

AN EXAMPLE OF BERGLUND-HÜBSCH MIRROR SYMMETRY FOR A CALABI-YAU COMPLETE INTERSECTION

STEFANO FILIPAZZI - FRANCO ROTA

We study an example of complete intersection Calabi-Yau threefold due to Libgober and Teitelbaum [11], and verify mirror symmetry at a cohomological level. Direct computations allow us to propose an analogue to the Berglund-Hübsch mirror symmetry setup for this example [2]. We then follow the approach of Krawitz to propose an explicit mirror map [9].

1. Introduction

This note is an account of a calculation which we carried out during the Summer School “Pragmatic 2015”, and represents a first step in our project of generalizing the Krawitz-Chiodo-Ruan cohomological isomorphisms to Berglund-Hübsch mirror pairs of Calabi-Yau complete intersections.

In 1993, Berglund and Hübsch describe a procedure to construct the mirror manifold of a large class of Calabi-Yau hypersurfaces in weighted projective space, using the formalism of Landau-Ginzburg models [2]. Later, Krawitz gives an explicit description of the mirror map between the state spaces of two

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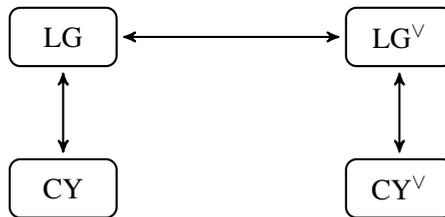
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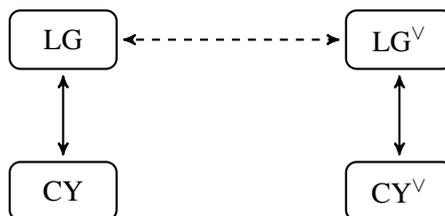
Berglund-Hübsch mirror LG models [9], and Borisov re-proves and generalizes his statements [3].

Since the work by Berglund and Hübsch, the role of Landau-Ginzburg models in mirror symmetry has increased in importance, particularly after Fan, Jarvis and Ruan defined their quantum invariants in [7] (FJRW theory), and extended the definition to an even more general setting in [8]. In a subsequent work, Chiodo and Nagel recognize the state space of Landau-Ginzburg isolated singularities defined in [7] as an instance of relative orbifold Chen-Ruan cohomology [4]. This allows to define the cohomology of hybrid Landau-Ginzburg models, and to identify it with the cohomology of Calabi-Yau complete intersections in weighted projective space. In this sense, [4] extends the LG/CY correspondence, which was proven to hold for hypersurfaces in [5], to the case of complete intersection Calabi-Yau. The situation in the hypersurface case can be summarized by the following diagram,



where an arrow indicates an isomorphism of the corresponding state spaces. Existence of the vertical arrows has been established in [5], while the horizontal one is the object of [9] and [3].

In the case of a complete intersection, [4] provides the vertical arrows of the diagram



by proving an orbifold version of the Thom isomorphism. A related result by Libgober [10] shows the invariance of the elliptic genus for complete intersections and other examples of GIT quotients.

The object of this work is to exhibit a horizontal arrow in a particular example by Libgober and Teitelbaum [11]. In this way, we aim at making progress towards a definition of a mirror symmetry construction for Calabi-Yau complete intersections of the same type as the Berglund-Hübsch construction. We

describe a complete intersection Calabi-Yau threefold and its mirror orbifold using an analogue of the Berglund-Hübsch formalism, and propose a generalization of Krawitz's mirror map.

The direction of this paper is suggested by the classical result of Batyrev and Borisov on cohomological mirror symmetry for Calabi-Yau complete intersections [1], although at the moment it is unclear how to explicitly relate to their work.

Starting from a paper by Libgober and Teitelbaum [11], we consider the complete intersection of two cubics in \mathbb{P}^5 together with its mirror. We recall the setup for hybrid Landau-Ginzburg models in Section 2. In Section 3, using the Chiodo-Nagel CY/LG-correspondence [4], we rephrase the example of [11] in terms of hybrid Landau-Ginzburg models. Then, in Section 4, we generalize the construction of Krawitz for the quintic Fermat hypersurface in \mathbb{P}^4 with the language proposed for complete intersections. Finally, in Section 5, we verify that the state spaces of the complete intersection and of its mirror are isomorphic, and present an explicit mirror map generalizing Krawitz's formula.

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2. Hybrid Landau-Ginzburg models

In this section, we recall the notion of hybrid Landau-Ginzburg models. Our setup follows closely the one in [4], whose notation we adopt for the most part. The main difference is that we consider a broader class of groups of symmetries. This is in line with the groups considered in [11].

Let $\mathbf{W} := \{W_1, \dots, W_r\}$ be a set of quasi-homogeneous polynomials of degrees d_1, \dots, d_r defining a complete intersection

$$X_{\mathbf{W}} := \{W_1 = \dots = W_r = 0\} \subset \mathbb{P}(w_1, \dots, w_n)$$

in weighted projective space. Assume furthermore that $X_{\mathbf{W}}$ is *non-degenerate*, i.e.

- the choice of weights w_i is unique;
- $X_{\mathbf{W}}$ is smooth outside the origin.

The *maximal group of diagonal symmetries* of $X_{\mathbf{W}}$ is the maximal subgroup $\Gamma_{max} \subset (\mathbb{C}^*)^n$ of diagonal matrices γ preserving $X_{\mathbf{W}}$. More explicitly, γ has diagonal entries of the form $(\gamma_1, \dots, \gamma_n) = \alpha \bar{\lambda} := (\alpha_1 \lambda^{w_1}, \dots, \alpha_n \lambda^{w_n})$, where $\lambda \in \mathbb{C}^*$, and $\alpha = \alpha \bar{1}$ is a vector whose entries are all non-zero. By definition, for any $1 \leq i \leq r$ and admissible choice of α , there exists $\beta_{i,\alpha} \in \mathbb{C}^*$ such that $W_i(\alpha_1 x_1, \dots, \alpha_n x_n) = \beta_{i,\alpha} W_i(x_1, \dots, x_n)$. By a *group of diagonal symmetries* we mean a subgroup of Γ_{max} .

Example 2.1. The group Γ_{max} contains the following special element. Let d denote the greatest common divisor of the d_i , and define

$$J := \begin{pmatrix} e^{2\pi i \frac{w_1}{d}} & & & \\ & \ddots & & \\ & & \ddots & \\ & & & e^{2\pi i \frac{w_n}{d}} \end{pmatrix}.$$

The element J is analogous to the generator of the group μ_d in the hypersurface case. The group $\langle J \rangle$ will play an important role later.

Example 2.2. Among the subgroups of Γ_{max} , one can consider the special linear group of diagonal symmetries of a complete intersection, denoted $SL(\mathbf{W})$. It contains elements of Γ_{max} of determinant 1.

Definition 2.3. Let γ be a diagonal symmetry acting on \mathbb{C}^n as a diagonal matrix with entries $\exp(2\pi i \gamma_i)$, with $\gamma_i \in [0, 1)$. Then, we denote γ by $(\gamma_1, \dots, \gamma_n)$. If we have $\gamma_i = \frac{c_i}{d}$ for all i , we write $(\gamma_1, \dots, \gamma_n) = \frac{1}{d}(c_1, \dots, c_n)$. The age of the element γ , denoted a_γ , is the sum $\sum_i \gamma_i$.

Unless otherwise stated, we will follow the classic literature (see [6, Ch. 3]), and assume that any group of diagonal symmetries Γ contains the torus

$$\Gamma_0 := \{ \bar{\lambda} = (\lambda^{w_1}, \dots, \lambda^{w_n}) \mid \lambda \in \mathbb{C}^* \}.$$

Then, Γ_0 is the connected component of the identity in Γ , and we denote by G the quotient Γ/Γ_0 .

Now, consider \mathbb{C}^{n+r} with coordinates $(x_1, \dots, x_n, p_1, \dots, p_r)$. We can extend the action of Γ to \mathbb{C}^{n+r} by setting

$$\alpha \bar{\lambda}(\mathbf{x}, \mathbf{p}) := (\alpha_1 \lambda^{w_1} x_1, \dots, \alpha_n \lambda^{w_n} x_n, \beta_{1,\alpha}^{-1} \lambda^{-d_1} p_1, \dots, \beta_{r,\alpha}^{-1} \lambda^{-d_r} p_r). \quad (1)$$

It is useful to introduce a notation for the fixed points of an element $\gamma \in \Gamma$. We set

$$\mathbb{C}_\gamma^n := \{\mathbf{x} \in \mathbb{C}^n \mid \gamma \cdot \mathbf{x} = \mathbf{x}\}, \quad n_\gamma := \dim \mathbb{C}_\gamma^n,$$

where the action of γ is restricted to the first set of coordinates. Similarly, we define

$$\mathbb{C}_\gamma^r := \{\mathbf{p} \in \mathbb{C}^r \mid \gamma \cdot \mathbf{p} = \mathbf{p}\}, \quad r_\gamma := \dim \mathbb{C}_\gamma^r.$$

The integer n_γ (respectively r_γ) counts the dimension of the space spanned by the \mathbf{x} (resp. \mathbf{p}) variables that are fixed by γ . For a polynomial $V \in \mathbb{C}[\mathbf{x}, \mathbf{p}]$, we let V_γ denote $V|_{\mathbb{C}_\gamma^n \times \mathbb{C}_\gamma^r}$.

For an element $V \in \mathbb{C}[\mathbf{x}, \mathbf{p}]$, its *chiral algebra* \mathcal{Q}_V is defined as

$$\mathcal{Q}_V := d\mathbf{x} \wedge d\mathbf{p} \otimes \text{Jac}(V),$$

where we are formally tensoring the *Jacobi ring*

$$\text{Jac}(V) := \frac{\mathbb{C}[x_1, \dots, x_n, p_1, \dots, p_r]}{\left(\frac{\partial V}{\partial x_1}, \dots, \frac{\partial V}{\partial x_n}, \frac{\partial V}{\partial p_1}, \dots, \frac{\partial V}{\partial p_r}\right)}$$

by the top form $d\mathbf{x} \wedge d\mathbf{p} := dx_1 \wedge \dots \wedge dx_n \wedge dp_1 \wedge \dots \wedge dp_r$. For brevity, in the following we will often omit the wedge symbol and write $d\mathbf{x}d\mathbf{p} := d\mathbf{x} \wedge d\mathbf{p}$. We assign bidegree $(D - k, k)$ to the elements of \mathcal{Q}_V which have degree k in the \mathbf{p} variables, where $D = n - r - 1$.

By construction, $\gamma^*(p_i W_i) = p_i W_i$ for all $\gamma \in \Gamma$ and all i . Then we can define the Γ -invariant function

$$\begin{aligned} \overline{W} : \mathbb{C}^{n+r} &\rightarrow \mathbb{C} \\ (\mathbf{x}, \mathbf{p}) &\mapsto p_1 W_1(\mathbf{x}) + \dots + p_r W_r(\mathbf{x}). \end{aligned}$$

Consider the open set in \mathbb{C}^{n+r} defined as $U_{LG} := \mathbb{C}^n \times (\mathbb{C}^r \setminus \{\mathbf{0}\})$. The quotient of U_{LG} by the action of \mathbb{C}^* is the total space of the vector bundle

$$[U_{LG}/\mathbb{C}^*] = \bigoplus_{j=1}^n \mathcal{O}_{\mathbb{P}(\mathbf{d})}(-w_j) =: \mathcal{O}_{\mathbf{d}}(-\mathbf{w}).$$

Let $M := \{\overline{W} = t_0\}$ denote the fiber over any point $t_0 \neq 0$.

Definition 2.4. *The datum $(\mathcal{O}_{\mathbf{d}}(-\mathbf{w}), \overline{W}, \Gamma)$ is a hybrid Landau-Ginzburg model with superpotential*

$$\overline{W} : [\mathcal{O}_{\mathbf{d}}(-\mathbf{w})/G] \rightarrow \mathbb{C}.$$

We define the generalized state space of the hybrid Landau-Ginzburg model $(\mathcal{O}_{\mathbf{d}}(-\mathbf{w}), \overline{W})$ as

$$\mathcal{H}_\Gamma^{p,q}(W_1, \dots, W_r) := H_{\text{CR}}^{p+r, q+r}([\mathcal{O}_{\mathbf{d}}(-\mathbf{w})/G], [F/G]),$$

where F is the quotient stack $[M/\mathbb{C}^*]$ inside $\mathcal{O}_{\mathbf{d}}(-\mathbf{w})$.

For any two non-negative integers $r \geq n$ we need to consider the bigraded ring $dt(-n) \otimes \mathbb{C}[t]/(t^{r-n})$, where the element $dt(-n) \otimes t^k$ has bidegree $(n+k, n+k)$. We will make use of the following theorem due to Chiodo and Nagel [4].

Theorem 2.5 ([4, Theorem 4.3]). *Let W_i , r and n be as introduced above. Let $\tilde{W} = \sum_i p_i W_i \in \mathbb{C}[\mathbf{x}, \mathbf{p}]$. Then, we have*

$$\mathcal{H}_\Gamma^*(W_1, \dots, W_r) = \bigoplus_{\gamma \in \Gamma} H_\gamma(-a_\gamma + r),$$

where H_γ with its double grading is given by

$$H_\gamma = \begin{cases} \left(\mathcal{Q}_{\tilde{W}_\gamma}\right)^\Gamma & \text{if } r_\gamma < n_\gamma \\ dt(-n_\gamma) \otimes \mathbb{C}[t]/(t^{r_\gamma - n_\gamma}) & \text{if } r_\gamma \geq n_\gamma. \end{cases} \tag{2a}$$

$$\tag{2b}$$

and carries the Tate twist $(-a_\gamma + r)$.

3. Description of the example

In this section, we introduce the complete intersection Calabi-Yau object of this work. This example was first described by Libgober and Teitelbaum [11].

Consider \mathbb{P}^5 with coordinates $[x_1 : x_2 : x_3 : X_1 : X_2 : X_3]$. Let $X \subset \mathbb{P}^5$ be the vanishing locus of the polynomials

$$W_1 = x_1^3 + x_2^3 + x_3^3 - 3\tau X_1 X_2 X_3$$

$$W_2 = X_1^3 + X_2^3 + X_3^3 - 3\tau x_1 x_2 x_3.$$

For generic choice of $\tau \in \mathbb{C}$, it is a Calabi-Yau threefold. Since we are in the standard projective space \mathbb{P}^5 , and we are considering two polynomials of degree 3, the group $\langle J \rangle$ introduced in Example 2.1 is isomorphic to μ_3 .

Introduce two new variables, p_1 and p_2 , and form the polynomial $W := p_1 W_1 + p_2 W_2$. Extend the action of the group $\langle J \rangle$ to $\mathbb{C}^6 \times \mathbb{C}^2$ by letting J act with weights $(1, \dots, 1, -3, -3)$. Similarly, following equation (1), we can extend the action of any group Γ of diagonal symmetries to the variables p_1 and p_2 .

According to the hypersurface case of Berglund-Hübsch mirror symmetry (see [6, Ch. 3]), to study the generalized state spaces of the Hybrid Landau-Ginzburg model induced by W_1 and W_2 , we have to make choices for a mirror set of polynomials and for a pair of groups. As suggested by Libgober and Teitelbaum, we choose the same set of polynomials for the mirror. Indeed, this is analogous to the Fermat hypersurface case. Then, let $\Gamma := \Gamma_0 \cdot \langle J \rangle$, and $\Gamma^T :=$

$\Gamma_{max} \cdot \text{SL}(\mathbf{W})$. This choice is in line with the Berglund-Hübsch prescription (see Rmk. 3.1).

By Theorem 2.5, the only elements of a group of diagonal symmetries Σ contributing to the generalized state space of $(\mathcal{O}_d(-\mathbf{w}), \overline{W}, \Sigma)$ are the elements fixing at least one of the coordinates $(x_1, x_2, x_3, X_1, X_2, X_3, p_1, p_2)$.

By direct inspection, one checks that the only elements of Γ that fix at least one of the variables are the elements of $\langle J \rangle$. Analogously, when considering Γ^T , one is left with those elements $\gamma \in \text{SL}(\mathbf{W})$ that fix at least one of the variables. These γ generate a subgroup $\Gamma' \subset \text{SL}(\mathbf{W})$ whose elements $g_{\lambda, \mu, \alpha, \beta, \delta, \varepsilon}$ act on $(x_1, x_2, x_3, X_1, X_2, X_3, p_1, p_2)$ via multiplication by

$$(\lambda \xi_3^\alpha \xi_9^\mu, \lambda \xi_3^\beta \xi_9^\mu, \lambda \xi_9^\mu, \lambda \xi_3^{-\delta} \xi_9^{-\mu}, \lambda \xi_3^{-\varepsilon} \xi_9^{-\mu}, \lambda \xi_9^{-\mu}, \lambda^{-3} \xi_3^{-\mu}, \lambda^{-3} \xi_3^\mu).$$

Here ξ_3 is a primitive third root of unity, ξ_9 is a primitive ninth root of unity, and $\lambda \in \mu_9$. Furthermore, the condition

$$\alpha + \beta = 3\mu = \delta + \varepsilon \pmod 3$$

is satisfied.

Remark 3.1. The group Γ^T also contains Γ_0 , and the quotient Γ^T/Γ_0 coincides with the group G_{81} described in [11]. We have $\Gamma/\Gamma_0 = \langle J \rangle$.

As a consequence of Libgober and Teitelbaum’s work [11] and an application of Chiodo and Nagel’s CY/LG-correspondence [4, Thm. 5.1], one knows that the generalized state spaces $\mathcal{H}_\Gamma^{p,q}(W_1, W_2)$ and $\mathcal{H}_{\Gamma^T}^{3-p,q}(W_1, W_2)$ have the same dimension. The object of this work is to further investigate this correspondence, and propose a possible mirror map. The main idea is summarized in the following.

Definition 3.2. We define an explicit mirror map $\mathcal{H}_{\Gamma^T}^{1,1}(W_1, W_2) \rightarrow \mathcal{H}_\Gamma^{2,1}(W_1, W_2)$, generalizing Krawitz’s map for hypersurfaces.

Remark 3.3. The mirror map in Definition 3.2 is expected to extend to a map $\mathcal{H}_{\Gamma^T}^*(W_1, W_2) \rightarrow \mathcal{H}_\Gamma^*(W_1, W_2)$. At this point, the main difficulty is to choose appropriate representatives among the 73 generators of $\mathcal{H}_{\Gamma^T}^{1,2}(W_1, W_2)$ in order to recognize a pattern similar to the one occurring among the generators of $\mathcal{H}_\Gamma^{2,1}(W_1, W_2)$, which are listed in Table 1.

4. Krawitz’s mirror map revisited

In this section, we discuss how the map mentioned in Definition 3.2 is related to the work of Krawitz. More precisely, we show how the approach suggested for

Calabi-Yau complete intersections is a generalization of the one already known for hypersurfaces.

We will focus on a concrete example, the Fermat quintic $X_W \subset \mathbb{P}^4$. This is given as the vanishing locus of the polynomial

$$W = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5.$$

First, we consider the group $\langle J \rangle$ as group of diagonal symmetries. Thus, we first consider the pair $(W, \langle J \rangle)$. Berglund-Hübsch mirror symmetry guarantees the existence of a mirror model, denoted by $(W^T, \langle J \rangle^T)$ [6, 9]. In our case, $W = W^T$, and $\langle J \rangle^T = \text{SL}(W^T) = \text{SL}(W)$ is a well known group, denoted by G_{125} .

Now, we would like to regard the hypersurface X_W as a “complete intersection of one hypersurface”. Therefore, we introduce one auxiliary variable p , and, as explained in equation (1), extend the action of the diagonal symmetries acting on X_W to p . We can then regard pW as a function on $\mathbb{C}^5 \times \mathbb{C}^*$, and consider the induced Landau-Ginzburg model.

Among the groups fixing X_W , there is $\mathbb{C}^* \cdot \langle J \rangle$. In particular, \mathbb{C}^* acts on x_i with weight 1, and on p with weight -5 . We can compute the corresponding state spaces. By Theorem 2.5, only the elements $\lambda \cdot J^a \in \mathbb{C}^* \cdot \langle J \rangle$ that fix some variable contribute to the computations. In this case, it is easy to check that $\lambda \in \mathbb{C}^*$ is forced to satisfy $|\lambda| = 1$. More precisely, λ is either 1, or a primitive fifth-root of unity. Therefore, we can identify the elements of $\mathbb{C}^* \cdot \langle J \rangle$ that give non-trivial contributions to cohomology with elements of $\langle J \rangle$. There are two possibilities: If $\gamma \in \langle J \rangle$ is the identity, then all variables are fixed, otherwise, just p remains unchanged.

The element $\gamma = \text{id}$ contributes to the state space with a summand of the form

$$\left(d\mathbf{p}d\mathbf{x} \otimes \frac{\mathbb{C}[p, x_1, \dots, x_5]}{(px_1^4, \dots, px_5^4, \sum_{i=1}^5 x_i^5)} \right)^{\mathbb{C}^*}.$$

Its elements carry a bigrading $(3 - k, k)$, where k is the degree of the variable p in the element. As $d\mathbf{p}d\mathbf{x}$ is \mathbb{C}^* invariant, and we are looking for \mathbb{C}^* -invariant elements, we need polynomial coefficients containing five among x_1, \dots, x_5 for every appearance of p . There is just a one-dimensional family in degree $(3, 0)$, there are spaces with dimension 101 in degree $(2, 1)$ and $(1, 2)$, and there is one more one-dimensional family in degree $(0, 3)$.

If γ acts by a primitive fifth-root of unity, we have $r_\gamma = 1$ and $n_\gamma = 0$. Therefore, each one of these elements contributes with a one-dimensional vector space, generated by one element, denoted by $1|g$.

Taking into account the grading of the above pieces, we recover the well known Hodge diamond

$$\begin{array}{cccc}
 & & & 1 \\
 & & & 1 \\
 & & 1 & 101 & & 101 & 1 \\
 & & & & & & 1 \\
 & & & & & & 1
 \end{array}$$

Now, we want to consider the mirror of $(pW, \mathbb{C}^* \cdot \langle J \rangle)$. As W is a polynomial of Fermat type, we will let the mirror polynomial be pW as well. As \mathbb{C}^* is the minimial group of diagonal symmetries containing \mathbb{C}^* , we consider the maximal group of symmetries fixing the polynomial pW . Let Γ be such group. It can be checked by direct computation that $\Gamma = \mathbb{C}^* \cdot G_{125}$. It acts as

$$(x_1, x_2, x_3, x_4, x_5, p) \mapsto (\lambda x_1, \lambda \xi^a x_2, \lambda \xi^b x_3, \lambda \xi^c x_4, \lambda \xi^{-a-b-c} x_5, \lambda^{-5} p),$$

where $\lambda \in \mathbb{C}^*$, ξ is a primitive fifth-root of unity, and $a, b, c \in \mathbb{Z}$.

Remark 4.1. In [4], Chiodo and Nagel consider the maximal group fixing W , and then they extend its action to pW . On the other hand, for our purposes, such a group turns out to be too rigid, and does not recover the right cohomologies.

As above, the only elements $\gamma \in \Gamma$ contributing to the state spaces are the ones where λ is a fifth-root of unity. In particular, p is fixed by any relevant element $\gamma \in \Gamma$. This highlights how the variable p is not necessary for the computations in the hypersurface case.

Since λ needs to be a fifth-root of unity, we can absorb it in the contribution of ξ . Therefore, any element γ contributing to the computations can be encoded by a 6-tuple

$$(a_1, a_2, a_3, a_4, a_5; 0),$$

where the action on x_i is given by multiplication by ξ^{a_i} . We will use this formalism as a slight modification of the one in Definition 2.3, in order to distinguish the p and the x variables.

Now, we are left with direct computations. We will proceed by cases.

- Assume that p is the only variable fixed. In such case, $r_\gamma = 1$, and $n_\gamma = 0$. Therefore, each element contributes with a one-dimensional vector space. There are 204 of these elements, and their ages groups them in four families of dimension 1, 101, 101 and 1 respectively. These compute the twisted sector of the cohomology ring.
- If p and some but not all of the x_i are fixed, there is no contribution.

that the diagonal symmetries have determinant 1. Indeed, the new group elements coming into the picture do not contribute to the computation of the state spaces. Recall that we consider the groups $\Gamma = \Gamma_0 \cdot \langle J \rangle$ and Γ^T acting on the same set of polynomials (see Section 3).

In the following, we first compute the state space associated to Γ . Then, we consider the one associated to Γ^T , and we conclude describing the mirror map.

5.1. The state spaces $\mathcal{H}_\Gamma^{p,q}(W_1, W_2)$

We follow the notation of Theorem 2.5. Notice that for $\gamma \in \Gamma$, if $r_\gamma = n_\gamma = 0$, then $H_\gamma = 0$ gives no contribution to the state space. Then, only the three elements of $\langle J \rangle \subset \Gamma$ give non-trivial contributions. We will proceed by cases.

- The non-trivial elements have ages 2 and 4 respectively. They both contribute with the cohomology of a projective line $\mathbb{P}(3, 3) \cong \mathbb{P}^1$, and form the twisted sectors of $\mathcal{H}_\Gamma^*(W_1, W_2)$.
- The contribution of the identity element is encoded in \mathcal{Q}_W^Γ . This ring carries a bigrading $(D - k, k)$, where $D = n - r - 1 = 3$, and $k \in \{0, 1, 2, 3\}$ is the degree in the variables p_1 and p_2 . The relations in $\text{Jac}(V)$ are:

$$\begin{aligned} &x_1^3 + x_2^3 + x_3^3 - 3\tau X_1 X_2 X_3, \\ &X_1^3 + X_2^3 + X_3^3 - 3\tau x_1 x_2 x_3, \\ &\quad p_1 x_i^2 - p_2 x_j x_k, \\ &\quad p_2 X_i^2 - p_1 X_j X_k, \end{aligned}$$

where all the i, j, k are assumed to be distinct. The form

$$d\mathbf{p}d\mathbf{x}d\mathbf{X} := dp_1 \wedge dp_2 \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge dX_1 \wedge dX_2 \wedge dX_3$$

is invariant under the action of Γ . Hence, the degree $(3, 0)$ component is generated by $d\mathbf{p}d\mathbf{x}d\mathbf{X} \otimes 1$. Table 1 contains a list of 73 independent generators of $\mathcal{H}_\Gamma^{2,1}(W_1, W_2)$. They determine the whole untwisted sector, which is completed by $\mathcal{H}_\Gamma^{1,2}(W_1, W_2)$ and $\mathcal{H}_\Gamma^{0,3}(W_1, W_2)$, whose dimensions are 73 and 1 respectively.

We can summarize the above computations with the following Hodge diamond.

$$\begin{array}{cccc}
 & & & 1 \\
 & & & 1 \\
 & & 1 & 73 & 73 & 1 \\
 & & & & & 1 \\
 & & & & & 1
 \end{array}$$

5.2. The state spaces $\mathcal{H}_{\Gamma^T}^{p,q}(W_1, W_2)$

Now, we consider the mirror side. We will list the elements of Γ^T , and study their contributions to the state spaces. As explained in Definition 2.3, we write $(\gamma_1, \dots, \gamma_8)$ if γ acts on $\mathbb{C}^6 \times \mathbb{C}^2$ as a diagonal matrix with entries $\exp(2\pi i \gamma_i)$.

As showed in Theorem 2.5, the contribution of an element γ has two different behaviors, depending on how r_γ and n_γ compare to each other. Therefore, we will proceed by cases.

- If $r_\gamma < n_\gamma$, we are in the case given by equation (2a). These elements are listed in Table 3. One of these elements is the identity of Γ^T . It corresponds to the untwisted sector of $\mathcal{H}_{\Gamma^T}^*(W_1, W_2)$, which is 4-dimensional and generated by powers of $d\mathbf{p}d\mathbf{x}d\mathbf{X} \otimes p_1x_1^3$ in the Jacobi ring.

The other four elements all give a two dimensional Jacobi ring, and behave in a similar way. For instance, $\gamma = \frac{1}{9}(0, 0, 0, 3, 3, 3; 0, 0)$ induces a direct summand $H_\gamma = \text{span}(d\mathbf{p}d\mathbf{x}d\mathbf{X} \otimes x_1^3, d\mathbf{p}d\mathbf{x}d\mathbf{X} \otimes x_2^3)$, which contributes as two-dimensional subspace of $\mathcal{H}_{\Gamma^T}^{1,1}(W_1, W_2)$. A second one of these elements contributes with a two-dimensional subspace of $\mathcal{H}_{\Gamma^T}^{1,1}(W_1, W_2)$. Similarly, the two remaining elements correspond to two two-dimensional subspaces of $\mathcal{H}_{\Gamma^T}^{2,2}(W_1, W_2)$.

- If $r_\gamma \geq n_\gamma$, we are in the case given by equation (2b). Each element in Table 2 contributes to the state space with $r_\gamma - n_\gamma$ elements. If $r_\gamma - n_\gamma = 2$, the two generators belong one to

$$\mathcal{H}_{\Gamma^T}^{a_\gamma - r + n_\gamma, a_\gamma - r + n_\gamma}(W_1, W_2),$$

and the other to

$$\mathcal{H}_{\Gamma^T}^{a_\gamma + 1 - r + n_\gamma, a_\gamma + 1 - r + n_\gamma}(W_1, W_2).$$

There are four such elements. The first one of them contributes with one generator to $\mathcal{H}_{\Gamma^T}^{0,0}(W_1, W_2)$, and with one to $\mathcal{H}_{\Gamma^T}^{1,1}(W_1, W_2)$. Similarly, a second element generates the top cohomology, contributing with one generator to $\mathcal{H}_{\Gamma^T}^{3,3}(W_1, W_2)$, and with one to $\mathcal{H}_{\Gamma^T}^{2,2}(W_1, W_2)$. Then, the two

remaining elements behave in the same way: Each one of them has one generator lying in $\mathcal{H}_{\Gamma^r}^{1,1}(W_1, W_2)$, and one in $\mathcal{H}_{\Gamma^r}^{2,2}(W_1, W_2)$.

On the other hand, if $r_\gamma - n_\gamma = 1$, there is only one generator, which belongs to

$$\mathcal{H}_{\Gamma^r}^{a_\gamma-r+n_\gamma, a_\gamma-r+n_\gamma}(W_1, W_2).$$

The last column in Table 2 counts the elements of the *same type*. For example, from the element $\frac{1}{9}(3, 3, 3, 3, 0, 6; 0, 0)$ we can obtain 5 more by permuting X_1, X_2 and X_3 via the action of the symmetric group S_3 . These new elements are considered of the same type as the starting one, and are not explicitly listed in Table 2. They are taken into account by counting 6 elements, including their representative, of the type of $\frac{1}{9}(3, 3, 3, 3, 0, 6; 0, 0)$.

We can summarize the above computations with the following Hodge diamond.

$$\begin{array}{ccccccc}
 & & & & & & 1 \\
 & & & & & & 73 \\
 & & & & & 1 & 1 \\
 & & 1 & 1 & & 1 & 1 \\
 & & & & & 73 \\
 & & & & & & 1
 \end{array}$$

5.3. The mirror map

We present the explicit isomorphism between the vector spaces $\mathcal{H}_{\Gamma^r}^{1,1}(W_1, W_2)$ and $\mathcal{H}_{\Gamma^r}^{2,1}(W_1, W_2)$ mentioned in Definition 3.2. Its main feature is that it respects the structure emerging from the lists in Tables 1, 3 and 2, where it becomes evident that these two vector spaces admit a further interesting subdivisions into direct summands.

We follow the notation in [9] and Theorem 2.5: Given an element σ of a group of diagonal symmetries Σ , we denote by $\omega|\sigma\rangle$ the element of the state space \mathcal{H}_Σ^* determined by $\omega \in H_\sigma$.

Now, we explain the mirror map via its action on some elements listed in Table 2, which represent the different possible behaviors. For the complete action of the mirror map, we refer to Table 4 in the Appendix.

First, we notice that there is a correspondence between direct summands in $\mathcal{H}_{\Gamma^r}^{1,1}(W_1, W_2)$ and $\mathcal{H}_{\Gamma^r}^{2,1}(W_1, W_2)$. For the reader's convenience, we will list them relying on Tables 1, 3 and 2.

- In $\mathcal{H}_{\Gamma}^{2,1}(W_1, W_2)$, we isolated six groups of generators, each one consisting of nine elements. These correspond to the six groups of nine elements $\gamma \in \Gamma^T$ having $r_\gamma = 1$, $n_\gamma = 0$, and $a_\gamma = 3$.
- The two groups of six generators of $\mathcal{H}_{\Gamma}^{2,1}(W_1, W_2)$ correspond to the two groups of six elements $\gamma \in \Gamma^T$ having $r_\gamma = 2$, $n_\gamma = 1$, and $a_\gamma = 2$.
- In Γ^T there are two elements with $r_\gamma = 2$, $n_\gamma = 3$ and $a_\gamma = 1$. These contribute with two two-dimensional vector spaces. These correspond to the subspaces $\langle p_1X_1^3, p_1X_2^3 \rangle$ and $\langle p_2x_1^3, p_2x_2^3 \rangle$ of $\mathcal{H}_{\Gamma}^{2,1}(W_1, W_2)$. Notice that, by the relations in the Jacobi ring, these spaces contain $p_1X_3^3$ and $p_2x_3^3$ respectively.
- To conclude, among the elements $\gamma \in \Gamma^T$ with $r_\gamma = 2$ and $n_\gamma = 0$, three of these contribute with a one-dimensional subspace of $\mathcal{H}_{\Gamma}^{1,1}(W_1, W_2)$ each. These will correspond to the three sporadic polynomials listed at the top of Table 1.

Before getting into the details of the mirror map, it is worth asking the following interesting question.

Question 5.1. *Do the aforementioned direct summands of $\mathcal{H}_{\Gamma}^{1,1}(W_1, W_2)$ and $\mathcal{H}_{\Gamma}^{2,1}(W_1, W_2)$ have an interpretation from a geometric or physical point of view?*

Now, we will proceed to describing the mirror map, analyzing each one of the cases listed above.

The two first groups of elements present a fundamental common feature, namely $r_\gamma - n_\gamma = 1$. This is essentially the only case occurring in the twisted sector for the hypersurface case [9]. As showed in equation (3), once we introduce a p in the hypersurface case, the exponents of the x_i contribute to the exponent of this new variable. We will follow this idea, bearing in mind that p_1 is related to the set of variables x_i , while p_2 to the set X_i .

For the elements with $r_\gamma - n_\gamma = 1$, we then propose the assignment

$$dt \left| \frac{1}{9}(b_1, b_2, b_3, c_1, c_2, c_3; a_1, a_2) \right\rangle \mapsto d\mathbf{p}d\mathbf{x}d\mathbf{X} \otimes p_1^{\min\{\lfloor \frac{b_j}{3} \rfloor\}} p_2^{\min\{\lfloor \frac{c_j}{3} \rfloor\}} \prod_{i=1}^3 x_i^{\lfloor \frac{b_i}{3} \rfloor - \min\{\lfloor \frac{b_j}{3} \rfloor\}} X_i^{\lfloor \frac{c_i}{3} \rfloor - \min\{\lfloor \frac{c_j}{3} \rfloor\}} |id\rangle. \tag{4}$$

Indeed, from Table 2, it is clear that exactly one set of variables has an action which is a rotation of at least $e^{\frac{2\pi i}{3}}$ on each single coordinate. We record this information choosing the corresponding variable p_i to appear in the assigned polynomial. Then, in order to choose the exponent appearing for each variable

in the polynomial in output, we compare the action within each group x_i and X_i . This is where the normalizing exponents $-\min\{\lfloor \frac{b_j}{3} \rfloor\}$ and $-\min\{\lfloor \frac{c_j}{3} \rfloor\}$ come from. Finally, we point out that looking at quantities of the form $\lfloor \frac{n}{3} \rfloor$ is consistent with distinguishing the action of third-roots of unity from the one of primitive ninth-roots.

As example of the assignments described in equation (4), we have

$$dt \left| \frac{1}{9}(3, 3, 3, 3, 0, 6; 0, 0) \right\rangle \mapsto d\mathbf{p}d\mathbf{x}d\mathbf{X} \otimes p_1X_1X_3^2|id\rangle,$$

and

$$dt \left| \frac{1}{9}(2, 2, 5, 6, 3, 6; 3, 0) \right\rangle \mapsto d\mathbf{p}d\mathbf{x}d\mathbf{X} \otimes p_2x_3X_1X_3|id\rangle.$$

Now, we can study how the mirror map acts on the subspaces corresponding to the elements $\gamma \in \Gamma^T$ with $r_\gamma = 2$, $n_\gamma = 3$, and $a_\gamma = 1$. The first element is

$$\frac{1}{9}(0, 0, 0, 3, 3, 3; 0, 0),$$

which contributes with the subspace

$$\langle d\mathbf{p}d\mathbf{x} \otimes x_1^3, d\mathbf{p}d\mathbf{x} \otimes x_2^3 \rangle.$$

Following the ideas already discussed, the output of the mirror map should carry the variable p_2 , in order to record the action on the X_i variables. Furthermore, it is natural to keep track of the two distinct generators $d\mathbf{p}d\mathbf{x} \otimes x_1^3$ and $d\mathbf{p}d\mathbf{x} \otimes x_2^3$ via their polynomial part. Therefore, we define

$$d\mathbf{p}d\mathbf{x} \otimes x_i^3 \left| \frac{1}{9}(0, 0, 0, 3, 3, 3; 0, 0) \right\rangle \mapsto d\mathbf{p}d\mathbf{x}d\mathbf{X} \otimes p_2x_i^3|id\rangle. \tag{5}$$

Notice that this assignment is valid also for $i = 3$.

Similarly, we define

$$d\mathbf{p}d\mathbf{X} \otimes X_i^3 \left| \frac{1}{9}(0, 0, 0, 3, 3, 3; 0, 0) \right\rangle \mapsto d\mathbf{p}d\mathbf{x}d\mathbf{X} \otimes p_2X_i^3|id\rangle \tag{6}$$

Notice that the assignments in equations (5) and (6) could be regarded as following an extension of the rule in equation (4), where we allow some degree of complexity in the polynomial part of the input as well.

Thus, we are left with assigning a mirror element for the three elements arising from $\gamma \in \Gamma^T$, with $r_\gamma = 2$ and $n_\gamma = 0$. In the case of the elements

$$dt \left| \frac{1}{9}(3, 3, 3, 6, 6, 6; 0, 0) \right\rangle, \quad dt \left| \frac{1}{9}(6, 6, 6, 3, 3, 3; 0, 0) \right\rangle$$

we can follow the algorithm in equation (4). In this way, we get

$$dt \left| \frac{1}{9}(3, 3, 3, 6, 6, 6; 0, 0) \right\rangle \mapsto d\mathbf{pdx}\mathbf{dX} \otimes p_2X_1X_2X_3|id\rangle,$$

and

$$dt \left| \frac{1}{9}(6, 6, 6, 3, 3, 3; 0, 0) \right\rangle \mapsto d\mathbf{pdx}\mathbf{dX} \otimes p_1x_1x_2x_3|id\rangle.$$

Now, we have to consider the element

$$tdt \left| \frac{1}{9}(3, 3, 3, 3, 3, 3; 0, 0) \right\rangle.$$

This element does not follow directly in the patter of equation (4). On the other hand, it comes from the most symmetric element $\gamma \in \Gamma^T \setminus \{0\}$. Analogously, there is a highly symmetric generator of $\mathcal{H}_\Gamma^{2,1}(W_1, W_2)$, namely

$$\begin{aligned} d\mathbf{pdx}\mathbf{dX} \otimes p_1X_1X_2X_3 &= d\mathbf{pdx}\mathbf{dX} \otimes p_2x_1x_2x_3 \\ &= d\mathbf{pdx}\mathbf{dX} \otimes p_2X_i^3 \\ &= d\mathbf{pdx}\mathbf{dX} \otimes p_1x_i^3. \end{aligned}$$

Given the common symmetries, we suggest the choice

$$tdt \left| \frac{1}{9}(3, 3, 3, 3, 3, 3; 0, 0) \right\rangle \mapsto d\mathbf{pdx}\mathbf{dX} \otimes p_1X_1X_2X_3|id\rangle.$$

The above assignments determine a bijective correspondence between a basis of $\mathcal{H}_{\Gamma^T}^{1,1}(W_1, W_2)$ and one of $\mathcal{H}_\Gamma^{2,1}(W_1, W_2)$, so we can extend the map by linearity. This completes the definition of the mirror map in Definition 3.2.

A. Tables

Here we collect the tables referenced above.

TABLE 1

Generator	#
$p_1X_1X_2X_3 = p_2x_1x_2x_3 = p_2X_i^3 = p_1x_i^3$	1
$p_1x_1x_2x_3 = p_1\sum X_i^3$	1
$p_2X_1X_2X_3 = p_2\sum x_i^3$	1
$p_1X_1^3$	1
$p_1X_2^3$	1
$p_2x_1^3$	1
$p_2x_2^3$	1
$p_1x_iX_jX_k$	9
$p_2x_iX_jX_k$	9
$p_1X_iX_jx_k = p_2X_i^2x_k$	9
$p_1x_i^2X_j = p_2x_jx_kX_j$	9
$p_1X_i^2x_j$	9
$p_2x_i^2X_j$	9
$p_1x_i^2x_j = p_2x_j^2x_k$	6
$p_1X_i^2X_j = p_2X_iX_k^2$	6

Table 1: A basis of $H_{\Gamma}^{2,1}(W_1, W_2)$. All indices i, j, k, l range from 1 to 3, distinct indices have distinct values. On every row there are all the possible ways of writing a generator, and the number of generators of that form.

TABLE 2

r_{γ}	n_{γ}	element type	a_{γ}	#
2	0	$\frac{1}{9}(3, 3, 3, 3, 3, 3; 0, 0)$	2	1
2	0	$\frac{1}{9}(6, 6, 6, 3, 3, 3; 0, 0)$	3	1
2	0	$\frac{1}{9}(3, 3, 3, 6, 6, 6; 0, 0)$	3	1
2	0	$\frac{1}{9}(6, 6, 6, 6, 6, 6; 0, 0)$	4	1
2	1	$\frac{1}{9}(3, 3, 3, 3, 0, 6; 0, 0)$	2	6
2	1	$\frac{1}{9}(3, 0, 6, 3, 3, 3; 0, 0)$	2	6
2	1	$\frac{1}{9}(6, 6, 6, 3, 0, 6; 0, 0)$	3	6
2	1	$\frac{1}{9}(3, 0, 6, 6, 6, 6; 0, 0)$	3	6
1	0	$\frac{1}{9}(2, 2, 5, 6, 3, 6; 3, 0)$	3	9
1	0	$\frac{1}{9}(6, 3, 6, 2, 2, 5; 0, 3)$	3	9
1	0	$\frac{1}{9}(3, 6, 3, 1, 7, 1; 0, 6)$	3	9
1	0	$\frac{1}{9}(1, 7, 1, 3, 6, 3; 6, 0)$	3	9
1	0	$\frac{1}{9}(3, 6, 3, 4, 4, 1; 0, 6)$	3	9
1	0	$\frac{1}{9}(4, 4, 1, 3, 6, 3; 6, 0)$	3	9
1	0	$\frac{1}{9}(6, 6, 3, 8, 2, 8; 0, 3)$	4	9
1	0	$\frac{1}{9}(8, 2, 8, 6, 6, 3; 3, 0)$	4	9
1	0	$\frac{1}{9}(6, 6, 3, 5, 5, 8; 0, 3)$	4	9
1	0	$\frac{1}{9}(5, 5, 8, 6, 6, 3; 3, 0)$	4	9
1	0	$\frac{1}{9}(7, 4, 7, 3, 3, 6; 6, 0)$	4	9
1	0	$\frac{1}{9}(3, 3, 6, 7, 4, 7; 0, 6)$	4	9

Table 2: Elements of Γ^T for which $r_{\gamma} \geq n_{\gamma}$.

TABLE 3

r_{γ}	n_{γ}	element	a_{γ}
2	6	$(0, 0, 0, 0, 0, 0; 0, 0)$	0
2	3	$\frac{1}{9}(0, 0, 0, 3, 3, 3; 0, 0)$	1
2	3	$\frac{1}{9}(3, 3, 3, 0, 0, 0; 0, 0)$	1
2	3	$\frac{1}{9}(0, 0, 0, 6, 6, 6; 0, 0)$	2
2	3	$\frac{1}{9}(0, 0, 0, 6, 6, 6; 0, 0)$	2

Table 3: Elements of Γ^T for which $r_{\gamma} < n_{\gamma}$.

TABLE 4

$H_{\Gamma^r}^{1,1}(W_1, W_2)$	\rightarrow	$H_{\Gamma^r}^{2,1}(W_1, W_2)$	$H_{\Gamma^r}^{1,1}(W_1, W_2)$	\rightarrow	$H_{\Gamma^r}^{2,1}(W_1, W_2)$
$dt _{\frac{1}{9}}(3, 3, 3, 3, 3, 0, 0)$	\mapsto	$p_1x_1^3 id$			
$dt _{\frac{1}{9}}(3, 3, 3, 6, 6, 6, 0, 0)$	\mapsto	$p_2X_1X_2X_3 id$	$dt _{\frac{1}{9}}(3, 6, 3, 1, 7, 1; 0, 6)$	\mapsto	$p_1x_2X_2^2 id$
$dt _{\frac{1}{9}}(6, 6, 6, 3, 3, 3, 0, 0)$	\mapsto	$p_1x_1x_2x_3 id$	$dt _{\frac{1}{9}}(3, 6, 3, 7, 1, 1; 0, 6)$	\mapsto	$p_1x_2X_1^2 id$
$dpx_1^3 _{\frac{1}{9}}(0, 0, 0, 3, 3, 3; 0, 0)$	\mapsto	$p_2x_1^3 id$	$dt _{\frac{1}{9}}(6, 3, 3, 1, 7, 1; 0, 6)$	\mapsto	$p_1x_1X_2^2 id$
$dpx_2^3 _{\frac{1}{9}}(0, 0, 0, 3, 3, 3; 0, 0)$	\mapsto	$p_2x_2^3 id$	$dt _{\frac{1}{9}}(6, 3, 3, 7, 1, 1; 0, 6)$	\mapsto	$p_1x_1X_1^2 id$
$dpx_3^3 _{\frac{1}{9}}(3, 3, 3, 0, 0, 0; 0, 0)$	\mapsto	$p_1X_1^3 id$	$dt _{\frac{1}{9}}(3, 3, 6, 1, 1, 7; 0, 6)$	\mapsto	$p_1x_3X_3^2 id$
$dpx_2^3 _{\frac{1}{9}}(3, 3, 3, 0, 0, 0; 0, 0)$	\mapsto	$p_1X_2^3 id$	$dt _{\frac{1}{9}}(3, 6, 3, 1, 1, 7; 0, 6)$	\mapsto	$p_1x_2X_3^2 id$
$dt _{\frac{1}{9}}(3, 3, 3, 3, 0, 6; 0, 0)$	\mapsto	$p_1X_1X_3^2 id$	$dt _{\frac{1}{9}}(6, 3, 3, 1, 1, 7; 0, 6)$	\mapsto	$p_1x_1X_3^2 id$
$dt _{\frac{1}{9}}(3, 3, 3, 0, 3, 6; 0, 0)$	\mapsto	$p_1X_2X_3^2 id$	$dt _{\frac{1}{9}}(3, 3, 6, 1, 7, 1; 0, 6)$	\mapsto	$p_1x_3X_2^2 id$
$dt _{\frac{1}{9}}(3, 3, 3, 0, 6, 3; 0, 0)$	\mapsto	$p_1X_2^2X_3 id$	$dt _{\frac{1}{9}}(3, 3, 6, 7, 1, 1; 0, 6)$	\mapsto	$p_1x_3X_1^2 id$
$dt _{\frac{1}{9}}(3, 3, 3, 6, 0, 3; 0, 0)$	\mapsto	$p_1X_1^2X_3 id$	$dt _{\frac{1}{9}}(3, 6, 3, 4, 4, 1; 0, 6)$	\mapsto	$p_1x_2X_1X_2 id$
$dt _{\frac{1}{9}}(3, 3, 3, 6, 3, 0; 0, 0)$	\mapsto	$p_1X_1^2X_2 id$	$dt _{\frac{1}{9}}(6, 3, 3, 4, 4, 1; 0, 6)$	\mapsto	$p_1x_1X_1X_2 id$
$dt _{\frac{1}{9}}(3, 3, 3, 3, 6, 0; 0, 0)$	\mapsto	$p_1X_1X_2^2 id$	$dt _{\frac{1}{9}}(3, 3, 6, 4, 1, 4; 0, 6)$	\mapsto	$p_1x_3X_1X_3 id$
$dt _{\frac{1}{9}}(0, 3, 6, 3, 3, 3; 0, 0)$	\mapsto	$p_2X_2X_3^2 id$	$dt _{\frac{1}{9}}(3, 3, 6, 1, 4, 4; 0, 6)$	\mapsto	$p_1x_3X_2X_3 id$
$dt _{\frac{1}{9}}(3, 0, 6, 3, 3, 3; 0, 0)$	\mapsto	$p_2X_1X_3^2 id$	$dt _{\frac{1}{9}}(3, 6, 3, 4, 1, 4; 0, 6)$	\mapsto	$p_1x_2X_1X_2 id$
$dt _{\frac{1}{9}}(6, 0, 3, 3, 3, 3; 0, 0)$	\mapsto	$p_2X_1^2X_3 id$	$dt _{\frac{1}{9}}(3, 6, 3, 1, 4, 4; 0, 6)$	\mapsto	$p_1x_2X_2X_3 id$
$dt _{\frac{1}{9}}(0, 6, 3, 3, 3, 3; 0, 0)$	\mapsto	$p_2X_2^2X_3 id$	$dt _{\frac{1}{9}}(6, 3, 3, 4, 1, 4; 0, 6)$	\mapsto	$p_1x_1X_1X_3 id$
$dt _{\frac{1}{9}}(3, 6, 0, 3, 3, 3; 0, 0)$	\mapsto	$p_2X_1X_2^2 id$	$dt _{\frac{1}{9}}(6, 3, 3, 1, 4, 4; 0, 6)$	\mapsto	$p_1x_1X_2X_3 id$
$dt _{\frac{1}{9}}(6, 3, 0, 3, 3, 3; 0, 0)$	\mapsto	$p_2X_1^2X_2 id$	$dt _{\frac{1}{9}}(3, 3, 6, 4, 4, 1; 0, 6)$	\mapsto	$p_1x_3X_1X_2 id$
$dt _{\frac{1}{9}}(2, 2, 5, 6, 3, 6; 3, 0)$	\mapsto	$p_2x_3X_1X_3 id$	$dt _{\frac{1}{9}}(1, 1, 7, 3, 3, 6; 6, 0)$	\mapsto	$p_2x_3^2X_3 id$
$dt _{\frac{1}{9}}(2, 2, 5, 3, 6, 6; 3, 0)$	\mapsto	$p_2x_3X_2X_3 id$	$dt _{\frac{1}{9}}(1, 7, 1, 3, 6, 3; 6, 0)$	\mapsto	$p_2x_2^2X_2 id$
$dt _{\frac{1}{9}}(2, 2, 5, 6, 6, 3; 3, 0)$	\mapsto	$p_2x_3X_1X_2 id$	$dt _{\frac{1}{9}}(1, 7, 1, 6, 3, 3; 6, 0)$	\mapsto	$p_2x_2^2X_1 id$
$dt _{\frac{1}{9}}(2, 5, 2, 6, 3, 6; 3, 0)$	\mapsto	$p_2x_2X_1X_3 id$	$dt _{\frac{1}{9}}(7, 1, 1, 3, 6, 3; 6, 0)$	\mapsto	$p_2x_1^2X_2 id$
$dt _{\frac{1}{9}}(2, 5, 2, 3, 6, 6; 3, 0)$	\mapsto	$p_2x_2X_2X_3 id$	$dt _{\frac{1}{9}}(7, 1, 1, 6, 3, 3; 6, 0)$	\mapsto	$p_2x_1^2X_1 id$
$dt _{\frac{1}{9}}(5, 2, 2, 6, 3, 6; 3, 0)$	\mapsto	$p_2x_1X_1X_3 id$	$dt _{\frac{1}{9}}(1, 1, 7, 3, 6, 3; 6, 0)$	\mapsto	$p_2x_3^2X_2 id$
$dt _{\frac{1}{9}}(5, 2, 2, 3, 6, 6; 3, 0)$	\mapsto	$p_2x_1X_2X_3 id$	$dt _{\frac{1}{9}}(1, 1, 7, 6, 3, 3; 6, 0)$	\mapsto	$p_2x_3^2X_1 id$
$dt _{\frac{1}{9}}(2, 5, 2, 6, 6, 3; 3, 0)$	\mapsto	$p_2x_2X_1X_2 id$	$dt _{\frac{1}{9}}(1, 7, 1, 3, 3, 6; 6, 0)$	\mapsto	$p_2x_2^2X_3 id$
$dt _{\frac{1}{9}}(5, 2, 2, 6, 6, 3; 3, 0)$	\mapsto	$p_2x_1X_1X_2 id$	$dt _{\frac{1}{9}}(7, 1, 1, 3, 3, 6; 6, 0)$	\mapsto	$p_2x_1^2X_3 id$
$dt _{\frac{1}{9}}(6, 3, 6, 2, 2, 5; 0, 3)$	\mapsto	$p_1x_1x_3X_3 id$	$dt _{\frac{1}{9}}(4, 4, 1, 3, 6, 3; 6, 0)$	\mapsto	$p_2x_1x_2X_2 id$
$dt _{\frac{1}{9}}(3, 6, 6, 2, 2, 5; 0, 3)$	\mapsto	$p_1x_2x_3X_3 id$	$dt _{\frac{1}{9}}(4, 4, 1, 6, 3, 3; 6, 0)$	\mapsto	$p_2x_1x_2X_1 id$
$dt _{\frac{1}{9}}(6, 3, 6, 2, 5, 2; 0, 3)$	\mapsto	$p_1x_1x_3X_2 id$	$dt _{\frac{1}{9}}(4, 1, 4, 3, 3, 6; 6, 0)$	\mapsto	$p_2x_1x_3X_3 id$
$dt _{\frac{1}{9}}(6, 3, 6, 5, 2, 2; 0, 3)$	\mapsto	$p_1x_1x_3X_1 id$	$dt _{\frac{1}{9}}(1, 4, 4, 3, 3, 6; 6, 0)$	\mapsto	$p_2x_2x_1X_3 id$
$dt _{\frac{1}{9}}(3, 6, 6, 2, 5, 2; 0, 3)$	\mapsto	$p_1x_2x_3X_2 id$	$dt _{\frac{1}{9}}(4, 4, 1, 3, 3, 6; 6, 0)$	\mapsto	$p_2x_1x_2X_3 id$
$dt _{\frac{1}{9}}(3, 6, 6, 5, 2, 2; 0, 3)$	\mapsto	$p_1x_2x_3X_1 id$	$dt _{\frac{1}{9}}(4, 1, 4, 3, 6, 3; 6, 0)$	\mapsto	$p_2x_1x_3X_2 id$
$dt _{\frac{1}{9}}(6, 6, 3, 2, 2, 5; 0, 3)$	\mapsto	$p_1x_1x_2X_3 id$	$dt _{\frac{1}{9}}(4, 1, 4, 6, 3, 3; 6, 0)$	\mapsto	$p_2x_1x_3X_1 id$
$dt _{\frac{1}{9}}(6, 6, 3, 2, 5, 2; 0, 3)$	\mapsto	$p_1x_1x_2X_2 id$	$dt _{\frac{1}{9}}(1, 4, 4, 3, 6, 3; 6, 0)$	\mapsto	$p_2x_2x_3X_2 id$
$dt _{\frac{1}{9}}(6, 6, 3, 5, 2, 2; 0, 3)$	\mapsto	$p_1x_1x_2X_1 id$	$dt _{\frac{1}{9}}(1, 4, 4, 6, 3, 3; 6, 0)$	\mapsto	$p_2x_2x_3X_1 id$

Table 4: The complete description of the mirror map on the generators of $\mathcal{H}_{\Gamma^r}^{1,1}$, grouped according to the subdivision described above.

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STEFANO FILIPAZZI

Department of Mathematics

University of Utah

e-mail: filipazz@math.utah.edu

FRANCO ROTA

Department of Mathematics

University of Utah

e-mail: rota@math.utah.edu