Krähmer, U. and Zieliński, B. (2006) *On piecewise trivial Hopf—Galois extensions.* Czechoslovak Journal of Physics, 56 (10-11). pp. 1221-1226. ISSN 0011-4626

ON PIECEWISE TRIVIAL HOPF-GALOIS EXTENSIONS

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Abstract. We discuss a noncommutative generalisation of compact principal bundles that can be trivialised relative to the finite covering by closed sets. In this setting we present bundle reconstruction and reduction.

1. Definitions

This note is an announcement of some results obtained in collaboration with P.M. Hajac and R. Matthes. They are concerned with a noncommutative generalisation of compact principal bundles that can be trivialised relative to the finite covering by closed sets. More details and proofs will be published in [11].

1.1. **Quantum principal bundles.** The bundles themselves become replaced by extensions of noncommutative algebras $B \subset P$ governed by (co)actions of Hopf algebras on P:

Definition 1. *Let H be a Hopf algebra. An H-Galois extension is an algebra extension $B \subset P$, where P is an H-comodule algebra with coaction $\rho : P \to P \otimes H$, $B = P^{H\text{-inv}} := \{a \in P \mid \rho(a) = a \otimes 1\}$ is the subalgebra of H-invariants, and the canonical map*

$$\text{can} : P \otimes_B P \to P \otimes H, \quad a \otimes a' \mapsto (a \otimes 1)\rho(a') \tag{1}$$

is bijective, where the unadorned $\otimes$ is the tensor product over the ground field k.

From the geometric point of view P and B play the roles of the total and the base space of the bundle, respectively, and H replaces the structure group. The Galois condition (1) corresponds classically to the freeness of the action of the structure group on the total space. For more explanation see e.g. [3, 13].

When developing the theory of these objects, the additional technical conditions of faithful flatness of P as B-module and bijectivity of the antipode in H turns out to be necessary in order to avoid certain pathologies [13]. A Hopf-Galois extension with this property will be referred to as *a principal H-extension* (a quantum principal bundle). A morphism of such extensions with fixed B and H is an H-colinear and left B-linear algebra morphism of the total spaces.

1.2. **Coverings of quantum spaces.** Local triviality of fibre bundles is a crucial ingredient throughout their theory and for example allows one to develop gauge theory in local coordinates. However, the concept of locality based upon open neighborhoods does not directly generalise to noncommutative geometry. On the other hand, closed subsets are naturally generalised by ideals (formed classically by the functions vanishing on the set), so several authors suggested to start instead with closed coverings [4, 5, 6]. The precise definition we will work with is the following:

Definition 2. *Let Ω be a finite set and $\{I_i\}_{i\in\Omega}$ be a family of ideals of an algebra B. Let $\pi_i : B \to B_i := B/I_i$, $\pi^i_j : B_i \to B_{ij} := B/(I_i + I_j)$ be the quotient maps, and define the homomorphism*

$$\pi : B \to B_c := \left\{ (b_i)_{i\in\Omega} \in \prod_{i\in\Omega} B_i \mid \pi^i_j(b_i) = \pi^j_i(b_j) \right\}, \quad b \longmapsto (\pi_i(b))_{i\in\Omega}. \tag{2}$$

Then the $\{I_i\}_{i\in\Omega}$ are said to form a weak covering *if π is injective, that is, if $\bigcap_{i\in\Omega} I_i = \{0\}$. If π is an isomorphism, then the weak covering is called a* covering *or is said to be* complete.

Thus if B is, say, the commutative C*-algebra $C(M)$ of continous functions on a compact Hausdorff space M, then coverings of B by closed ideals correspond bijectively to coverings of M by closed subsets.

An integral domain does not possess any finite weak covering which corresponds geometrically to the fact that an infinite irreducible topological space admits by very definition no finite closed covering. Similarly, a C*-algebra has no finite covering by closed ideals if it is primitive, that is, admits a faithful irreducible representation, since this implies that it is prime, that is, every two nonzero closed ideals have nontrivial intersection. (In fact Dixmier proved also the converse for separable algebras). Concerning the completeness property we remark that weak coverings consisting of closed ideals in a C*-algebra or only of two ideals are always coverings (the first statement follows since $I \cap J = IJ$ for closed ideals in a C*-algebra). See [5] for an example of a covering that is not complete.

1.3. **Glueing quantum spaces.** The operation of gluing of algebras is defined as a fibre product: Given families $\{B_i\}_{i\in\Omega}$ and $\{B_{ij}\}_{i,j\in\Omega}$ of algebras and algebra epimorphisms $\pi^i_j : B_i \to B_{ij}$ such that $B_{ii} = B_i$, $\pi^i_i = \mathrm{id}_{B_i}$ and $B_{ij} = B_{ji}$ one can define an algebra B_c as in eq. (2). This is then called the gluing of the B_i's along the π^i_j's, and the kernels I_i of the canonical maps $\pi_i : B_c \to B_i$ define a covering of B_c [5].

Note that here the maps π_i are not necessarily surjective. That is, the data B_i, B_{ij}, π^i_j which define the gluing B_c do *not* in general coincide with the data of the resulting covering (i.e. we usually have $B_i \not\simeq B_c/I_i$, etc.). However, here is a sufficient criterion for surjectivity of the π_i [5]:

Proposition 1. *Given gluing data B_i, B_{ij}, π^i_j as above, the morphisms $\pi_i : B_c \to B_i$ are epimorphisms, provided that the following conditions are satisfied:*

1. $\pi^i_j(\ker \pi^i_k) = \pi^j_i(\ker \pi^j_k)$ *for all $i, j, k \in \Omega$.*
2. *The isomorphisms $\theta^{ij}_k : B_i/(\ker \pi^i_j + \ker \pi^i_k) \to B_{ij}/\pi^i_j(\ker \pi^i_k)$, $[b_i] \mapsto [\pi^i_j(b_i)]$ satisfy $(\theta^{ij}_k)^{-1} \circ \theta^{ji}_k = (\theta^{ik}_j)^{-1} \circ \theta^{ki}_j \circ (\theta^{kj}_i)^{-1} \circ \theta^{jk}_i$.*
3. *For all $i \in \Omega$, $\beta \subset \Omega\backslash\{i\}$ and $k \in \Omega\backslash\beta$, $k \neq i$, we have $\bigcap_{j\in\beta}(\ker \pi^i_j + \ker \pi^i_k) = (\bigcap_{j\in\beta} \ker \pi^i_j) + \ker \pi^i_k$.*

In fact conditions 1. and 2. are necessary, and 3. is automatic in the setting of C*-algebras. Thus the conditions are optimal in the applications we have in mind.

1.4. **A sheaf-theoretic picture.** In classical topology the idea of covering and gluing is intrinsically connected with the concept of sheaves. Although our coverings are analogues of closed coverings, the above can indeed be viewed alternatively as follows: Define

$$X := \Omega \times \Omega, \quad U_i := \{(i,j),(j,i) \mid j \in \Omega\} \subset X, \quad i \in \Omega \tag{3}$$

and consider X with the topology generated by the U_i. That is, the open sets are the unions of finite intersections of the U_i. We have $U_i \cap U_j = \{(i,j),(j,i)\}$, and these are pairwise disjoint. The data of a gluing as described above are now encoded in the sheaf F of rings on X with $F(U_i) := B_i, F(U_i \cap U_j) := B_{ij}$ (the sheaf property determines $F(U)$ for any other open set). In particular, $F(X) = B_c$. This sheaf is of combinatorial rather than of geometric nature, but it can be used to streamline some formulations. For example, the problem solved by Proposition 1 can be restated as the question whether F is a flasque (flabby) sheaf.

In addition, these considerations show that the approach of [4, 5, 6] is compatible with the one of [12]. Therein, quantum spaces are introduced as topological spaces equipped with a sheaf of noncommutative rings, and the above simply tells that an algebra equipped with a covering defines a quantum space in this sense.

1.5. **Piecewise triviality.** Tensor products $P = B \otimes H$ are obvious candidates for the notion of trivial quantum principal bundles. But one can also allow for more noncommutativity and work with crossed products $B \rtimes H$ whenever B can be equipped with the structure of an H-module algebra. This is needed for example to cover examples like the Heegaard-type quantum 3-spheres [1, 9].

Furthermore, if $\pi : X \to M$ is a fibre bundle say with M, X compact Hausdorff, and $I \subset C(M)$ is the ideal of functions vanishing on a closed subset $A \subset M$, then $\pi^{-1}(A) \subset X$ corresponds to the ideal generated by I in the algebra $C(X)$.

Hence the following definition extends the classical notion of triviality over a closed covering to principal extensions. To point out that we are not working with open coverings, we speak of piecewise rather than of local triviality.

Definition 3. *A principal H-extension $B \subset P$ is* piecewise trivial *if there exists a cover $\{I_i\}_{i\in\Omega}$ of B, an H-module algebra structure on each $B_i = B/I_i$, and isomorphisms of principal H-extensions $\chi_i : P_i \to B_i \rtimes H$, where $P_i := P/J_i$ and $J_i := PI_iP$ is the ideal generated in P by I_i.*

It is not difficult to prove that a classical locally trivial principal fibre bundle over a compact Hausdorff space is piecewise trivial. However, the converse is not true as a counterexample due to P.F. Baum shows (see [2]).

In the noncommutative geometry piecewise triviality is an equivariant version of fibre products. Principal extensions obtained by gluing trivial extensions such as the Heegaard-type quantum 3-spheres [1, 8, 9] provide nice examples that will be studied in detail in [11]. On the other hand, B might not admit a covering at all which happens if it is an integral domain, or a simple algebra or a primitive C*-algebra. In particular this applies to the standard quantum Hopf fibration based on the compact quantum group $SU_q(2)$ viewed as a principal extension over the standard Podleś quantum sphere S^2_q.

We point out that Definition 3 is a slight refinement of the one given e.g. in [6] where it was part of the assumptions that the covering of the total space algebra is indeed a covering. However, this somehow unnatural requirement can in fact be deduced in some cases and in particular when B is a C*-algebra:

Proposition 2. *Let $B \subset P$ be a piecewise trivial principal H-extension over a* C**-algebra B with $\{I_i\}_{i\in\Omega}$ as in Definition 3. Then $J_i = I_iP$, and $\{J_i\}_{i\in\Omega}$ is a covering of P. In particular, Definition 3 is then equivalent to the one used in* [6].

We remarked that M. Pflaum defines in [12] quantum spaces as topological spaces equipped with a sheaf of rings, and we pointed out that this approach to quantum spaces is compatible with the one through coverings of algebras. Consequently, Pflaum introduces quantum principal bundles not globally as algebra extensions, but as sheaves over the same topological space. Proposition 2 now provides compatibility also of the notions of quantum principal bundles:

Corollary 1. *A piecewise trivial principal H-extension over a* C^**-algebra is a quantum principal bundle in the sense of* [12].

2. Applications

In this section we shortly describe two quantum analogues of classical geometric constructions whose generalisation involves the concept of piecewise triviality.

2.1. **Transition functions and bundle reconstruction.** As discussed in length in [4, 6, 12, 14], the machinery of transition functions can be developed more or less straightforwardly in the setting of quantum principal bundles: If $B \subset P$ is a piecewise trivial principal H-extension with respect to the covering $\{I_i\}_{i\in\Omega}$ of B, then one has a family of linear maps

$$\tau_{ij} : H \to B_{ij}, \quad i,j \in \Omega, \tag{4}$$

that are classically dual to the transition functions of the bundle and satisfy

$$\begin{gathered}(h_{(1)} \triangleright \pi^i_j(b_i))\tau_{ij}(h_{(2)}) = \tau_{ij}(h_{(1)})(h_{(2)} \triangleright \pi^i_j(b_i)),\\ \tau_{ij}(1) = 1, \quad \tau_{ji}(h) = h_{(1)} \triangleright \tau_{ij}(Sh_{(2)})), \quad \tau_{ij}(hg) = \tau_{ij}(h_{(1)})(h_{(2)} \triangleright \tau_{ij}(g)),\\ \pi^{ij}_k(\tau_{ij}(h)) = \pi^{ik}_j(\tau_{ik}(h_{(1)}))\pi^{kj}_i(\tau_{kj}(h_{(2)})),\end{gathered} \tag{5}$$

where $h, g \in H,\ b_i \in B_i,\ i,j,k \in \Omega$. Here $h_{(1)} \otimes h_{(2)}$ is Sweedler's shorthand notation for the coproduct $\Delta(h)$, $h \in H$, and S is the antipode of H, and π^{ij}_k is the canonical projection $B_{ij} \to B_{ijk} = B/(I_i + I_j + I_k)$. Now we have:

Proposition 3. *Let $\{I_i\}_{i\in\Omega}$ be a covering of B and assume that all B_i's are H-module algebras such that $\ker \pi^i_j$ is an H-submodule of B_i. Then*

$$\tilde{P} = \{(b_i \otimes h_i)_{i\in\Omega} \in \prod_{i\in\Omega} B_i \rtimes H \mid \pi^i_j(b_i)\tau_{ij}(h_{i(1)}) \otimes h_{i(2)} = \pi^j_i(b_j) \otimes h_j\} \tag{6}$$

is an H-comodule algebra with $P^{H\text{-inv}} \simeq B$ if $\tau_{ij} : H \to B_{ij}$ satisfy the above properties. If τ_{ij} arise from a piecewise trivial principal H-extension P, then $\tilde{P}$ is isomorphic to P via

$$\chi : P \ni a \mapsto (\chi_i(a_i))_{i\in\Omega} \in \tilde{P}, \tag{7}$$

where $\chi_i : P_i \to B_i \rtimes H$ is as in Definition 3 and a_i is the image of a in P_i.

2.2. **Reductions of piecewise trivial quantum principal bundles.** The following defines the quantum replacement for a reduction of the structure group of a principal bundle.

Definition 4. *Let $B \subset P$ be a principal H-extension and $K \subset H$ be a Hopf ideal. Then a K-reduction of P is a principal H/K-extension P/J over B for some ideal $J \subset P$ which is an H/K-subcomodule.*

K plays the role of the ideal of functions vanishing on the subgroup to which we reduce and J that of the ideal of functions vanishing on the reduced bundle.

A trivial extension $P = B \otimes H$ can obviously be reduced to any quotient H/K of H, but in general the question of reducibility is unclear. However, for principal extensions which are piecewise tensor products we can prove the following:

Proposition 4. *Let $B \subset P$ be a principal H-extension over a* C^**-algebra B which is piecewise isomorphic to $B_i \otimes H$ over some covering $\{I_i\}_{i\in\Omega}$ of B, and let K be a Hopf ideal in H. Denote by ${}^{H/K\text{-inv}}H$ the invariants of the natural left coaction of H/K on H. Write for brevity $D^+ = {}^{H/K\text{-inv}}H \cap \ker\varepsilon$, where ε denotes the counit in H. Then the ideal*

$$J = P\chi^{-1}((1_{B_i} \otimes D^+)_{i\in\Omega}) \tag{8}$$

with $\chi : P \to \tilde{P}$ as in eq. (7) defines a piecewise trivial K-reduction of P, provided that the transition functions $\tau_{ij} : H \to B_{ij}$, $i, j \in \Omega$, vanish on K.

However, this result does not extend to arbitrary piecewise trivial principal extensions which are trivialised to crossed products (cf. [7]).

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