On diffraction within a dielectric medium as an example of the Minkowski formulation of optical momentum

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Abstract: The Abraham-Minkowski dilemma relates to the disputed value of the optical momentum within a dielectric medium and whether the free-space value should be divided (Abraham) or multiplied (Minkowski) by the refractive index. Although undoubtedly simplistic, these two approaches provide intuitive insight to many subtle problems in optical physics. This paper reviews a modified version of the Einstein box argument that supports an Abraham formulation, then considers diffraction within a dielectric medium and shows it supports a simple Minkowski formulation, i.e. that the optical momentum should be multiplied by the refractive index.

OCIS codes: (050.1940) Diffraction; (260.2110) Electromagnetic optics

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#102536 - $15.00 USD Received 8 Oct 2008; revised 13 Nov 2008; accepted 20 Nov 2008; published 2 Dec 2008

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1. Introduction

A simple understanding of the optical momentum within a dielectric of refractive index n has been sought intermittently over the last 100 years and is characterized by the alternative formulations of Abraham and Minkowski [1, 2, 3, 4, 5]. Although neither of these formulations are dependent upon quantization, it is usual to express and consider them in terms of single photons with energy $\hbar\omega$ and free-space momentum of $\hbar k_0$, where ω and k_0 are the optical frequency and free-space wavenumber respectively. The Abraham formulation [6] is equivalent to proposing a photon momentum within the dielectric, of refractive index n, of $\hbar k/n$. This formulation can be derived from a number of different starting points ranging from various forms of the energy-force tensor [1] to the *Gedankenexperiment* of the "Einstein Box" [7]. The latter approach, non-intuitively, arising from a conservation of angular momentum within general relativity [8]. By contrast, the Minkowski formulation [9] is equivalent to proposing a photon momentum within the dielectric of $n\hbar k_0$. This formulation can also be reasoned from various starting positions, ranging from an alternative form of the energy-force tensor [1] to, most intuitively, a simple implementation of the de Broglie relationship. The de Broglie relationship states that the momentum p is equal to h/λ, where the optical wavelength λ is reduced within the dielectric by a factor n. Hence the de Broglie relationship suggests a photon momentum of nh/λ_0.

Whilst both of these formulations are arguably gross simplifications, they do provide a common language by which the dilemma can be discussed. More pertinently, invariably one or the other does predict the correct outcome in the vast majority of experiments (see below). This paper repeats a variation of the Einstein box argument, supporting Abraham, and presents a simple argument based upon diffraction, supporting Minkowski. Superficially it seems that both arguments are correct, hence the dilemma. These examples perhaps establish two test cases against which more sophisticated formulations can be tested, with a requirement that such formulations need to be shown to reduce to the Abraham or Minkowski interpretations in these extremes.

2. The Einstein Box revisited

The Einstein box *Gedankenexperiment* considers a single photon passing through a block of transparent material of length L and refractive index n. It assumes that the surfaces of the block have perfect anti-reflection coatings, so that there are no Fresnel reflections and associated recoil forces. Furthermore, it assumes that there is no dispersion or absorption within the block. It avoids taking any decision on the interpretation of photon momentum by considering instead the energy transport. Upon transmission of the photon through the glass block it is delayed compared to free-space propagation by a distance $(n-1)L$. To compensate for its energy of $\hbar\omega$ being delayed by $(n-1)L$, the mass M of the glass block is advanced by a distance Δz. Relating

#102536 - $15.00 USD Received 8 Oct 2008; revised 13 Nov 2008; accepted 20 Nov 2008; published 2 Dec 2008

the mass of the block, via $E = Mc^2$, to its energy equivalent one obtains [10]

$$\Delta z = \hbar\omega(n-1)L/Mc^2 = \hbar k_0(n-1)L/Mc \tag{1}$$

This displacement of the block, in the direction of the optical propagation, is that which should occur for every transmitted photon (see Fig. 1(a)). This Einstein box result is superficially the same as that which is obtained by assuming that the photon momentum within the block is reduced to a value $\hbar k_0/n$. In this case, the conservation of the momentum on entry of the photon results in a small momentum acquired by the block, setting it into motion. Upon exit, the exchange is reversed and the block comes to a halt; note that the block is in motion only while the photon is within the dielectric.

By contrast, a simplistic application of the Minkowski momentum (which is larger than the free-space value, i.e. $n > 1$) gives a predicted displacement of the block which is different in both magnitude and direction. This apparent discrepancy can be potentially resolved be recognizing that the Minkowski momentum itself includes a mechanical momentum of the medium [5] and therefore that the Minkowski term cannot in itself be used to predict the motion of the block. However, as shown below, the Einstein box argument can be reformulated without the need of an explicit medium.

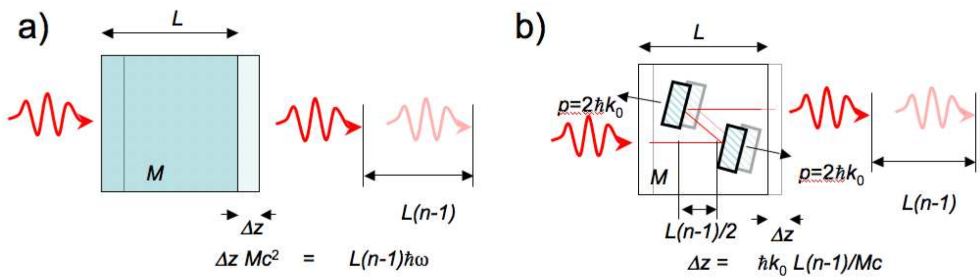

Fig. 1. a) Shows the traditional Einstein box *Gedanken*experiment which equates the energy delay of the photon to the mass energy advance of the block b) Shows an equivalent example where the delay arises from a free-space delay line and the recoil force acting on the mirrors causes the block displacement. Both a) and b) are consistent with the Abraham formulation of optical momentum.

Resorting to general relativity to justify the continuity of energy flow is perhaps not intuitive, and is in fact unnecessary. If one considers the block as a “black box” for which the only thing known is that the transmitted light is delayed, the block can be represented as a simple optical delay line (see Fig. 1(b)), where the fold-angle is extremely small so that the transverse displacement is negligible. A single photon on reflection from the first mirror exerts a recoil force $2\hbar k_0$, setting the block into motion with a velocity $2\hbar k_0/M$. The block is then halted after a time $(n-1)L/2c$ by the reflection from a second mirror. This start-stop motion results in a displacement of the block of $\Delta z = \hbar k_0(n-1)L/Mc$, i.e. identical to the original Einstein box and consistent with the Abraham formulation. The concept of a delay line as a representation of increased refractive index is clearly simplistic but does have close linkage to the understanding of Bragg reflection and propagation through air spaced photonic crystals [11]. Furthermore, this delay line approach eliminates the uncertain role of dielectric surfaces, dispersion and other properties of the medium.

Having presented a rational justification for an Abraham formulation, it is useful to present an equally convincing support of Minkowski. This can be provided by considering the diffraction from a single slit of width Δx. The uncertainty principle can be used to approximate how this

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lateral confinement introduces a lateral spread in the optical momentum of the diffracted light, $\Delta p_x \approx \hbar/\Delta x$. (Strictly speaking the sinc function of the diffraction pattern has an ill-defined standard deviation and hence cannot be specified by the uncertainty principle. However, within this argument it is sufficient that the uncertainty principle can be used to give the scaling of the diffraction pattern, which is valid for all slit profiles.)

For a perfectly collimated input beam, the angular position of the first diffraction minimum is $\theta = \lambda_0/\Delta x$, which can be approximated by the ratio between the spread in the lateral momentum and the axial momentum. The angular spread of the zero-order maximum is therefore $\Delta\theta \approx 2\Delta p_x/p_z$, where p_z is the axial component of the photon linear momentum. For small diffraction angles, $p_0 = p_z$ and the angular spread approximates to $2\Delta p_x/p_0$. Thus the angular spread behind a slit of width Δx is given by (see Fig. 2(a))

$$\Delta\theta = 2\hbar/\Delta x p_0. \tag{2}$$

Now consider the case when the whole laboratory, including the slit and screen, is filled by a medium of refractive index n. The wavelength of the light is reduced by a factor n and it follows that the lateral scale of the diffraction pattern is similarly reduced $\Delta\theta' = 2\hbar/n\Delta x p_0$ (see Fig. 2(b)). However, Δx is unchanged by the addition of the medium and hence Δp_x is set without any dependance upon n. Hence, to give the observed diffractive spreading it is p_z, i.e. p_0 that must increase by a factor proportional to n. It follows that the linear momentum of the light within the dielectric medium is $n\hbar k_0$ – the Minkowski formulation. A similar argument can also be applied to the case of double-slit diffraction [10].

Note that within this diffraction argument, care was taken to ensure that both the slit and screen are within the medium. Even if the light is incident from outside the medium, it is collimated upon entry so that the transverse momentum is not affected by any issues associated with the interface. Consequently, the transverse momentum of the light as incident on the slit is unambiguously zero. The spread in transverse momentum introduced by the finite width of the slit is independent of the refractive index of the medium with which it is filled. Similarly, although there may be ambiguity over the optical momentum transferred to the detector, this has no bearing upon the interpretation of the results, since it is the position of detection from which the ratio of axial to transverse momenta, $2\Delta p_x/p_0$, is inferred.

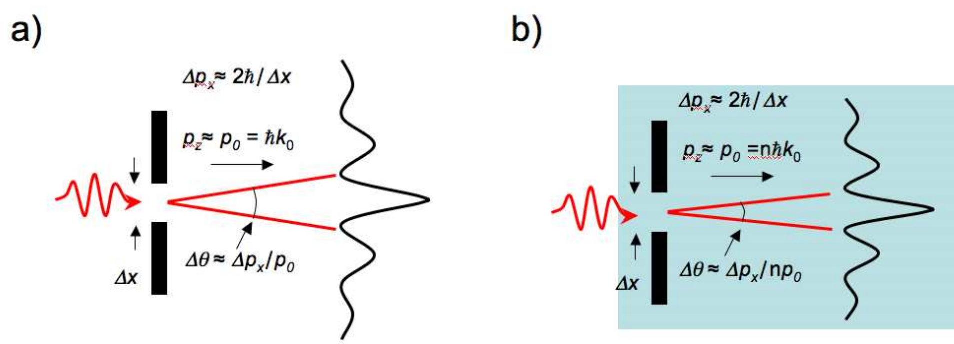

Fig. 2. a) Shows how in single slit diffraction the width of the slit introduces a transverse moment spread, the ratio of which to the axial momentum gives the diffractive, angular spreading of the light b) Shows how increasing the refractive index of the medium reduces the angular spreading, which is consistent with an increase in the axial optical momentum and the Minkowski formulation.

3. Conclusion and Discussion

This paper does not in any way propose a solution to the Abraham-Minkowski dilemma. Instead, it seeks to re-affirm that in simple *Gedankenexperiments* one formulation gives a correct prediction whilst the other, superficially, does not. The literature has numerous papers proposing various treatments but meaningful experiments are rare. The Minkowski formulation seems to be most appropriate when the wavevector and wavelength of the light is central to predicting the outcome of experiment, usually the refractive index refers to the phase velocity. The Minkowski formulation is also in agreement with experiments involving the recoil of individual atoms from a cold gas cloud [12], with the apparent stretching of cells positioned between counter-propagating laser beams [13], the dragging of electrons by pulses of light within a conductor [14, 15] and in the rotational case the transfer of angular momentum to an RF antenna within a fluid-filled waveguide [16]. The Abraham formulation seems most applicable to situations where it is the energy transport or Poynting vector of the light which dominates, and usually the refractive index refers to the group velocity. Clear experimental evidence is harder to determine but it is worth noting that the Abraham-like predictions of the Einstein box are also in agreement with calculations of the Lorentz force in both linear [17, 18] and rotational forms [19, 20].

Various earlier experiments were also reported, including those trying to measure the recoil force acting on a submerged mirror [21]. Most recent has been the observation that the light emanating from a fiber seems to cause the tip to bend in a direction indicative of an Abraham result [22]. However, it seems that all these experiments are both extremely challenging and ultimately open to multiple interpretations. In one interpretation of the recoil on a submerged mirror it is possible to obtain either the Abraham or the Minkowski result by varying the phase shift of the reflection [23]. Certainly on an atomic/molecular level of an individual dipole, it seems as if the alternative forms of the force arising from gradients in field and dipole moment reduce to give identical forces [24].

It seems likely that, providing they are applied correctly, both the Abraham and Minkowski formulations or indeed variants thereof are all potentially capable of giving the observed result. Hence, the Abraham-Minkowski dilemma is not one of formal consistency, but one of appropriate interpretation. The challenge is to understand in which situations to apply which formulation if the correct result is to be simply obtained.

Acknowledgments

I wish to thank Masud Mansuripur, Stephen Barnett, Sonja Franke-Arnold and the referees for helpful suggestions made with respect to this manuscript. MJP thanks the Leverhulme Trust for financial support.

#102536 - $15.00 USD Received 8 Oct 2008; revised 13 Nov 2008; accepted 20 Nov 2008; published 2 Dec 2008