + \mathsf{K}'_{011}\psi'_{,2}, \tag{104}$$

$$\psi_{,1} = \psi'_{,1}, \tag{105}$$

the first of which corresponds to continuity of displacement.

Hence the problem is reduced to solving equations (94) and (95) in $0 < x_2 < h$, equations (96) and (97) in $x_2 < 0$, and equation (98) in $x_2 > h$ using the boundary conditions (99), (100) and (101) at $x_2 = h$, and (102), (103), (104) and (105) at $x_2 = 0$.

4.2.2. Wave propagation. We again study Love-type waves in the same form as in the previous section and consider harmonic solutions of the form

$$\phi = P\exp[\mathrm{i}\,(skx_2 + kx_1 - \omega t)], \quad 0 < x_2 < h, \tag{106}$$

$$\psi = Q\exp[\mathrm{i}\,(skx_2 + kx_1 - \omega t)], \quad 0 < x_2 < h, \tag{107}$$

$$\phi' = P'\exp\left(s'kx_2 + \mathrm{i}kx_1 - \mathrm{i}\omega t\right), \quad x_2 < 0, \tag{108}$$

$$\psi' = Q'\exp\left(s'kx_2 + \mathrm{i}kx_1 - \mathrm{i}\omega t\right), \quad x_2 < 0, \tag{109}$$

$$\psi^* = R\exp\left(s^*kx_2 + \mathrm{i}kx_1 - \mathrm{i}\omega t\right), \quad x_2 > h, \tag{110}$$

with the conditions $\mathrm{Re}(s') > 0$ and $\mathrm{Re}(s^*) < 0$ for the solutions to decay as $x_2 \to -\infty$ and $x_2 \to \infty$, respectively.

Substituting the solutions (108) and (109) in equations (96) and (97), we obtain

$$\left(-\mathcal{A}'_{01313} + s'^2\mathcal{A}'_{02323} + \rho'v^2\right)P' + \mathrm{i}s'\left(\mathcal{C}'_{013|1} - \mathcal{C}'_{023|2}\right)Q' = 0, \tag{111}$$

$$\mathrm{i}s'\left(\mathcal{C}'_{023|2} - \mathcal{C}'_{013|1}\right)P' + \left(\mathsf{K}'_{022} - s'^2\mathsf{K}'_{011}\right)Q' = 0, \tag{112}$$

wherein the wave speed v is defined as $v = \omega/k$.

For non-trivial solutions of P' and Q', the determinant of the coefficients of the above equations should be zero which gives

$$\mathcal{A}'_{02323}\mathsf{K}'_{011}s'^4 + \{\mathsf{K}'_{011}\left(\rho'v^2 - \mathcal{A}'_{01313}\right) - \mathsf{K}'_{022}\mathcal{A}'_{02323} + \left(\mathcal{C}'_{023|2} - \mathcal{C}'_{013|1}\right)^2\}s'^2 - \mathsf{K}'_{022}\left(\rho'v^2 - \mathcal{A}'_{01313}\right) = 0. \tag{113}$$

Let s'_1 and s'_2 be the two solutions satisfying the condition $\mathrm{Re}(s') > 0$, then we note that the condition $s'^2_1 s'^2_2 \geq 0$ gives an upper bound on the wave speed, which we express in the form

$$\rho'v^2 \leq \mathcal{A}'_{01313}. \tag{114}$$

With the two possible values of s', the relevant general solutions for ϕ' and ψ' are

$$\phi' = \left(P'_1\mathrm{e}^{s'_1kx_2} + P'_2\mathrm{e}^{s'_2kx_2}\right)\exp[\mathrm{i}\,(kx_1 - \omega t)], \tag{115}$$

$$\psi' = \left(Q'_1\mathrm{e}^{s'_1kx_2} + Q'_2\mathrm{e}^{s'_2kx_2}\right)\exp[\mathrm{i}\,(kx_1 - \omega t)], \tag{116}$$

where P'_j and Q'_j are related by (112) as

$$Q'_j = \frac{-\mathrm{i}s'_j\left(\mathcal{C}'_{023|2} - \mathcal{C}'_{013|1}\right)}{\left(\mathsf{K}'_{022} - s'^2_j\mathsf{K}'_{011}\right)}P'_j, \quad j = 1, 2. \tag{117}$$

Substituting the solutions (106) and (107) into equations (94) and (95), we obtain

$$\left(-\mathcal{A}_{01313} - s^2\mathcal{A}_{02323} + \rho v^2\right)P - s\left(\mathcal{C}_{013|1} - \mathcal{C}_{023|2}\right)Q = 0, \tag{118}$$

$$s\left(\mathcal{C}_{013|1} - \mathcal{C}_{023|2}\right)P + \left(\mathsf{K}_{011}s^2 + \mathsf{K}_{022}\right)Q = 0. \tag{119}$$

For non-trivial solutions for P and Q, the determinant of the coefficients should be zero, which gives

$$\mathcal{A}_{02323}\mathsf{K}_{011}s^4 + \{\mathsf{K}_{011}\left(\mathcal{A}_{01313} - \rho v^2\right) + \mathcal{A}_{02323}\mathsf{K}_{022} - \left(\mathcal{C}_{013|1} - \mathcal{C}_{023|2}\right)^2\}s^2 + \mathsf{K}_{022}\left(\mathcal{A}_{01313} - \rho v^2\right) = 0. \tag{120}$$

Let the solutions of this equation be s_1, s_2, s_3 and s_4. Then the general solutions for ϕ and ψ may be written in the form

$$\phi = \left(P_1\mathrm{e}^{\mathrm{i}s_1kx_2} + P_2\mathrm{e}^{\mathrm{i}s_2kx_2} + P_3\mathrm{e}^{\mathrm{i}s_3kx_2} + P_4\mathrm{e}^{\mathrm{i}s_4kx_2}\right)\exp[\mathrm{i}\left(kx_1 - \omega t\right)], \tag{121}$$

$$\psi = \left(Q_1\mathrm{e}^{\mathrm{i}s_1kx_2} + Q_2\mathrm{e}^{\mathrm{i}s_2kx_2} + Q_3\mathrm{e}^{\mathrm{i}s_3kx_2} + Q_4\mathrm{e}^{\mathrm{i}s_4kx_2}\right)\exp[\mathrm{i}\left(kx_1 - \omega t\right)], \tag{122}$$

where P_j and Q_j are related by (119) as

$$Q_j = \frac{-s_j\left(\mathcal{C}_{013|1} - \mathcal{C}_{023|2}\right)}{\left(\mathsf{K}_{011}s_j^2 + \mathsf{K}_{022}\right)}P_j, \quad j = 1, 2, 3, 4. \tag{123}$$

Substituting the solution (110) into equation (98), we obtain $s^{*2} = 1$, and to satisfy the condition $\mathrm{Re}(s^*) < 0$, we take $s^* = -1$. Hence

$$\psi^* = R\exp\left(-kx_2 + \mathrm{i}kx_1 - \mathrm{i}\omega t\right). \tag{124}$$

Substituting the modified solutions (115), (116), (121), (122) and (124) into the boundary conditions (99), (100) and (101) at $x_2 = h$, and (102), (103), (104) and (105) at $x_2 = 0$, we obtain

$$\mathcal{A}_{02323}\sum_{j=1}^{4}P_js_j\mathrm{e}^{\mathrm{i}s_jkh} - \mathcal{C}_{023|2}\sum_{j=1}^{4}Q_j\mathrm{e}^{\mathrm{i}s_jkh} + H_3^*R\mathrm{e}^{-kh} = 0, \tag{125}$$

$$\sum_{j=1}^{4}Q_j\mathrm{e}^{\mathrm{i}s_jkh} - R\mathrm{e}^{-kh} = 0, \tag{126}$$

$$\mathrm{i}\left(\mathcal{C}_{013|1} - H_3^*\right)\sum_{j=1}^{4}P_j\mathrm{e}^{\mathrm{i}s_jkh} + \mathrm{i}\mathsf{K}_{011}\sum_{j=1}^{4}Q_js_j\mathrm{e}^{\mathrm{i}s_jkh} + \frac{1}{\mu_0}R\mathrm{e}^{-kh} = 0, \tag{127}$$

$$\sum_{j=1}^{4}P_j - \sum_{j=1}^{2}P'_j = 0, \tag{128}$$

$$\mathrm{i}\mathcal{A}_{02323}\sum_{j=1}^{4}P_js_j - \mathrm{i}\mathcal{C}_{023|2}\sum_{j=1}^{4}Q_j - \mathcal{A}'_{02323}\sum_{j=1}^{2}P'_js'_j + \mathrm{i}\mathcal{C}'_{023|2}\sum_{j=1}^{2}Q'_j = 0, \tag{129}$$

$$\mathrm{i}\mathcal{C}_{013|1}\sum_{j=1}^{4}P_j + \mathrm{i}\mathsf{K}_{011}\sum_{j=1}^{4}Q_js_j - \mathrm{i}\mathcal{C}'_{013|1}\sum_{j=1}^{2}P'_j - \mathsf{K}'_{011}\sum_{j=1}^{2}s'_jQ'_j = 0, \tag{130}$$

$$\sum_{j=1}^{4}Q_j - \sum_{j=1}^{2}Q'_j = 0. \tag{131}$$

Using the relations (117) and (123) between P_j–Q_j and P'_j–Q'_j, we can modify the above equations to

$$\sum_{j=1}^{4}\left[\mathcal{A}_{02323}+\frac{\mathcal{C}_{023|2}\left(\mathcal{C}_{013|1}-\mathcal{C}_{023|2}\right)}{\mathsf{K}_{011}s_j^2+\mathsf{K}_{022}}\right]s_j\mathrm{e}^{\mathrm{i}s_jkh}P_j+H_3^*R\,\mathrm{e}^{-kh}=0, \tag{132}$$

$$\sum_{j=1}^{4}s_j\frac{\left(\mathcal{C}_{013|1}-\mathcal{C}_{023|2}\right)}{\mathsf{K}_{011}s_j^2+\mathsf{K}_{022}}\mathrm{e}^{\mathrm{i}s_jkh}P_j+R\,\mathrm{e}^{-kh}=0, \tag{133}$$

$$\mathrm{i}\sum_{j=1}^{4}\left[\mathcal{C}_{013|1}-H_3^*-s_j^2\frac{\mathsf{K}_{011}\left(\mathcal{C}_{013|1}-\mathcal{C}_{023|2}\right)}{\mathsf{K}_{011}s_j^2+\mathsf{K}_{022}}\right]\mathrm{e}^{\mathrm{i}s_jkh}P_j+\frac{1}{\mu_0}R\,\mathrm{e}^{-kh}=0, \tag{134}$$

$$\sum_{j=1}^{4}P_j-\sum_{j=1}^{2}P'_j=0, \tag{135}$$

$$\begin{aligned}\sum_{j=1}^{4}\left[\mathcal{A}_{02323}+\frac{\mathcal{C}_{023|2}\left(\mathcal{C}_{013|1}-\mathcal{C}_{023|2}\right)}{\mathsf{K}_{011}s_j^2+\mathsf{K}_{022}}\right]s_jP_j&\\+\mathrm{i}\sum_{j=1}^{2}\left[\mathcal{A}'_{02323}-\frac{\mathcal{C}'_{023|2}\left(\mathcal{C}'_{013|1}-\mathcal{C}'_{023|2}\right)}{\mathsf{K}'_{011}s_j'^2-\mathsf{K}'_{022}}\right]s'_jP'_j&=0,\end{aligned} \tag{136}$$

$$\sum_{j=1}^{4}\frac{\mathsf{K}_{022}\mathcal{C}_{013|1}+s_j^2\mathsf{K}_{011}\mathcal{C}_{023|2}}{\mathsf{K}_{011}s_j^2+\mathsf{K}_{022}}P_j+\sum_{j=1}^{2}\frac{\mathsf{K}'_{022}\mathcal{C}'_{013|1}-s_j'^2\mathsf{K}'_{011}\mathcal{C}'_{023|2}}{\mathsf{K}'_{011}s_j'^2-\mathsf{K}'_{022}}P'_j=0, \tag{137}$$

$$\sum_{j=1}^{4}\frac{\left(\mathcal{C}_{013|1}-\mathcal{C}_{023|2}\right)}{\mathsf{K}_{011}s_j^2+\mathsf{K}_{022}}s_jP_j-\mathrm{i}\sum_{j=1}^{2}\frac{\left(\mathcal{C}'_{013|1}-\mathcal{C}'_{023|2}\right)}{\mathsf{K}'_{011}s_j'^2-\mathsf{K}'_{022}}s'_jP'_j=0. \tag{138}$$

These are seven equations for the seven constants $P_1, P_2, P_3, P_4, P'_1, P'_2$ and R. For non-trivial solutions, the determinant of the matrix formed by their coefficients should be zero. This condition gives the secular equation for the problem. We now illustrate the results for particular constitutive laws.

4.2.3. Application to a Mooney–Rivlin magnetoelastic material. In the underlying configuration, the boundary conditions require that $H_3^*=H_3=H'_3$. Thus $B_3^*=\mu_0H_3$, $B_3=0.5\lambda_3^2H_3/(l+m\lambda_3^2)$, $B'_3=0.5\lambda_3'^2H_3/(l'+m'\lambda_3'^2)$. Also, we have $\mathcal{C}_{023|2}=\mathcal{C}_{013|1}=2mB_3$. Hence the governing equations (91) and (92) reduce to

$$\mathcal{A}_{01313}u_{3,11}+\mathcal{A}_{02323}u_{3,22}=\rho u_{3,tt}, \tag{139}$$

$$\mathsf{K}_{022}\psi_{,11}+\mathsf{K}_{011}\psi_{,22}=0. \tag{140}$$

On substituting the harmonic solutions (106) and (107) in the above equations we get one value of s^2 for each of the mechanical and magnetic equations, say s_1^2 and s_2^2, respectively, i.e.

$$s_1^2=\frac{\rho v^2-\mathcal{A}_{01313}}{\mathcal{A}_{02323}},\quad s_2^2=-\frac{\mathsf{K}_{022}}{\mathsf{K}_{011}}. \tag{141}$$

Since the equations are decoupled these need not be the same, although in general there will be a coupling of the mechanical and magnetic effects through the boundary conditions.

When the mechanical and magnetic fields are combined the general solution may be written

$$\phi = \left(P^+ e^{is_1 kx_2} + P^- e^{-is_1 kx_2}\right)\exp[i(kx_1 - \omega t)], \quad 0 < x_2 < h, \tag{142}$$

$$\psi = \left(Q^+ e^{is_2 kx_2} + Q^- e^{-is_2 kx_2}\right)\exp[i(kx_1 - \omega t)], \quad 0 < x_2 < h, \tag{143}$$

$$\phi' = P' \exp\left(s_1' kx_2 + ikx_1 - i\omega t\right), \quad x_2 < 0, \tag{144}$$

$$\psi' = Q' \exp\left(s_2' kx_2 + ikx_1 - i\omega t\right), \quad x_2 < 0, \tag{145}$$

$$\psi^* = R \exp\left(-kx_2 + ikx_1 - i\omega t\right), \quad x_2 > h, \tag{146}$$

where

$$s_1'^2 = \frac{\mathcal{A}'_{01313} - \rho' v^2}{\mathcal{A}'_{02323}}, \quad s_2'^2 = \frac{\mathsf{K}'_{022}}{\mathsf{K}'_{011}}. \tag{147}$$

After substituting these into the seven boundary conditions (99)-(105) we find that $P^+ + P^- = P'$, $Q^+ + Q^- = Q'$, $Q^+ e^{is_2 kh} + Q^- e^{-is_2 kh} = Re^{-kh}$, and the remaining four boundary conditions expressed in terms of P^+, P^-, Q^+, Q^- are

$$s_1 \mathcal{A}_{02323}(P^+ - P^-) + i\mathcal{A}'_{02323} s_1'(P^+ + P^-) + (\mathcal{C}'_{023|2} - \mathcal{C}_{023|2})(Q^+ + Q^-) = 0, \tag{148}$$

$$(\mathcal{C}_{013|1} - \mathcal{C}'_{013|1})(P^+ + P^-) + \mathsf{K}_{011} s_2(Q^+ - Q^-) + i\mathsf{K}'_{011} s_2'(Q^+ + Q^-) = 0, \tag{149}$$

$$\begin{aligned} s_1 \mathcal{A}_{02323}(P^+ e^{is_1 kh} - P^- e^{-is_1 kh}) - \mathcal{C}_{023|2}(Q^+ e^{is_2 kh} + Q^- e^{-is_2 kh}) \\ + H_3^*(Q^+ e^{is_2 kh} + Q^- e^{-is_2 kh}) = 0, \end{aligned} \tag{150}$$

$$\begin{aligned} (\mathcal{C}_{013|1} - H_3^*)(P^+ e^{is_1 kh} + P^- e^{-is_1 kh}) + \mathsf{K}_{011} s_2(Q^+ e^{is_2 kh} - Q^- e^{-is_2 kh}) \\ - i\mu_0^{-1}(Q^+ e^{is_2 kh} + Q^- e^{-is_2 kh}) = 0. \end{aligned} \tag{151}$$

We plot the variation of the non-dimensionalized wave speed $\zeta = \rho v^2/\mu(0)$ against the non-dimensionalized wave number kh in Figure 5 to study the effects of magnetic field and deformation. Values of the material constants listed in (88) are used for the numerical calculations. In general, the wave speed decreases with an increase in the wave number and in the magnetic field B_3. Considering a plane strain deformation ($\lambda_3 = 1$), a compression represented by the stretch λ_1 parallel to the surface in the direction of wave propagation tends to increase the wave speed.

Equation (139) is the same as that obtained for the pure elastic case in Section 4.1.2, and if the incremental magnetic field vanishes the problem reduces to a purely mechanical problem to solve for u_3. However, in the presence of a magnetic field vanishing of the incremental magnetic field (so that $Q^+ = Q^- = 0$) in general forces $u_3 = 0$. There is an exception to this if both coefficients $\mathcal{C}_{013|1} - H_3^*$ and $\mathcal{C}_{013|1} - \mathcal{C}'_{013|1}$ vanish. For the considered material we have

$$\mathcal{C}_{013|1} - H_3^* = -2l\lambda_3^{-2} B_3, \quad \mathcal{C}_{013|1} - \mathcal{C}'_{013|1} = (lm'\lambda_3'^2 - l'm\lambda_3^2)H_3. \tag{152}$$

Thus, for a purely mechanical wave to propagate in the presence of a magnetic field we must have $l = 0$ and either $l' = 0$ or $m = 0$. If both l and m vanish then the layer is not a magnetic material. In either case it is easy to show that the wave speed does not depend on the value of the magnetic field since, for the considered model, $\mathcal{A}_{01313}$ and $\mathcal{A}_{02323}$ are independent of B_3.

Similarly, if $u_3 = 0$, i.e. $P^+ = P^- = 0$, then in general a purely magnetic wave cannot exist except when both $\mathcal{C}_{023|2} - \mathcal{C}'_{023|2}$ and $\mathcal{C}_{023|2} - H_3^*$ are zero. For the Mooney–Rivlin model we have

$$\mathcal{C}_{023|2} - H_3^* = 2l\lambda_3^{-2} B_3, \quad \mathcal{C}_{023|2} - \mathcal{C}'_{023|2} = \left(\frac{m\lambda_3^2}{l + m\lambda_3^2} - \frac{m'\lambda_3'^2}{l' + m'\lambda_3'^2}\right) H_3. \tag{153}$$

If we take the deformation in the layer and the bulk half space to be the same, i.e. $\lambda_3 = \lambda_3'$, then for a purely magnetic wave to propagate we must have $l = 0$ and either $l' = 0$ or $m = m' = 0$. Vanishing of both l and m will make the layer non-magnetic.

In order to consider the case in which there is coupling through the equations we specialize the constitutive law to a version of the neo-Hookean solid.

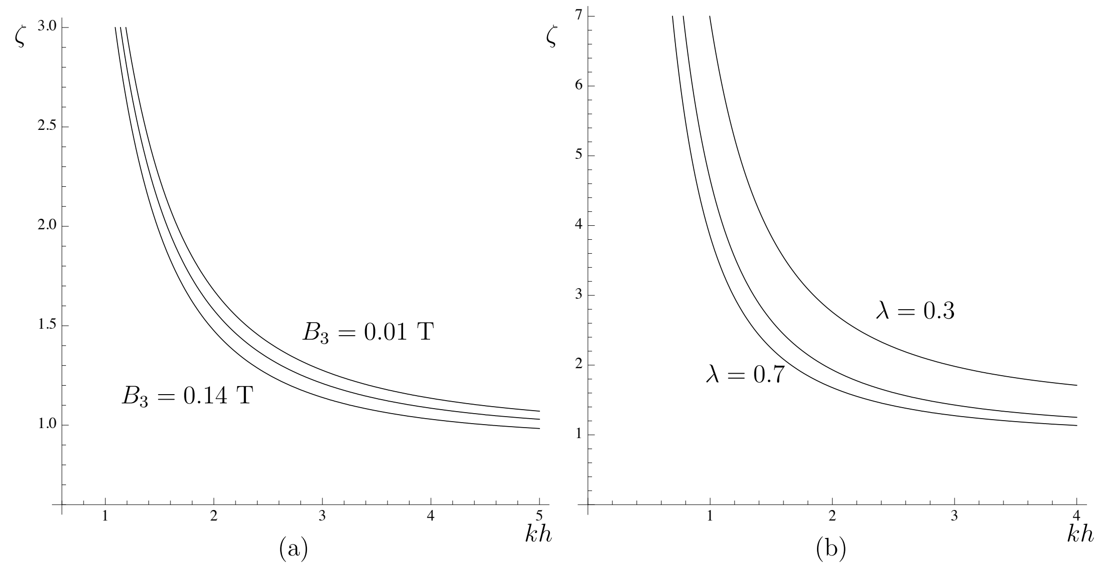

FIGURE 5. First mode of the dispersion curves $\zeta = \rho v^2/\mu(0)$ vs. kh for a Mooney–Rivlin type material in the presence of an out-of-plane magnetic field B_3. $\lambda = \lambda_1 = 1/\lambda_2, \lambda_3 = 1$. (a) $\lambda_1 = 0.7$, $B_3 = 0.01, 0.1, 0.14$ T; (b) $B_3 = 0.01$ T, $\lambda_1 = 0.3, 0.5, 0.7$.

4.2.4. Application to a neo-Hookean type magnetoelastic material. We consider a generalization of the neo-Hookean energy function for the magnetoelastic case which is a slight modification of the one used by Dorfmann and Ogden [6] and given by

$$\Omega = \frac{\mu}{2}(1 + \alpha I_4)(I_1 - 3) + lI_4 + mI_5 + qI_6, \tag{154}$$

where $\mu(I_4) = \mu(1 + \alpha I_4)$ is a shear modulus that varies with the magnetic field, and α, l, m and q are magnetoelastic coupling parameters. For this function, the relevant components of the moduli tensors are

$$\begin{aligned} \mathcal{A}_{01313} = \mu\lambda_1^2\,(1 + \alpha I_4) + 2\lambda_1^2 B_3^2 q, \quad \mathcal{A}_{02323} &= \mu\lambda_2^2\,(1 + \alpha I_4) + 2\lambda_2^2 B_3^2 q, \\ \mathcal{C}_{013|1} = 2B_3[m + (\lambda_1^2 + \lambda_3^2)q], \quad \mathcal{C}_{023|2} &= 2B_3[m + (\lambda_2^2 + \lambda_3^2)q], \\ \mathsf{K}_{011} &= \lambda_1^{-2}\,[\mu\alpha(I_1 - 3) + 2l] + 2m + 2q\lambda_1^2, \\ \mathsf{K}_{022} &= \lambda_2^{-2}\,[\mu\alpha(I_1 - 3) + 2l] + 2m + 2q\lambda_2^2. \end{aligned} \tag{155}$$

For this model, we use equations (132)–(138) to study the variation of the non-dimensionalized wave speed $\zeta = \rho v^2/\mu$ with the underlying magnetic field and deformation. We use the following values of the material parameters for the numerical calculations

$$\begin{aligned} &\mu = 2.6 \times 10^5\,\mathrm{N/m^2}, \quad \mu' = 2\mu, \quad \rho' = 2\rho, \quad \alpha = 2, \quad \alpha' = 0.7, \\ &l\mu_0 = 2, \quad l'\mu_0 = 1.7, \quad m\mu_0 = 2, \quad m'\mu_0 = 0.7, \quad q\mu_0 = 2, \quad q'\mu_0 = 0.1. \end{aligned} \tag{156}$$

The equations are dispersive and we obtain an infinite number of wave modes. We plot ζ against the non-dimensionalized wave number kh for the first modes in Figures 6 and 7.

In general the wave speeds decrease with increasing wave number and a higher magnetic field tends to increase the wave speed for the material described by the generalized neo-Hookean model. Considering an underlying deformation of plane strain ($\lambda_3 = 1$), a larger stretch λ_1 parallel to the surface in the direction of wave propagation tends to increase the wave speed. When a plane strain in

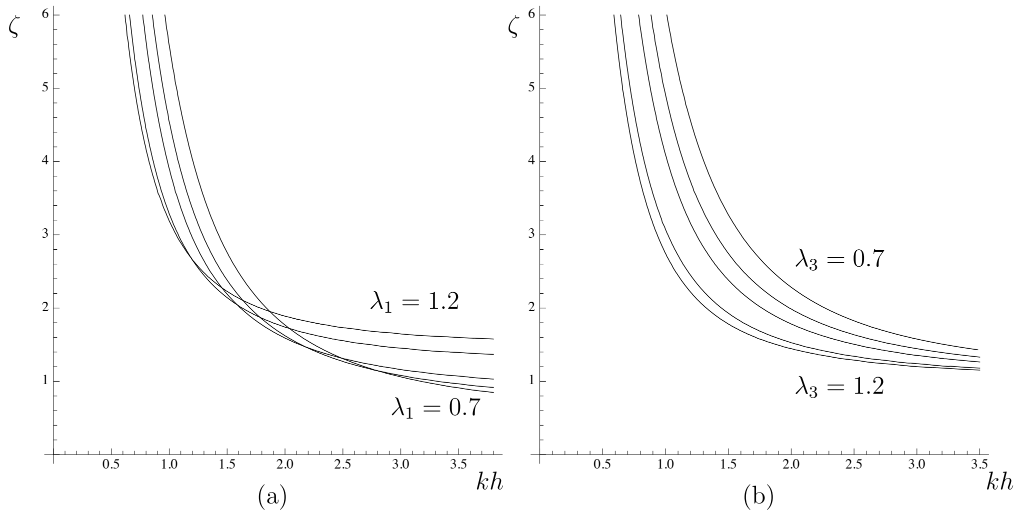

FIGURE 6. First mode dispersion curves $\zeta = \rho v^2/\mu$ vs. kh for a neo-Hookean type material in the presence of an out-of-plane magnetic field $B_3 = 0.03$ T: (a) $\lambda_3 = 1, \lambda_2^{-1} = \lambda_1 = 0.7, 0.8, 0.9, 1.1, 1.2$; (b) $\lambda_1 = 1, \lambda_2^{-1} = \lambda_3 = 0.7, 0.8, 0.9, 1.1, 1.2$.

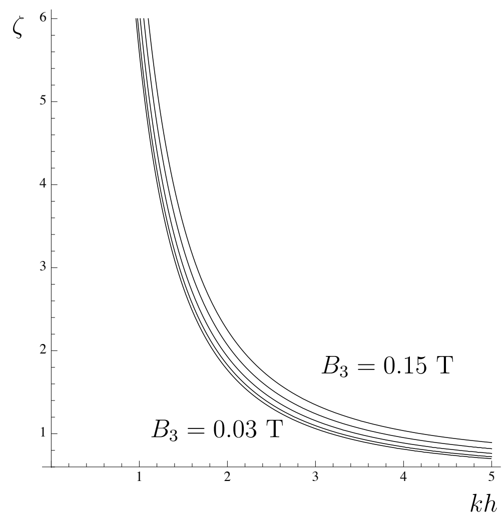

FIGURE 7. First mode dispersion curves $\zeta = \rho v^2/\mu$ vs. kh for a neo-Hookean type material in the presence of an out-of-plane magnetic field B_3. $\lambda_1 = 0.7 = \lambda_2^{-1}, \lambda_3 = 1$; $B_3 = 0.03, 0.06, 0.09, 0.12, 0.15$ T.

the plane perpendicular to the wave propagation direction is considered ($\lambda_1 = 1$), a larger principal stretch λ_3 in the out-of-plane direction tends to decrease the wave speed.

5. Shear horizontal surface waves without a layer

We now consider a magnetoelastic half-space without a layer and seek the possibility of waves with an out-of-plane displacement component. Waves of this type, first described by Parekh [22, 23], are similar to the Bleustein–Gulyaev waves in electroelasticity and do not have a counterpart in pure elasticity. We consider the two cases of in-plane and out-of-plane directions of the underlying magnetic induction.

5.1. $\mathbf{B} = (B_1, B_2, 0)$

The relevant governing equations are (42), (43), and (44) in $x_2 < 0$ with the boundary condition (52) at $x_2 = 0$. We consider solutions of the type (57) and (58) and substitute into the boundary conditions to obtain

$$(\mathrm{i}s' A' - B')P' = 0. \tag{157}$$

This cannot be satisfied since for a non-trivial wave we must have $P' \neq 0$, but also, since A' and B' are real, and, by strong ellipticity $A' > 0$ the real part of s' must vanish. Therefore such a mode of wave propagation does not exist when the underlying magnetic field is in-plane.

5.2. $\mathbf{B} = (0, 0, B_3)$

In this case, we consider the governing equations (96) and (97) in $x_2 < 0$, and equation (98) in $x_2 > 0$ to solve with the boundary conditions (99), (100) and (101) at $x_2 = 0$. We consider solutions similar to (115), (116) and (124). Substituting into the boundary conditions we obtain

$$\mathcal{A}'_{02323}(s'_1 P'_1 + s'_2 P'_2) - \mathrm{i}\mathcal{C}_{023|2}(Q'_1 + Q'_2) + \mathrm{i}H^*_3 R = 0, \tag{158}$$

$$Q'_1 + Q'_2 - R = 0, \tag{159}$$

$$\mathrm{i}(\mathcal{C}_{013|1} - H^*_3)(P'_1 + P'_2) + \mathsf{K}_{011}(s'_1 Q'_1 + s'_2 Q'_2) + \mu_0^{-1} R = 0, \tag{160}$$

while equation (112) gives the relations

$$\mathrm{i}s'_1 \left(\mathcal{C}'_{023|2} - \mathcal{C}'_{013|1}\right) P'_1 + \left(\mathsf{K}'_{022} - s'^2_1 \mathsf{K}'_{011}\right) Q'_1 = 0, \tag{161}$$

$$\mathrm{i}s'_2 \left(\mathcal{C}'_{023|2} - \mathcal{C}'_{013|1}\right) P'_2 + \left(\mathsf{K}'_{022} - s'^2_2 \mathsf{K}'_{011}\right) Q'_2 = 0, \tag{162}$$

Here s'_1 and s'_2 are the solutions of equation (113) satisfying the criterion $\mathrm{Re}(s') > 0$.

For non-trivial solutions for P'_1, P'_2, Q'_1, Q'_2 and R, the determinant of their coefficients should be zero which gives an equation to solve for the wave speed. We therefore illustrate the results in Figure 8 by considering again the modified neo-Hookean energy function defined in (154). The non-dimensionalized wave speed $\zeta = \rho' v^2/\mu'(0)$ is plotted against the underlying axial stretch for different values of the underlying magnetic field.

For a plane strain deformation ($\lambda_3 = 1$) denoted in Figure 8(a), it is observed that a stretch parallel to the direction of wave propagation λ_1 tends to increase the wave speed. A higher underlying magnetic field also increases the wave speed.

For the plane strain deformation when there is no compression or extension parallel to the wave propagation direction ($\lambda_1 = 1$) as shown in Figure 8(b), a critical value of $\lambda_3 = \lambda_c$ is observed at which the wave speed becomes independent of the underlying magnetic field B_3. The critical stretch λ_c depends on the parameters of the energy function used. When $\lambda_3 < \lambda_c$ the wave speed decreases with an increase in B_3 while in the region $\lambda_3 > \lambda_c$ the wave speed increases with an increase in B_3. For large values of compression (small λ_3) ζ goes to zero which coincides with the onset of instability in the material. The wave speed increases with an increase in λ_3 and reaches an asymptotic value dependent on the underlying magnetic field B_3.

When there is no underlying deformation, for the considered model we have $\mathcal{C}'_{013|1} = \mathcal{C}'_{023|2}$ and $\mathsf{K}'_{011} = \mathsf{K}'_{022}$. Equation (113) can be factorized to obtain the roots $s'^2_1 = 1 - \rho' v^2/\mathcal{A}_{01313}$ and $s'^2_2 = 1$. This results in (162) becoming identically zero and hence the above procedure yields no solution for the wave speed. So in this case we consider the solutions

$$\phi' = P' \exp\left(s'_1 k x_2 + \mathrm{i}k x_1 - \mathrm{i}\omega t\right), \quad x_2 < 0, \tag{163}$$

$$\psi' = Q' \exp\left(s'_2 k x_2 + \mathrm{i}k x_1 - \mathrm{i}\omega t\right), \quad x_2 < 0, \tag{164}$$

$$\psi^* = R \exp\left(-k x_2 + \mathrm{i}k x_1 - \mathrm{i}\omega t\right), \quad x_2 > 0, \tag{165}$$

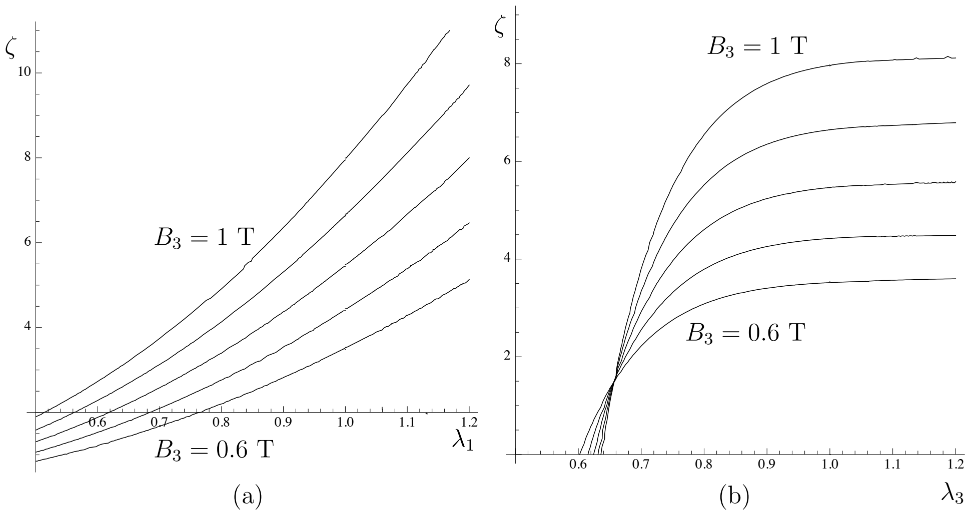

FIGURE 8. Variation of $\zeta = \rho' v^2/\mu'(0)$ with the underlying deformation and the underlying magnetic field for a Bleustein–Gulyaev type wave in a neo-Hookean type solid. $B_3 = 0.6, 0.7, 0.8, 0.9, 1$ T; (a) $\lambda = \lambda_1 = \lambda_2^{-1}, \lambda_3 = 1$; (b) $\lambda_1 = 1, \lambda_3 = \lambda_2^{-1} = \lambda$.

where the boundary conditions are (99)–(101), applied on $x_2 = 0$. These yield $R = Q'$ and

$$\mathcal{A}'_{02323} s'_1 P' + \mathrm{i}(H_3^* - \mathcal{C}'_{023|2})Q' = 0 \tag{166}$$

$$-\mathrm{i}(H_3^* - \mathcal{C}'_{013|1})P' + (\mu_0^{-1} + s'_2 \mathsf{K}'_{011})Q' = 0, \tag{167}$$

Requiring a non-trivial solution yields the following explicit formula for the wave speed:

$$\rho' v^2 = \mathcal{A}'_{01313} - \frac{(H_3^* - \mathcal{C}'_{013|1})^4}{(\mu_0^{-1} + \mathsf{K}'_{011})^2 \mathcal{A}'_{01313}}. \tag{168}$$

The value of the wave speed thus obtained for the linear elastic case is consistent with those illustrated in Figure 8.

The Mooney–Rivlin type energy function given by equation (84) requires special treatment, and we follow the procedure as above for the linear elastic case and obtain an explicit formula of the wave speed

$$\rho' v^2 = \mathcal{A}'_{01313} - \frac{(H_3^* - \mathcal{C}'_{013|1})^4}{(\mu_0^{-1} + \sqrt{\mathsf{K}'_{011}\mathsf{K}'_{022}})^2 \mathcal{A}'_{02323}}, \tag{169}$$

When the specific forms of the Mooney–Rivlin constants are substituted (for the case $n = 1$ for illustration), we get

$$\rho' v^2/\mu' = \lambda_1^2 - \frac{16 l^4 \lambda_3^{-8} B_3^4}{[\mu_0^{-1} + 2\sqrt{(m + l\lambda_1^{-2})(m + l\lambda_2^{-2})}]^2 \lambda_2^2 \mu'^2}. \tag{170}$$

The above formula suggests that there is an upper bound on the underlying magnetic field for the wave speed to be real. When evaluated for no underlying deformation this reduces to

$$\rho' v^2/\mu' = 1 - \frac{16 l^4 B_3^4}{[\mu_0^{-1} + 2(l + m)]^2 \mu'^2}. \tag{171}$$

Appendix A. Magnetoelastic moduli tensors

When referred to the principal axes of the left Cauchy–Green tensor $\mathbf{b} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$ with principal stretches $\lambda_1, \lambda_2, \lambda_3$ and components (B_1, B_2, B_3) of the magnetic induction $\mathbf{B}$ the components of $\boldsymbol{\mathcal{A}}_0, \boldsymbol{\mathcal{C}}_0$ and K_0 are given explicitly for an incompressible material as, for $i \neq j \neq k \neq i$,

$$\begin{aligned}\mathcal{A}_{0iiij} &= 4B_iB_j\lambda_i^2\{\Omega_6 + \Omega_{15} + (\lambda_j^2+\lambda_k^2)\Omega_{25} + +(\lambda_i^2+\lambda_j^2)[\Omega_{16} + (\lambda_j^2+\lambda_k^2)\Omega_{26}] \\ &+ \lambda_j^2\lambda_k^2B_i^2[\Omega_{55} + (3\lambda_i^2+\lambda_j^2)\Omega_{56} + 2\lambda_i^2(\lambda_i^2+\lambda_j^2)\Omega_{66}]\},\end{aligned}$$

$$\begin{aligned}\mathcal{A}_{0ijij} &= 2\lambda_i^2\{\Omega_1 + \lambda_k^2\Omega_2 + B_i^2\lambda_j^2\lambda_k^2\Omega_5 + \lambda_j^2\lambda_k^2(2B_i^2\lambda_i^2 + B_i^2\lambda_j^2 + B_j^2\lambda_i^2)\Omega_6 \\ &+ 2B_i^2B_j^2\lambda_j^2\lambda_k^2[\Omega_{55} + 2(\lambda_i^2+\lambda_j^2)\Omega_{56} + (\lambda_i^2+\lambda_j^2)^2\Omega_{66}]\},\end{aligned}$$

$$\begin{aligned}\mathcal{A}_{0iijk} &= 4B_jB_k\lambda_i^2\{\Omega_{15} + (\lambda_j^2+\lambda_k^2)(\Omega_{25}+\Omega_{16}) + (\lambda_j^2+\lambda_k^2)^2\Omega_{26} \\ &+ B_i^2\lambda_j^2\lambda_k^2[\Omega_{55} + (I_1+\lambda_i^2)\Omega_{56} + 2\lambda_i^2(\lambda_j^2+\lambda_k^2)\Omega_{66}]\},\end{aligned}$$

$$\mathcal{A}_{0ijki} = \mathcal{A}_{0ijik} = 2B_jB_k\{\lambda_i^2\Omega_6 + 2B_i^2[\Omega_{55} + (I_1+\lambda_i^2)\Omega_{56} + (I_2+\lambda_i^4)\Omega_{66}]\},$$

$$\mathcal{A}_{0jiki} = 2B_jB_k\{\Omega_5 + I_1\Omega_6 + 2B_i^2[\Omega_{55} + (I_1+\lambda_i^2)\Omega_{56} + (I_2+\lambda_i^4)\Omega_{66}]\},$$

$$\begin{aligned}\mathcal{C}_{0ii|i} &= 4B_i\{\Omega_5 + 2\lambda_i^2\Omega_6 + \Omega_{14} + \lambda_i^2\Omega_{15} + \lambda_i^4\Omega_{16} + (\lambda_j^2+\lambda_k^2)(\Omega_{24} + \lambda_i^2\Omega_{25} + \lambda_i^4\Omega_{26}) \\ &+ B_i^2\lambda_j^2\lambda_k^2[\Omega_{45} + \lambda_i^2\Omega_{55} + \lambda_i^4\Omega_{56} + 2\lambda_i^2(\Omega_{46} + \lambda_i^2\Omega_{56} + \lambda_i^4\Omega_{66})]\},\end{aligned}$$

$$\begin{aligned}\mathcal{C}_{0ii|j} &= 4B_j\lambda_i^2\lambda_j^{-2}\{\Omega_{14} + \lambda_j^2\Omega_{15} + \lambda_j^4\Omega_{16} + (\lambda_j^2+\lambda_k^2)(\Omega_{24} + \lambda_j^2\Omega_{25} + \lambda_j^4\Omega_{26}) \\ &+ B_i^2\lambda_j^2\lambda_k^2[\Omega_{45} + \lambda_j^2\Omega_{55} + \lambda_j^4\Omega_{56} + 2\lambda_i^2(\Omega_{46} + \lambda_j^2\Omega_{56} + \lambda_j^4\Omega_{66})]\},\end{aligned}$$

$$\begin{aligned}\mathcal{C}_{0ij|i} &= 2B_j\{\Omega_5 + (\lambda_i^2+\lambda_j^2)\Omega_6 + 2B_i^2\lambda_j^2\lambda_k^2[\Omega_{45} + \lambda_i^2\Omega_{55} + \lambda_i^4\Omega_{56} \\ &+ (\lambda_i^2+\lambda_j^2)(\Omega_{46} + \lambda_i^2\Omega_{56} + \lambda_i^4\Omega_{66})]\},\end{aligned}$$

$$\mathcal{C}_{0ij|k} = 4B_iB_jB_k\lambda_i^2\lambda_j^2[\Omega_{45} + \lambda_k^2\Omega_{55} + \lambda_k^4\Omega_{56} + (\lambda_i^2+\lambda_j^2)(\Omega_{46} + \lambda_k^2\Omega_{56} + \lambda_k^4\Omega_{66})],$$

$$\begin{aligned}\mathsf{K}_{0ii} &= 2\lambda_i^{-2}\{\Omega_4 + \lambda_i^2\Omega_5 + \lambda_i^4\Omega_6 + 2B_i^2\lambda_j^2\lambda_k^2[\Omega_{44} + \lambda_i^2\Omega_{45} + \lambda_i^4\Omega_{46} \\ &+ \lambda_i^2(\Omega_{45} + \lambda_i^2\Omega_{55} + \lambda_i^4\Omega_{56}) + \lambda_i^4(\Omega_{46} + \lambda_i^2\Omega_{56} + \lambda_i^4\Omega_{66}]\},\end{aligned}$$

$$\begin{aligned}\mathsf{K}_{0ij} &= 4B_iB_j\lambda_k^2[\Omega_{44} + \lambda_i^2\Omega_{45} + \lambda_i^4\Omega_{46} + \lambda_j^2(\Omega_{45} + \lambda_i^2\Omega_{55} + \lambda_i^4\Omega_{56}) \\ &+ \lambda_j^4(\Omega_{46} + \lambda_i^2\Omega_{56} + \lambda_i^4\Omega_{66})].\end{aligned}$$

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Prashant Saxena
School of Mathematics & Statistics
University of Glasgow
University Gardens
Glasgow G12 8QW, UK
e-mail: p.saxena.1@research.gla.ac.uk

Ray W. Ogden
School of Engineering
University of Aberdeen
King's College
Aberdeen AB24 3UE, UK
e-mail: r.ogden@abdn.ac.uk