Popular matchings in the Marriage and Roommates problems*

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Abstract. Popular matchings have recently been a subject of study in the context of the so-called *House Allocation Problem*, where the objective is to match applicants to houses over which the applicants have preferences. A matching M is called *popular* if there is no other matching M' with the property that more applicants prefer their allocation in M' to their allocation in M. In this paper we study popular matchings in the context of the *Roommates Problem*, including its special (bipartite) case, the *Marriage Problem*. We investigate the relationship between popularity and stability, and describe efficient algorithms to test a matching for popularity in these settings. We also show that, when ties are permitted in the preferences, it is NP-hard to determine whether a popular matching exists in both the Roommates and Marriage cases.

1 Introduction

Background. Stable matching problems have a long history, dating back to the seminal paper of Gale and Shapley [9], and these problems continue as an area of active research among computer scientists, mathematicians and economists [13, 25]. An instance of the classical Stable Marriage problem (SM) involves sets of n men and n women, and each person has a strict order of preference (their *preference list*) over all of the members of the opposite sex. A *stable* matching M is a set of n disjoint man-woman pairs such that no man m and woman w who do not form a pair prefer each other to their partners in M. The Stable Roommates problem (SR) is the generalisation of SM to the non-bipartite case, where each person has a strict order of preference over all of the others.

Gale and Shapley [9] showed that every instance of SM admits a stable matching, and such a matching can be found in $O(n^2)$ time, whereas, by contrast, some SR instances admit no stable matching. Irving [15] gave an $O(n^2)$ time algorithm to find a stable matching in an SR instance, when one exists.

A wide range of extensions of these fundamental problems have been studied. For instance, the existence results and efficient algorithms extend to the case where preference lists are *incomplete*, i.e., when participants can declare some of the others to be unacceptable as partners. In this case, both the Gale-Shapley

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algorithm and Irving's algorithm can be adapted to run in $O(m)$ time, where m is the sum of the lengths of the preference lists [13]. Furthermore, in this case, it is known that all stable matchings have the same size and match exactly the same people [10, 13]. If, in addition, *ties* are permitted in an individual's preferences, then the situation becomes more complex. Here, a stable matching always exists, but different stable matchings can have different sizes, and it is NP-hard to find a stable matching of maximum (or minimum) size [16, 18]. In the Roommates case it is NP-complete to determine whether a stable matching exists (even if preference lists are complete) [24].

Here, we are interested in the Marriage and Roommates scenarios, where each participant expresses preferences over some or all of the others, but we focus on matchings that are *popular* rather than stable. A matching M is *popular* if there is no other matching M' with the property that more participants prefer M' to M than prefer M to M'. M is *strongly popular* if, for any other matching M', more participants prefer M to M' than prefer M' to M. These concepts were introduced in the Marriage context by Gärdenfors [11].

Recently, popular matchings have been studied in the context of the so-called House Allocation problem (HA). An instance of HA involves a set of applicants and a set of houses. Each applicant has a strict order of preference over the houses that are acceptable to him, but houses have no preference over applicants. Abraham et al. [1] described an $O(n + m)$ time algorithm to find a popular matching, if one exists, in an instance of HA, where n is the total number of applicants and houses, and m is the total number of acceptable applicant-house pairs. In the case that ties are allowed in the preference lists, they gave an $O(\sqrt{n}m)$ time algorithm. These results motivated the present study.

The contribution of this paper. Our prime focus in this paper is the problem of finding popular matchings in the Roommates and Marriage contexts. In Section 2 we formalise the problem descriptions and give the necessary terminology and notation. In Section 3 we focus on strict preferences. We describe some basic properties of popular matchings, and the more restrictive strongly popular matchings, and their relation to stable matchings. We give a linear time algorithm to test for and to find a strongly popular matching for Roommates instances without ties. We show that, given a Roommates instance (with or without ties) and a matching M, we can test whether M is popular in $O(\sqrt{n\alpha(n,m)}m \log^{3/2} n)$ time (where α is the inverse Ackermann's function), and in the Marriage case we show how this can be improved to $O(\sqrt{n}m)$ time. This latter result generalises a previous $O(\sqrt{n}m)$ algorithm for the special case where preference lists may include ties and are *symmetric* (i.e., a man m ranks a woman w in kth place if and only if w ranks m in kth place) [26]. In Section 4 we first investigate which of the results of Section 3 can be extended to the case of ties. Then we establish an NP-completeness result for the problem of determining whether a popular matching exists for a Marriage (or Roommates) instance with ties. We conclude with some open problems in Section 5.

Related work. Gärdenfors [11] introduced the notions of a *(strong) majority* *assignment*, which is equivalent to a (strongly) popular matching in our terminology. He proved that every stable matching is popular in the Marriage case with strict preferences. Also, he showed that a strongly popular matching is stable in the Marriage case, even if there are ties in the preference lists.

The results of Abraham et al. [1] mentioned above led to a number of subsequent papers exploring further aspects and extensions of popular matchings in HA. Manlove and Sng [19] studied the extension in which each house has a *capacity*, the maximum number of applicants that can be assigned to it in any matching, and gave a $O(\sqrt{C}n_1 + m)$ time algorithm for this variant, where C is the sum of the capacities of the houses and n_1 is the number of applicants. Mestre [22] gave a linear time algorithm for a version of the problem in which each applicant has an associated weight. This algorithm, which assumes that all houses have capacity 1, was extended by Sng and Manlove [27] to the case where houses can have non-unitary capacities. Mahdian [17] showed that, for random instances of HA, popular matchings exist with high probability if the number of houses exceeds the number of applicants by a small constant multiplicative factor. Abraham and Kavitha [2] studied a dynamic version of HA allowing for applicants and houses to enter and leave the market, and for applicants to arbitrarily change their preference lists. They showed the existence of a 2-step *voting path* to compute a new popular matching after every such change, assuming that a popular matching exists. McCutchen [20] focused on instances of HA for which no popular matching exists, defining two notions of 'near popularity', and proving that, for each of these, finding a matching that is as near to popular as possible is NP-hard. Huang et al. [14] built upon the work of McCutchen with a study of approximation algorithms in the context of near popularity. McDermid and Irving [21] characterised the structure of the set of popular matchings for an HA instance, and gave efficient algorithms to count and enumerate the popular matchings, and to find several kinds of optimal popular matchings.

In voting theory, a well-established concept of *majority equilibrium* is the following. Let $S = \{1, 2, \ldots, n\}$ be a society of n individuals, and let X be a set of alternatives. Each individual $i \in S$ has a preference order, $\geq_i$, on X. An alternative $x \in X$ is called a *weak Condorcet winner* if for every $y \in X$ distinct of x, $|\{i \in S : x >_i y\}| \geq |\{i \in S : y >_i x\}|$; x is a *strong Condorcet winner* if for every $y \in X$ distinct of x, $|\{i \in S : x >_i y\}| > |\{i \in S : y >_i x\}|$. It is easy to see that if the set of alternatives is the set of all possible matchings of the individuals then a matching is a weak (respectively strong) Condorcet winner if and only if it is popular (respectively strongly popular). Therefore recent papers of Chen et al., see e.g. [4], which deal with the problems of finding a weak and strong Concordet winner for special graph models, are related to our work.

2 Problem descriptions, terminology and notation

Since the Roommates problem can be seen as an extension of the Marriage problem, we introduce our notation and terminology in the former setting. An instance I of the *Roommates Problem* (RP) comprises a set of *agents* $A =$

$\{a_1, \ldots, a_n\}$. For each agent a_i there is a subset A_i of $A \setminus \{a_i\}$ containing a_i's *acceptable* partners, and a_i has a linear order over A_i, which we refer to as a_i's *preference list*. If a_j precedes a_k in a_i's preference list, we say that a_i *prefers* a_j to a_k. We are also interested in the extension of RP, called the *Roommates Problem with Ties* (RPT), in which preference lists may contain tied entries, so that a_i prefers a_j to a_k if and only if a_j is a strict predecessor of a_k in a_i's preference list. We say that agent a_i is *indifferent* between a_j and a_k if a_j and a_k are tied in his preference list.

An instance I of RP may also be viewed as a graph $G = (A, E)$ where $\{a_i, a_j\}$ forms an edge in E if and only if a_i and a_j are each *acceptable* to the other. We assume that G contains no isolated vertices, and we let $m = |E|$. We refer to G as the *underlying graph* of I. A *matching* in I is a set of disjoint edges in the underlying graph G.

An instance of the *Marriage Problem with Ties* (MPT) may be viewed as an instance of RPT in which the underlying graph G is bipartite. The *Marriage Problem* (MP) is the analogous restriction of RP. In either case, the two sets of the bipartition are known as the *men* and the *women*.

Let I be an instance of RPT. Let $\mathcal{M}$ denote the set of matchings in I, and let $M \in \mathcal{M}$. Given any $a_i \in A$, if $\{a_i, a_j\} \in M$ for some $a_j \in A$, we say that a_i is *matched* in M and $M(a_i)$ denotes a_j, otherwise a_i is *unmatched* in M.

We define the preferences of an agent over matchings as follows. Given two matchings M and M' in $\mathcal{M}$, we say that an agent a_i *prefers* M' *to* M if either (i) a_i is matched in M' and unmatched in M, or (ii) a_i is matched in both M' and M and prefers $M'(a_i)$ to $M(a_i)$. Let $P(M', M)$ denote the set of agents who prefer M' to M, and let $I(M', M)$ be the set of agents who are indifferent between M' and M (i.e., $a_i \in I(M', M)$ if and only if either (i) a_i is matched in both M' and M and either (a) $M'(a_i) = M(a_i)$ or (b) a_i is indifferent between $M'(a_i)$ and $M(a_i)$, or (ii) a_i is unmatched in both M' and M). Then $P(M, M')$, $P(M', M)$ and $I(M', M)$ $(=I(M, M'))$ partition A.

A *blocking pair* with respect to a matching $M \in \mathcal{M}$ is an edge $\{a_i, a_j\} \in E \backslash M$ such that each of a_i and a_j prefers $\{\{a_i, a_j\}\}$ to M. A matching is *stable* if it admits no blocking pair. As observed earlier, an instance of RP or RPT may or may not admit a stable matching, whereas every instance of MP or MPT admits at least one such matching.

Given two matchings M and M' in $\mathcal{M}$, define $D(M, M') = |P(M, M')| - |P(M', M)|$. Clearly $D(M, M') = -D(M', M)$. We say that M is *more popular than* M', denoted $M \succ M'$, if $D(M, M') > 0$. M is *popular* if $D(M, M') \geq 0$ for all matchings $M' \in \mathcal{M}$. Also M is *strongly popular* if $D(M, M') > 0$ for all matchings $M' \in \mathcal{M} \backslash \{M\}$.[1]

Furthermore, for a set of agents $S \subseteq V(G)$, let $P_S(M, M')$ denote the subset of S whose members prefer M to M'. Let $D_S(M, M') = |P_S(M, M')| - |P_S(M', M)|$. We say that $M \succ_S M'$ if $D_S(M, M') > 0$. We will also use the

[1] In fact it is not difficult to see that M is popular if $D(M, M') \geq 0$ for all maximal matchings $M' \in \mathcal{M}$, and M is *strongly popular* if $D(M, M') > 0$ for all maximal matchings $M' \in \mathcal{M} \backslash \{M\}$.

standard notation $M|_S$ for the restriction of a matching M to the set of agents S, where $v \in S$ is considered to be unmatched in $M|_S$ if he is matched in M but $M(v)$ is not in S.

3 The case of strict preferences

In this section we investigate popular matchings in instances of RP and MP where, by definition, every agent's preference list is strictly ordered.

3.1 Relationships between strongly popular, popular and stable matchings

Let $S_1, S_2, \ldots S_k$ be a partition of $V(G)$. Then for any two matchings M and M', $P(M, M') = \cup_{i=1}^{k} P_{S_i}(M, M')$ and $D(M, M') = \sum_{i=1}^{k} D_{S_i}(M, M')$ by definition. We will prove some useful lemmas by using the above identity for two particular partitions.

First, let us consider the component-wise partition of $V(G)$ for the symmetric difference of two matchings M and M'. For each component G_i of $M \oplus M'$ let $C_i = V(G_i)$. We consider the following equation:

$$D(M, M') = \sum_{i=1}^{k} D_{C_i}(M, M') \tag{1}$$

Note that if $|C_i| = 1$ then it must be the case that $D_{C_i}(M, M') = 0$ since the corresponding agent is either unmatched in both M and M' or has the same partner in M and M'.

Lemma 1 *For a given instance of* RP*, a matching M is popular if and only if, for any other matching M', $D_{C_i}(M, M') \geq 0$ for each component G_i of $M \oplus M'$, where $C_i = V(G_i)$.*

The proof of Lemma 1 can be found in our corresponding technical report [3]. A similar statement holds for strongly popular matchings, as follows.

Lemma 2 *For a given instance of* RP*, a matching M is strongly popular if and only if, for any other matching M', $D_{C_i}(M, M') > 0$ for each component G_i of $M \oplus M'$, where $C_i = V(G_i)$ and $|C_i| \geq 2$.*

See [3] for the proof of Lemma 2. Now, let M, M' be any two matchings and let $F = M' \backslash M = \{e_1, e_2, \ldots, e_k\}$. Further let $X \subseteq V(G)$ be the set of agents covered by F and let $\bar{X} = V(G) \setminus X$. Considering the partition $\{E_1, E_2, \ldots E_k, \bar{X}\}$, where E_i represents the end vertices of the edge e_i, we have

$$D(M', M) = \sum_{i=1}^{k} D_{E_i}(M', M) + D_{\bar{X}}(M', M). \tag{2}$$

This identity leads to the following lemma.

Lemma 3 *Suppose that we are given an instance of* RP *and two matchings M and M'.*

a) If $M' \succ M$ then M' must contain an edge that is blocking for M.
b) If M is stable then M is popular.
c) If M is stable and M' is popular then M' covers all the vertices that M covers, implying $|M'| \geq |M|$, and $D_{E_i}(M', M) = 0$ for each $e_i \in M' \setminus M$ (i.e., in each pair corresponding to an edge of $M' \setminus M$ exactly one agent prefers M' to M and the other prefers M to M').

The proof of Lemma 3 can be found in [3].

We note that the result of Lemma 3(b) was proved by Gärdenfors [11] for MP. It is straightforward to verify that if I is an instance of RP and M is a strongly popular matching in I then M is the only popular matching in I. This implies that an instance of RP admits at most one strongly popular matching.

The following proposition was proved by Gärdenfors [11] for MP. Here we generalise the result to the RP context.

Proposition 4 *Let I be an instance of* RP *and let M be a strongly popular matching in I. Then M is stable in I.*

Proof. If M is not stable then let $\{a_i, a_j\}$ be a blocking pair of M. Let M' be a matching formed from M as follows: (i) remove the edge $\{a_i, M(a_i)\}$ if a_i is matched in M, (ii) remove the edge $\{a_j, M(a_j)\}$ if a_j is matched in M, then (iii) add the edge $\{a_i, a_j\}$. Then $|P(M', M)| = 2$ whilst $|P(M, M')| \leq 2$, contradicting the strong popularity of M. Hence M is stable in I. □

Lemma 3(b) and Proposition 4 thus give the following chain of implications involving properties of a matching M in an instance I of RP:

strongly popular ⇒ stable ⇒ popular ⇒ maximal

Examples to illustrate the following facts are given in [3]. An instance of RP may not admit a popular matching, but a popular matching may exist even if the instance does not admit a stable matching. Moreover, a unique stable matching (which is also a unique popular matching) is not necessarily strongly popular. Therefore the converse to each of the above implications is not true in general. In the case of MP, examples are given to show that a popular matching can be larger than a stable matching, a maximum cardinality matching need not be popular, and the relation $\succ$ can cycle (even if a stable matching exists).

3.2 Testing for and finding a strongly popular matching

We begin this section by giving an $O(m)$ algorithm that tests a given stable matching for strong popularity. Let I be an instance of RP and let M be a stable matching in I. Define the graph $H_M = (A, E_M)$, where

$$E_M = \left\{ \{a_i, a_j\} \in E : \begin{array}{l} a_i \text{ is unmatched in } M \text{ or prefers } a_j \text{ to } M(a_i) \vee \\ a_j \text{ is unmatched in } M \text{ or prefers } a_i \text{ to } M(a_j) \end{array} \right\}.$$

Lemma 5 *Let I be an instance of* RP *and let M be a stable matching in I. Let H_M be the graph defined above. Then M is strongly popular in I if and only if H_M contains no alternating cycle or augmenting path relative to M.*

Proof. Strong popularity implies that no such alternating cycle or augmenting path exists in H_M relative to M by Lemma 2. This is because if M' is the matching obtained by switching edges along this alternating path (or cycle) and C_i denotes the set of agents involved then it would be the case that $D_{C_i}(M, M') \leq 0$ in this component. On the other hand, suppose that M is stable but not strongly popular, i.e., there is a matching M' such that $D(M', M) = 0$. The statements of Lemma 3(c) hold in this case too by the very same argument used in the proof of that result. The fact that M' covers all the vertices that are covered by M means that each component of $M' \oplus M$ is either an alternating cycle or an augmenting path. And since $D_{E_i}(M', M) = 0$ for each $e_i \in M' \setminus M$, every edge in $M' \setminus M$ must belong to H_M. □

Based on Lemma 5 we can give a linear time algorithm for the problem of finding a strongly popular matching as indicated by the following theorem.

Theorem 6 *Given an instance I of* RP*, we may find a strongly popular matching or report that none exists in $O(m)$ time.*

Proof. We firstly test whether I admits a stable matching in $O(m)$ time [13, Section 4.5.2]. If no such matching exists, I does not admit a strongly popular matching by Proposition 4. Now suppose that I admits a stable matching M. Then I admits a strongly popular matching if and only if M is strongly popular. For, suppose that I admits a strongly popular matching $M' \neq M$. Then M' is certainly popular, and M is popular by Lemma 3(b), a contradiction to the fact that if a strongly popular matching exists then no other popular matching exists for the instance. By Lemma 5, M is strongly popular if and only if H_M contains no augmenting path or alternating cycle relative to M. Clearly H_M has $O(n)$ vertices and $O(m)$ edges. We may test for the existence of each of these structures in $O(m)$ time (see [5, 7] and [6] respectively). □

3.3 Testing for popularity

In order to test a matching M in a given instance of RP for popularity, we form a weighted graph H_M as follows. The vertices of H_M are $A \cup A'$, where $A' = \{a'_1, \ldots, a'_n\}$. The edges of H_M are $E \cup E' \cup E''$, where $E' = \{\{a'_i, a'_j\} : \{a_i, a_j\} \in E\}$ and $E'' = \{\{a_i, a'_i\} : 1 \leq i \leq n\}$. For each edge $\{a_i, a_j\} \in E$, we define $\delta_{i,j}$ as follows:

$$\delta_{i,j} = \begin{cases} 0, & \text{if } \{a_i, a_j\} \in M \\ \frac{1}{2}, & \text{if } a_i \text{ is unmatched in } M \text{ or prefers } a_j \text{ to } M(a_i) \\ -\frac{1}{2}, & \text{otherwise} \end{cases}$$

For each edge $\{a_i, a_j\} \in E$, we define the weight of $\{a_i, a_j\}$ in H_M to be $\delta_{i,j} + \delta_{j,i}$. Similarly, for each edge $\{a'_i, a'_j\} \in E'$, we define the weight of $\{a'_i, a'_j\}$ in H_M to be $\delta_{i,j} + \delta_{j,i}$. Finally, for each edge $\{a_i, a'_i\} \in E''$, we define the weight of $\{a_i, a'_i\}$ in H_M to be -1 if a_i is matched in M, and 0 otherwise. It is clear that the weight of each edge belongs to the set $\{-1, 0, 1\}$.

In what follows, given a matching M in G, we define M' to be a matching in H_M such that $M' = \{\{a'_i, a'_j\} : \{a_i, a_j\} \in M\}$.

Lemma 7 *Let I be an instance of* RP *and let M be a matching in I. Let H_M be the weighted graph defined above. Then M is popular if and only if a maximum weight perfect matching in H_M has weight 0.*

Proof. Let M_1 be any matching in I, and let A_{M_1} denote the agents in A who are matched in M_1. Define the matching

$$S(M_1) = M_1 \cup M'_1 \cup \{\{a_i, a'_i\} : a_i \in A \backslash A_{M_1}\}.$$

We claim that $wt(S(M_1)) = D(M_1, M)$, where $wt(M^{\sim})$ is the weight of a matching $M^{\sim}$ in H_M. To show this let $M''_1 = \{\{a_i, a'_i\} : a_i \in A \setminus A_{M_1}\}$. Also let $X = M_1 \setminus M$. Define n_-, n_0, n_+ to be the numbers of edges of weight $-1, 0, 1$ in X respectively. Also define n''_- to be the number of edges of weight -1 in M''_1. Then $wt(S(M_1)) = 2(n_+ - n_-) - n''_-$. Also $|P(M_1, M)| = n_0 + 2n_+$ and $|P(M, M_1)| = n_0 + 2n_- + n''_-$. So $wt(S(M_1)) = D(M_1, M)$ as claimed. Now suppose that a maximum weight perfect matching in H_M has weight 0. Suppose M is not popular. Then there is a matching M_1 such that $D(M_1, M) > 0$. But $wt(S(M_1)) = D(M_1, M)$, a contradiction.

Conversely suppose that M is popular. By the above claim, $wt(S(M)) = D(M, M) = 0$. Suppose that $S(M)$ is not a maximum weight perfect matching in H_M. Let M^* be a perfect matching in H_M such that $wt(M^*) > 0$. Then either $S(M_1)$ or $S(M_2)$ has positive weight, where $M_1 = M^*|_A$ and $M_2 = \{\{a_i, a_j\} : \{a'_i, a'_j\} \in M^*|_{A'}\}$. Hence by the above claim, it follows that either M_1 or M_2 respectively is more popular than M, a contradiction. □

Theorem 8 *Given an instance I of* RP *and a matching M in I, we can test whether M is popular in $O(\sqrt{n\alpha(n,m)}m \log^{3/2} n)$ time.*

Proof. Clearly H_M has $O(n)$ vertices and $O(m)$ edges. The current fastest algorithm for finding a maximum weight perfect matching in a weighted graph with weights $\{-1, 0, 1\}$ has complexity $O(\sqrt{n\alpha(n,m)}m \log^{3/2} n)$ [8]. □

It is clear that a perfect matching M^* of positive weight exists in H_M if and only if H_M admits an alternating cycle (relative to $S(M)$) of positive weight. It is an open question whether testing for such an alternating cycle is possible in a better running time than finding a maximum weight perfect matching in the general case.

However, this is possible in the MP case. First we observe that if G is bipartite then H_M is also bipartite. Then the problem of finding an alternating cycle of positive weight can be reduced to the problem of finding a directed cycle of positive weight in D_M, where D_M is a directed graph obtained by orienting the edges of H_M in the following way: all the edges of $S(M)$ are directed from the men to the women and all the other edges are directed from the women to the men. The problem of finding a directed cycle of positive weight in a directed graph with weights $\{-1, 0, 1\}$ (or reporting that none exists) can be solved in $O(\sqrt{n}m)$ time by the algorithm of Goldberg [12]. This implies the following result.

Theorem 9 *Given an instance I of* MP *and a matching M in I, we can test whether M is popular in $O(\sqrt{n}m)$ time.*

4 The case of preferences with ties

In this section we consider popular matchings in instances of RPT and MPT.

4.1 Some results extended to the case of ties

It is not hard to see that Proposition 4 continues to hold in the presence of ties. However, one of the key differences is that stability no longer necessarily implies popularity, so that, in particular, it is not necessarily the case that an instance of MPT admits a popular matching (this is illustrated with an example in [3]).

The algorithm for testing the popularity of a matching in an instance of RP can be extended to the ties case in a natural way, namely by setting $\delta_{i,j}$ to be 0 if $\{a_i, a_j\} \in M$ or if a_i is indifferent between a_j and $M(a_i)$. As a result we will have weights $\{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\}$ in H_M but the technique and the complexity of the popularity checking algorithm (in both the RPT and MPT cases) does not change. On the other hand, the algorithm for finding a strongly popular matching no longer works for the case of ties.

However we can still check the strong popularity of a matching, using a similar technique to that used for popularity checking, in the following way. A matching M is strongly popular if and only if every perfect matching in H_M, excluding $S(M)$, has negative weight. That is, if every perfect matching $M^* \neq S(M)$ in H_M has weight at most $-\frac{1}{2}$. We can reduce this decision problem to another maximum weight perfect matching problem, where $w_\varepsilon(e) = w(e) - \varepsilon = -\varepsilon$ for every edge $e \in S(M)$ (and $w_\varepsilon(e) = w(e)$ for every edge e of H_M not in $S(M)$) where $\varepsilon < \frac{1}{2n}$. Here $S(M)$ is the only perfect matching of weight 0 for w if and only if the maximum weight of a perfect matching in H_M is $-\varepsilon n$ for w_ε.

To facilitate description of some of the subsequent results in this section, we introduce some shorthand notation for variants of the popular matching problem, as follows:

POP-MPT/RPT: the problem of determining whether a popular matching exists, given an instance of MPT/RPT;

PERFECT-POP-MPT/RPT: the problem of determining whether there exists a perfect popular matching, given an instance of MPT/RPT.

4.2 Popular matching in the Marriage Problem with Ties

We next show that the problem of deciding whether a perfect popular matching exists, given an instance of MPT, is NP-complete.

Theorem 10 PERFECT-POP-MPT *is NP-complete.*

Proof (The construction only).

We reduce from EXACT-MM, that is the problem of deciding, given a graph G and an integer K, whether G admits a maximal matching of size exactly K. EXACT-MM is NP-complete even for subdivision graphs of cubic graphs [23]. Suppose that we are given an instance I of EXACT-MM with the above restriction on a graph $G = (A \cup B, E)$, where $A = \{u_1, \ldots, u_{n_1}\}$ and $B = \{v_1, \ldots, v_{n_2}\}$ satisfying $3n_1 = 3|A| = 2|B| = 2n_2$. We construct an instance I' of PERFECT-POP-MPT with a graph $G' = (U \cup V, E')$, where U and V are referred to as women and men respectively, as follows. Initially we let $U = A$ and $V = B$.

The *proper part* of I' is the exact copy of I such that all neighbours of each agent $u_i \in A$ (and $v_j \in B$) are in a tie in u_i's (and v_j's) preference list. The agents of the proper part are called *proper agents*. For each edge $\{u_i, v_j\} \in E(G)$, we create two vertices, $s_{i,j} \in V$ and $t_{i,j} \in U$ with three edges, $\{u_i, s_{i,j}\}, \{s_{i,j}, t_{i,j}\}, \{t_{i,j}, v_j\}$ in E', where u_i (and v_j) prefers her (his) proper neighbours to $s_{i,j}$ (to $t_{i,j}$) respectively, $s_{i,j}$ prefers u_i to $t_{i,j}$, whilst $t_{i,j}$ is indifferent between $s_{i,j}$ and v_j. Moreover, for a given $u_i \in A$, all agents of the form $s_{i,j}$ such that $\{u_i, v_j\} \in E$ are tied in u_i's list in I', and similarly, for a given $v_j \in B$, all agents of the form $t_{i,j}$ such that $\{u_i, v_j\} \in E$ are tied in v_j's list in I'. We complete the construction by adding two sets of garbage collectors to V and U, namely $X = \{x_1, \ldots, x_{n_1-K}\}$ of size $n_1 - K$ and $Y = \{y_1, \ldots, y_{n_2-K}\}$ of size $n_2 - K$, respectively, such that these sets of agents appear in a tie at the end of each proper agent's list. That is, each $u_i \in A$ has the members of X in a tie at the tail of her list and each $v_j \in B$ has the members of Y in a tie at the tail of his list. The members of the garbage collectors are indifferent between the proper agents.

We need to show that I admits a maximal matching of size K if and only if I' admits a perfect popular matching. This part of the proof can be found in [3]. □

It is possible to extend the result of Theorem 10 to the case where we do not require a popular matching to be perfect. This leads to the following result, the proof of which can be found in [3].

Theorem 11 POP-MPT *is NP-complete.*

These results imply the NP-completeness of PERFECT-POP-RPT and POP-RPT.

5 Open problems

In this paper we proved that the problem of finding a perfect popular matching (or reporting that none exists) given an MPT instance is NP-hard, and that the problem remains NP-hard even if we merely seek a popular matching (of arbitrary size). However, the complexity of the problem of constructing a maximum cardinality popular matching in an MP instance remains open. The other main open problem is whether finding a popular matching (or reporting that none exists) is possible in polynomial time for an instance of RP. A third open problem is the complexity of finding a strongly popular matching (or reporting that none exists), for an instance of RPT. Finally we remark that the above-mentioned NP-hardness results were established for MPT instances with incomplete lists, and it is open as to whether the same results hold for complete lists.

Our results and the main open problems are summarised in Table 1.

The problem of finding a popular matching that is	Marriage instances		Roommates instances	
	strict	with ties	strict	with ties
arbitrary	P [9, 11]	**NPC**	*open*	**NPC**
maximum	*open*	**NPC**	*open*	**NPC**

Table 1. Complexity results for problems of finding popular matchings

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