



Biro, P., Fleiner, T., Irving, R.W. and Manlove, D.F. (2010) *The College Admissions problem with lower and common quotas*. *Theoretical Computer Science*, 411 (34-36). pp. 3136-3153. ISSN 0304-3975

<http://eprints.gla.ac.uk/38576/>

Deposited on: 30 September 2010

# The College Admissions problem with lower and common quotas

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## Abstract

We study two generalised stable matching problems motivated by the current matching scheme used in the higher education sector in Hungary. The first problem is an extension of the College Admissions problem in which the colleges have lower quotas as well as the normal upper quotas. Here, we show that a stable matching may not exist and we prove that the problem of determining whether one does is NP-complete in general. The second problem is a different extension in which, as usual, individual colleges have upper quotas, but in addition, certain *bounded* subsets of colleges have common quotas smaller than the sum of their individual quotas. Again, we show that a stable matching may not exist and the related decision problem is NP-complete. On the other hand we prove that, when the bounded sets form a nested set system, a stable matching can be found by generalising, in non-trivial ways, both the applicant-oriented and college-oriented versions of the classical Gale-Shapley algorithm. Finally, we present an alternative view of this nested case using the concept of choice functions, and with the aid of a matroid model we establish some interesting structural results for this case.

## 1 Introduction

The *College Admissions* (or *Hospitals / Residents*) problem was introduced by Gale and Shapley [10]. They gave a linear time algorithm that always finds a stable matching. Roth [19] discovered that the very same method had already been implemented in 1952 in the National Resident Matching Program, the centralised matching scheme that coordinates junior doctor recruitment in the US. Since then, similar matching schemes have been organised in many countries to allocate graduating medical students to hospital posts (hence the alternative name for the problem), and these matching schemes are widely used for other professions as well. Gusfield and Irving [12] and Roth and Sotomayor [22] provide the classical results and background material for this problem.

Regarding the original context, the Gale-Shapley algorithm is also used in handling higher education admissions in a number of countries, including Spain [18], Turkey [4] and Hungary [5] (whilst a different method is used for medicine and related subjects in

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\*Supported by EPSRC grant EP/E011993/1.

†Supported by OTKA grant K69027.

Germany [7]). Moreover, the same kind of admission systems have been introduced for secondary schools in, amongst others, Boston [2], New York [1] and again Hungary [5].

We define the College Admissions problem (CA) as follows. We are given a bipartite graph  $G(A \cup C, E)$ , where  $A = \{a_1, a_2, \dots, a_n\}$  is the set of applicants and  $C = \{c_1, c_2, \dots, c_m\}$  is the set of colleges. If an applicant  $a_i$  applies to a college  $c_j$  and  $c_j$  finds  $a_i$  acceptable then  $(a_i, c_j) \in E$ . Every applicant has a strict order of preference over the colleges she applies to, and every college also has a strict order of preference over its acceptable applicants, so each applicant  $a_i$  and each college  $c_j$  has a *preference list*, denoted by  $P(a_i)$  and  $P(c_j)$ , respectively. Furthermore, each college  $c_j$  has a positive integer *quota*  $q(c_j)$ , this being the maximum number of applicants that it can accept.

A *matching* is a set of edges in the graph, such that every applicant is matched to at most one college and no college has more assignees than its quota. For a matching  $M$ , if an applicant  $a_i$  is matched in  $M$  then let  $M(a_i)$  denote the college to which  $a_i$  is assigned, and let  $M(c_j)$  denote the set of applicants assigned to college  $c_j$ . A matching is *stable* if, for every acceptable applicant-college pair  $(a_i, c_j)$  that is not in the matching, either  $a_i$  is matched and she prefers  $M(a_i)$  to  $c_j$ , or  $c_j$  has a full quota of applicants that it prefers to  $a_i$ . In this case, we say that the pair  $(a_i, c_j)$  is *dominated* by  $M$ . An applicant and a college form a *blocking pair* if it is not dominated by the matching.

Admission to higher education establishments in Hungary has been organised by a centralised matching scheme since 1985. The applicants submit a preference list over the particular courses of study they are interested in. In addition to the name of the institution and the field of study, they have to indicate whether they are applying for state financed or privately financed study<sup>1</sup>, but here, we consider these entities simply as distinct *colleges* for simplicity.

Basically, it is the Gale-Shapley algorithm that produces a kind of stable matching every year by matching more than 100,000 students. However, there are at least three special features in the scheme that are also interesting in a theoretical sense.

The first feature, which was studied in [5], is the presence of *ties* in the system. The colleges rank their applicants according to their academic scores and/or their results in the entrance exams; the solution is based on the concept of so-called *stable score-limit*. This is equivalent to a stable matching if there are no ties, i.e., if two students cannot have the same score at the same college.

The second feature, which is the subject of Section 2 of this paper, is the condition of *lower quotas*. In addition to upper quotas, which are part of the classical College Admissions problem, here, every college may have a lower quota as well, that is the minimum number of students that must be admitted if the college is to be open. We will show that a stable matching may not exist in this case; moreover, the problem of deciding whether a stable matching exists is NP-complete in general. We also mention some relaxed versions of this problem, and we describe the current heuristics used in the Hungarian application. Here we note that in a recent paper [13], Hamada *et al.* studied a related problem with similar motivations. The difference between their model and ours is that they assume that each applicant has a complete preference list (i.e., the underlying bipartite graph is complete) and they also require the matching to satisfy all lower quotas (i.e., no college can be closed in their model). They proved that the problem of finding a matching with the minimum number of blocking pairs under these conditions is NP-hard (moreover it is not approximable within  $n^{1-\varepsilon}$  for any positive constant  $\varepsilon$ ).

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<sup>1</sup>An applicant may rank first a state financed course of study in economics at university A, then secondly another state financed course of study in economics at university B, and thirdly a privately financed course of study in economics at the first university A. So the fee status is included in the preferences of the applicants in this way.

The third feature, studied in Sections 3, 4 and 5, is the problem of *common quotas*. In this case, in addition to the individual quotas of the colleges, particular sets of colleges can have common quotas smaller than the sum of their individual quotas. This means that the number of students assigned to colleges in a given set cannot exceed the common quota for that set. A solution is said to be stable if, for every acceptable applicant-college pair  $(a_i, c_j)$ , if  $a_i$  is not admitted to  $c_j$  then either  $a_i$  is admitted to a preferred college, or  $c_j$  has a full quota of better applicants, or there is a set of colleges  $C_k$  such that  $c_j \in C_k$  and the common quota of  $C_k$  is filled by better applicants than  $a_i$ . (This implies that all of the colleges in the set  $C_k$  judge the students on the same basis – i.e., have preference lists that are consistent.) In Section 3, we show that a stable matching may not exist under these conditions, and that it is NP-complete to determine whether one does. On the other hand, we show in Section 4 that, for nested set systems, the problem becomes solvable in polynomial time. We describe applicant-oriented and college-oriented Gale-Shapley type algorithms that find stable matchings, which we prove to be optimal / pessimal for the applicants and colleges, respectively. Finally, in Section 5, we study the problem from the perspective of *choice functions* and present some results on the structure of stable matchings that follow from an appropriate matroid model.

## 2 Lower quotas

In Hungary, higher education institutions can declare lower quotas for each of their particular areas of study. If the number of assigned students is less than this quota for a particular area then the course has to be cancelled for that year. In general, this creates an interval that ensures a reasonable number of students every year in order to make the course viable. For some specialist areas of study this lower quota may be very small. In this section we consider the complexity of the College Admissions problem with lower quotas. We prove that for a given instance of this problem, a stable matching need not exist and deciding whether one does exist is NP-complete.

### 2.1 Problem definition

Suppose we are given an instance of the College Admissions problem in which, additionally, each college  $c_j \in C$  has a *lower quota*  $l(c_j)$ . In the presence of lower quotas, to avoid ambiguity we refer to  $q(c_j)$  as defined previously as  $c_j$ 's *upper quota*, also denoting  $q(c_j)$  by  $u(c_j)$ . We assume that  $l(c_j) \leq u(c_j)$  for all  $c_j \in C$ .

A *matching* in this context requires that every college  $c_j$  satisfies  $|M(c_j)| = 0$  or  $l(c_j) \leq |M(c_j)| \leq u(c_j)$ . We say that  $c_j$  is *closed* if  $|M(c_j)| = 0$ , and *open* otherwise. If  $|M(c_j)| < u(c_j)$  then we say that  $c_j$  is *undersubscribed*, otherwise, if  $|M(c_j)| = u(c_j)$  then we say that  $c_j$  is *full*.

A matching is *stable* if the following two conditions are satisfied:

- a) (*no blocking pair*) there is no open college  $c_j$  and applicant  $a_i$  such that  $(a_i, c_j) \in E$ ,  $c_j$  is either undersubscribed or prefers  $a_i$  to a member of  $M(c_j)$ , and  $a_i$  is either unmatched or prefers  $c_j$  to  $M(a_i)$ ;
- b) (*no blocking coalition*) there is no closed college  $c_j$  (*blocking college*) and a set of  $l(c_j)$  applicants, each of whom is either unmatched (and finds  $c_j$  acceptable) or prefers  $c_j$  to her assigned college.

Let CA-LQ denote the problem of deciding whether an instance of the College Admissions problem with lower quotas admits a stable matching.

For a given instance of CA, each college admits the same number of applicants in all stable matchings, by the so-called ‘‘Rural Hospitals’’ theorem, first proved by Gale and Sotomayor [11]. Therefore, if every college achieves its lower quota for an instance of CA, then the set of stable matchings remains the same in the corresponding CA-LQ instance as well. To show this, it is enough to see that a college that achieves its lower quota in a stable matching for the CA instance would be a blocking college if closed in any matching with no blocking pair for the corresponding CA-LQ instance. However, if some college does not achieve its lower quota in the stable matchings for the CA instance then a stable matching may not exist for the CA-LQ instance, as we show by the following example.

## 2.2 An unsolvable instance

### Example 1

We have two applicants  $a_1, a_2$  and two colleges  $c_1, c_2$ , whose preference lists are given below. In this example, and henceforth throughout this section of the paper, in the preference list of a given college  $c_j$ , the integers following the first and second colons are  $l(c_j)$  and  $u(c_j)$  respectively.

$$\begin{array}{ll} P(a_1) : c_1 c_2 & P(c_1) : 2 : 2 : a_1 a_2 \\ P(a_2) : c_2 c_1 & P(c_2) : 1 : 1 : a_1 a_2 \end{array}$$

Here, one college must be closed, since we have only two applicants and the sum of the lower quotas is three. If  $c_2$  is closed and  $M(c_1) = \{a_1, a_2\}$  then  $(a_2, c_2)$  is a blocking coalition. Suppose now that  $c_1$  is closed. If  $M(c_2) = \{a_2\}$  then  $(a_1, c_2)$  is a blocking pair. Otherwise, if  $M(c_2) = \{a_1\}$  then  $c_1$  is a blocking college with  $\{a_1, a_2\}$ .

## 2.3 Complexity results

We now prove that CA-LQ is NP-complete. To do so, we reduce from the NP-complete problem COM SMTI [16]. This problem is defined as follows: we are given an instance  $I$  of the stable marriage problem [10] with  $n$  men and  $n$  women, where preference lists may include ties and may be incomplete (i.e., a given person may not find all members of the opposite sex to be acceptable). In this context a matching is a set of mutually acceptable (man,woman) pairs such that no person appears in more than one pair. A matching  $M$  is *stable* if there is no mutually acceptable (man,woman) pair, each of whom is unmatched or prefers the other to his/her partner in  $M$ . The question is whether  $I$  admits a *complete* stable matching, i.e., a stable matching of size  $n$ . COM SMTI is NP-complete, even if each man’s list is strictly ordered, and each woman  $w_j$ ’s list is either strictly ordered or is a tie comprising two men, at least one of whom ranks  $w_j$  in first place [16, Theorem 2].

**Theorem 1.** *CA-LQ is NP-complete, even if no upper quota exceeds 3.*

*Proof.* The problem is clearly in NP. To show NP-hardness, we reduce from the NP-complete restriction of COM SMTI as described above. Let  $I$  be an instance of this problem, in which  $U = \{m_1, \dots, m_n\}$  is the set of men, and  $W = \{w_1, \dots, w_n\}$  is the set of women. Let  $W_0 \subseteq W$  be the set of women whose preference list in  $I$  is a tie of length 2. We create an instance  $I'$  of CA-LQ as follows. Every man in  $I$  corresponds to an applicant in  $I'$ , and every woman in  $W \setminus W_0$  corresponds to a college with lower and upper quota 1 in  $I'$ ; the preference lists of these entities in  $I'$  are initially identical to the corresponding preference lists in  $I$ . Now let  $w_j \in W_0$  and suppose that the two men  $m_{j,1}$  and  $m_{j,2}$  are tied in  $w_j$ ’s list in  $I$ . In  $I'$  we create two colleges  $w_j^1$  and  $w_j^2$ , and two additional applicants  $a_j^1$  and  $a_j^2$ .

The preference lists of these applicants and colleges, and the lower and upper quotas of these colleges, are as follows:

$$\begin{array}{ll} P(a_j^1) : w_j^2 w_j^1 & P(w_j^1) : 3 : 3 : a_j^1 m_{j,1} a_j^2 \\ P(a_j^2) : w_j^1 w_j^2 & P(w_j^2) : 3 : 3 : a_j^2 m_{j,2} a_j^1 \end{array}$$

Finally, we replace  $w_j$  by  $w_j^1$  in  $m_{j,1}$ 's list in  $I'$ , and similarly we replace  $w_j$  by  $w_j^2$  in  $m_{j,2}$ 's list in  $I'$ . We claim that  $I$  has a complete stable matching if and only if  $I'$  has a stable matching.

For, suppose that  $I$  has a complete stable matching  $M$ . Create a matching  $M'$  in  $I'$  as follows. Initially let  $M' = M$ . Suppose  $(m_i, w_j) \in M'$  for some woman  $w_j \in W_0$ , where  $m_i = m_{j,r}$  for some  $r \in \{1, 2\}$ . Replace  $(m_i, w_j)$  in  $M'$  by  $(a_j^1, w_j^r)$ ,  $(a_j^2, w_j^r)$  and  $(m_{j,r}, w_j^r)$ . It may be verified that  $M'$  is a stable matching in  $I'$ .

Conversely suppose that  $M'$  is a stable matching in  $I'$ . Create a matching  $M$  in  $I$  as follows. Initially let  $M = M'$ . Suppose that  $(m_{j,r}, w_j^r) \in M'$  for some  $r \in \{1, 2\}$ , where  $m_{j,r} = m_i$  for some  $m_i \in U$ . Then  $\{(a_j^1, w_j^r), (a_j^2, w_j^r)\} \subseteq M$  as  $l(w_j^r) = 3$ . Replace  $(a_j^1, w_j^r)$ ,  $(a_j^2, w_j^r)$ ,  $(m_{j,r}, w_j^r)$  in  $M$  by  $(m_i, w_j)$ . It is clear that  $M$  is a matching in  $I$ , since the lower quotas imply that at most one of  $w_j^1, w_j^2$  can be open, for each  $w_j \in W_0$ . Moreover, the stability of  $M'$  in  $I'$  implies that, for each  $w_j \in W_0$ , exactly one of  $w_j^1, w_j^2$  is open, since at least one of  $m_{j,1}, m_{j,2}$  ranks  $w_j$  in first place in  $I$ . Hence it follows that  $M$  is a complete stable matching in  $I$ .  $\square$

## 2.4 Heuristics and relaxed problems

At the beginning of Section 2, we noted that lower quotas are specified by higher education institutions in Hungary for the areas of study that they offer. However, the results of the previous subsection indicate that the problem of finding a stable matching (if one exists) in such a context is hard. In this subsection we briefly describe heuristics that are used in Hungary to cope with the complexity of this problem. These heuristics themselves motivate some variants of CA-LQ, whose complexity we also consider here.

In the heuristics currently in use in Hungary, a generalised version of the Gale-Shapley algorithm is used several times. After each run, one college is closed and the Gale-Shapley algorithm is called again for the reduced instance until all the remaining colleges achieve their lower quotas. The college to be closed is selected from those colleges that have not reached their lower quota. Moreover, the chosen college is one for which the ratio of the number of assigned applicants to the lower quota is minimum.

Although this natural heuristic gives a matching that is stable for the remaining open colleges, there may still be a blocking college. This motivates a relaxed variant of the problem, where we omit the second stability condition and forget about blocking colleges. We refer to the resulting (weaker) stability criterion as *pairwise stability*. A pairwise stable matching always exists; a natural question is to maximise the number of assigned applicants. Let MAX-PS-CA-LQ denote the problem of finding a maximum size pairwise stable matching for an instance of CA-LQ.

Furthermore, let  $I$  be an instance of CA-LQ. Define a college  $c_i$  in  $I$  to be *popular* if  $|M_0(c_i)| \geq l(c_i)$ , where  $M_0$  is a stable matching<sup>2</sup> in the instance  $I'$  of CA obtained from  $I$  by disregarding the lower quotas. College  $c_i$  is *unpopular* otherwise. As a third variant of CA, in a given instance  $I$  of CA-LQ we can try to find a pairwise stable matching  $M$  in  $I$ , such that no popular college is closed in  $M$ . Such a matching  $M$  in  $I$  is called a

<sup>2</sup>By the Rural Hospitals Theorem [11], each college has the same number of assignees in the stable matchings of a given CA instance.

*popular pairwise stable matching.* A popular pairwise stable matching always exists in  $I$ , and in fact the heuristic described above will find such a matching. So here, again we can consider the problem of finding a maximum popular pairwise stable matching in  $I$ , denoted by MAX-POP-CA-LQ. This problem is a restricted version of MAX-PS-CA-LQ, since we have the extra requirement that no popular college can be closed. Also MAX-POP-CA-LQ is a more general variant of CA-LQ, because in the latter problem, unpopular colleges can be blocking.

In our corresponding technical report [6] we show that the decision problems related to both MAX-PS-CA-LQ and MAX-POP-CA-LQ are NP-complete, even if each lower and upper quota is equal to 3. However, we remark that the complexity of all of the above three problems, namely CA-LQ, MAX-PS-CA-LQ and MAX-POP-CA-LQ, remains open for lower quotas at most 2.

### 3 Common quotas: hardness results

Until 2007, in the Hungarian matching scheme, there were separate quotas for state financed and privately financed courses of study in each field at each university, and the common quotas of fields applied to state financed places only. As from 2007, there has been a common quota for each field at each university, and still a common national quota for state financed places for each field of study.<sup>3</sup> This latter case may cause difficulties, because a stable matching may not exist in this setting, as is shown in Example 2 below. The main result in this section is that the problem of deciding whether a stable matching exists, given an instance of the College Admissions problem with common quotas, is NP-complete.

#### 3.1 Problem definition

For an instance of CA, we return to using the symbol  $q$  to represent (upper) quotas, so we have  $q : C \rightarrow \mathbb{N}$ . Let  $\mathcal{C} \subseteq 2^C$  be a *set system of colleges*, comprising the so-called *bounded* sets of colleges. We extend the domain of  $q$  to include every  $C_k \in \mathcal{C}$ ; let  $q(C_k)$  be the *common quota* of  $C_k$ .

A matching with respect to common quotas is a matching of the CA instance in which no bounded set of colleges has more assignees than its common quota. A matching  $M$  is *stable*, if for every acceptable applicant-college pair  $(a_i, c_j) \notin M$ , either  $a_i$  is matched and she prefers  $M(a_i)$  to  $c_j$ , or  $c_j$  fills its quota with better applicants than  $a_i$ , or there is a bounded set of colleges  $C_k$  such that  $c_j \in C_k$  and the common quota of  $C_k$  is filled by better applicants than  $a_i$ . To make the definition meaningful, there has to be a preference list for every bounded set of colleges, which we consider as a *master list*. So we suppose that if  $c_i \in C_k$  and  $a_j \in P(c_i)$  then  $a_j \in P(C_k)$  (where  $P(C_k)$  denotes the preference list of bounded set  $C_k$ ), and conversely, if  $a_j \in P(C_k)$  then there must be a college  $c_i \in C_k$  such that  $a_j \in P(c_i)$ . Furthermore, if  $c_i \in C_k$  then  $c_i$  prefers  $a_j$  to another applicant  $a_r$  if and only if  $C_k$  prefers  $a_j$  to  $a_r$ . We note that it is a consequence of the previous sentence that there is some consistency regarding the preferences of bounded sets with common colleges, namely, if  $C_k \cap C_l \neq \emptyset$  then an applicant  $a_j$  precedes another applicant  $a_r$  in the master list of  $C_k$  if and only if  $a_j$  precedes  $a_r$  in  $C_l$ 's master list.

We denote the decision problem of determining whether a stable matching exists for an instance of CA with common quotas by CA-CQ.

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<sup>3</sup>For example, for CS studies at Budapest University of Technology and Economics (BME) there were 450 places for state-financed study and 50 places for privately financed study, whilst there was a common quota of, say, 3000 places for state financed CS studies in Hungary. Since 2007, there is only a common quota of 500 for CS studies at BME and still the same national quota for state financed CS studies.

### 3.2 An unsolvable instance and complexity results

First we give an example to show that a stable matching may not exist, and another example for which the sets of admitted applicants differ in two stable matchings. By using these examples as gadgets, we will prove that CA-CQ is NP-complete.

#### Example 2.

We are given three applicants  $a_1, a_2, a_3$ , four colleges  $c_1, c_2, c_3, c_4$  and two sets of colleges  $\{c_1, c_2\}, \{c_2, c_3\}$  with common quotas. In this example, and from this point onwards, the integer after the first colon in the preference list of each college (and each set of colleges) denotes the (common) quota. The preference lists and the quotas are as follows :

$$\begin{array}{lll}
 P(a_1) & : & c_1 \ c_4 \\
 P(a_2) & : & c_2 \\
 P(a_3) & : & c_4 \ c_3 \\
 P(c_1) & : & 1 : a_1 \\
 P(c_2) & : & 1 : a_2 \\
 P(c_3) & : & 1 : a_3 \\
 P(\{c_1, c_2\}) & : & 1 : a_2 \ a_1 \\
 P(\{c_2, c_3\}) & : & 1 : a_3 \ a_2 \\
 P(c_4) & : & 1 : a_1 \ a_3
 \end{array}$$

Here, no stable matching exists. To see this, suppose for a contradiction that  $M$  is a stable matching. If  $a_1$  is unmatched then  $(a_1, c_4)$  is blocking. Otherwise, if  $M(a_1) = c_1$  then it must be the case that  $(a_3, c_4) \in M$ , but then  $(a_2, c_2)$  is blocking. Finally, if  $M(a_1) = c_4$  then  $(a_3, c_3) \in M$ , otherwise this pair would be blocking. But this implies  $(a_2, c_2) \notin M$ , so  $(a_1, c_1)$  is blocking, a contradiction.<sup>4</sup>

Finally we remark that if we remove  $a_2$  from the instance, then it becomes solvable, since  $\{(a_1, c_1), (a_3, c_4)\}$  is a stable matching. We will use this fact in the proof of Theorem 2 below.

#### Example 3.

We are given four applicants  $a_1, a_2, a_3, a_4$ , six colleges  $c_1, c_2, c_3, c_4, c_5, c_6$  and four sets of colleges  $\{c_1, c_2\}, \{c_2, c_3\}, \{c_4, c_5\}, \{c_5, c_6\}$  with common quotas. The preference lists and the quotas are as follows.

$$\begin{array}{lll}
 P(a_1) & : & c_1 \\
 P(a_2) & : & c_5 \ c_2 \\
 P(a_3) & : & c_3 \ c_6 \\
 P(a_4) & : & c_4 \\
 P(c_1) & : & 1 : a_1 \\
 P(c_2) & : & 1 : a_2 \\
 P(c_3) & : & 1 : a_3 \\
 P(\{c_1, c_2\}) & : & 1 : a_2 \ a_1 \\
 P(\{c_2, c_3\}) & : & 1 : a_2 \ a_3 \\
 P(c_4) & : & 1 : a_4 \\
 P(c_5) & : & 1 : a_2 \\
 P(c_6) & : & 1 : a_3 \\
 P(\{c_4, c_5\}) & : & 1 : a_2 \ a_4 \\
 P(\{c_5, c_6\}) & : & 1 : a_3 \ a_2
 \end{array}$$

Here, there are precisely two distinct stable matchings,  $\{(a_1, c_1), (a_2, c_5), (a_3, c_3)\}$  and  $\{(a_2, c_2), (a_3, c_6), (a_4, c_4)\}$ . To see this, it is enough to observe that both  $a_2$  and  $a_3$  must be matched in a stable matching, since either  $(a_2, c_2)$  or  $(a_3, c_6)$  would be blocking otherwise. Under this restriction, only the above two maximal matchings are possible, and it is easy to check that both of them are stable. This example shows that the sets of admitted applicants may differ for two stable matchings.

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<sup>4</sup>We note that this situation can be realised in the Hungarian application. Colleges  $c_1$  and  $c_2$  can correspond to privately financed and state financed versions of a particular field of study at some faculty with a common quota, whilst  $c_3$  can correspond to state financed study in the same area (e.g. Swahili language) at another faculty, with a common national quota for  $c_2$  and  $c_3$  respectively. Finally,  $c_4$  can be any other field of study in some different faculty.

**Theorem 2.** CA-CQ is NP-complete, even if every quota is 1 and no college is included in more than 2 bounded sets.

*Proof.* The problem is clearly in NP. To show NP-hardness, we reduce from the NP-complete restriction of COM SMTI as described in Section 2.3. Let  $I$  be an instance of this problem, in which  $U = \{m_1, \dots, m_n\}$  is the set of men, and  $W = \{w_1, \dots, w_n\}$  is the set of women. Let  $W_0 \subseteq W$  be the set of women whose preference list in  $I$  is a tie of length 2. We create an instance  $I'$  of CA-CQ, with applicant set  $A$  and college set  $C$ , as follows.

First we create the so-called *proper part* of  $I'$ . Every man  $m_i \in U$  in  $I$  gives rise to an applicant  $a_i \in A$ , and every woman  $w_j$  in  $W \setminus W_0$  gives rise to a college  $c_j$  in  $C$  with quota 1; the preference lists of these agents in  $I'$  are derived directly from the corresponding preference lists in  $I$ , except that a woman in  $W_0$  appearing in the preference list of a man  $m_i$  gives rise to a college in the preference list of applicant  $a_i$  as described below. Now let  $w_j \in W_0$  and suppose that the two men  $m_{j,1}$  and  $m_{j,2}$  are tied in  $w_j$ 's list in  $I$ . In  $I'$  we create a gadget corresponding to the instance of Example 3. We create two additional applicants  $b_j^1, b_j^2$ , six colleges  $c_j^1, c_j^2, c_j^3, c_j^4, c_j^5, c_j^6$  where four sets of colleges  $\{c_j^1, c_j^2\}, \{c_j^2, c_j^3\}, \{c_j^4, c_j^5\}, \{c_j^5, c_j^6\}$  have common quotas with the following preferences:

$$\begin{array}{llll}
& P(c_j^1) & : & 1 : a_{j,1} & & P(c_j^4) & : & 1 : a_{j,2} \\
& P(c_j^2) & : & 1 : b_j^1 & & P(c_j^5) & : & 1 : b_j^1 \\
P(b_j^1) & : & c_j^5 & c_j^2 & & P(c_j^6) & : & 1 : b_j^2 \\
P(b_j^2) & : & c_j^3 & c_j^6 & & P(\{c_j^4, c_j^5\}) & : & 1 : b_j^1 a_{j,2} \\
& P(c_j^3) & : & 1 : b_j^2 & & P(\{c_j^5, c_j^6\}) & : & 1 : b_j^2 b_j^1 \\
& P(\{c_j^1, c_j^2\}) & : & 1 : b_j^1 a_{j,1} & & & & & \\
& P(\{c_j^2, c_j^3\}) & : & 1 : b_j^1 b_j^2 & & & & & 
\end{array}$$

Let  $A_1 = \{a_i : 1 \leq i \leq n\}$ ,  $A_2 = \{b_j^1, b_j^2 : w_j \in W_0\}$ . Finally, in the position where  $w_j$  appears in the list of  $m_{j,1}$  we put  $c_j^1$  in  $a_{j,1}$ 's list, and similarly the occurrence of  $w_j$  in the list of  $m_{j,2}$  leads to  $c_j^4$  in  $a_{j,2}$ 's list. We show that  $I$  has a complete stable matching  $M$  if and only if the proper part of  $I'$  has a stable matching  $M'$  in which every applicant  $a_i \in A_1$  is matched to some college. This natural correspondence is the following. Whenever  $(m_i, w_j) \in M$ , with  $w_j \in W \setminus W_0$ , let  $(a_i, c_j) \in M'$ , and for every  $w_j \in W_0$  let

$$(m_{j,1}, w_j) \in M \iff \{(a_{j,1}, c_j^1), (b_j^2, c_j^3), (b_j^1, c_j^5)\} \in M'$$

and

$$(m_{j,2}, w_j) \in M \iff \{(a_{j,2}, c_j^4), (b_j^2, c_j^6), (b_j^1, c_j^2)\} \in M'.$$

It is straightforward to verify that the completeness and stability of  $M$  implies the stability of  $M'$  and ensures that  $M'$  covers every applicant  $a_i \in A_1$ .

In the other direction, if  $M'$  is a stable matching that covers every applicant  $a_i \in A_1$  then it must be the case that either  $\{(a_{j,1}, c_j^1), (b_j^2, c_j^3), (b_j^1, c_j^5)\} \subseteq M'$ , or  $\{(a_{j,2}, c_j^4), (b_j^2, c_j^6), (b_j^1, c_j^2)\} \subseteq M'$  for every pair of applicants  $a_{j,1}$  and  $a_{j,2}$  such that  $m_{j,1}$  and  $m_{j,2}$  are tied in the list of  $w_j$  for some  $w_j \in W_0$ . This is because exactly one of the pairs  $(a_{j,1}, c_j^1)$ ,  $(a_{j,2}, c_j^4)$  must belong to  $M'$ . They cannot both belong to  $M'$  for stability reasons, and if neither of them were to belong to  $M'$  then, because both  $(a_{j',1}, c_{j'}^1)$  and  $(a_{j',2}, c_{j'}^4)$  cannot belong to  $M'$  for any other  $w_{j'} \in W_0$ ,  $M'$  could not cover every  $a_i \in A_1$ , a contradiction. Therefore we can create  $M$  from  $M'$  in the opposite direction from that described, so the correspondence (between the matchings of  $I$  and  $I'$  with the required properties) is one-to-one, the stability of  $M'$  implies the stability of  $M$  and the assumption that  $M'$  covers every  $a_i \in A_1$  implies the completeness of  $M$ .

To complete the reduction, we construct the *additional part* of  $I'$  as follows. For every  $a_i \in A_1$  we add a gadget which is equivalent to the instance defined in Example 2. To be precise, for every  $a_i \in A_1$ , we create two applicants  $z_i^1, z_i^2$  and four colleges  $d_i^1, d_i^2, d_i^3, d_i^4$  with common quotas for the sets of colleges  $\{d_i^1, d_i^2\}$  and  $\{d_i^2, d_i^3\}$ . The preference lists and the quotas are as follows.

$$\begin{array}{ll}
P(z_i^1) & : \quad d_i^1 \ d_i^4 \\
P(z_i^2) & : \quad d_i^4 \ d_i^3 \\
P(d_i^1) & : \quad 1 : z_i^1 \\
P(d_i^2) & : \quad 1 : a_i \\
P(d_i^3) & : \quad 1 : z_i^2 \\
P(\{d_i^1, d_i^2\}) & : \quad 1 : a_i \ z_i^1 \\
P(\{d_i^2, d_i^3\}) & : \quad 1 : z_i^2 \ a_i \\
P(d_i^4) & : \quad 1 : z_i^1 \ z_i^2
\end{array}$$

Finally, we attach  $d_i^2$  to the end of  $a_i$ 's preference list. We claim that  $I$  has a complete stable matching if and only if  $I'$  has a stable matching. To prove this we observe that on the one hand, in any stable matching for  $I'$ , every  $a_i \in A_1$  must be matched to a college in the proper part of  $I'$ , since otherwise the corresponding additional gadget would contain a blocking pair as described in Example 2. On the other hand, if every  $a_i \in A_1$  is matched to a college in the proper part of  $I'$  then this matching can be extended to the additional part, ensuring stability, by adding  $\{(z_i^1, d_i^1), (z_i^2, d_i^4)\}$  to the matching in each gadget, as we remarked in Example 2.  $\square$

## 4 Common quotas: nested set systems

We say that the set system  $\mathcal{C}$  is *nested* if, for every pair  $S, S'$  of sets in  $\mathcal{C}$  such that  $S \cap S' \neq \emptyset$ , we have either  $S \subseteq S'$  or  $S' \subseteq S$ . In this section we consider a special case of CA-CQ in which the bounded sets form a nested set system.

The Student-Project Allocation problem (SPA), studied by Abraham *et al.* [3], is such a special case of CA-CQ. In SPA, students seek to be matched to projects, and each project is proposed by a single supervisor. In addition to a quota for each project (typically 1), each supervisor has a quota, so the projects proposed by a given supervisor form a bounded set. It is immediate that the bounded sets form a nested set system in this case.

We note that the bounded sets formed a nested set system in the Hungarian application until 2007, because each faculty had separate quotas for state financed and privately financed studies and the common quota of any particular field of study applied to state financed places only. In this section we give two polynomial-time algorithms for CA-CQ with nested set systems. The first algorithm is *applicant-oriented* and the second is *college-oriented*; these produce stable matchings that are, in the first case, unequivocally optimal for the applicants, and in the second case, in a more limited sense that will be explained in Section 4.3, optimal for the colleges.

### 4.1 Definitions

We suppose that each set consisting of a single college is bounded (by its quota), and we assume that each bounded set  $S$  of colleges of cardinality greater than 1 has a quota that is less than the sum of the quotas of any sets in  $\mathcal{C}$  that form a partition of  $S$  (otherwise  $S$  would be redundant as a bounded set.) A *group* is a maximal bounded set, i.e., a bounded set that is contained in no other. So the assumption that bounded sets are nested implies that the groups form a partition of the set of colleges. We denote by  $G(c)$  the group containing the college  $c$ . If  $S \subset S'$  and there exists no bounded set  $S^*$  satisfying

$S \subset S^* \subset S'$  then  $S$  is a *child* of  $S'$  and  $S'$  is the *parent* of  $S$ , the terminology reflecting the tree-structured hierarchies of bounded sets.

Let  $S \in \mathcal{C}$ . For a given matching  $M$ ,  $M(S)$  denotes the set of assignees of the colleges in  $S$ . Recall that  $M(a)$  denotes the assigned college of applicant  $a$  ( $M(a)$  is null if  $a$  is unassigned). Relative to  $M$ , a bounded set  $S$  is *full* if the number of applicants assigned to colleges in that set,  $|M(S)|$ , is equal to the quota of the set,  $q(S)$ , otherwise it is *undersubscribed*. A college is *free* if no set containing it is full, and is *constrained* otherwise. So a free college can admit at least one additional applicant without violating any quota restrictions. For a constrained college  $c$  let the *critical set of  $c$*  relative to  $M$ , denoted by  $cs_M(c)$ , be the innermost, i.e., minimal with respect to  $\subseteq$ , full bounded set that contains  $c$ . A constrained college can admit an additional applicant only if another applicant is rejected by some college in its critical set. The concepts *free*, *constrained* and *critical set* are defined also, in the obvious way, for any bounded set.

As observed in Section 3.1, it is implicit in the definition of stability that colleges in a given group have consistent preferences over applicants. Hence we assume that associated with each group is a strictly ordered master preference list containing all the applicants who are acceptable to at least one college of that group. The preference list of a college  $c$  is inherited from the master preference list of  $G(c)$ .

## 4.2 Applicant-oriented algorithm

In the applicant-oriented algorithm each applicant ‘proposes’ to the colleges on her preference list, in turn, as in the classical Gale-Shapley algorithm. In a *step* of the algorithm, when applicant  $a$  proposes to college  $c$ , the outcome depends initially on whether  $c$  is free. If so,  $c$  (provisionally) accepts  $a$ . Otherwise,  $a$  must compete for a place at  $c$ . The applicant who may be displaced by  $a$  is not necessarily currently assigned to  $c$ . Let  $S$  be the critical set of  $c$  in  $M$ . If  $a$  is preferred, on the master list of  $G(c)$ , to the least preferred assignee  $b$  of some college  $d$  in  $S$  then  $b$  is rejected by  $d$ , enabling  $a$  to be accepted by  $c$ . Otherwise  $c$  rejects  $a$ . As usual, this process continues until every applicant is either assigned, or has been rejected by every acceptable college.

A pseudocode version of the algorithm appears in Figure 1.

**Lemma 3.** (i) *If a college  $c$  becomes constrained at some point during execution of the applicant-oriented algorithm, then it is never again free.*

(ii) *If  $M$  is a matching in which  $c$  is constrained, and  $M'$  is the matching obtained after the subsequent step of the applicant-oriented algorithm, then the least preferred assignee of  $cs_{M'}(c)$  cannot be worse than the least preferred assignee of  $cs_M(c)$ .*

*Proof.* (i) At any point when  $c$  has a critical set  $S$ , the only way that  $S$  can become undersubscribed is as a result of the rejection by some college in  $S$  of an existing assignee  $a$ . If this leaves  $S$  undersubscribed, it must be because some superset  $S'$  of  $S$  is full, and the rejection of  $a$  was because a college in  $S' \setminus S$  gained a new more highly ranked assignee. So  $c$  remains constrained (by  $S'$ ).

(ii) We first observe that if  $S \subset S'$  then it is immediate that the worst assignee of  $S$  is at least as highly ranked as the worst assignee of  $S'$  (in the appropriate master list).

Suppose that, in the relevant step of the algorithm, applicant  $b$  applies to college  $d$ . If  $b$  is rejected then the matching remains the same, so that the result holds trivially. On the other hand, if  $d$  accepts the proposal of  $b$ , let  $S$  be the critical set of  $c$  in matching  $M$ , i.e.  $cs_M(c) = S$ , and let  $S' = cs_{M'}(c)$ . We denote  $cs_M(d)$  by  $T$ , when  $d$  is constrained, and consider four cases.

```

while (there is an applicant  $a$  who is unmatched
      and who has not proposed to all of her acceptable colleges) {
   $c =$  the first college on  $a$ 's preference list to which she has not proposed;
  if ( $c$  is free)
    assign  $a$  to  $c$ ;
  else {
    let  $S$  be the critical set of  $c$ ;
    if ( $a$  is preferred, on the master list of  $G(c)$ , to the
        worst applicant  $b$  who is assigned to a college in  $S$ ) {
      let  $d$  be the college to which  $b$  is currently assigned;
      remove  $(b, d)$  from the matching; // i.e.  $b$  is rejected by  $d$ 
      add  $(a, c)$  to the matching;
    }
    else
       $a$  is rejected by  $c$ ;
  }
}

```

Figure 1: Applicant-oriented algorithm

Case (a)  $d$  was free or  $T \cap S = \emptyset$ . In this case it is easy to see that  $S = S'$  and  $M(S) = M(S')$ , so the result follows at once.

Case (b)  $T \subset S$ . Then  $c \notin T$ , otherwise the fact that  $T$  is full would contradict the assumption that  $S = cs_M(c)$ . Hence  $cs_{M'}(c) = S$ . The transformation from  $M$  to  $M'$  affects only colleges in  $T$ , so  $M(S) \setminus M(T) = M'(S) \setminus M'(T)$ . Furthermore the lowest-ranked applicant in  $M'(T)$  must be ranked more highly than the lowest ranked applicant in  $M(T)$ . It follows that the lowest-ranked applicant in  $M'(S)$  must be ranked at least as high as the lowest ranked applicant in  $M(S)$ .

Case (c)  $T = S$ . Here, we have  $|M(S)| = |M'(S)| = q(S)$  and the worst applicant in  $M'(S)$  is ranked more highly than the worst applicant in  $M(S)$ . In this case, it is possible that  $S' \subset S$ , but then, by our first observation, the worst applicant in  $M'(S')$  is ranked at least as highly as the worst applicant in  $M'(S)$ , and therefore, is ranked more highly than the worst applicant in  $M(S)$ .

Case (d)  $S \subset T$ . If the rejected applicant was not in  $M(S)$  then it is immediate that  $S = S'$  and  $M(S) = M(S')$ . Otherwise, it must be the case that  $|M'(S)| < |M(S)|$  and also  $|M'(S^*)| < |M(S^*)|$  for every  $S^*$  where  $S \subset S^* \subset T$  and  $d \notin S^*$ . Furthermore, for every  $S^*$  such that  $S \subset S^* \subset T$  and  $d \in S^*$ ,  $|M'(S^*)| = |M(S^*)| < q(S^*)$  implying  $S' = T$ . Since  $S'$  was the critical set of  $d$ , and its worst assignee, who was also the worst assignee of  $S$ , was rejected, it follows that the worst applicant in  $M'(S')$  is ranked more highly than the worst applicant in  $M(S)$ .  $\square$

It is straightforward to verify that both parts of Lemma 3 are true, not only for a single college  $c$ , but for any bounded set  $S$  of colleges.

**Theorem 4.** (i) *The matching  $M$  found by the applicant-oriented algorithm is stable.*  
(ii) *In  $M$ , each applicant has the best assignment possible in any stable matching.*

*Proof.* (i) Suppose that  $M$  is not stable, and that  $(a, c)$  is a blocking pair. Then  $a$  must have been rejected by  $c$  during the execution of the algorithm. So at that point,  $c$  was constrained and the worst assignee  $b$  of its critical set must be ranked higher than  $a$  on the master list of  $G(c)$ . By Lemma 3,  $c$  remains constrained in the subsequent steps of the algorithm and the worst assignee of its critical set can never be ranked lower than  $b$ ,

and hence not lower than  $a$ , a contradiction.

(ii) Now suppose that  $(a, c) \in M$ , and that there is another stable matching  $M'$  with  $(a, c') \in M'$ , where  $a$  prefers  $c'$  to  $c$ . So  $(a, c')$  is a *stable pair*, i.e., a pair that is matched in some stable matching, and  $c'$  must have rejected  $a$  during the execution of the algorithm. Assume, without loss of generality, that this was the first rejection involving a stable pair.

Let  $X$  be the matching at the point in the algorithm just before a college  $c^*$  received a proposal from an applicant  $a^*$  that resulted in the rejection of applicant  $a$  by college  $c'$ . Let  $S = cs_X(c^*)$  and let  $X^* = X \cup (a^*, c^*)$ , i.e.  $X^*$  is the matching  $X$  augmented with the additional pair  $(a^*, c^*)$ . We will show that there exists an applicant  $b$  and a college  $d$  such that

- $d = X^*(b)$ , and
- $b \notin M'(d)$ , and
- either  $d$  is free in  $M'$  or  $S \subseteq cs_{M'}(d)$ .

To see this, first we note that  $|M'(S)| \leq q(S) = |X(S)|$ . Moreover,  $|X^*(S)| = q(S) + 1$ , since the quota of  $S$  is exceeded after adding the new pair, and  $|X^*(S_i)| \leq q(S_i)$  for each  $S_i \subset S$ , since  $S = cs_X(c^*)$ , so that  $S$  is the innermost set for which the quota is exceeded. This implies that one of the children of  $S$ , say  $S_2$ , satisfies  $|M'(S_2)| < |X^*(S_2)| \leq q(S_2)$ . By the same reasoning, we can construct a sequence of sets  $S = S_1 \supset S_2 \supset \dots \supset S_k = \{d\}$  such that for every  $i$ ,  $S_{i+1}$  is a child of  $S_i$  and  $S_i$  satisfies  $|M'(S_i)| < |X^*(S_i)| \leq q(S_i)$ . Finally, let  $b$  be any applicant from  $X^*(d) \setminus M'(d)$ . By the construction, it is straightforward to verify that  $b$  and  $d$  satisfy the required conditions.

Now, we are in a position to complete the proof. We first note that  $b$  must be more highly ranked than  $a$ , for otherwise  $d$  would have rejected  $b$  instead of  $c'$  rejecting  $a$ . Hence, if  $b$  is unmatched in  $M'$  or prefers  $d$  to  $M'(b)$  then, since  $S \subseteq cs_{M'}(d)$ , it follows that  $(b, d)$  is a blocking pair for  $M'$ . On the other hand, if  $b$  prefers  $M'(b)$  to  $d$  then  $b$  must have been rejected by  $M'(b)$  earlier than  $a$  was rejected by  $c'$ , a contradiction.  $\square$

**Corollary 5.** *All executions of the applicant-oriented algorithm yield the same stable matching.*

## Complexity analysis

We now establish that the complexity of the applicant-oriented algorithm is  $O(kL + pn)$  where  $L$  is the number of acceptable (applicant, college) pairs,  $k$  is the maximum level of nesting of bounded sets,  $n$  is the number of applicants and  $p$  is the number of bounded sets. The first term comes from the fact that the main loop of the algorithm is executed  $O(L)$  times, and, as we establish below, with the exception of updating worst assignees, every step within the loop can be implemented to run in  $O(k)$  time. The second term represents the total amount of time for updating worst assignees.

We represent the bounded sets by a forest structure  $F$  in which each node corresponds to a bounded set. Each individual college is a leaf node, and the parent of a node representing the bounded set  $C$  is the node representing the parent of  $C$ . For each individual college  $c$  and bounded set  $C$  create a boolean value, accessible in constant time, that records whether  $c$  belongs to  $C$ . We assume that the number of colleges is  $O(n)$ , and that the data is provided in a form that allows these structures to be set up in  $O(pn)$  time.

For each college, maintain a pointer to the leaf node representing it. In each node of  $F$ , representing, say, the bounded set  $C$ , we store

- a pointer to its parent, to allow us to move up the tree;
- the (common) quota of  $C$ ;
- the number of applicants currently assigned to colleges in  $C$ ;
- a pointer to the preference list of the group containing  $C$ , and a *worst assignee* pointer, which references the last entry in that list representing an applicant who is currently assigned to a college in  $C$ .

For efficiency reasons we adopt a ‘lazy’ approach to updating the worst assignee pointer. We ensure that it is accurate whenever the bounded set in question is full, since it is only in these circumstances that the pointer is needed. It is initialised to point to a dummy entry at the end of the relevant master list.

To determine whether a college  $c$  is constrained, and if so to find its critical set, it is enough to follow the path of parent pointers from the leaf node representing  $c$  until a node is reached for which the number of assignees is equal to the quota, or until the root of the tree is reached without this condition being satisfied. This can be done in  $O(k)$  time.

When a pair  $(a, c)$  is added to, or deleted from, the matching, follow the path from the leaf representing  $c$  to the root, incrementing or decrementing the number of assignees at each node, as appropriate. This can be achieved in  $O(k)$  time.

If the bounded set  $C$  represented by a node becomes full as a result of incrementing the number of assignees, the worst assignee pointer is updated. By the analogue of Lemma 3(ii) for an arbitrary bounded set, the worst assignee of  $C$  must be at least as good as the last time  $C$  was full, so this update can be accomplished by scanning backwards in the appropriate master preference list, from that former worst assignee, until we reach an applicant assigned to a college in  $C$ . Because of the initialisation of the worst assignee pointer, this works also in the case where  $C$  becomes full for the first time. For each node in the forest, each entry in the appropriate master list is visited at most once during such scans, so the total time for all such updates is the sum, taken over all trees in the forest, of the number of nodes times the length of the master list, which is  $O(pn)$ .

### 4.3 College-oriented algorithm

The college-oriented algorithm is a little more subtle. As we would expect, the basis is a sequence of offers from colleges to applicants. Each college  $c$  makes offers to applicants in the order in which they appear in the master list of  $G(c)$ , and never makes an offer to the same applicant twice. As always, when an applicant receives an offer, she (provisionally) accepts it if she is unmatched at that point, otherwise she accepts it if and only if it is a better offer than the one she currently holds.

At each step of the algorithm an offer is initiated by an undersubscribed group  $G$  that contains a free college with an *untried* applicant (i.e., an applicant to whom it has not yet offered a place). Let  $a$  be the highest ranking applicant in the master list of  $G$  who is an untried applicant for some free college in  $G$ . Among these free colleges, let  $c$  be the one most preferred by  $a$ ; then  $c$  offers a place to  $a$  at that point. We say that the pair  $(c, a)$  constitutes the *leading offer* at that point.

A pseudocode version of the algorithm appears in Figure 2.

To establish the correctness of this algorithm, we again refer to the hierarchical forest of bounded sets, identifying each bounded set with the node representing it.

The following lemma is easy to prove.

**Lemma 6.** *If, at some step during the execution of the algorithm, a college  $d$  in group  $G$  offers a post to an applicant  $b$ , and (a) some other superior applicant  $a$  is not offered a*

```

while (some group  $G$  of colleges is undersubscribed,
      and contains a free college with an untried applicant) {
  let  $(c, a)$  constitute the leading offer at this point;
  mark  $a$  as a ‘tried’ applicant for  $c$ ;
  if ( $a$  is unassigned)
    add  $(a, c)$  to the matching;
  else if ( $a$  prefers  $c$  to her current assigned college  $d$ ) {
    remove  $(a, d)$  from the matching; //  $d$  is rejected by  $a$ 
    add  $(a, c)$  to the matching;
  }
  else
     $a$  rejects  $c$ ;
}

```

Figure 2: College-oriented algorithm

post by a college  $c$  in  $G$  where  $a$  is not marked as a ‘tried’ applicant for  $c$ , or (b)  $b$  is not offered a post by a preferred college  $c$  where  $b$  is not marked as a ‘tried’ applicant for  $c$ , then  $c$  must be constrained, and its critical set at that point is not on the path from  $d$  to the root of the tree.

Now we can establish the key properties of the college-oriented algorithm.

- Theorem 7.** (i) *The matching  $M$  produced by the college-oriented algorithm is stable.*  
(ii) *If an applicant  $a$  is matched to a college  $c$  in  $M$  then there is no stable matching in which  $a$  is unmatched, or matched to a college lower on her preference list, i.e.,  $M$  is the applicant-pessimal stable matching.*  
(iii) *If a college  $c$  is matched to applicants  $a_1, \dots, a_k$  in  $M$ , then there is no applicant more highly ranked than any of  $a_1, \dots, a_k$  who is matched to  $c$  in any stable matching.*

*Proof.* (i) Suppose that  $M$  is not stable, and that it is blocked by an applicant  $a$  and college  $c$ . Then  $c$  cannot have offered a place to  $a$ , because applicants can only improve as the algorithm progresses, so this would contradict the assumption that  $(a, c)$  is a blocking pair for  $M$ . It follows that  $c$  must have a critical set, say  $S$ , on termination of the algorithm.

If  $(a, c)$  is a blocking pair then there is some college  $d$  in  $S$  and some applicant  $b$  who is assigned to  $d$  in  $M$  such that either  $b$  is inferior to  $a$ , or  $b$  is equal to  $a$  and  $a$  prefers  $c$  to  $d$ . If  $b$  is inferior to  $a$  then it might be the case that  $d$  is equal to  $c$ . However, in this case it is easy to see the contradiction, because when  $c$  offered a place to  $b$ ,  $c$  must have offered to the more preferred applicant  $a$  and hence  $a$  must be eventually matched to a college at least as good as  $c$ . Suppose now that  $d$  is not equal to  $c$  and  $b$  is the worst assignee of  $S$  (where  $b$  could be equal to  $a$ ). Let  $\mathcal{P}$  denote the path from the node representing  $d$  to the root of the tree containing it. It is immediate that the node representing  $S$  lies on  $\mathcal{P}$ . Further let  $\mathcal{Q}$  denote the path from the node representing  $c$  to the node representing  $S$ .

At the step when  $d$  offered a place to  $b$ , it follows from Lemma 6 that  $c$ ’s critical set must have been a descendant node of  $S$  not on  $\mathcal{P}$ , say  $T$ . Let  $M^*$  denote the matching at the point just before  $d$  offered a place to  $b$ . Since  $d$  is free in  $M^*$ , it follows that every node on  $\mathcal{P}$  is undersubscribed, implying in particular that  $|M^*(S)| < q(S)$ . On the other hand,  $T$  is the critical set of  $c$  in  $M^*$ , which implies  $|M^*(T)| = q(T)$ . Further let  $M' = M \setminus \{(b, d)\}$ . The fact that  $S$  is the critical set of  $c$  in  $M$  implies that every node below  $S$  on  $\mathcal{Q}$ , including  $T$ , is undersubscribed in  $M$  and therefore also in  $M'$ , namely  $q(T) > |M'(T)|$ .

From the above observations we have  $|M'(S)| = q(S) - 1 \geq |M^*(S)|$ . Let us consider the children of  $S$ . It must be the case that either

a1) there exists a child  $T_1$  of  $S$  such that  $T \subset T_1 \subset S$  and  $|M^*(T_1)| \leq |M'(T_1)|$ , or

b1) there exists a child  $S_1$  of  $S$  such that  $T \not\subset S_1 \subset S$  and  $|M^*(S_1)| < |M'(S_1)|$ .

This is because if  $|M^*(T_1)| > |M'(T_1)|$  holds for  $T_1$ , where  $T_1$  is the child of  $S$  on the path from  $S$  to  $T$ , then there must be some other child of  $S$ , say  $S_1$  (not on the path from  $S$  to  $T$ ), for which  $|M^*(S_1)| < |M'(S_1)|$ . Note that  $T_1$  cannot be equal to  $T$ , since  $|M^*(T)| = q(T) > |M'(T)|$ .

In case b1) we can construct a sequence of sets  $S \supset S_1 \supset S_2 \supset \dots \supset S_k = d^*$  such that for every  $i$ ,  $S_{i+1}$  is a child of  $S_i$  and  $S_i$  satisfies  $|M^*(S_i)| < |M'(S_i)|$ . This means that  $d^*$  is a free college in  $M^*$  and  $d^*$  has an assignee  $b^*$  in  $M'(S) \setminus M^*(S)$  who must be better than  $b$  (since  $b$  was the worst assignee of  $S$  in  $M$ ), leading to a contradiction since  $d^*$  should have offered a place to  $b^*$  at the point where  $d$  offered a place to  $b$ .

In case a1) we use the same argument for  $T_1$  as we did for  $S$ . It must be the case that either

a2) there exists a child  $T_2$  of  $T_1$  such that  $T \subset T_2 \subset T_1$  and  $|M^*(T_2)| \leq |M'(T_2)|$ , or

b2) there exists a child  $S_1$  of  $T_1$  such that  $T \not\subset S_1 \subset T_1$  and  $|M^*(S_1)| < |M'(S_1)|$ .

As before,  $T_2$  cannot be equal to  $T$ .

Again, in case b2) we can construct a sequence of sets  $S \supset T_1 \supset S_1 \supset S_2 \supset \dots \supset S_k = d^*$  leading to a similar contradiction. On the other hand, in case a2) we continue with the same reasoning for  $T_2$  leading to subcases a3) and b3), and so on.

Continuing in this way, after at most  $j$  steps, where  $j$  is the length of the path  $\mathcal{Q}$ , we are led to an inevitable contradiction, since case a) cannot arise at the  $j$ th step.

(ii) Suppose that there is a stable matching  $M'$  in which  $a$  is unmatched, or is matched to a college lower than  $c$  on her preference list. Suppose, without loss of generality, that, when  $c$  offered a post to  $a$ , this was the first time that an applicant received an offer that was superior to her status in some stable matching. Call such an offer a *superior* offer. In order to avoid  $(a, c)$  being a blocking pair for  $M'$ , the critical set of  $c$  in  $M'$ , say  $S = cs_{M'}(c)$ , must be full in  $M'$  with applicants ranked more highly than  $a$ . Suppose that just before  $c$  offered a place to  $a$  we had a matching  $M^*$ . Since  $c$  was free at that moment, we have  $|M^*(S)| < |M'(S)| = q(S)$ . This implies that one of the children of  $S$ , say  $S_2$ , satisfies  $|M^*(S_2)| < |M'(S_2)| \leq q(S_2)$ . By the same reasoning, we can construct a sequence of sets  $S = S_1 \supset S_2 \supset \dots \supset S_k = d$  such that for every  $i$ ,  $S_{i+1}$  is a child of  $S_i$  and  $S_i$  satisfies  $|M^*(S_i)| < |M'(S_i)| \leq q(S_i)$ . Finally, let  $b$  be any applicant from  $M'(d) \setminus M^*(d)$ . It follows from the construction that  $d$  is free in  $M^*$  and that  $b$  is more highly ranked than  $a$ . So when  $c$  offered a place to  $a$ ,  $b$  must have been matched to a college  $e$  in  $M^*$  that she prefers to  $d$ . Therefore  $b$  had already received an offer from a college ( $e$ ) that is preferred to the college ( $d$ ) she is matched to in the stable matching  $M'$ , which contradicts the assumption that  $c$ 's offer to  $a$  was the first superior offer.

(iii) Suppose that  $(a, c)$  is in  $M$ , and that  $(b, c) \notin M$  is in some stable matching  $M'$ , with  $b$  more highly ranked than  $a$ . Then, by part (ii),  $b$  must prefer  $c$  to  $M(b)$  or  $b$  is unmatched in  $M$ , and it is immediate that  $(b, c)$  is a blocking pair for  $M$ , a contradiction. □

The following is an immediate corollary of Theorem 7(ii).

**Corollary 8.** *All executions of the college-oriented algorithm yield the same stable matching.*

Note that Theorem 7(iii) does not claim that, in matching  $M$ , each college has the 'best' set of assigned applicants that it can have in any stable matching. This need not

be true. For example, a college can be assigned a particular set  $A$  of applicants in the matching found by the college-oriented algorithm, and a superset of  $A$  in some other stable matching. This also indicates that, in contrast to the classical case, the number of applicants assigned to a college may vary between one stable matching and another. These points are illustrated in the following example. (Despite this fact, we will use the term “college-optimal” for the matching obtained by a general kind of college-oriented algorithm in Section 5, however the expression “applicant-pessimal” is probably more accurate in this case.)

**Example 4.**

We have five applicants  $a_1, \dots, a_5$  and three colleges  $c_1, \dots, c_3$ . The only bounded set, in addition to the individual colleges, is  $\{c_1, c_2\}$ , so this set and the set  $\{c_3\}$  are the groups. The preferences of the applicants, quotas of the bounded sets, and master lists of the groups are as shown:

$$\begin{array}{ll}
 P(a_1) & : \quad c_2 \\
 P(a_2) & : \quad c_3 \ c_1 \\
 P(a_3) & : \quad c_3 \ c_1 \\
 P(a_4) & : \quad c_2 \ c_3 \\
 P(a_5) & : \quad c_2 \ c_3 \\
 P(c_1) & : \quad 2 : a_2 \ a_3 \\
 P(c_2) & : \quad 3 : a_1 \ a_4 \ a_5 \\
 P(\{c_1, c_2\}) & : \quad 3 : a_1 \ a_2 \ a_3 \ a_4 \ a_5 \\
 P(c_3) & : \quad 2 : a_5 \ a_4 \ a_3 \ a_2
 \end{array}$$

There are two stable matchings, as follows:

$$\begin{aligned}
 M_1 &= \{(a_1, c_2), (a_2, c_3), (a_3, c_3), (a_4, c_2), (a_5, c_2)\}, \\
 M_2 &= \{(a_1, c_2), (a_2, c_1), (a_3, c_1), (a_4, c_3), (a_5, c_3)\}.
 \end{aligned}$$

Matching  $M_1$  is the applicant-optimal stable matching, found by the applicant-oriented algorithm, and  $M_2$  is the applicant-pessimal stable matching, found by the college-oriented algorithm. Note that colleges  $c_1$  and  $c_2$  have different numbers of assigned applicants in the two matchings. For college  $c_2$ , in particular, the assignees in  $M_1$  form a superset of those in  $M_2$ , though the additional assignees,  $a_4$  and  $a_5$ , are ranked lower than  $a_1$ , its only assignee in  $M_2$ , as must be the case according to Theorem 7(iii).

**Complexity analysis**

The number of loop iterations is  $O(L)$ , where  $L$  is the number of acceptable applicant-college pairs, since no college makes more than one offer to the same applicant. At each loop iteration, we have to decide which college makes an offer to which applicant. This can be decided by a traversal of the forest  $F$  of bounded sets. During this traversal, when we encounter a node representing a bounded set that is full, we need not enter any of its subtrees, since no free college can thereby be reached. When we reach a leaf node that represents a free college  $c$ , we can locate its next untried applicant  $a$  (if any), and determine whether an offer from  $c$  to  $a$  is a better candidate for leading offer than any previously seen, all in constant time, with suitable data structures. Also, as in the applicant-oriented algorithm, whenever a pair  $(a, c)$  is added to or removed from the matching, we update the count of assignees in each node that is an ancestor of the leaf node representing  $c$ . Hence each loop iteration can be completed in  $O(m)$  time, the time for the forest traversal, giving an overall time bound of  $O(mL)$  for the college-oriented algorithm.

We saw earlier that the complexity of the applicant-oriented algorithm is  $O(kL + pn)$ , where  $k$  is the maximum level of nesting of bounded sets,  $n$  is the number of applicants and  $p$  is the number of bounded sets. Since  $k = O(m)$ ,  $p = O(m)$  and  $n = O(L)$ , it follows

that the complexity bound that we have derived for the applicant-oriented algorithm is at least as good as that for the college-oriented algorithm.

## 5 CA-CQ with nested set systems: choice functions and a matroid model

From Example 4, it is clear that certain well-known properties of the classical College Admissions problem do not carry over directly to the case of CA-CQ with nested set systems; for example, that each college should have the same number of assigned applicants in all stable matchings. However, it turns out that there are analogues of many of the structural aspects of the classical problem. To investigate these, we adopt in this section a more general model based on choice functions and matroids. Recall that  $G = (A \cup C, E)$  denotes the bipartite graph with applicants and colleges as colour classes, and acceptable pairs forming the set  $E$  of edges. In this section we work with the edges rather than the vertices of  $G$ , so for this reason, for a set  $E'$  of edges and set  $V'$  of vertices (and for vertex  $v$ ),  $E'(V')$  (and  $E'(v)$ ) will denote the set of those edges of  $E'$  that are incident with some vertex of  $V'$  (and with vertex  $v$ , respectively). In particular, if  $M$  is a stable matching and  $C_i$  is a bounded set,  $M(C_i)$  stands for all those edges with an end vertex that is a college in  $C_i$ . (Hitherto, by  $M(C_i)$  we meant the corresponding set of applicants.)

### 5.1 Choice functions

We define two specific choice functions on the edge set  $E$  of  $G$ . A *choice function* on ground set  $E$  is a mapping  $Q$  that assigns to each subset  $X$  of  $E$  a subset  $Q(X)$  of  $X$ . (Note that in the Economics literature choice functions are often defined in a much more restrictive way, that is,  $Q$  is a *preference induced choice function on  $E$*  if there is a (well-) order  $\prec$  on all subsets of  $E$  such that  $Q(X)$  is the  $\prec$ -minimal subset of  $X$ .) A subset  $X$  of  $E$  is called  *$Q$ -independent* if  $Q(X) = X$ .

The choice function of the applicants is denoted by  $Q_A$  and for subset  $X$  of  $E$ ,  $Q_A(X)$  is the set of those edges in  $X$  that the applicants would choose if they could select freely from  $X$ , ignoring all quotas on the college side. Formally, every applicant  $a$  that has at least one edge (application) in  $X$  chooses her most preferred edge from  $X$ , and  $Q_A(X)$  is the set of edges selected in this way. An *applicant-independent* set of edges is a set  $X$  of edges for which  $Q_A(X) = X$ .

For the choice function  $Q_C$  of colleges,  $Q_C(X)$  denotes the set of edges that the colleges would accept if they could freely select from  $X$  and could accept several edges from the same applicant. More precisely, subset  $Q_C(X)$  of  $X$  is determined by the following algorithm. Order the bounded sets as  $\mathcal{C} = \{C_1, C_2, \dots, C_m\}$  such that if  $C_i \subseteq C_j$  then  $i \leq j$  holds. (In other words, take a linear extension of the partial order on  $\mathcal{C}$  given by set inclusion.) Let  $X_0 := X$ , and for  $i = 1, 2, \dots, m$  let  $X_i$  denote the set obtained from  $X_{i-1}$  after applying the quota of  $C_i$ . So  $X_{i-1}(C_i)$  is the set of edges of  $X_{i-1}$  that are incident with some college in  $C_i$ . Let  $X'_i$  denote the set of those edges of  $X_{i-1}(C_i)$  that are not amongst the best  $q(C_i)$  edges of this set. Now let  $X_i := X_{i-1} \setminus X'_i$  and let  $Q_C(X) := X_m$ . To construct  $X'_i$ , in addition to the quota  $q(C_i)$ , we also need the preference order on  $X_{i-1}(C_i)$ . Although we have a preference order for  $C_i$  on the applicants (that is, for the vertices), here we need one on the edges. The difference is that  $X_{i-1}(C_i)$  may contain several edges from the same applicant-vertex. We use the following preference order: if two edges in  $X_{i-1}(C_i)$  correspond to different applicants then the original preference order of  $C_i$  is used. To compare two edges from the same applicant, we use the applicant's preference order. Therefore we have the following partial order on the edges, denoted by

$\preceq_E$ : an edge  $(a, c) \in E(G)$  is preferred to  $(b, d) \in E(G)$  if either applicant  $a$  is preferred to applicant  $b$  by the bounded set containing both  $c$  and  $d$  or if  $a = b$  and college  $c$  is preferred to college  $d$  by applicant  $a$ . We call subset  $X$  of  $E$  *college-independent* if  $Q_C(X) = X$ . Note that in this case each bounded set  $C_i$  contains at most  $q(C_i)$  edges from  $X$ .

Clearly, if  $X$  is a set of edges then  $Q_A(X) = X$  if and only if each applicant has at most one edge in  $X$ , and  $Q_C(X) = X$  holds if and only if  $X$  satisfies all quotas of  $\mathcal{C}$ . Consequently, subset  $M$  of  $E$  is a matching if and only if  $M$  is both applicant-independent and college-independent. Let edge  $(a, c) = e$  of  $E$  be an edge from applicant  $a$  to college  $c$ . We say that subset  $X$  of  $E$  *dominates edge  $e$  from the applicant side* if  $e \notin Q_A(X \cup \{e\})$ . This means that  $X$  contains an edge between some applicant-vertex  $a$  and college-vertex  $c'$  such that  $a$  prefers  $c'$  to  $c$ . Subset  $X$  of  $E$  *dominates edge  $e$  from the college side* if  $e \notin Q_C(X \cup \{e\})$ . This means that there is a bounded set  $C_i$  such that  $c \in C_i$  and  $Q_C(X \cup \{e\})$  contains  $q(C_i)$  different edges, each of which is better than  $e$  according to the preference order of  $C_i$ . We say that subset  $X$  of  $E$  *dominates subset  $Y$  of  $E$  from the college (applicant) side* and denote this fact by  $X \preceq_C Y$  ( $X \preceq_A Y$ ) if each element of  $Y \setminus X$  is dominated by  $X$  from the college (applicant) side. Subset  $X$  *dominates  $e$*  if  $X$  dominates  $e$  from the applicant side or from the college side (or from both). If each edge of  $Y$  is dominated by  $X$  then we say that  $X$  *dominates  $Y$* . The above definitions are also valid for any arbitrary choice function  $Q$  on  $E$ , in particular subset  $X$  of  $E$   *$Q$ -dominates* subset  $Y$  of  $E$  (in notation  $X \preceq_Q Y$ ) if  $X$   $Q$ -dominates each element  $y$  of  $Y \setminus X$ , that is, if  $y \notin Q(X \cup \{y\})$ .

From these definitions, it is not difficult to see that a matching  $M$  is stable if and only if the set of those edges that  $M$  dominates is  $E \setminus M$ . That is, each edge outside  $M$  is dominated by  $M$ , but  $M$  does not dominate any edge of  $M$ , hence  $M$  is both applicant and college-independent. Edge  $e$  *blocks* matching  $M$  if  $e$  is not dominated by  $M$ . More generally, if  $G$  is a bipartite graph with edge set  $E$  and colour classes  $C$  and  $A$ , and we are given choice functions  $Q_C$  and  $Q_A$  on  $E$ , then subset  $M$  of  $E$  is a *matching* if  $M = Q_C(M) = Q_A(M)$  holds, that is, if  $M$  is both  $Q_A$ -independent and  $Q_C$ -independent. Matching  $M$  is *stable* if  $M$  dominates all edges of  $E \setminus M$ , that is, for any  $e \notin M$  we have  $e \notin Q_C(M \cup \{e\})$  or  $e \notin Q_A(M \cup \{e\})$ .

We show that our choice functions  $Q_A$  and  $Q_C$  have two important properties. A choice function  $Q$  on  $E$  is *comonotone* if the mapping of unchosen elements  $X \mapsto X \setminus Q(X)$  is monotone, that is, if  $X \subseteq Y \subseteq E$  implies  $X \setminus Q(X) \subseteq Y \setminus Q(Y)$ , or, in other words  $\overline{Q}(X) \subseteq \overline{Q}(Y)$ , where for choice function  $Q$ ,  $\overline{Q}(X) := X \setminus Q(X)$  denotes the unselected elements. Roughly speaking, if some option  $x$  is ignored then it will still be ignored from a greater choice set. Choice function  $Q$  is *increasing* if  $X \subseteq Y \subseteq E$  implies  $|Q(X)| \leq |Q(Y)|$ .

To justify the above properties for  $Q_A$  and  $Q_C$ , we prove that choice function  $Q_C$  is closely related to matroids. (The interested reader is referred to the second volume of the book of Schrijver [23] that contains everything we need about matroids.) We recall that subset  $E'$  of edges is college-independent if each bounded set  $C_i$  contains at most  $q(C_i)$  edges from  $E'$ . Let  $\mathcal{I}_C \subseteq 2^E$  denote the system of college-independent sets of edges.

**Theorem 9.** *Set system  $\mathcal{I}_C$  forms the independent set of a matroid on set  $E$  of edges, that is  $\mathcal{M}_C = (E, \mathcal{I}_C)$  is a matroid.*

*Proof.* We use induction on the number  $m$  of bounded sets. If  $m = 1$  then we have only one bounded set, and as each college is a bounded set by itself,  $(E, \mathcal{I}_C)$  is a uniform matroid of rank  $q(C_1)$ .

For  $m > 1$ , let  $C^1, C^2, \dots, C^k$  be the groups, that is the inclusionwise maximal bounded sets. Let  $\mathcal{I}_{C^i}$  denote the system of those college-independent edge sets that only contain edges with college vertices from  $C^i$ . If  $k > 1$  then  $\mathcal{I}_{C^i}$  forms the independent set of a

matroid by the induction hypothesis. This means that  $(E, \mathcal{I}_C)$  is the direct sum of these matroids, hence it is a matroid, as we claimed.

On the other hand, if  $k = 1$ , that is, if  $C^1 = C_m$  is the unique group, then let  $\mathcal{I}'$  denote the set of those edges that are college-independent for the set  $C_1, C_2, \dots, C_{m-1}$  of bounded sets. By the induction hypothesis,  $(E, \mathcal{I}')$  is a matroid, hence its  $q(C_m)$ -truncation  $(E, \mathcal{I}_C)$  is also a matroid.  $\square$

Note that the proof of Theorem 9 also implies that “applicant-independent” subsets of edges (i.e., those subsets of the edges that contain at most one application from each applicant) form a matroid. The reason is that these subsets can be regarded as college independent where the bounded sets are the singletons and each quota is 1.

Observe that choice function  $Q_C$  always selects a college-independent set of edges. Moreover, for any set  $X$  of edges,  $Q_C(X)$  has the property that it is an inclusionwise maximal college-independent subset of  $X$ : for each unselected element  $x$  of  $X \setminus Q_C(X)$  subset  $\{x\} \cup Q_C(X)$  is not college-independent, because there is some bounded  $C_i$  such that  $Q_C(X)$  contains  $q(C_i)$  edges to  $C_i$  with the property that all of these  $q(C_i)$  elements are preferred to  $x$ .

This observation has two consequences. On the one hand,  $|Q_C(X)| = r_{\mathcal{M}_C}(X)$  is the  $\mathcal{M}_C$ -rank of  $X$  for any  $X$ . As  $\mathcal{M}_C$  is a matroid,  $r_{\mathcal{M}_C}(X) \leq r_{\mathcal{M}_C}(Y)$  whenever  $X \subseteq Y$ . This implies that  $Q_C$  (and hence  $Q_A$ ) is increasing.

On the other hand, the above observation means that  $Q_C(X)$  can be constructed by the greedy algorithm in the following way. Fix a linear order  $e_1, e_2, \dots$  of  $E$  in such a way that if two edges,  $e_i$  and  $e_j$  belong to the same bounded set then  $i < j$  if and only if edge  $e_i$  is preferred to  $e_j$  (according to the partial order  $\preceq_E$ ). Let  $E_i := \{e_j : j \leq i\}$  and  $X_i := X \cap E_i$ . The greedy algorithm constructs each  $Q_C(X_i)$  in such a way that  $Q_C(X_0) := \emptyset$  and

$$Q_C(X_{i+1}) := \begin{cases} Q_C(X_i) \cup \{e_{i+1}\} & \text{if } Q_C(X_i) \cup \{e_{i+1}\} \text{ is independent} \\ Q_C(X_i) & \text{otherwise.} \end{cases}$$

To prove comonotonicity of  $Q_C$ , we have to show that if for edge  $x$  of  $X$  we have  $x \notin Q_C(X)$  then  $x \notin Q_C(Y)$  holds whenever  $X \subseteq Y$ . But this follows immediately from the greedy algorithm: an edge  $x$  is thrown away by the greedy algorithm during the construction of  $Q_C(Z)$  if and only if  $x$  is spanned in  $\mathcal{M}_C$  by those elements of  $Z$  that precede  $x$ . So if  $x$  is not selected from  $X$  then  $x$  will not be selected from  $Y$  either, and this means that  $Q_C$  (and hence  $Q_A$  also) is comonotone.

Comonotone and increasing choice functions have some interesting properties. The following three statements, namely Lemmas 10, 11 and Theorem 12 are well-known in lattice theory. We include their proofs for the sake of completeness.

**Lemma 10.** *If choice function  $Q$  on  $E$  is comonotone and increasing then*

$$Q(X) \subseteq Y \subseteq X \Rightarrow Q(X) = Q(Y) \text{ holds.} \quad (1)$$

*Proof.* If  $Q(X) \subseteq Y \subseteq X$  then  $Y \setminus Q(Y) \subseteq X \setminus Q(X)$  by comonotonicity, hence  $Q(X) \subseteq Q(Y)$ . The increasing property of  $Q$  implies  $|Q(Y)| \leq |Q(X)|$ , and with the previous relation this implies that  $Q(X) = Q(Y)$ .  $\square$

An important consequence of Property (1) is that for any comonotone and path-independent choice function  $Q$  on  $E$ , relation  $\preceq_Q$  is a partial order on  $Q$ -independent subsets of  $E$ . We need the following useful lemma to prove this.

**Lemma 11.** *If  $Q$  is a comonotone choice function with Property (1) on  $E$  and  $X$  and  $Y$  are subsets of  $E$  then  $X \preceq_Q Y$  (that is,  $X$  dominates subset  $Y$  of  $E$ ) if and only if  $Q(X \cup Y) = Q(X)$ .*

*Proof.* Assume first that  $X \preceq_Q Y$ . For any  $y \in Y \setminus X$  we have that  $y \in \overline{Q}(X \cup \{y\})$ , hence by monotonicity of  $\overline{Q}$  it follows that  $(Y \setminus X) \subseteq \overline{Q}(X \cup Y)$ , that is,  $Q(X \cup Y) \subseteq X \subseteq X \cup Y$ . Now property (1) implies that  $Q(X) = Q(X \cup Y)$ .

Suppose now that  $Q(X) = Q(X \cup Y)$  holds and let  $y \in Y \setminus X$ . Property (1) and  $Q(X) \subseteq X \cup \{y\} \subseteq X \cup Y$  yields that  $Q(X \cup \{y\}) = Q(X \cup Y) = Q(X)$ , hence each element of  $Y \setminus X$  is dominated by  $X$ , that is,  $X \preceq_Q Y$ .  $\square$

This observation allows us to prove that  $\preceq_Q$  is a partial order on  $Q$ -independent sets.

**Theorem 12.** *If  $Q$  is a comonotone choice function with Property (1) on  $E$  then  $\preceq_Q$  is a partial order on  $Q$ -independent subsets of  $E$ .*

*Proof.* For any subset  $X$  of  $E$  we have  $Q(X) = Q(X \cup X)$ . This means  $X \preceq_Q X$  by Lemma 11, hence  $\preceq_Q$  is reflexive. Assume that  $X$  and  $Y$  are  $Q$ -independent  $E$  such that  $X \preceq_Q Y$  and  $Y \preceq_Q X$  holds. Lemma 11 shows that  $X = Q(X \cup Y) = Y$ . This justifies the antisymmetry of  $\preceq_Q$ . For transitivity, assume that  $X, Y$  and  $Z$  are  $Q$ -independent and  $X \preceq_Q Y \preceq_Q Z$ . This means that  $Q(X \cup Y) = X$  and  $Q(Y \cup Z) = Y$  by Lemma 11 and the  $Q$ -independence. Monotonicity of  $\overline{Q}$  implies that  $Z \setminus Y \subseteq \overline{Q}(Y \cup Z)$  and  $Y \setminus X \subseteq \overline{Q}(X \cup Y)$ , hence  $(Z \setminus Y) \cup (Y \setminus X) \subseteq \overline{Q}(X \cup Y \cup Z)$ , that is,  $Q(X \cup Y \cup Z) \subseteq X \subseteq X \cup Y \cup Z$ . Now property (1) of  $Q$  yields that  $Q(X \cup Y \cup Z) = Q(X) = X$ , and from  $Q(X \cup Y \cup Z) = X \subseteq X \cup Z \subseteq X \cup Y \cup Z$  it follows again by property (1) that  $Q(X \cup Z) = Q(X \cup Y \cup Z) = X$ , that is,  $X \preceq_Q Z$ . So  $\preceq_Q$  is indeed a partial order on  $Q$ -independent sets.  $\square$

So far in this section we have proved that the CA-CQ problem with nested bounded sets can be formulated as a stable matching problem with increasing comonotone choice functions. It is not difficult to see that if  $Q$  is an increasing comonotone choice function then  $Q$  is *path-independent*, that is, for any  $X$  and  $Y$  in the domain of  $Q$  we have  $Q(X \cup Y) = Q(Q(X) \cup Q(Y))$ . Roth proved the following theorem for preference induced comonotone choice functions that are clearly path-independent [20, 21].

**Theorem 13.** *Assume that  $G = (A \cup C, E)$  is a bipartite graph between sets  $A$  of applicants and  $C$  of colleges and edges of  $E$  represent applications. If choice functions  $Q_C$  of colleges and  $Q_A$  of applicants are both comonotone with property (1) then there is a stable matching, that is, a subset  $S$  of  $E$  such that  $Q_A(S) = Q_C(S) = S$  and each edge  $e \in E \setminus S$  is dominated by  $S$ .*

A possible proof of Theorem 13 is contained in [9], where a generalisation of the deferred acceptance algorithm is also described. It is slightly different from the one we need later, so here we describe a proof and an algorithm.

*Proof.* If choice function  $Q$  is comonotone then  $\overline{Q}$  is monotone by the definition of comonotonicity. Let us define function  $F$  on subsets of  $E$  by  $F(X) := E \setminus \overline{Q_A}(E \setminus \overline{Q_C}(X))$ . First we prove a connection between stable matchings and fixed points of  $F$ .

Assume that  $S$  is a stable matching. Let  $S_C := \{e \in E : e \in \overline{Q_C}(S \cup \{e\})\}$  denote the set of edges that  $S$  dominates from the college side. By the stability of  $S$ , each edge in  $E \setminus (S \cup S_C)$  is dominated by  $S$  from the applicant side. The monotonicity of  $\overline{Q_C}$  implies for  $S' = S \cup S_C$  that  $S_C \subseteq \overline{Q_C}(S')$ , hence  $Q_C(S') \subseteq S \subseteq S'$  holds. By the property (1) of  $Q_C$  this means that  $Q_C(S') = Q_C(S) = S$ , where the latter equation comes from the fact that  $S$  is a stable matching hence college-independent. In particular,

$\overline{Q_C}(S') = S_C$  holds. The above argument for choice function  $Q_A$  shows that  $Q_A(S'') = S$ , where  $S'' = S \cup S_A$  and  $S_A$  is the set of edges that  $S$  dominates from the applicant side. As  $S$  is a stable matching, each element outside  $S'$  is dominated from the applicant side, hence  $Q(S'') = S \subseteq E \setminus S_C \subseteq S \cup S_A = S''$ , and property (1) of  $Q_A$  implies that  $Q_A(E \setminus S_C) = Q_A(S'') = S$  thus  $\overline{Q_A}(E \setminus S_C) = (E \setminus S_C) \setminus S = E \setminus (S_C \cup S) = E \setminus S'$ . So

$$F(S') = F(S \cup S_C) = E \setminus \overline{Q_A}(E \setminus \overline{Q_C}(S')) = E \setminus \overline{Q_A}(E \setminus S_C) = E \setminus (E \setminus S') = S'$$

So each stable matching  $S$  determines a fixed point of  $F$ .

Now assume that  $F(S') = S'$  is a fixed point. By denoting  $Q_C(S')$  by  $S$  we have  $\overline{Q_C}(S') = S' \setminus S$ . We show that  $S$  is a stable matching. By property (1) of  $Q_C$ ,  $Q_C(S') = S \subseteq S'$  implies that  $Q_C(S) = Q_C(S') = S$  and for  $x \in S' \setminus S$  we have  $Q(S') \subseteq S \cup \{x\} \subseteq S'$ , so  $Q_C(S \cup \{x\}) = Q_C(S') = S$ , that is  $S$  dominates each edge of  $S' \setminus S$  from the college side.

From  $F(S') = S'$  we get that

$$F(S') = S' = E \setminus \overline{Q_A}(E \setminus \overline{Q_C}(S')) = E \setminus \overline{Q_A}((E \setminus S') \cup S),$$

hence  $E \setminus S' = \overline{Q_A}((E \setminus S') \cup S)$ , or, in other words  $S = Q_A((E \setminus S') \cup S)$ . This means that for any element  $x$  of  $E \setminus S'$  we have  $Q_A((E \setminus S') \cup S) = S \subseteq S \cup \{x\} \subseteq (E \setminus S') \cup S$  so property (1) implies that  $S = Q_A(S \cup \{x\})$ . The conclusion is that  $S$  is a stable matching.

In what follows we prove that  $F$  is monotone. If  $X \subseteq Y$  then  $\overline{Q_C}(X) \subseteq \overline{Q_C}(Y)$  follows by the monotonicity of  $\overline{Q_C}$ , thus  $E \setminus \overline{Q_C}(X) \supseteq E \setminus \overline{Q_C}(Y)$ . The monotonicity of  $\overline{Q_A}$  implies that  $\overline{Q_A}(E \setminus \overline{Q_C}(X)) \supseteq \overline{Q_A}(E \setminus \overline{Q_C}(Y))$ , hence  $F(X) = E \setminus \overline{Q_A}(E \setminus \overline{Q_C}(X)) \supseteq E \setminus \overline{Q_A}(E \setminus \overline{Q_C}(Y)) = F(Y)$  holds, proving that  $F$  is indeed monotone.

This follows that  $\emptyset \subseteq F(\emptyset) \subseteq F(F(\emptyset)) \subseteq \dots$  is a chain of increasing sets. As ground-set  $E$  is finite, after some iterations we find a fixed point  $S^0 = F(F(\dots(F(\emptyset))\dots))$  of  $F$ , such that  $S_0 := Q_C(S^0)$  is a stable matching.  $\square$

Actually, the set  $S^0$  we constructed at the end of the above proof is an inclusionwise minimal fixed point of  $F$  because if  $F(S^*) = S^*$  is another fixed point then  $\emptyset \subseteq S^*$ , hence  $F(\emptyset) \subseteq F(S^*) = S^*$ , so  $F(F(\emptyset)) \subseteq F(S^*) = S^*$ , and so on. In other words, stable matching  $S_0$  dominates all those edges from the student side that some stable matching may dominate from the student side, that is, each applicant receives the best assignment that she can have in a stable matching. In other words,  $Q_C(S)$  is the so-called *applicant-optimal* stable matching in general. This is the solution that we constructed also with the applicant-oriented algorithm in Section 4.2 for the particular choice functions of our model CA-CQ with nested set systems.

Note that there is a college-optimal stable matching, as well. To construct it, we start the iteration of  $F$  with  $E$ , rather than with  $\emptyset$ . So we get a decreasing chain  $F(E) \supseteq F(F(E)) \supseteq F(F(F(E))) \dots$  of subsets of  $E$  that stabilises at a fixed point  $S^1 = F(F(\dots F(E)\dots)) = F(F(\dots F(F(S^1))\dots)) = F(S^1)$ . This  $S^1$  is the inclusionwise maximal fixed point of  $F$  because if  $F(S^*) = S^*$  is another fixed point of  $F$  then from  $S^* \subseteq E$  by monotonicity it follows that  $S^* = F(F(\dots F(S^*)\dots)) \subseteq F(F(\dots F(E)\dots)) = S^1$ . That is, if a stable matching dominates some edge  $e$  from the college side then  $e$  is certainly dominated by stable matching  $S_1 := Q_C(S^1)$  from the college side. We call  $S_1$  the *college-optimal* stable matching. In particular, for the special choice functions of our model CA-CQ with nested set systems, this solution is the applicant-pessimal stable matching that the college-oriented algorithm finds in Section 4.3.

How do we construct the above applicant- and college-optimal stable matchings? It is simple: we iterate function  $F$  starting from  $E$  or from  $\emptyset$ . Let  $E_0 := E$  and for  $i = 0, 1, 2, \dots$  let  $E_{i+1} := F(E_i)$ . As soon as  $E_{i+1} = E_i$  we can determine college-optimal stable matching

$S_1 = Q_C(E_i)$ . We know that  $E_0 \supseteq E_1 \supseteq \dots$  is a decreasing chain, so to construct  $E_{i+1}$  from  $E_i$  we have to find elements of  $E_i \setminus E_{i+1}$  that we have to delete from  $E_i$ . From the formula  $E_{i+1} := F(E_i) = E \setminus \overline{Q_A}(E \setminus \overline{Q_C}(E_i))$  it follows that  $E_{i+1} = E_i \setminus \overline{Q_A}(E \setminus \overline{Q_C}(E_i))$ , so at the  $(i+1)$ st iteration we delete  $\overline{Q_A}(E \setminus \overline{Q_C}(E_i)) = \overline{Q_A}(Q_C(E_i) \cup (E \setminus E_i))$  from  $E_i$ .

Note that the iteration of  $F$  is not exactly the Gale-Shapley algorithm. The Gale-Shapley algorithm corresponds to the recursive definition  $E'_{i+1} := E'_i \setminus \overline{Q_A}(Q_C(E'_i))$ . However, from property (1) of  $Q_A$  and  $Q_C$  it is not difficult to prove that  $E_i = E'_i$  holds for all  $i$ .

## 5.2 Structural results

Several interesting properties about the structure of stable matchings are known. Donald Knuth [15] attributes to John Conway the observation that stable marriages have a lattice structure: if each man picks the better assignment out of two stable matchings then another stable matching is created in which each woman gets the worse husband from the two. Using linear programming tools, Teo and Sethuraman [24] proved a generalisation of this, namely, that if  $k$  stable matchings are given and each man selects the  $l$ th best partner then a stable matching is created in which each woman receives her  $(k+1-l)$ th husband. Later this result was further generalised by Fleiner [8] for the many-to-many model and the proof was based on the lattice property. Even later, Klaus and Klijn [14] found the same proof for the many-to-one case.

Another interesting consequence of the lattice structure is the ‘‘Rural hospitals theorem’’ of Gale and Sotomayor [11] stating that if in the college admissions problem some college  $c$  does not fill up its quota in a stable matching then college  $c$  receives the same set of applicants in each stable matching. Our goal in this section is to extend the above properties to the CA-CQ problem. Note that the ‘‘natural’’ CA-CQ extension of the ‘‘Rural hospitals theorem’’ does not hold as we saw in Example 4. (There college  $c_1$  may receive as many students as its quota in one stable matching and gets no students at all in another one. The example also shows that in contrast to the case of CA, in the CA-CQ problem it is not true that in the college-optimal stable matching each college gets the best set of students.)

Fleiner studied stable matching problems with comonotone set functions in [9]. Corollaries 26 and 27 of [9] yield the following result.

**Theorem 14.** *Assume that  $G = (A \cup C, E)$  is a bipartite graph with colour classes  $C$  and  $A$  and that  $\mathcal{M}_C$  and  $\mathcal{M}_A$  are two matroids on  $E$ . Let  $Q_C$  and  $Q_A$  be increasing comonotone choice functions on  $E$  that are defined by the greedy algorithm on matroids  $\mathcal{M}_C$  and  $\mathcal{M}_A$  respectively, for some linear orders.*

(i) *If  $M_1, M_2, \dots, M_k$  are stable matchings then  $Q_C(\bigcup_{i=1}^k M_i)$  and  $Q_A(\bigcup_{i=1}^k M_i)$  are also stable matchings.*

(ii) *For any two stable matchings  $M$  and  $M'$ ,  $\text{span}_{\mathcal{M}_C}(M) = \text{span}_{\mathcal{M}_C}(M')$  and  $\text{span}_{\mathcal{M}_A}(M) = \text{span}_{\mathcal{M}_A}(M')$  holds.*

Obviously, Theorem 14 holds for the CA-CQ problem. But observe that it is more general than that. Theorem 14 is also true if (say) applicants can be assigned to several colleges up to certain personal quotas, and there can be nested bounded sets on the applicant side as well, just as for colleges. Theorem 13 implies for this case, too, that there exists a stable matching, and Theorem 14 concerns the structure of these stable matchings.

The first part of Theorem 14 implies our earlier observation on the applicant-optimal and college-optimal stable matchings, namely, if  $X$  is the union of all stable matchings then  $M_C := Q_C(X)$  and  $M_A := Q_A(X)$  are stable matchings. As  $M_C$  and  $M_A$  dominate

any other stable matching from the college side and from the applicant side, respectively, it follows that  $M_C$  is the college optimal stable matching and  $M_A$  is the applicant-optimal one. This observation implies Theorem 4 (ii) and Theorem 7 (ii) and (iii).

But the first part of Theorem 14 has an even more interesting consequence, as this generalises the lattice property of stable marriages. If the colleges freely choose from the union of the edge sets of two stable matchings  $M_1$  and  $M_2$ , then by Theorem 14 another stable matching is created that we denote by  $M_1 \vee M_2$ . Clearly,  $M_1 \vee M_2$  dominates both  $M_1$  and  $M_2$  from the college side. Moreover, if a stable matching  $M$  dominates both  $M_1$  and  $M_2$  from the college side then  $M$  dominates  $M_1 \cup M_2$  from the college side, hence  $M$  also dominates  $M_1 \vee M_2$  from the college side. From this fact it follows that  $M_1 \vee M_2$  is the least upper bound of  $M_1$  and  $M_2$  as  $\preceq_C$  is a partial order on stable matchings by Theorem 12.

A similar proof shows that for stable matchings  $M_1$  and  $M_2$  stable matching  $M_1 \wedge M_2 := Q_A(M_1 \cup M_2)$  is the least upper bound of  $M_1$  and  $M_2$  for partial order  $\preceq_A$ . The following lemma gives a relation the above two partial orders.

**Lemma 15.** *Assume that  $G = (A \cup C, E)$  is a bipartite graph with colour classes  $A$  and  $C$  and that  $\mathcal{M}_A$  and  $\mathcal{M}_C$  are two matroids on  $E$ . Let  $Q_A$  and  $Q_C$  be comonotone choice functions with property (1) on  $E$  that are defined by the greedy algorithm on  $\mathcal{M}_A$  and  $\mathcal{M}_C$  respectively, for some linear orders. Then for any two stable matchings  $M_1$  and  $M_2$  relation  $M_1 \preceq_A M_2$  is equivalent to  $M_2 \preceq_C M_1$ , or, in other words partial orders  $\preceq_A$  and  $\preceq_C$  are opposite on stable matchings.*

*Proof.* If  $M_1$  and  $M_2$  are stable matchings then each of them must dominate the other one. If  $M_1$  dominates each element of  $M_2 \setminus M_1$  from the applicant side then  $M_2$  must dominate each element of  $M_1 \setminus M_2$  from the college side, and vice versa. The lemma directly follows from this observation.  $\square$

Lemma 15 implies that partial order  $\preceq_C$  forms a lattice on stable matchings with lattice operations  $\vee$  and  $\wedge$ : we have seen that  $M_1 \vee M_2$  is the  $\preceq_C$ -least upper bound and  $M_1 \wedge M_2$  is the  $\preceq_A$ -least upper bound, hence by Lemma 15  $M_1 \wedge M_2$  is the  $\preceq_C$ -greatest lower bound of  $M_1$  and  $M_2$ .

The lattice structure of stable matchings allows us to prove an extension of the result of Teo and Sethuraman [24]. Our proof is essentially the same as the one in [8] that has also been found by Klaus and Klijn [14] for the many-to-one case.

**Theorem 16.** *Assume that the CA-CQ problem is given by bipartite graph  $G = (A \cup C, E)$  with colour classes  $A$  and  $C$ , a nested system  $\mathcal{C}$  of bounded sets and quotas  $q : \mathcal{C} \rightarrow \mathbb{N}$ . Let  $M_1, M_2, \dots, M_n$  be arbitrary stable matchings and  $1 \leq k \leq n$  an arbitrary integer. If each applicant chooses her  $k$ -th best assignment out of the  $n$  assignments provided by stable matchings  $M_i$  then the set  $M$  of these edges is a stable matching.*

*Proof.* Let each applicant  $a$  order the  $n$  stable matchings according to her preference as  $M_a^1, M_a^2, \dots, M_a^n$  such that  $a$  prefers  $M_a^1$  the best and  $M_a^n$  the least. Let

$$M := \bigwedge_{a \in A} M_a = Q_A\left(\bigcup_{a \in A} M_a\right) \quad \text{where} \quad M_a := \bigvee_{i=1}^k M_a^i = Q_C\left(\bigcup_{i=1}^k M_a^i\right).$$

Clearly, each of  $M$  and the  $M_a$ 's are a stable matchings by Theorem 14 (i). Observe that  $M_a$  dominates each of  $M_a^1, M_a^2, \dots, M_a^k$  from the college side, hence  $M_a$  is dominated by these matchings from the applicant side. This means that in  $M_a$  each applicant  $a'$  of  $A$  receives her worst assignment out of her assignments in  $M_a^1, M_a^2, \dots, M_a^k$ . In particular,

applicant  $a$  receives her  $k$ th best assignment out of the ones given by  $M_1, M_2, \dots, M_n$ . Any other applicant  $a'$  gets the  $k$ th best out of  $k$  assignments which is an assignment that is not better than the one represented by  $M_{a'}^k$ . As  $M$  is constructed by letting the applicants to choose from  $\bigcup_{a \in A} M_a$ , each applicant  $a$  will choose her  $k$ th best assignment represented by  $M_a^k$ . That is,  $M$  is the stable matching described in the theorem.  $\square$

The second part of Theorem 14 talks about the span on a subset of a matroid. Recall that an element  $e$  of some matroid  $\mathcal{M}$  is spanned by subset  $E'$  of  $\mathcal{M}$  if either  $e \in E'$  or there is an independent subset  $E^*$  of  $E'$  such that any proper subset of  $E^* \cup \{e\}$  is independent. This is equivalent to saying that the rank  $r_{\mathcal{M}}(E')$  of  $E'$  equals the rank  $r_{\mathcal{M}}(E' \cup \{e\})$ , that is, in case of (say)  $\mathcal{M}_C$ , we have  $|Q_C(E')| = |Q_C(E' \cup \{e\})|$ . This latter formulation implies that if some stable matching  $M$  dominates  $e$  from the college side then  $e \in \text{span}_{\mathcal{M}_C}(M)$ . Recall that  $E'(a)$ ,  $E'(c)$  and  $E'(C_i)$  denote the set of edges of  $E'$  that belong to applicant  $a$ , to college  $c$  and to some bounded set of colleges  $C_i$ , respectively.

What does it mean that some set  $E'$  of edges spans a certain edge  $e$  in the CA-CQ problem? Clearly, for matroid  $\mathcal{M}_A$  it happens if and only if applicant-vertex  $a$  of edge  $e$  is incident with an edge ( $e$  or some other) of  $E'$ , i.e., if  $|E'(a)| \geq 1$ . For  $\mathcal{M}_C$  this means that either  $e \in E'$  or there is a bounded set  $C_i$  containing  $c$  such that  $|E'(C_i)| \geq q(C_i)$ . Recall that bounded set  $C_i$  is full relative to matching  $M$  if  $|M(C_i)| = q(C_i)$ , and otherwise  $C_i$  is undersubscribed. Recall that a bounded set is free if it is not contained in a full bounded set. We call bounded set  $C_i$  *weakly free relative to  $E'$*  if no bounded set that properly contains  $C_i$  is full relative to  $E'$ . (In particular, a weakly free set relative to  $E'$  can be full relative to  $E'$ .) Bounded set  $C_i$  is *essential relative to  $E'$*  if  $C_i$  is weakly free and no bounded subset of  $C_i$  is full relative to  $E'$ . (In particular an essential bounded set cannot be full relative to  $E'$ .) We have the following generalisation of the ‘‘Rural hospitals theorem’’.

**Theorem 17.** *Let  $M$  be a stable matching in an instance of CA-CQ. If bounded set  $C_i$  is weakly free relative to  $M$  then for each stable matching  $M'$  we have  $|M(C_i)| = |M'(C_i)|$ . Moreover, if  $C_i$  is essential relative to  $M$  then  $M(C_i) = M'(C_i)$ .*

*In other words, if a bounded set is not properly contained in a full bounded set then it is incident with the same number of edges in each stable matching. Moreover, if this bounded set does not contain a full bounded set then it is incident with the same set of edges in each stable matching.*

Note that Theorem 17 remains true if applicants also have a college-type choice function as the proof only uses Theorem 14 and the structure of matroid  $\mathcal{M}_C$ .

*Proof.* We prove Theorem 17 by induction on  $i$ . (Recall that we fixed an order  $C_1, C_2, \dots$  of bounded sets such that a superset always has a greater index.) If  $C_1$  is not weakly free relative to  $M$  then we have nothing to prove. If  $C_1$  is free (that is weakly free and undersubscribed) relative to  $M$  then  $C_1$  is essential relative to  $M$  as  $C_1$  does not contain any other bounded sets. This means that  $\text{span}_{\mathcal{M}_C}(M) \cap E(C_1) = M(C_1)$ . So if  $M'$  is another stable matching then by Theorem 14,

$$M(C_1) = \text{span}_{\mathcal{M}_C}(M) \cap E(C_1) = \text{span}_{\mathcal{M}_C}(M') \cap E(C_1) = M'(C_1)$$

just as we claimed. Now suppose that  $C_1$  is full relative to  $M$  and assume for a contradiction that  $|M(C_1)| \neq |M'(C_1)|$ . From  $|M(C_1)| = q(C_1) \geq |M'(C_1)|$ , it follows that  $|M'(C_1)| < q(C_1)$ , hence  $C_1$  is undersubscribed in stable matching  $M'$ . But we just have proved that in this case  $C_1$  cannot be full in any stable matching, a contradiction.

Now assume that Theorem 17 is true for each  $C_j$  with  $j < i$  and we prove it for  $C_i$ . If  $C_i$  is an inclusionwise minimal bounded set then the proof is exactly the same as for

$C_1$  above. Otherwise  $C_i = C^1 \cup C^2 \cup \dots \cup C^k$ , where  $C^1, C^2, \dots, C^k$  are the inclusionwise maximal bounded subsets of  $C_i$  that are disjoint by the nested property. We may assume that  $C_i$  is weakly free relative to  $M$  as otherwise we have nothing to prove.

Suppose first that  $C_i$  is undersubscribed relative  $M$ . Clearly, each of  $C^1, C^2, \dots, C^k$  is weakly free relative to  $M$ , so Theorem 17 holds for them:  $|M(C^t)| = |M'(C^t)|$  for any stable matching  $M'$ . But this means that

$$|M(C_i)| = \sum_{t=1}^k |M(C^t)| = \sum_{t=1}^k |M'(C^t)| = |M'(C_i)|.$$

If  $C_i$  is essential relative to  $M$  then all of  $C^1, C^2, \dots, C^k$  are also essential relative to  $M$ , hence

$$M(C_i) = \bigcup_{t=1}^k M(C^t) = \bigcup_{t=1}^k M'(C^t) = M'(C_i)$$

holds. This finishes the case when  $C_i$  is undersubscribed in  $M$ .

It remains to settle the case when  $C_i$  is full relative to  $M$ . If, indirectly  $C_i$  is undersubscribed in some other stable matching  $M'$  then the above argument also holds for  $M'$  instead of  $M$ . This yields in particular that  $C_i$  cannot be full relative to any stable matching, contradicting the existence of  $M$ .  $\square$

Theorem 17 is a genuine generalisation of the ‘‘Rural hospitals theorem’’ as in the absence of common quotas, each college  $c$  is a weakly free bounded set by itself, so each stable matching assigns the same number of applicants to  $c$ . Moreover, if college  $c$  does not fill up its quota in some stable matching then bounded set  $\{c\}$  is essential hence  $c$  gets the same set of applicants in each stable matching.

Actually, in the college admissions problem each college is a group by itself. Interestingly, in the CA-CQ problem groups behave somewhat similarly to colleges in the College Admissions problem.

**Corollary 18.** *If  $C_i$  is a group in the CA-CQ problem then any stable matching assigns the same number of edges to  $C_i$ .*

*Proof.* A group is weakly free by definition, so the corollary follows from Theorem 17.  $\square$

### 5.3 Comparison between the direct and the matroid approaches

In Section 4, we constructed and studied two direct algorithms for solving the College Admissions problem with common quotas in the case of nested set systems, whilst in Section 5, we solved the same problem by a more general approach using choice functions and matroids. The reader might wonder why both descriptions are necessary. Here, we would like to answer this hypothetical question.

We proved the existence of a stable matching by each method, and we also showed that the two main variants of each method produce the applicant-optimal and the applicant-pessimal solutions, respectively. This latter fact ensures that the corresponding variants lead to the very same results. The reason for studying both methods, beside the obvious interest in having both a direct and a general argument, is the following. On the one hand, the direct algorithms have complexities which are not achievable by the general method. On the other hand, the structural results which are straightforward by the matroid model would be difficult to prove directly. A further advantage of the general approach is that, as we noted, similar results can be verified for such more general settings where the choice function on the the applicants’ side is more complicated than the one in our model.

We note that the original algorithm of Gale and Shapley [10] and its variant studied by McVitie and Wilson [17] have a similar relation to each other as our general and direct method. Both produce the same results, but the  $O(L)$  running time is achievable only with the latter variant, where the proposals are made one by one rather than simultaneously.

## Acknowledgments

We would like to thank the Hungarian Higher Education National Institute and one of its representatives in particular, Gábor Varjasy, for providing us information about the higher education matching scheme, including a detailed description of their current algorithm and the real data of the 2008 matching run. Also, we would like to thank to an anonymous referee for helpful comments.

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