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Minimum uncertainty states of angular momentum and angular position

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Abstract. The states of linear momentum that satisfy the equality in the Heisenberg uncertainty principle for position and momentum, that is the intelligent states, are also the states that minimize the uncertainty product for position and momentum. The corresponding uncertainty relation for angular momentum and angular position, however, is more complicated and the intelligent states need not be the constrained minimum uncertainty product states. In this paper, we investigate the differences between the intelligent and the constrained minimum uncertainty product states for the angular case by means of instructive approximations, a numerical iterative search and the exact solution. We find that these differences can be quite significant for particular values of angular position uncertainty and indeed may be amenable to experimental measurement with the present technology.

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Contents

1. Introduction

It is well known that the spin angular momentum of a photon associated with circular polarization of a light beam can be found with one of the values $\pm\hbar$. On the other hand, the orbital angular momentum associated with a helical phase front can take a range of values $l\hbar$ per photon, where l is any integer [1]. Recently, it has been shown that the orbital angular momentum of a single photon can be measured [2], and this has led to the experimental confirmation of the uncertainty relation for angular momentum and angular position [3].

The uncertainty relation for angular momentum and angular position is much less well known than the Heisenberg uncertainty principle for linear momentum and position, which is fundamental to quantum mechanics. The latter states that the product of the uncertainties in linear position and momentum has a lower bound such that [4]

$$\Delta x \Delta p \geqslant \hbar/2. \tag{1}$$

The states that satisfy the equality in an uncertainty relation are sometimes referred to as intelligent states [5, 6]. The intelligent states for (1) have Gaussian probability distributions of both position and momentum. Because of the state independence of the right-hand side of (1), for any given Δx, the intelligent states are also the states that minimize the uncertainty product on the left-hand side of (1).

The uncertainty relation between angular momentum and angular position is more complicated than for linear momentum and position (1). If we consider, for example, a bead sliding on a circular wire of known large diameter r, then we might be able to write the angular momentum uncertainty ΔL as $r\Delta p$ and the angular position uncertainty $\Delta\theta$ as $\Delta x/r$. From (1) we might then write $\Delta L\Delta\theta \geqslant \hbar/2$ and find the intelligent states to have Gaussian probability distributions in angle and angular momentum. The difficulty with this simple argument of course is that, unlike the linear position, the angular position takes values only over a finite range of size 2π. Angular positions that differ by 2π represent the same physical state. Thus $\Delta\theta$ must have an upper bound and the relation $\Delta L \Delta\theta \geqslant \hbar/2$ must fail for sufficiently small values of ΔL. For the same reason, no probability distributions of angle can be exactly Gaussian. At best, if this angle probability distribution is sufficiently sharp, that is, if the angular position is sufficiently well defined, the distribution can be approximately Gaussian.

The problems associated with the periodicity of the angular position and probability distributions on a circle were considered by Judge and Lewis [7]. They found that the lower bound on the uncertainty product must be state dependent, of a form given by (4) below. This state dependence leads to the interesting situation in which an intelligent state, even though it has uncertainties that satisfy the equality in (4), need not be a state that minimizes the uncertainty product for a given angular position uncertainty or for a given angular momentum uncertainty [8].

When dealing with an uncertainty relation where the lower bound is state dependent, the question as to what constitutes a minimum uncertainty state can be asked in different ways. Firstly, there is the question about the global minimum. Clearly, here the angular momentum states give this minimum. Beyond this point, one can search for minimum uncertainty states under various additional constraints. One possibility is to consider only states that fulfil the equality in the uncertainty relation, that is, the intelligent states. Another interesting constraint is to consider only states with a given uncertainty in angular position and find which of these minimize the uncertainty product. A third possibility is to consider only states with a given uncertainty in angular momentum. As it turns out, we find that the states that minimize the uncertainty product under the constraint of a given uncertainty in angular position are the same as those that minimize the uncertainty product for a given uncertainty in angular momentum. In the following, we will refer to these states as constrained minimum uncertainty product (CMUP) states.

The experimental confirmation of the uncertainty principle for angular momentum and angular position [3] was carried out for intelligent states, with the uncertainty product plotted against the uncertainty in angle. Because of the state-dependence of the uncertainty bound, a state other than the intelligent states may give an uncertainty product smaller than that measured in [3]. In this paper we examine this question and identify the states that give the minimum uncertainty product. Interestingly, we find that the difference in uncertainty product between intelligent and CMUP states should be measurable with present technology.

2. Intelligent states

In units for which $\hbar = 1$, it is usual to represent the operator for the z-component of angular momentum as [4][5]

$$\hat{L}_z = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\varphi}. \tag{2}$$

We shall refer to the corresponding representation as the angle representation. In this representation, the angle operator $\hat{\varphi}$ is the multiplicative operator $Y(\varphi) = \varphi + 2n\pi$, where n

[5] This representation makes certain assumptions about the differentiability of the angle wavefunction and needs to be used with care. We present a brief analysis of this point in appendix C.

is an integer chosen so that $Y(\varphi)$ has a value within a selected 2π range. $Y(\varphi)$ is a sawtooth function of φ that rises as φ increases but drops sharply by an amount 2π at 2π intervals. This restricts the angle eigenvalues to lie within a particular 2π range as would be expected from the impossibility of distinguishing physically between two states of angle differing by 2π. In this paper we choose the 2π range to be $[-\pi, \pi)$. $Y(\varphi)$ can be expressed in terms of φ plus a series of unit step functions [7], from which follows the commutator

$$[\hat{L}_z, \hat{\varphi}] = -\mathrm{i}\left\{1 - 2\pi \sum_{n=-\infty}^{\infty} \delta[\varphi - (2n+1)\pi]\right\} \tag{3}$$

in the angle representation. From the commutator we can find the uncertainty relation by use of Robertson's general expression [9]. Expressing the expectation value of this operator in terms of the angle wavefunction $\psi(\varphi)$ that is normalized in a 2π interval, allows us to write the corresponding uncertainty relation as

$$\Delta L_z \Delta\varphi \geqslant \tfrac{1}{2}|1 - 2\pi P(\pi)|, \tag{4}$$

where $(\Delta L_z)^2 = \langle \hat{L}_z^2 \rangle - \langle \hat{L}_z \rangle^2$ and $P(\pi) = |\psi(\pi)|^2$ is the probability density for finding the system at the angle π. The uncertainty relation (4) has also been derived in [10] for physical states, that is states with finite moments of angular momentum, by use of the angular momentum representation. In that derivation, it arises as a natural consequence of the rigorous commutation relation between the angular momentum and angle operators.

It should be noted that the second term on the right-hand side of (4) depends on our choice of 2π range or window. In general if we choose the range to be $[\theta_0, \theta_0 + 2\pi)$, we would replace $P(\pi)$ by $P(\theta_0)$, which is equal to $P(\theta_0 + 2\pi)$. Thus an intelligent state for a particular choice of θ_0 need not be an intelligent state for another choice. The same will be true for the minimum uncertainty product states discussed later. This may seem somewhat surprising compared with the linear case, where the property of being an intelligent state is independent of the choice of origin of the coordinate system. The reason, however, can be seen as follows. Let us represent the angle probability distribution as a periodic series of peaks at 2π intervals. Then choosing a 2π window that is centred on one peak will give a smaller variance than choosing a window with half of one peak at one end and half of the next peak at the other.

The angular uncertainty relation (4) is of particular interest because it is an example of a case where the intelligent states, which satisfy the equality, do not necessarily minimize the uncertainty product on the left-hand side. This is because the second term on the right-hand side is itself state-dependent. Thus there may be states that, for a given uncertainty $\Delta\varphi$ or ΔL_z, yield a smaller uncertainty product than do the intelligent states but, of course, obey the inequality in (4). This is possible only because $P(\pi)$ for such states is larger than for the intelligent states. This reduces the right-hand side of the uncertainty relation compared with that for the intelligent states, allowing the uncertainty product on the left-hand side to be smaller than that for the intelligent states while still obeying the uncertainty relation.

The intelligent states $|g\rangle$ satisfy the eigenvalue equation [6, 11]

$$(\hat{L}_z - \mathrm{i}\gamma\hat{\varphi})|g\rangle = \mu|g\rangle, \tag{5}$$

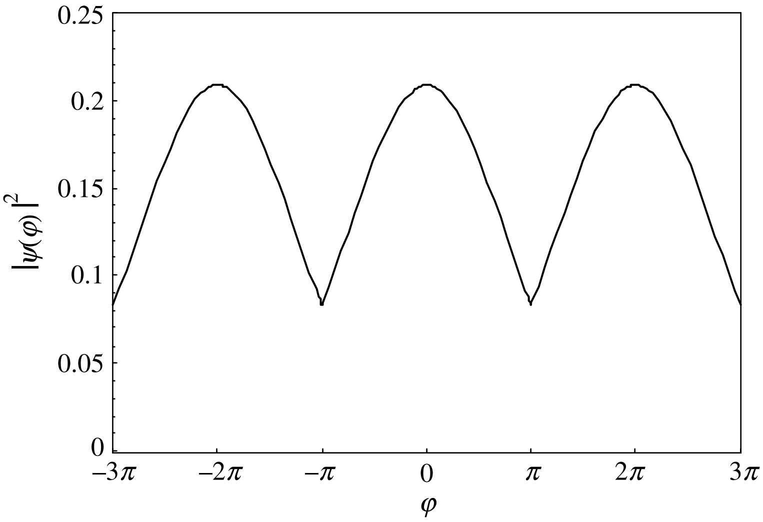

Figure 1. The angular probability distribution for the intelligent state corresponding to an angular uncertainty of $\Delta\varphi = 1.6$. The distribution is Gaussian truncated so as to fit within a chosen 2π range and repeated with a periodicity of 2π.

where γ is real. Without loss of generality, we can restrict ourselves to intelligent states with zero mean angle and angular momentum [3]. Acting on (5) from the left with $\langle g|$ then gives $\mu = 0$, so we have

$$(\hat{L}_z - \mathrm{i}\gamma\hat{\varphi})|g\rangle = 0. \tag{6}$$

In the angle representation $\hat{\varphi} = Y(\varphi)$ so (6) becomes, from (2),

$$\frac{\mathrm{d}\psi(\varphi)}{\mathrm{d}\varphi} = -\gamma Y(\varphi)\psi(\varphi). \tag{7}$$

The solution of this differential equation is

$$\psi(\varphi) = N\exp\left[-\gamma\int Y(\varphi)\,\mathrm{d}\varphi\right] = N\exp\left[-\frac{\gamma}{2}Y^2(\varphi)\right], \tag{8}$$

with N being a normalization constant. Noting that $Y(\varphi + 2n\pi) = Y(\varphi)$, we see that ψ is a Gaussian function between $-\pi$ and π which is duplicated between π and 3π and so on, forming a periodic function with cusps at 2π intervals (see figure 1). It is the same result as was derived by use of the angular momentum representation and obtained experimentally in [3].

We note that the action of the unitary angular momentum shift operator $\exp(\mathrm{i}\hat{\varphi}\,k)$ on the state $|g\rangle$ will not change $|\psi(\varphi)|^2$ and will thus leave $\Delta\varphi$ unaltered. This operation will, however, shift the angular momentum distribution uniformly by an integer amount k [10] but this will not change ΔL_z. Thus the state $\exp(\mathrm{i}\hat{\varphi}k)|g\rangle$ will also be an intelligent state with a non-zero mean angular momentum.

3. Minimum uncertainty product states

In this section, we derive the expression for the states that minimize the uncertainty product $\Delta L_z \Delta\varphi$ either for a given $\Delta\varphi$ or for a given ΔL_z. These are the constrained minimum uncertainty product states. We consider an angular-momentum decomposition

$$|f\rangle = \sum_{m=-\infty}^{\infty} b_m |m\rangle, \tag{9}$$

where $|m\rangle$ is an eigenstate[6] of $\hat{L}_z$ with $m = 0, \pm 1, \pm 2, \ldots$. In appendix A.1 we show that in seeking the CMUP states with a constraint in the angle variance we can assume the b_m to be real and that $\langle \hat{L}_z \rangle = \langle \hat{\varphi} \rangle = 0$. Therefore, the variances simplify to $\langle \hat{L}_z^2 \rangle$ and $\langle \hat{\varphi}^2 \rangle$. Interestingly, as shown in appendix A.2, exactly the same assumptions can be made in minimizing the uncertainty product for a given angular momentum variance. It follows that the states minimizing the uncertainty product will be of the same form whether we fix $\Delta\varphi$ or ΔL_z.

We require the state $|f\rangle$ that minimizes $\langle \hat{L}_z^2 \rangle \langle \hat{\varphi}^2 \rangle$ subject to the normalization constraint $\langle f|f\rangle = 1$. We approach this by the method of undetermined multipliers [8, 12]. The basic equation is the vanishing of a linear combination of the variations $\delta\langle \hat{L}_z^2 \rangle$, $\delta\langle \hat{\varphi}^2 \rangle$ and $\delta\langle f|f\rangle$ with the coefficients being the multipliers. This is the equation we obtain whether we minimize $\langle \hat{L}_z^2 \rangle$ for a fixed $\langle \hat{\varphi}^2 \rangle$ or whether we minimize $\langle \hat{\varphi}^2 \rangle$ for a fixed $\langle \hat{L}_z^2 \rangle$ and the state $|f\rangle$ that satisfies this equation will minimize $\langle \hat{L}_z^2 \rangle \langle \hat{\varphi}^2 \rangle$ for either a given $\langle \hat{\varphi}^2 \rangle$ or a given $\langle \hat{L}_z^2 \rangle$. As the b_m are all real we can introduce undetermined multipliers λ and μ and use the arguments of [12] to obtain an eigenvalue equation for the CMUP state $|f\rangle$

$$(\hat{L}_z^2 + \lambda \hat{\varphi}^2)|f\rangle = \mu |f\rangle. \tag{10}$$

Before looking for an analytical solution, it is instructive to examine some of its general properties. To this end we consider the equation in the angular momentum representation, where we find that we can obtain some good analytic approximations.

3.1. General properties of CMUP states

We start by noticing that the CMUP states have a symmetric probability distribution, $|\psi(\varphi)|^2 = |\psi(-\varphi)|^2$ centred at $\varphi = 0$. This property is discussed further in appendix A.

A second property of the CMUP states can be obtained as follows. In the angular momentum representation, we can express $\hat{\varphi}^2$ when operating on the space of physical states as

$$\hat{\varphi}^2 = \frac{\pi^2}{3} + 2 \sum_{\substack{m,m' \\ m \neq m'}} \frac{(-1)^{m-m'}}{(m-m')^2} |m'\rangle\langle m|. \tag{11}$$

This can be obtained either by using the approach of [10] or from the cosine series for φ^2, as used in [12] for the phase of light, together with the cosine angle operators. Using (9) we find

[6] It is an accident of history that the integer associated with the z-component of angular momentum is denoted by m in quantum theory but l in optics.

that (10) becomes, in the angular momentum representation,

$$(m^2 - \mu)b_m + \lambda\langle m|\hat{\varphi}^2|f\rangle = 0. \tag{12}$$

This gives the equation

$$\left(m^2 - \mu + \frac{\pi^2}{3}\lambda\right) b_m + 2\lambda \sum_{m'\neq m} \frac{(-1)^{m-m'}}{(m-m')^2} b_{m'} = 0. \tag{13}$$

Consider the form of this equation for very large $|m|$. Clearly, the first term will tend to $m^2 b_m$. In addition, in order for $\hat{L}_z^2|f\rangle$ to be normalizable we require that b_m must fall off faster than m^{-2}. If we turn to the summation, we see that this will be dominated by values of $|m'| \ll |m|$. It follows that we can approximate $(m - m')^{-2}$ by m^{-2}. We are then left with an equation for the b_m that is valid for very large $|m|$:

$$m^2 b_m = -\frac{(-1)^m}{m^2} 2\lambda \sum_{m'\neq m} (-1)^{m'} b_{m'}. \tag{14}$$

The summation is independent of m and hence for very large $|m|$

$$b_m = \frac{(-1)^m}{m^4} A, \tag{15}$$

where A is some constant. We note that this is precisely the form of the perturbative solutions that we obtain below in (16) and (22). Expression (15) implies that $\hat{L}_z^4|f\rangle$ will not be normalizable and that the eighth moment of the angular momentum will be infinite for a CMUP state. This is associated with discontinuities in the higher derivatives of $\psi(\varphi)$. To be specific, the amplitudes (15) are associated with a discontinuity at $\varphi = \pi$ of $\frac{\mathrm{d}^3\psi}{\mathrm{d}\varphi^3}$. As discussed in appendix C, these states show the minimum degree of regularity needed for the CMUP states.

3.2. *States with small ΔL*

It is clear that an eigenstate of (10) for $\lambda = 0$ is just an angular momentum state $|m\rangle$ and thus the CMUP state with zero mean angular momentum is just the $m = 0$ state $|0\rangle$. This is an extreme case of a CMUP state with zero angular momentum variance with an associated uncertainty product of zero. This state is also an intelligent state. Taking $|0\rangle$ as our unperturbed state, we can find the eigenstates of (10) for small λ by perturbation theory. The matrix elements of $\hat{\varphi}^2$ between angular momentum states are easily obtained from (11). Substitution into the standard first-order perturbation expression [4] yields the result

$$|f\rangle = |0\rangle - 2\lambda \sum_{m\neq 0} \frac{(-1)^m}{m^4} |m\rangle. \tag{16}$$

These states will have non-zero but small angular momentum variances. The coefficients b_m of the angular momentum representation and the wavefuction $\psi(\varphi)$ of the angle representation form a finite Fourier transform pair [13]

$$\psi(\varphi) = \frac{1}{\sqrt{2\pi}} \sum_m b_m \exp(\mathrm{i}m\varphi), \tag{17}$$

$$b_m = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \psi(\varphi) \exp(-\mathrm{i}m\varphi)\, \mathrm{d}\varphi. \tag{18}$$

From (17) we find the angle wavefunction for the first-order perturbation solution as

$$\psi(\varphi) = \frac{1}{\sqrt{2\pi}} \left[1 + \frac{\lambda}{6} \left(\frac{\varphi^2}{2} - \pi^2\varphi^2 + \frac{7\pi^4}{30} \right) \right]. \tag{19}$$

The values of ΔL_z and $\Delta\varphi$ are easily obtained from (16) and (19). We find to first order in λ

$$\Delta\varphi = \frac{\pi}{\sqrt{3}} \left(1 - \frac{8\pi^4}{315}\lambda \right), \tag{20}$$

$$\Delta L_z \Delta\varphi = \frac{\lambda\pi^4}{\sqrt{3}} \sqrt{\frac{8}{945}} \left(1 - \frac{8\pi^4}{315}\lambda \right). \tag{21}$$

Remarkably, this first-order approximation, depicted by a dotted line in figure 2, is sufficient to show the difference between the CMUP states and the intelligent states and is quite accurate over more than half the range of values of $\Delta\varphi$, i.e. $1 < \Delta\varphi < \pi/\sqrt{3}$.

Second-order perturbation theory can be used to extend the validity of the small-λ approximation. To second order we obtain the state

$$|f\rangle = \left(1 - 4\lambda^2 \frac{\pi^8}{9450} \right) |0\rangle + \sum_{m, m\neq 0} \left[-2\lambda \frac{(-1)^m}{m^4} + 4\lambda^2 \frac{(-1)^m}{m^4} \left(\frac{\pi^4}{45} + \frac{4\pi^2}{3m^2} - \frac{15}{m^4} \right) \right] |m\rangle. \tag{22}$$

We note that for large m, coefficients of $|m\rangle$ decrease as m^{-4}. This state is normalized just to second order in λ and, to be accurate, it needs to be renormalized. The resulting uncertainty product is shown by a dashed line in figure 2.

3.3. States with large ΔL

The opposite approximation to that above is for states with large $\langle L_z^2 \rangle$, with the state of zero angle, for which $\langle L_z^2 \rangle$ is infinite, being the extreme case. For this extreme case, $P(\pi) = 0$ and so, from (4), the uncertainty product is $1/2$. In the angular momentum representation, (10) yields, with (9)

$$\lambda \langle m|\hat{\varphi}^2|f\rangle = (\mu - m^2) b_m, \tag{23}$$

where, from (11),

$$\langle m|\hat{\varphi}^2|f\rangle = \frac{\pi^2}{3} b_m + 2 \sum_{p>0} \frac{(-1)^p}{p^2} (b_{m+p} + b_{m-p}). \tag{24}$$

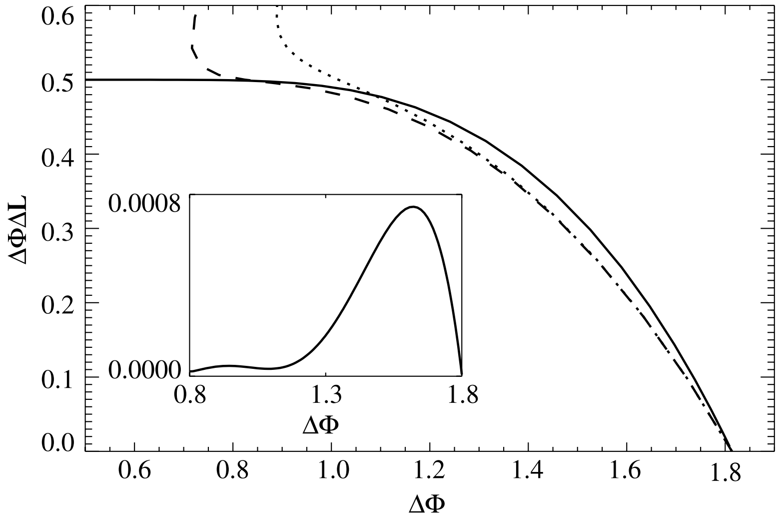

Figure 2. The uncertainty product $\Delta L_z\,\Delta\varphi$ plotted against $\Delta\varphi$ for the first-order perturbative (······), second-order perturbative (- - - -) and intelligent states (——). At this resolution, the uncertainty product for the exact solution is indistinguishable from the second-order perturbative value for $\Delta\varphi > 0.9$ and from 0.5 for smaller values of $\Delta\varphi$. The inset shows the difference in the product of the uncertainties obtained with the sum of Gaussians and with the exact solution (38), that is $(\Delta\varphi\,\Delta L_z)^G - (\Delta\varphi\,\Delta L_z)^E$, plotted against $\Delta\varphi$.

To solve equation (23), we use an ansatz based on the approximation (B.11) from appendix B, that is

$$\langle m|\hat{\varphi}^2|f\rangle \approx -\frac{\mathrm{d}^2 b(m)}{\mathrm{d}m^2}, \tag{25}$$

where $b(m)$ is a continuous envelope curve for which $b(m) = b_m$ when m is an integer. Our procedure is to use (25) to find a solution of (23) for various values of λ and then test this solution for consistency, by checking that (25) is actually satisfied.

Substituting (25) into (23) gives the differential equation for the envelope as

$$\lambda\frac{\mathrm{d}^2 b(m)}{\mathrm{d}m^2} = (m^2 - \mu)b(m). \tag{26}$$

The solution with the minimum variance of m that satisfies the boundary condition that $b(m) \to 0$ as m becomes infinite is the Gaussian

$$b(m) = N_1 \exp\left(-\frac{m^2}{2\sigma^2}\right), \tag{27}$$

where $\sigma^2 = \mu = \lambda^{1/2}$ and N_1 is a constant to be determined by normalization. Thus, for integer m,

$$b_m = N_1 \exp\left(-\frac{m^2}{2\sigma^2}\right) \tag{28}$$

is our solution of (23), subject to consistency of our ansatz.

To test the ansatz, we substitute (28) into the exact expression (24) and obtain

$$\langle m|\hat{\varphi}^2|f\rangle = \left[\frac{\pi^2}{3} + 4\sum_{p>0}\frac{(-1)^p}{p^2}\cosh\left(\frac{pm}{\sigma^2}\right)\exp\left(-\frac{p^2}{2\sigma^2}\right)\right]b_m. \tag{29}$$

A numerical comparison of (29) and

$$-\frac{\mathrm{d}^2 b(m)}{\mathrm{d}m^2} = \left(\frac{1}{\sigma^2} - \frac{m^2}{\sigma^4}\right)b_m \tag{30}$$

shows that the accuracy of our ansatz (25) is excellent, that is a small fraction of 1%, for $\sigma \geqslant 2$ provided $|m| \leqslant 6\sigma$. Even down to $\sigma^2 = 0.5$, the accuracy is still 10%. The contribution of values of $|m| > 6\sigma$ to $\langle m^2\rangle$ will be negligible, so our solution (27) with $\sigma^2 = \lambda^{1/2}$ will give an accurate value for the uncertainty product for sufficiently broad angular momentum distributions. For $\sigma = 1$ we need $|m| \leqslant 4$, for $\sigma = 0.5$ we need $|m| \leqslant 1$. For $\sigma \leqslant 0.4$, no value of m gives consistency.

To find the angle wavefunction from (17), it is simple to write first

$$b_m = \int_{-\infty}^{\infty} b(x)\,\delta(m-x)\,\mathrm{d}x, \tag{31}$$

with $b(x) \propto \exp\left[-x^2/(2\sigma^2)\right]$. The final result is

$$\psi(\varphi) = N_2 \sum_{n=-\infty}^{\infty} \exp\left[-\frac{(\varphi^2+2n\pi)^2}{2/\sigma^2}\right]. \tag{32}$$

This is a sum of Gaussians centred at angles 2π apart. For large σ^2, the values of $\psi(\varphi)$ in the range between $-\pi$ and π reduces to the form of a single Gaussian with $n = 0$. For this case, $P(\pi)$ is effectively zero and the uncertainty product is 1/2.

3.4. Interpolation for all values of ΔL

Our ansatz is quite good for larger values of $\langle \hat{L}_z^2\rangle$, giving the correct value of $\Delta\varphi$ to be well within 1% for values of $\Delta\varphi$ from zero up to about 0.75. We can improve on this accuracy and also interpolate between this value and the values of $\Delta\varphi$ for which perturbation theory is accurate by noting that we can relax the requirement $\sigma^2 = \mu = \lambda^{1/2}$ for the Gaussian representing b_m. We note that as long as

$$\langle m|\hat{\varphi}^2|f\rangle = \alpha m^2 + \xi \tag{33}$$

for all m, where α and ξ are any constants, $|f\rangle$ is a CMUP state. A state for which (33) is approximately true for some value of α and of ξ, for the range of values of m for which b_m are not negligible, will approximate a CMUP state. Thus the Gaussian state (28) will be a good approximation to a CMUP state as long as (33) with $\langle m|\hat{\varphi}^2|f\rangle$ given by (29) is approximately true for some value of α and of ξ for the range of values of m that, for example, make significant contributions to ΔL_z. For large ΔL_z, we have seen that values of α and ξ such that $\sigma^2 = \mu = \lambda^{1/2}$

satisfy (33) for a range of m more than sufficient to ensure that the important values of b_m are quite accurate. As ΔL_z becomes smaller, this range of m decreases but we can find different values of α and ξ that ensure that (33) is reasonably satisfied for the non-negligible values of b_m. In the limit where ΔL_z is very small, only the values of b_m with $m = 0,\ \pm 1$ will contribute significantly and we can always choose α and ξ such that (33) holds exactly for these values of m. The accuracy of representing b_m by a Gaussian for values of ΔL_z where first-order perturbation theory applies can be seen as follows. Expression (16) shows that the contribution to $\langle \hat{L}_z^2 \rangle$ from the $m = \pm 2$ terms is only $1/64$ of that from the $m = \pm 1$ terms, so a mismatch for these terms will have only a very minor effect. By fitting a Gaussian to have the values $b_0,\ b_1$ and b_{-1}, we find from (16) that for such a Gaussian

$$\sigma^2 = -\frac{1}{2\ln(2\lambda)}. \tag{34}$$

From the above discussion, a Gaussian state in the angular momentum representation or, equivalently, a sum of Gaussians in the angle representation should provide a good analytic approximation to a CMUP state across the whole range of values of $\Delta\varphi$. We see below how good such states are in providing an approximation to the uncertainty product.

3.5. Exact CMUP states

We now construct the exact solution of equation (10) by working in the angle representation and making the substitutions

$$x = \sqrt{2}\lambda^{1/4}\varphi, \tag{35}$$

$$a = -\frac{\mu}{2\sqrt{\lambda}}. \tag{36}$$

With $\hat{L}_z^2$ represented by $-\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2}$, (10) becomes, for values of φ between $-\pi$ and π,

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} - \left(\frac{x^2}{4} + a\right)\psi = 0. \tag{37}$$

The solution for the angle wavefunction ψ will be an even function in this range so that we can write it in the form

$$\begin{aligned}\psi &= \exp(-x^2/4)M\left(\frac{1}{2}a + \frac{1}{4}, \frac{1}{2}, \frac{1}{2}x^2\right)\\ &= \exp(-x^2/4)\Bigg[1 + \left(a+\frac{1}{2}\right)\frac{x^2}{2!} + \left(a+\frac{1}{2}\right)\left(a+\frac{5}{2}\right)\frac{x^4}{4!}\\ &\quad + \left(a+\frac{1}{2}\right)\left(a+\frac{5}{2}\right)\left(a+\frac{9}{2}\right)\frac{x^6}{6!} + \cdots\Bigg],\end{aligned} \tag{38}$$

where M is a confluent hypergeometric function [14]. The evenness ensures that $\psi(-\pi) = \psi(\pi)$, which is required from periodicity and the need for continuity of the function. We can also apply

Table 1. Uncertainty product for four values of the angle variance. The superscripts E, G, I and II refer to the exact solution, the sum of Gaussians, first-order and second-order perturbative solutions, respectively.

$\Delta\varphi$	0.8130954	1.317335	1.633695	1. 809022
$(\Delta\varphi\Delta L_z)^E$	0.498188	0.389542	0.181522	0.0054894678
$(\Delta\varphi\Delta L_z)^G$	0.498210	0.389739	0.182268	0.0055360491
$(\Delta\varphi\Delta L_z)^{\mathrm{I}}$	–	0.392661	0.181683	0.0054894713
$(\Delta\varphi\Delta L_z)^{\mathrm{II}}$	–	0.389559	0.181522	0.0054894678

the boundary condition that the first derivative is continuous at $\varphi = \pi$ (see appendix C). This, together with the fact that the wavefunction is periodic, requires that the first derivative should vanish here. The solution giving the smallest $\Delta\varphi$ will be that whose first minimum occurs when $\varphi = \pi$. The corresponding values of a will be between 0 and -0.5. (When $a = -0.5$, ψ becomes a Gaussian, that is, its first minimum is at infinity; when $a = 0$, the first minimum is at $x = 0$.)

The requirement to have the first minimum of ψ at $\varphi = \pi$ leads to

$$\lambda = \frac{x_0^4}{4\pi^4} \tag{39}$$

and

$$\varphi = \frac{\pi}{x_0} x, \tag{40}$$

so

$$\langle\hat{\varphi}^2\rangle = \frac{\pi^2}{x_0^2}\langle\hat{x}^2\rangle \tag{41}$$

for values of x between $-x_0$ and x_0, where x_0 is the position of the first minimum of $\psi(x)$. The values of ψ can be normalized numerically in this range and the value of $\langle\hat{x}^2\rangle$ and hence $\langle\hat{\varphi}^2\rangle$ computed. The uncertainty product can then be obtained from (10), giving

$$\begin{aligned}\langle\hat{\varphi}^2\rangle\langle\hat{L}_z^2\rangle &= \langle\hat{\varphi}^2\rangle(\mu - \lambda\langle\hat{\varphi}^2\rangle)\\ &= \langle\hat{x}^2\rangle\left(-a - \tfrac{1}{4}\langle\hat{x}^2\rangle\right),\end{aligned} \tag{42}$$

where we have used (41) and (36), and the uncertainty product is the square root of (42). Varying the choice of a allows the calculation of uncertainty products corresponding to a range of $\Delta\varphi$ values.

In table 1 we compare, for a range of $\Delta\varphi$, values of uncertainty products calculated from the exact solution and from the use of sums of Gaussians as CMUP states. We also include the uncertainty products calculated from perturbation theory. For large angle variances, in the last column of table 1, we note the very high precision of the perturbative approximation in relation to the exact solution. As expected, these solutions do not minimize the uncertainty product accurately for small angle variances. On the other hand, the sum of Gaussians is a good approximation to the exact solution for a wide range of angle variances, as shown in figure 2. The maximum deviation between the sum of Gaussians and the exact solution uncertainty product is less than 8×10^{-4}.

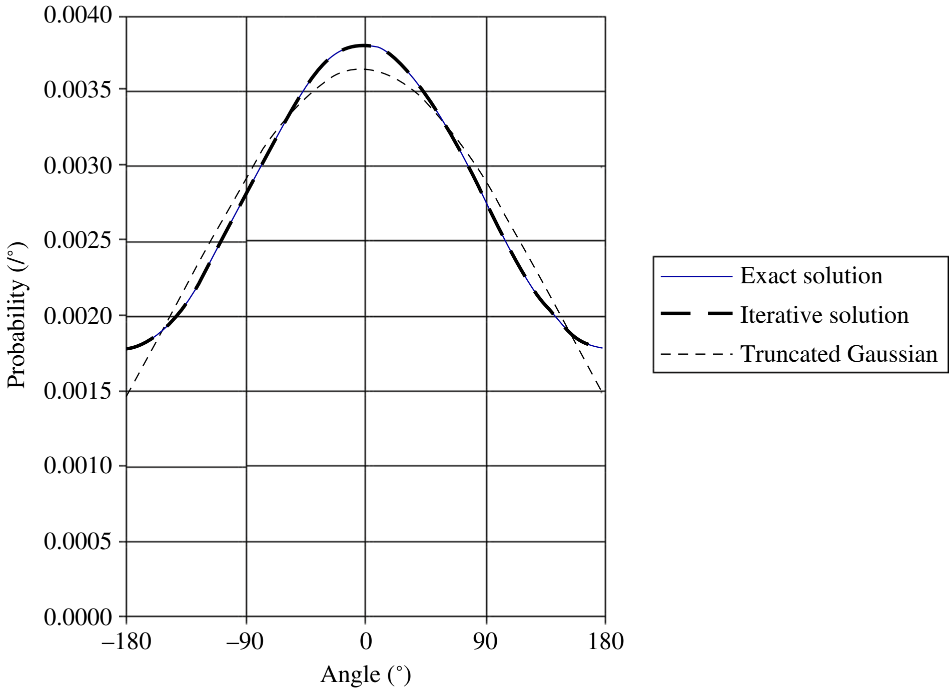

Figure 3. The various forms of the angular wavefunction corresponding to $\Delta\varphi = 1.60186058$. The exact solution (38) and iterative solution agree to 11 significant figures and are clearly distinct from the intelligent state (8).

3.6. Numerical iterative search for CMUP states

To confirm our identification of the correct form of the minimum uncertainty product state, we performed an iterative numerical search for the angle wavefunction that gives the smallest uncertainty product. The overall approach is to set a specific value of $\Delta\varphi$ and then search to find the function giving the smallest possible value of ΔL_z. To reduce the search time required to find the optimum angle wavefunction, we assumed that this function is symmetric and that it increases monotonically from the edge to the centre. The iterative algorithm is seeded with a triangular-shaped function. Upon each iteration, the gradient at a random position within the function is itself randomized and the baseline of the modified function offset to obtain the required standard deviation $\Delta\varphi$. The resulting function is then Fourier-transformed to give the distribution in angular momentum and the corresponding standard deviation ΔL_z from which the uncertainty product is calculated. If the uncertainty product is reduced then the most recent change to the gradient is kept, if not it is discarded. Even using a personal computer, 10 such iterations can be trialled each second with a period of several hours being sufficient to obtain an optimized function. From the inset in figure 2, we see that the largest difference predicted between the uncertainty product corresponding to an overlapping Gaussian wavefunction and the minimum uncertainty product state occurs near a value of $\Delta\varphi \approx 1.6$. The exact solution, equation (38), predicts that the CMUP state with a standard deviation $\Delta\varphi = 1.60186058$ gives an uncertainty product of $\Delta\varphi\Delta L_z = 0.208848427271$. We take this value of $\Delta\varphi$ as the target for the iterative search. Our algorithm obtains an angle probability distribution which agrees with that obtained from the exact solution to 11 significant figures, as shown in figure 3. The angular momentum

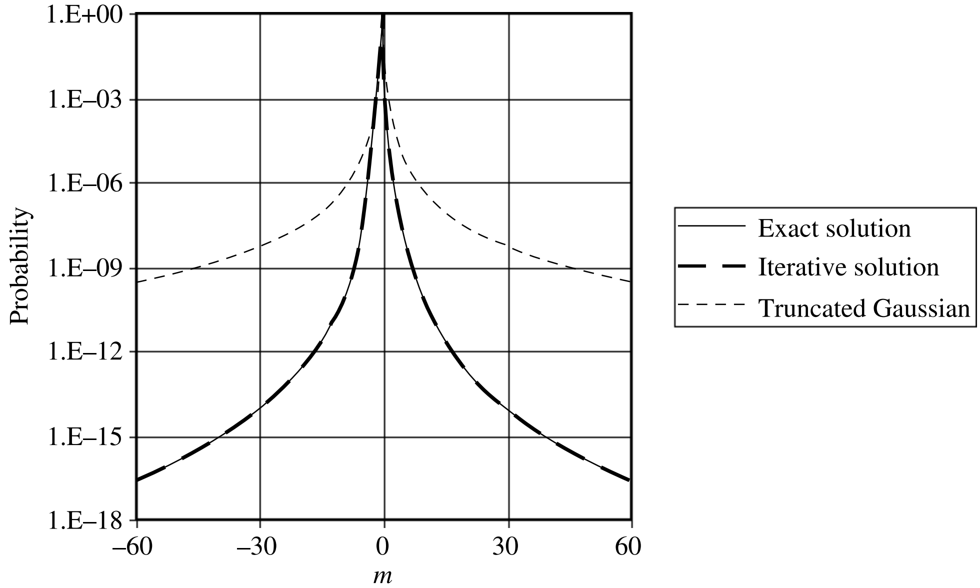

Figure 4. The probability distribution in angular momentum corresponding to the wavefunctions in figure 3.

distributions, as given by the finite Fourier transforms of the iterative and exact solutions, are again numerically indistinguishable over the range $|m| \leqslant 60$, corresponding to a range of probabilities in excess of 16 orders of magnitude. Perhaps the most convincing evidence that the iterative form matches that of the exact solution is that, over the same numerical range, both probability distributions fall with a power dependence of m^{-8}. This is precisely in accord with our asymptotic result (15). The angular momentum probability distributions for our iterative and exact solutions, together with that for the intelligent state, are presented in figure 4.

4. Conclusion

In this paper we have examined the states that minimize the uncertainty product of angular position and angular momentum either for a given variance in angle or for a given variance in angular momentum. We have established that both constraints result in the same CMUP states and that they differ from the intelligent states, that is the states satisfying the equality in the uncertainty relation $\Delta L_z \Delta\varphi \geqslant \frac{1}{2}|1 - 2\pi P(\pi)|$. The constrained minimum uncertainty product (CMUP) states yield a smaller uncertainty product than the intelligent states because they have a larger probability density $P(\pi)$ at the edge of the 2π range than the intelligent states. This allows the uncertainty product to be less than that for the intelligent states while still exceeding $\frac{1}{2}|1 - 2\pi P(\pi)|$. The exact solution for the CMUP states was checked to a very high precision by an iterative minimization algorithm. We also have found that analytic perturbation approximations are useful over a significant part of the range of possible variances $\Delta\varphi$, including the region of the largest deviation from the corresponding intelligent state expression. An analytic approximation that is quite accurate over the whole range of values of $\Delta\varphi$ is a state with a Gaussian distribution in the angular momentum representation. In the angle representation, this corresponds to a wavefunction that is a periodic sum of Gaussians. This behaviour contrasts strongly with the situation for linear momentum and position, for which the intelligent states and CMUP states are identical and have Gaussian distributions in both position and momentum.

In the experimental verification [3] of the uncertainty relation for angular momentum and angular position, the most accurate measurements of the uncertainty product were in the range of $\Delta\varphi$ values between 1.0 and 1.5. It is fortunate that the largest discrepancy between the uncertainty products for the intelligent and the minimum product states occurs around $\Delta\varphi \approx 1.4$, as can be seen from figure 2. This means that the difference between intelligent states and minimum uncertainty product states for angular position and angular momentum should be within the reach of experimental test.

Acknowledgments

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Appendix A. Demonstration of the adequacy of considering real b_m

In this appendix, we show that the minimum uncertainty product $\Delta L_z \Delta\varphi$ can be obtained by considering real angular momentum amplitudes b_m, both in the case of a given angle variance and of a given angular momentum variance.

A.1. Minimizing the uncertainty product for a given $\Delta\varphi$

We begin by writing our angle wavefunction in the form

$$\psi(\varphi) = \chi(\varphi)\mathrm{e}^{\mathrm{i}\alpha(\varphi)}, \tag{A.1}$$

where $\chi(\varphi)$ is a real function. It is straightforward to show, using the continuity of $\psi(\varphi)$ and of its derivatives, that

$$\langle \hat{L}_z \rangle = \int_{-\pi}^{\pi} \frac{\mathrm{d}\alpha}{\mathrm{d}\varphi}\chi^2 \mathrm{d}\varphi, \tag{A.2}$$

$$\Delta L_z^2 = -\int_{-\pi}^{\pi} \chi \frac{\mathrm{d}^2\chi}{\mathrm{d}\varphi^2}\,\mathrm{d}\varphi + \left\langle \left(\frac{\mathrm{d}\alpha}{\mathrm{d}\varphi} - \left\langle \frac{\mathrm{d}\alpha}{\mathrm{d}\varphi} \right\rangle \right)^2 \right\rangle. \tag{A.3}$$

Clearly, the variance in the angular momentum is minimized by choosing $\mathrm{d}\alpha/\mathrm{d}\varphi$ to be a constant so that the second term in (A.3) is zero. The requirement that the wavefunction should be continuous tells us that $\mathrm{d}\alpha/\mathrm{d}\varphi$ must be an integer and (A.2) leads us to identify it as the mean angular momentum. Without loss of generality, but for the sake of definiteness, we choose this mean value to be zero so that α is simply a constant. We have now established that for any given angle probability distribution, and hence for any given angular uncertainty, the corresponding minimum angular momentum variance will be obtained for a wavefunction of the form (A.1) with constant α and we have chosen solutions with $\langle \hat{L}_z \rangle = 0$. We can readily obtain solutions with non-zero mean angular momentum by multiplying our wavefunction by $\exp(\mathrm{i}\bar{m}\varphi)$, where $\bar{m} = \langle \hat{L}_z \rangle$.

We now turn our attention to the expansion (17) which we rewrite as

$$\chi(\varphi) = \frac{1}{\sqrt{2\pi}} \sum_m b_m \mathrm{e}^{-\mathrm{i}\alpha} \exp(\mathrm{i}m\varphi). \tag{A.4}$$

The fact that $\chi(\varphi)$ must be real requires that

$$b_{-m}\mathrm{e}^{-\mathrm{i}\alpha} = b_m^* \mathrm{e}^{\mathrm{i}\alpha}. \tag{A.5}$$

A second relationship between our angular momentum amplitudes follows from the fact that we are seeking the minimum of $\langle \hat{L}_z^2 \rangle$, which is insensitive to the sign of the angular momentum m. It follows that if (A.4) is a state that minimizes this variance, then so too will be the state with each b_m replaced by b_{-m}:

$$\tilde{\chi}(\varphi) = \frac{1}{\sqrt{2\pi}} \sum_m b_{-m} \mathrm{e}^{-\mathrm{i}\alpha} \exp(\mathrm{i}m\varphi). \tag{A.6}$$

If both (A.4) and (A.6) are satisfactory states then, by the linearity of quantum mechanics, so is any superposition of them. It suffices then to consider only symmetric and antisymmetric combinations of these two states, that is $\chi \pm \tilde{\chi}$. These correspond to (A.4) with $b_{-m} = b_m$ for the symmetric case and $b_{-m} = -b_m$ for the antisymmetric case. If we choose the arbitrary phase α in (A.4) to be zero in the symmetric case and $\pi/2$ in the antisymmetric case then, using (A.5), we find in both cases that the b_m are real. It is straightforward to show that both of these states have symmetric angular probability distributions and hence that $\langle \hat{\varphi} \rangle = 0$.

In summary, this short analysis has demonstrated that in seeking to minimize the angular momentum uncertainty for a given angular uncertainty, it suffices to consider states with real angular momentum amplitudes b_m and with a mean angle of zero.

A.2. Minimizing the uncertainty product for given ΔL_z

We start by noting that ΔL_z is independent of a shift of the angular coordinate so $\psi(\varphi)$ and $\psi(\varphi - \Phi)$ will have the same angular momentum variance. The associated angular variance is

$$\Delta\varphi^2 = \int_{-\pi}^{\pi} |\psi(\varphi - \Phi)|^2 \varphi^2 \,\mathrm{d}\varphi - \left(\int_{-\pi}^{\pi} |\psi(\varphi - \Phi)|^2 \varphi \,\mathrm{d}\varphi \right)^2. \tag{A.7}$$

Differentiation with respect to Φ shows that this variance will be minimized by choosing Φ such that $\langle \hat{\varphi} \rangle = 0$. Hence, minimizing $\Delta\varphi^2$ corresponds to minimizing $\langle \hat{\varphi}^2 \rangle$. For the same reasons as given in [12] for minimizing the optical phase variance, we can minimize $\langle \hat{\varphi}^2 \rangle$ by choosing all the b_m to be real and positive or zero. It is interesting to note that we can also obtain the maximum $\Delta\varphi^2$ by setting $\langle \hat{\varphi} \rangle = 0$. It transpires that for an angle distribution peaked at $\varphi = 0$, we find a minimum of $\Delta\varphi^2$ while for an angle distribution peaked at $\varphi = \pm\pi$ we find a maximum.

Appendix B. Derivation of equation (25)

In the angular momentum representation, the angle operator when acting on physical states can be written as [10]

$$\hat{\varphi} = -\mathrm{i} \sum_{\substack{m,\, m' \\ m \neq m'}} \frac{(-1)^{m'-m}}{m'-m} |m\rangle\langle m'| \tag{B.1}$$

with eigenvalues between $-\pi$ and π.

For the state $|f\rangle = \sum_m b_m |m\rangle$, we have

$$\langle m|\mathrm{i}\hat{\varphi}|f\rangle = \sum_{p\neq 0} (-1)^p \frac{b_{m+p} - b_{m-p}}{2p}. \tag{B.2}$$

Consider a function

$$F_m(x) = \frac{b(m+x) - b(m-x)}{2x}, \tag{B.3}$$

where m and x are continuous variables and $b(m) = b_m$ when m is an integer.

Writing

$$\int_{-\infty}^{\infty} F_m(x)\,\mathrm{d}x = 2 \sum_{p\,\mathrm{odd}} \frac{b(m+p) - b(m-p)}{2p} + \varepsilon_1 \tag{B.4}$$

$$= 2 \sum_{p\,\mathrm{even}} \frac{b(m+p) - b(m-p)}{2p} + \varepsilon_2. \tag{B.5}$$

(B.5) can also be written as

$$\int_{-\infty}^{\infty} F_m(x)\,\mathrm{d}x = 2F_m(0) + 2 \sum_{\substack{p\,\mathrm{even} \\ p\neq 0}} \frac{b(m+p) - b(m-p)}{2p} + \varepsilon_2. \tag{B.6}$$

Hence, from (B.4) and (B.6) we have

$$\sum_{p\,\mathrm{odd}} \frac{b(m+p) - b(m-p)}{2p} - \sum_{\substack{p\,\mathrm{even} \\ p\neq 0}} \frac{b(m+p) - b(m-p)}{2p} = F_m(0) + (\varepsilon_2 - \varepsilon_1)/2 \tag{B.7}$$

so that

$$\langle m|\mathrm{i}\hat{\varphi}|f\rangle = -F_m(0) - (\varepsilon_2 - \varepsilon_1)/2. \tag{B.8}$$

We find $F_m(0)$ from

$$F_m(0) = \lim_{x\to 0} F_m(x) = \frac{\mathrm{d}b(m)}{\mathrm{d}m}, \tag{B.9}$$

which follows from (B.3). Thus

$$\langle m|\mathrm{i}\hat{\varphi}|f\rangle \approx -\frac{\mathrm{d}b(m)}{\mathrm{d}m} \tag{B.10}$$

provided $|F_m(0)| \gg |(\varepsilon_2 - \varepsilon_1)|/2$. The validity of this condition depends on how broad and smooth the function is and also on the value of m.

A parallel derivation starting with expression (11) for $\hat{\varphi}^2$ in place of (B.1) yields

$$\langle m|\hat{\varphi}^2|f\rangle \approx -\frac{\mathrm{d}^2 b(m)}{\mathrm{d}m^2}. \tag{B.11}$$

Appendix C. Angle representation of the angular momentum

The replacement (with units in which $\hbar = 1$)

$$\hat{L}_z \to -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\varphi} \tag{C.1}$$

can sometimes lead to difficulties. In particular, problems will arise when the action of $\hat{L}_z$ on $|\psi\rangle$ does not lead to a normalizable state.

It is helpful to begin by establishing some asymptotic properties of periodic functions of the angle φ. Consider a function ψ defined by the Fourier series

$$\psi(\varphi) = (2\pi)^{-1/2} \sum_{m=-\infty}^{\infty} c_m \mathrm{e}^{\mathrm{i}m\varphi}. \tag{C.2}$$

We do not demand at present that this should necessarily be a wavefunction. We are interested in the asymptotic properties of the c_m, that is its form as $|m| \to \infty$.

C.1. Divergent functions

Consider first a function, ψ, which contains a delta function at some given angle. The Fourier components of the non-delta-function part will decay away as $|m| \to \infty$, but the delta-function part has constant components. Hence for functions containing a delta function we have

$$c_m \to \alpha \mathrm{e}^{-\mathrm{i}m\varphi_0} \quad \text{as } |m| \to \infty, \tag{C.3}$$

where α is a complex constant and φ_0 is the position of the delta function. Functions containing such delta function components cannot, of course, represent wave functions as they are not square-integrable.

C.2. Discontinuous functions

Consider next a function that contains a discontinuity at some given angle. Such a function might be associated with transmission through a mask having a sharp edge. Such a discontinuity can be described in terms of a Heaviside step function. The derivative, with respect to φ of such a step function is a delta function and hence the Heaviside function is the integral of a delta function. Hence, we can write our discontinuous function as

$$\chi(\varphi) = \int \psi(\varphi)\,\mathrm{d}\varphi, \tag{C.4}$$

where ψ is a function containing a delta function. It follows that for functions containing a discontinuity

$$c_m \to \frac{\beta}{m}\mathrm{e}^{-\mathrm{i}m\varphi_0} \quad \text{as } |m| \to \infty, \tag{C.5}$$

where β is a complex constant and φ_0 is the position of the discontinuity.

Functions containing such discontinuities will represent square-integrable wavefunctions but the associated mean square angular momentum will be divergent:

$$\langle \hat{L}_z^2 \rangle = \infty. \tag{C.6}$$

It is clear that the wavefunctions we seek must contain neither delta functions nor discontinuities.

C.3. Functions having discontinuities of gradient

Consider, finally, a function (such as the truncated Gaussian) which contains a discontinuity of gradient. The derivative, with respect to φ of such a discontinuous gradient is a Heaviside function and it follows that a discontinuity of gradient is the integral of a Heaviside function. It then follows, from the reasoning given above, that for functions containing a discontinuity of gradient

$$c_m \to \frac{\gamma}{m^2}\mathrm{e}^{-\mathrm{i}m\varphi_0} \quad \text{as } |m| \to \infty, \tag{C.7}$$

where γ is a complex constant and φ_0 is the position of the discontinuity of gradient. Such functions will represent square-integrable wavefunctions with finite mean-squared angular momentum.

The argument presented above can be readily extended to higher-order inverse powers of m. This leads us to identify the m^{-3} dependence with discontinuity in the second derivative and the m^{-4} dependence with discontinuity in the third derivative of the wavefunction. The latter occurs for the exact CMUP states (38).

We are now in a position to analyse the success and failing of the differential representation of the angular momentum operator (C.1). In seeking the minimum uncertainty state, we required the solution of the eigenvalue equation:

$$(\hat{L}_z - \bar{l} + \mathrm{i}\lambda\hat{\varphi})|\psi\rangle = 0. \tag{C.8}$$

In order to find this, we used the replacement (C.1) to give a differential equation for $\psi(\varphi)$. This procedure can only work if the solution that is obtained has $-\mathrm{i}\frac{\mathrm{d}\psi}{\mathrm{d}\varphi}$ as a valid wavefunction so that $\hat{L}_z|\psi\rangle$ can be correctly represented by $\psi(\varphi)$ in (C.8). The truncated Gaussian solution has a discontinuity of gradient so that its derivative has a discontinuity. Such a state has a divergent mean-squared momentum, but it is square integrable and so can provide a mathematically sensible probability distribution and hence an acceptable wavefunction. In showing that the solution of the differential equation obtained from (C.8) has nothing worse than a discontinuity of gradient, we are verifying that it is an acceptable solution.

Finding the minimum product state led us to seek the solution of the operator equation

$$(\hat{L}_z^2 + \lambda\hat{\varphi}^2)|\psi\rangle = \mu|\psi\rangle. \tag{C.9}$$

Applying the differential representation (C.1) leads to a number of possible solutions including the truncated Gaussian solution. This solution, however, has discontinuities of gradient, but for this problem, the solution will be acceptable only if $-\mathrm{d}^2\psi/\mathrm{d}\varphi^2$ is a valid wavefunction so that $\hat{L}_z^2|\psi\rangle$ can be correctly represented by $\psi(\varphi)$ in (C.9). The second derivative of a function containing a discontinuity in gradient has a delta-function and so will not be square integrable and cannot be a valid wavefunction. It follows that we need a solution that has a continuous gradient everywhere and this, together with the periodicity of the wavefunction requires that $\mathrm{d}\psi/\mathrm{d}\varphi = 0$ at $\varphi = \pm\pi$. As seen in section 3.1, CMUP states have, not only a continuous gradient, but also a continuous second-order derivative. The latter follows from the evenness and periodicity of these states. Irregularities appear only in the third- and higher-order derivatives.

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