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Angular diffraction

B Jack, M J Padgett and S Franke-Arnold[1]

Department of Physics and Astronomy, SUPA, University of Glasgow,
Glasgow G12 8QQ, UK
E-mail: s.franke-arnold@physics.gla.ac.uk

Abstract. Angle and angular momentum are linked by a Fourier transformation. A restriction of the angular range within an optical beam profile therefore generates orbital angular momentum (OAM) sidebands on the transmitted light. We interpret this phenomenon as angular diffraction: a mask that blocks light within one or several angular ranges is the angular analogue of a single slit or a diffraction grating, respectively. In the OAM spectra of light transmitted through angular masks we observe the typical sinc^2 envelope known from single slit diffraction, and in particular the suppression of OAM sidebands. This is the angular analogy to missing orders observed for a linear diffraction grating.

Contents

[1] Author to whom any correspondence should be addressed.

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1. Introduction

During the last two decades the investigation of the orbital angular momentum (OAM) of light [1]–[3] has evolved from rather fundamental studies to applications, reaching from quantum information [4, 5], through interaction with matter [6] and Bose condensates [7], free-space communication with large data capacity [8] to potential applications in astronomy [9]. A light beam can carry OAM if its phase fronts are not perpendicular to its direction of propagation. The local Poynting vectors [10], and therefore the direction of energy flow, have an azimuthal component, resulting in an OAM around the propagation direction. Light modes with a phase term $\exp(i\ell\phi)$, where ϕ is the azimuthal angle within the beam profile, correspond to an OAM of $\ell\hbar$ per photon. One should note that light modes are also characterized by a radial dependence, e.g. the p-value specifying the radial nodes of Laguerre–Gauss modes. The properties of angular diffraction hold for all forms of radial profile and therefore the p-value has no bearing for this work.

As the azimuthal angle ϕ is a periodic function, its Fourier conjugate, the OAM, is a discrete variable, and the linking Fourier relationship is given by [11]

$$A_\ell = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi \, \psi(\phi) \exp(-i\ell\phi), \tag{1}$$

$$\psi(\phi) = \sum_{\ell=-\infty}^{+\infty} A_\ell \exp(i\ell\phi). \tag{2}$$

Equation (2) simply expresses the light mode in terms of its spiral harmonics $\exp(i\ell\phi)$. An angular restriction of the light profile modifies the OAM spectrum, generating new OAM sidebands, without, of course, changing the net or average OAM of the beam. This has been observed for classical light beams [11] and it is also at the base of various theoretical descriptions [12, 13]. In particular, it has been observed that the OAM spectrum of a light beam transmitted through a hard-edged angle aperture has a sinc^2 envelope. In this paper, we present an intuitive explanation of this OAM spectrum, interpreting the transmission through an angle mask as *angular diffraction*. In the conventional single, or multiple, slit experiment a restriction in linear position of a light beam causes interference between the various Fourier components. This interference modifies the transverse linear momentum, i.e. the wavevectors of the light field, and causes the characteristic diffraction pattern in the far field. The angular analogue of a single slit is a mask containing an angular Heavyside step function, and the OAM spectrum of the transmitted beam is enveloped by a sinc^2 function, as shown in figure 1.

2. Theoretical predictions

We consider a mask that has a uniform transmission of t_1 within an angle β and a different transmission of t_2 elsewhere,

$$M(\phi) = \begin{cases} t_1 & \text{for } 0 \leqslant \phi < \beta, \\ t_2 & \text{for } \beta \leqslant \phi < 2\pi. \end{cases} \tag{3}$$

The mask may be an amplitude mask, where the transmission takes on real values between 0 and 1, or a phase mask that delays the wavefront in a certain angular range β. The wavefunction of the transmitted beam is $\psi(\phi) = M(\phi)\psi_0(\phi)$, where we assume the initial mode $\psi_0(\phi)$ to be
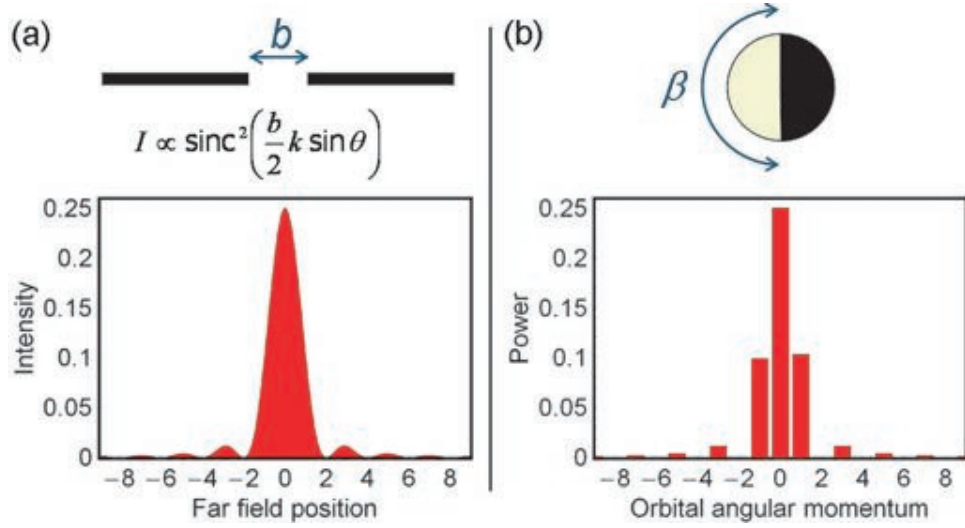
Figure 1. Linear and angular diffraction: (a) a restriction of position due to a single slit causes a sinc^2 envelope of the light intensity in the transverse linear momentum or the far field position. (b) Similarly, restriction of the angular position results in a sinc^2 envelope of the OAM spectrum.

the pure spiral harmonic mode $\psi_0 \exp(i\ell_0\phi)$ with an OAM of $\ell_0\hbar$. The OAM spectrum of the transmitted beam is given by the Fourier transform of the wavefunction (1):

$$A_\ell = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}\phi\, M(\phi) \mathrm{e}^{-i\ell\phi} \psi_0\, \mathrm{e}^{i\ell_0\phi}. \tag{4}$$

The integration can easily be evaluated to

$$A_{\ell+\ell_0} = \psi_0 \begin{cases} \frac{\beta}{2\pi}(t_1 - t_2)\mathrm{e}^{-i(\beta/2)\ell}\mathrm{sinc}(\frac{\beta}{2}\ell), & \text{for } \ell \neq 0, \\ \frac{\beta}{2\pi}(t_1 - t_2) + t_2, & \text{for } \ell = 0. \end{cases} \tag{5}$$

This is equivalent to expressing the mask in terms of its spiral harmonics $\exp(-i\ell\phi)$. Note that the relative weighting of the OAM modes, other than for the initial mode, is independent of the values of t_1 and t_2, and in particular of whether it is an amplitude or a phase mask.

The OAM spectrum for an absorption mask with $t_1 = 1$ and $t_2 = 0$ simplifies to

$$|A_{\ell+\ell_0}|^2 = |\psi_0|^2 \left(\frac{\beta}{2\pi}\right)^2 \mathrm{sinc}^2\left(\frac{\beta}{2}\ell\right), \tag{6}$$

reminiscent of the diffraction pattern of a conventional single slit experiment (compare figure 1). While the diffraction pattern of a single slit is the continuous intensity variation of linear momentum, or position in the far field, angular diffraction results in a sinc^2 envelope of discrete OAM modes. This mirrors the nature of the Fourier relations: the unbounded and continuous variables of position and momentum are linked by Fourier transforms, while the periodically defined angular position and discrete angular momentum are connected by the Fourier transform (1) and series (2).

The angular analogue of a double slit is a mask with two symmetrically placed opening angles, and a grating of N slits corresponds to an angle mask with N-fold rotational symmetry. The individual opening angle β is analogue to the single slit width, b, and the repetition angle

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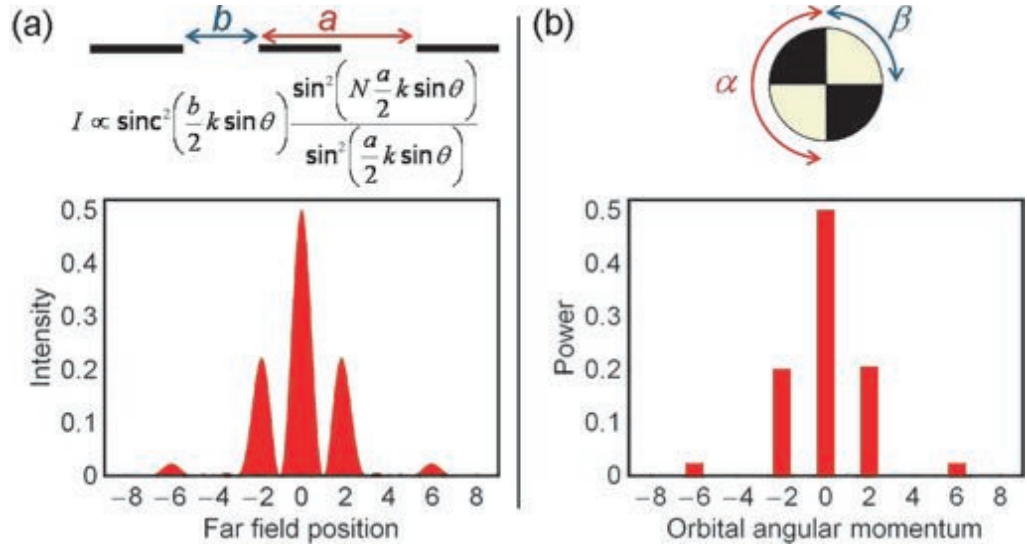
Figure 2. Linear and angular double slit diffraction. (a) For a slit separation a of double the slit width b every second peak of the sinc^2 diffraction pattern is suppressed. (b) Similarly, for an angular mask with 2-fold symmetry every uneven OAM is missing in the transmitted light. Additionally, the rational fraction between repetition angle $\alpha = \pi$ and opening angle $\beta = \pi/2$ causes the cancellation of the OAM sidebands at $\pm 4, \pm 8, \ldots$.

α to the separation between the individual slits, a, i.e. the grating constant, see figure 2. As one may expect, in this case the sinc^2 envelope of the single slit or opening angle is multiplied with a function that describes interference between the slits/opening angles. Due to the periodicity of the angle mask, the repetition angle α is directly linked to the number of opening angles N in a 2π radian range by $\alpha = 2\pi/N$. An angular diffraction grating with N opening angles of width β is described by a mask

$$M(\phi) = \begin{cases} 1, & \text{for } n\alpha \leqslant \phi < (n\alpha + \beta) \\ 0, & \text{else} \end{cases}, \tag{7}$$

where $n = 0, 1, \ldots, N - 1$ and $\beta < \alpha$.

As before we can find the spiral harmonics of the transmitted beam from the Fourier transform (1):

$$
\begin{aligned}
A_{\ell + \ell_0} &= \frac{1}{2\pi} \sum_{n=0}^{N-1} \int_{n\alpha}^{n\alpha + \beta} d\phi M(\phi) e^{-i\ell\phi} \psi_0 \\
&= \psi_0 \frac{\beta}{2\pi} e^{-i\ell\beta/2} \operatorname{sinc}\left(\frac{\beta}{2}\ell\right) \sum_{n=0}^{N-1} e^{-i\ell 2\pi n/N} \\
&= \psi_0 \frac{\beta}{2\pi} e^{-i\ell\beta/2} e^{-i\ell\alpha(N-1)/2} \operatorname{sinc}\left(\frac{\beta}{2}\ell\right) \frac{\sin(N(\alpha/2)\ell)}{\sin((\alpha/2)\ell)},
\end{aligned}
\tag{8}
$$

in analogy to diffraction off a conventional multiple slit experiment, see figure 2. A mask with 2-fold symmetry causes cancellation of every odd OAM component. More generally, in an N-fold geometry only every Nth OAM component survives. For rational fractions between repetition
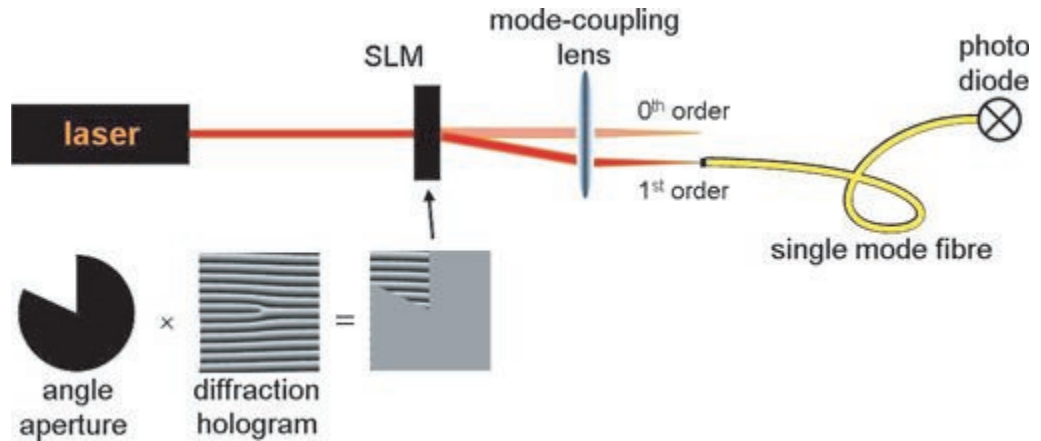
Figure 3. Experimental setup of the angular diffraction experiment.

angle α and opening angle β, the zeroes of the sinc^2 envelope coincide with OAM sidebands that are allowed by the mask symmetry. This is an exact analogy to the missing orders that occur in conventional diffraction patterns for rational fractions between slit width b and slit separation a [14].

3. Experiment and results

We have confirmed these results experimentally with the setup shown in figure 3. An angular aperture mask is placed into a HeNe laser beam with a Gaussian profile, and the generated OAM spectrum of the transmitted light is analysed in terms of the transmitted spiral harmonics, typically over a range from $\ell = -12$ to $+12$. We use spatial light modulators (SLMs) both for preparing the state with the angular aperture and for analysing the resulting OAM modes [15]. SLMs act as reconfigurable refractive elements, allowing to control the complex amplitude of the diffracted beam. If the index of the analysing ℓ-forked hologram is opposite to that of the incoming mode, planar wave fronts with on-axis intensity are generated in the first diffraction order. The on-axis intensity efficiently couples into a single mode fibre and can be measured with a photodiode. In the experiment, we display the angle aperture and the OAM analysis on the same SLM as indicated in figure 3.

Figure 4 shows the OAM spectrum for the angular analogue of single slit diffraction, an aperture mask with a single opening angle. If no aperture is placed in the beam the OAM is not modified, and accordingly we detect almost all of the light in the state with zero OAM (figure 4(a)). Figures 4(b)–(d) show the characteristic sinc^2 OAM spectrum for blocking half, a third and a sixth of the beam, respectively. As in a conventional single slit experiment, destructive interference within the transmitted light results in zeroes in the diffraction pattern. A complete cancellation of OAM modes occurs if the opening angle β is a rational fraction of 2π. For a mask that transmits only half (a third) of the beam, modes with $\ell = \pm 2n$ ($\ell = \pm 3n$) with n being a positive integer, disappear in the OAM spectrum, as shown in figures 4(b) and (c), respectively.

The angular analogue to multiple slit diffraction is shown in figures 5(a)–(c), displaying the OAM spectra of masks with 2, 3 and 4 symmetrically placed blocked beam areas, each with an angular width of $\pi/3$. An N-fold repetition of an angular pattern results in a spreading of
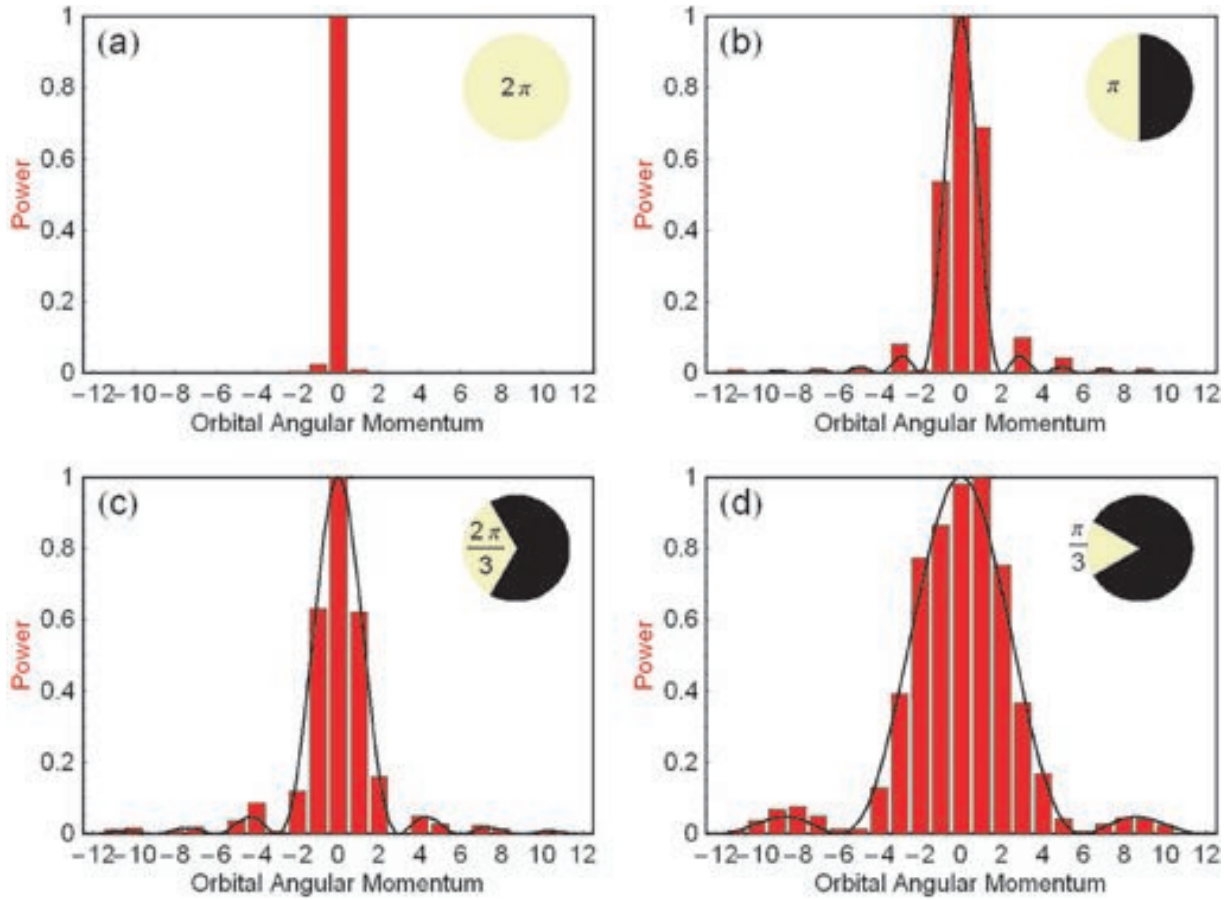
Figure 4. OAM contributions measured without mask (a), and transmitted through a mask that blocks one half (b), one-third (c) and one-sixth (d) of the incoming beam. The spectral intensities are background corrected and displayed in arbitrary units with the highest spectral component set to unity. The envelope of the OAM spectrum is given by the typical sinc^2 of single slit diffraction.

the OAM modes. The sinc^2 envelope of the single aperture of angular width $\pi/3$ is combined with the sinusoidal pattern arising from interference between the individual angular opening angles. Figure 5(d) shows how an N-fold symmetric mask causes constructive interference only for OAM modes with the same rotational symmetry: the top row displays the phase profile of OAM modes with $\ell = 0, 1, 2$ and 3. The second and third rows demonstrate the effect of a 2- and 3-fold symmetric mask: only if the transmitted beam areas have equal phases is the OAM sideband present in the OAM spectrum.

Alternatively, angular phase gratings can be realized by inducing a π phase delay for an angular region of the light beam. The benefit of phase gratings is their higher transmission efficiency, and they furthermore reduce the zero-order transmission, resulting in stronger sidebands. The zero-order component is completely suppressed for masks that delay half of the light by π (figure 6(a)) and otherwise it is reduced compared to an absorption mask with equal geometry (figure 6(b)).

We note that this description can easily be generalized to more complicated transmission functions. An aperture with any transmission function will generate an OAM spectrum envelope
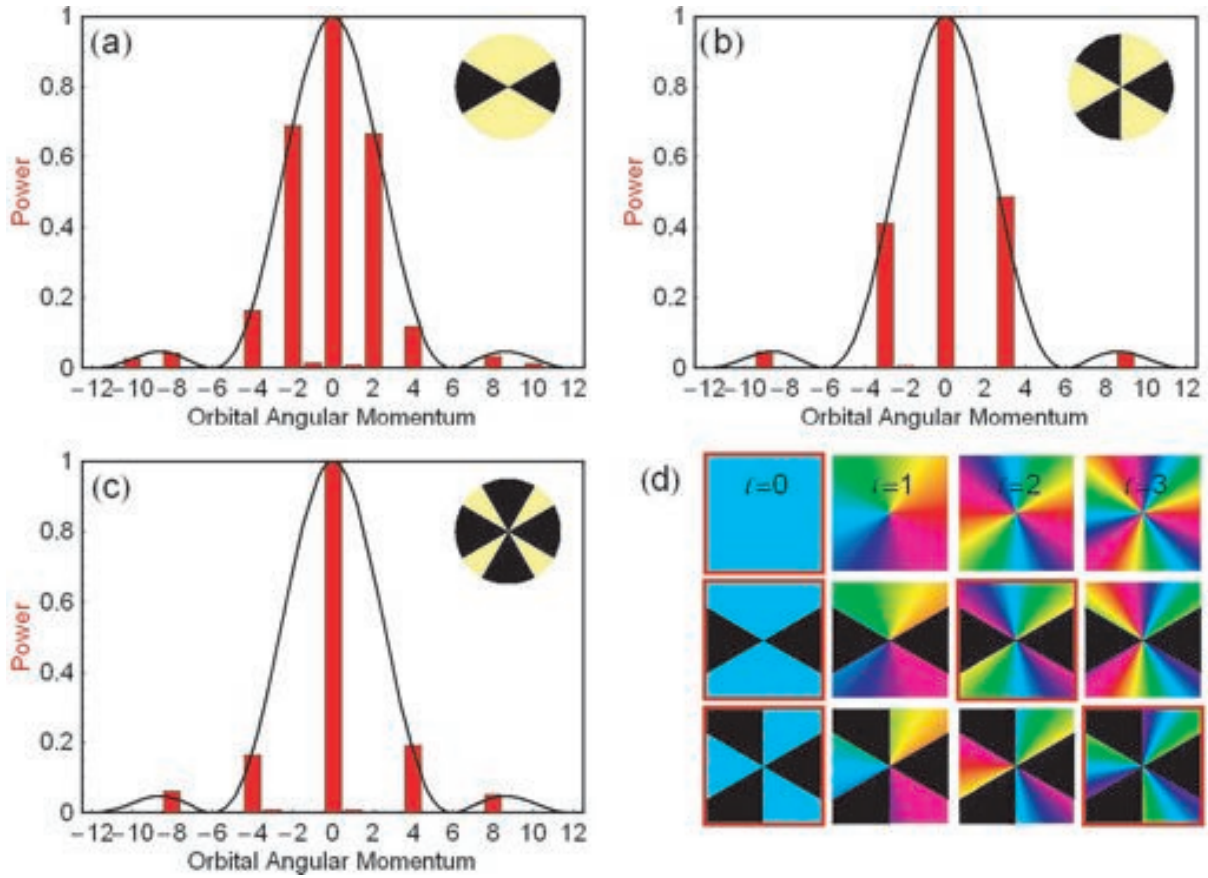
Figure 5. Angular diffraction patterns from masks with multiple symmetric blocked angles of $\pi/3$: OAM spectrum resulting from a mask with 2-fold symmetry (a), 3-fold symmetry (b) and 4-fold symmetry (c). Note the additional suppression of $\ell = \pm 6$ in (b). An OAM mode is present in the OAM spectrum of the transmitted light only if its symmetry matches the mask symmetry (d).

given by the Fourier transform of this transmission function, e.g. a Gaussian envelope for a Gaussian angular transmission function [12], or discrete sidebands for a sinusoidal variation of the angular transmission. Similar techniques were considered for the rotation of single atom wavefunctions trapped in sinusoidally varying light profiles [16]. Alternatively, one could envisage masks that approximate phase plates to a given order [17]. A mask that approximates for example an $\ell = 1$ spiral phase plate to the 6th order, i.e. in 6 phase steps, will result in a transmitted light beam with the main contribution at $\ell = 1$ and sidebands at $\ell = 1 \pm 6n$, where n is a positive integer.

4. Conclusions

In this paper, we have presented the angular analogue to conventional diffraction from single and multiple slits, realized with angular masks. We find a direct correspondence between linear and angular diffraction, with the difference that linear diffraction patterns are continuous functions, whereas angular diffraction generates discrete patterns recorded in the OAM spectrum. This approach offers an intuitive understanding of transmission through angular masks. Angular

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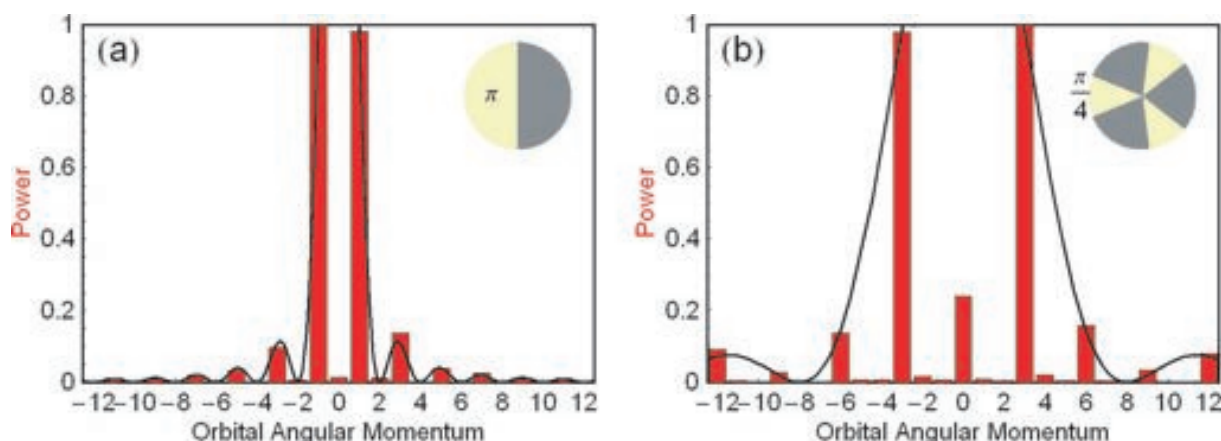
Figure 6. Angular diffraction from phase masks. Apart from the zero order the OAM spectrum is identical to amplitude masks. If equal fractions of the beam are delayed by a phase of π as in (a), the zero order is totally suppressed; otherwise it is reduced (b).

diffraction is a direct demonstration of the Fourier relation between the angular momentum and angular position. We have compared our calculations with experimental data and found excellent agreement.

Acknowledgment

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