



Lim, B. and Rota, F. (2022) Characteristic classes and stability conditions for projective Kleinian orbisurfaces. *Mathematische Zeitschrift*, 300(1), pp. 827-849. (doi: [10.1007/s00209-021-02805-8](https://doi.org/10.1007/s00209-021-02805-8))

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CHARACTERISTIC CLASSES AND STABILITY CONDITIONS FOR PROJECTIVE KLEINIAN ORBISURFACES

BRONSON LIM AND FRANCO ROTA

Abstract. We construct Bridgeland stability conditions on the derived category of smooth quasi-projective Deligne–Mumford surfaces whose coarse moduli spaces have ADE singularities. This unifies the construction for smooth surfaces and Bridgeland’s work on Kleinian singularities. The construction hinges on an orbifold version of the Bogomolov–Gieseker inequality for slope semistable sheaves on the stack, and makes use of the Toën–Hirzebruch–Riemann–Roch theorem.

1. Introduction

Bridgeland Stability. Stability conditions were introduced by Bridgeland in [5] following work of Douglas on Π -stability [12]. Since then, the problem of constructing stability conditions has been investigated successfully for triangulated categories \mathcal{D} coming from a variety of different sources. In fact, stability conditions are completely classified if \mathcal{D} is the derived category of a smooth curve (see [20] and references therein) and there is a procedure to construct stability conditions on derived categories of smooth projective surfaces (a first construction appears in [6], and is then generalized in [1]. See also the survey [21] for a thorough account on the matter). On a related line of investigation, Bridgeland studies the stability manifold of categories associated with the class of ADE surface singularities [7]. The most recent results on this matter concern threefolds [2], [3], [18].

In this work, we extend the general construction for surfaces to $\mathcal{D} = D^b(\mathrm{Coh}(\mathcal{S}))$, where \mathcal{S} is the canonical stack associated with a projective surface S with ADE singularities. The main result of the paper is Theorem 4.7: it unifies the construction for smooth surfaces [1] and Bridgeland’s work on Kleinian singularities [7]. To prove Theorem 4.7 we develop a strengthening of the Bogomolov–Gieseker inequality for slope semistable sheaves on \mathcal{S} , and make use of the Toën–Hirzebruch–Riemann–Roch theorem [23].

Notation and conventions. Throughout, we work over the field of complex numbers. We denote by $\pi: \mathcal{S} \rightarrow S$ the canonical stack associated with a surface S with isolated quotient singularities, and by $f: \tilde{S} \rightarrow S$ its minimal resolution. For a finite subgroup $G \subset \mathrm{SL}_2$, we denote by $\rho_0 = \mathbb{1}$ its trivial representation, and by ρ_i , $i = 1, \dots, M$, its non-trivial ones. We set $N := |G|$.

If X is a smooth scheme or algebraic stack, $D(X) := D^b(\mathrm{Coh}(X))$ denotes the bounded derived category of coherent sheaves on X .

2010 *Mathematics Subject Classification.* 14E16; 14A20, 14F05.

Key words and phrases. Kleinian singularities, orbisurfaces, Bogomolov inequality, Bridgeland stability conditions.

Summary of results. Let \mathcal{S} be the canonical stack associated to a projective surface S with a unique Kleinian singularity, and denote by \tilde{S} its minimal resolution. Let $\iota: BG \rightarrow \mathcal{S}$ be the associated residual gerbe. If H is an ample divisor on \mathcal{S} , we define the slope of a sheaf E on \mathcal{S} :

$$\mu(E) = \frac{H \cdot \text{ch}_1(E)}{\text{rk}(E)},$$

and its discriminant $\Delta(E) := \text{ch}_1(E)^2 - 2 \text{ch}_0(E) \text{ch}_2(E)$ (the Chern classes are the ones introduced in [27]). Tilting with respect to slope, we construct a heart of a bounded t-structure $\text{Coh}^b(\mathcal{S})$, for $b \in \mathbb{R}$.

To define a suitable central charge, we need to take into account the orbifold cohomology of \mathcal{S} . We have $[\mathbf{L} \iota^* E] = \sum_{i=0}^M a_i \rho_i$, and define the *orbifold Chern character* of E as

$$\text{ch}_{orb}(E) = (\text{ch}(E), a_0, \dots, a_M).$$

Toën's version of the Riemann Roch theorem (Theorem 2.9) involves a function

$$(1) \quad \delta(E) = \sum_i a_i T_i$$

where the T_i are rational coefficients that depend on G . For $w \in \mathbb{C}$ and $\gamma \in \mathbb{R}$, let $Z_{w,\gamma}$ be the function

$$Z_{w,\gamma}(E) = Z(\text{ch}_{orb}(E)) = -\text{ch}_2(E) + w \text{ch}_0(E) + \gamma \cdot \delta(E) + iH \cdot \text{ch}_1(E).$$

The main result of the paper is Theorem 4.7:

Theorem 1.1. *Let $N := |G|$, and choose parameters $\gamma \in (0, \frac{1}{N-1})$ and $w \in \mathbb{C}$ such that:*

- (i) $\text{Re } w > -\frac{(\text{Im } w)^2}{H^2} + (2 + \gamma)D - (1 + \gamma)^2$;
- (ii) $\text{Re } w > \frac{1}{2} \frac{(\text{Im } w)^2}{H^2} - \gamma(D - \frac{N-1}{N}) > 0$.

Then, the pair $(Z_{w,\gamma}, \text{Coh}^{-\text{Im } w}(\mathcal{S}))$ is a stability condition on \mathcal{D} .

The proof hinges on a Bogomolov–Gieseker type inequality involving the *orbifold discriminant*: this is defined through the McKay equivalence $\Phi: \mathcal{D} \xrightarrow{\sim} D^b(\text{Coh}(\tilde{S}))$ [8], as the form

$$\Delta_{orb}(E) := \Delta(\Phi(E)).$$

We obtain Theorem 3.1, which strengthens the usual Bogomolov–Gieseker inequality.

Theorem 1.2. *Let E be a μ_H -semistable sheaf on \mathcal{S} . Then, $\Delta_{orb}(E) \geq 0$.*

The form Δ_{orb} also plays a crucial role in proving the support property, as it turns out to be negative definite on the kernel of $Z_{w,\gamma}$ (Lemma 4.14).

In Section 5 we study wall-crossing for objects of class $[\mathcal{O}_x]$, where $x \in \mathcal{S}$ is a closed point with trivial stabilizer. As a result, we find stability conditions σ and σ_0 such that the corresponding moduli spaces coincide with S and \tilde{S} , respectively, and that the wall-crossing morphism

$$M_{\sigma_0}([\mathcal{O}_x]) \rightarrow M_{\sigma}([\mathcal{O}_x])$$

is exactly the minimal resolution of S (Proposition 5.3). Section 5.1 further illustrates the relation between our construction, quiver stability [15], and [7].

Remark 1.3. In light of the equivalence $\mathcal{D} \xrightarrow{\sim} D^b(\text{Coh}(\tilde{S}))$, this work can be compared to some previous results on surfaces. In fact, it is closely related to [24], where the authors construct stability conditions on smooth surfaces admitting a curve of negative self-intersection. Theorem 4.7 overlaps with [24, Theorem 5.4] in the A_2 singularity case.

The heart used in [24] is constructed by tilting coherent sheaves twice (this construction also appears in [22]), while working on the stack appears to be a more natural choice, as it only requires one tilt, in accordance with the expectation for surfaces.

Acknowledgements. We wish to thank Aaron Bertram and Michael Wemyss for the many fruitful conversations on this topic.

2. Preliminaries

2.1. Kleinian orbisurfaces.

Definition 2.1. An *orbisurface* is a smooth and proper Deligne-Mumford surface such that the stacky locus has codimension 2.

For any orbisurface \mathcal{S} , and geometric point $s \in \mathcal{S}$, there is an étale local chart near s :

$$j_s: [U/\text{st}(s)] \rightarrow \mathcal{S}$$

where $U \subset \mathbb{A}^2$ is open and $\text{st}(s)$ is the stabilizer group of s acting through GL_2 . The mapping j_s induces a closed embedding

$$j_s: [*/\text{st}(s)] \rightarrow \mathcal{S}$$

called the *residual gerbe* at s . We denote by BG the quotient stack $[/math>$

An orbisurface is *Kleinian* if for each $s \in \mathcal{S}$, the stabilizer group acts through SL_2 . And an orbisurface is an A_{N-1} -orbisurface if it is Kleinian and the non-trivial stabilizer groups are cyclic of order N .

Let S be a surface with Kleinian singularities. Then there exists a Kleinian orbisurface \mathcal{S} and a map $\pi: \mathcal{S} \rightarrow S$ such that:

- the restriction $\mathcal{S} \setminus \pi^{-1}(\text{Sing}(S)) \rightarrow S \setminus \text{Sing}(S)$ is an isomorphism;
- π is universal among all dominant, codimension preserving maps to S .

The stack \mathcal{S} is called the *canonical stack* associated with the surface S , see [13].

A line bundle on \mathcal{S} is *ample* if it is the pullback of an ample line bundle on the coarse space S . An orbisurface is *projective* if the coarse moduli is projective.

Example 2.2. The weighted projective plane $\mathbb{P}_{1,1,N}$ has canonical stack the stacky weighted projective plane

$$\mathbf{P}_{1,1,N} = [(\mathbb{C}_{1,1,N}^3 \setminus \{0\})/\mathbb{C}_m]$$

where the subscript indicates the weights of the \mathbb{C}^* -action. That is, $\lambda \in \mathbb{C}^*$ acts by $\lambda(x, y, z) = (\lambda x, \lambda y, \lambda^N z)$. There is a unique stacky point where x and y are zero with residual gerbe $B\mu_N$. Thus the stacky weighted projective plane is a projective A_{N-1} -orbisurface.

Example 2.3. The local model for a surface with an A_{N-1} singularity is the hypersurface

$$S = \{x^2 + y^2 + z^N = 0\}$$

in \mathbb{C}^3 . The canonical stack is the A_{N-1} -orbisurface

$$\mathcal{S} = [\mathbb{C}^2/\mu_N]$$

where $\lambda \in \mu_N$ acts via $\lambda(u, v) = (\lambda u, \lambda^{-1}v)$.

Although we are primarily interested in the case where there is a unique stacky point, the following example should be kept in mind.

Example 2.4. Let A be an Abelian surface and let $\mu_2 = \langle -1 \rangle$ act on A via negation, i.e. $-1 \cdot a = -a$ for all $a \in A$. There are sixteen fixed points of this action. Thus the quotient stack $[A/\mu_2]$ is an A_1 -orbisurface with sixteen residual gerbes of type $B\mu_2$.

2.2. The derived McKay correspondence. A Kleinian orbisurface \mathcal{S} can be interpreted as a *stacky resolution of singularities* of its coarse moduli space S . The derived McKay correspondence [8] exhibits an equivalence Φ between the derived category $D(\mathcal{S})$ and that of the minimal resolution $f: \tilde{S} \rightarrow S$ of S .

Let $\tilde{\mathcal{C}}$ be the abelian subcategory of $\text{Coh}(\tilde{S})$ consisting of sheaves E such that $\mathbf{R}f_*(E) = 0$, and define a torsion pair:

$$\begin{aligned}\tilde{\mathcal{T}}_0 &:= \{T \in \text{Coh}(\tilde{S}) \mid \mathbf{R}^1 f_*(T) = 0\}; \\ \tilde{\mathcal{F}}_0 &:= \{F \in \text{Coh}(\tilde{S}) \mid f_*(F) = 0, \text{Hom}(\tilde{\mathcal{C}}, F) = 0\}.\end{aligned}$$

The heart of the bounded t -structure on $D(\tilde{S})$ obtained by tilting $\text{Coh}(\tilde{S})$ along the pair above is denoted ${}^0\text{Per}(\tilde{S}/S)$, its objects are called *perverse sheaves*. The reader is referred to [4] and [26] for the details on this construction.

The derived McKay correspondence, Φ , satisfies:

$$\Phi(\text{Coh}(\mathcal{S})) \simeq \langle \tilde{\mathcal{F}}_0[1], \tilde{\mathcal{T}}_0 \rangle = {}^0\text{Per}(\tilde{S}/S).$$

More explicitly, suppose S has a unique singular point p . Let \mathcal{S} be the associated canonical stack and, abusing notation, p the lift of the point p to \mathcal{S} . Denote by C the fundamental cycle of $\tilde{S} \rightarrow S$, and by C_i its irreducible components. Then we have

$$\begin{aligned}\Phi(\mathcal{O}_{\mathcal{S}}) &= \mathcal{O}_{\tilde{S}}; \\ \Phi(\mathcal{O}_p) &= \omega_C[1]; \\ \Phi(\mathcal{O}_p \otimes \rho_i) &= \mathcal{O}_{C_i}(-1), \quad i = 1, \dots, M.\end{aligned}$$

We fix a quasi-inverse Φ^{-1} of Φ and write $\mathcal{F}_0 := \Phi^{-1}(\tilde{\mathcal{F}}_0[1])$ and $\mathcal{T}_0 := \Phi^{-1}(\tilde{\mathcal{T}}_0)$, so that

$$\text{Coh}(\mathcal{S}) = \langle \mathcal{F}_0, \mathcal{T}_0 \rangle.$$

Moreover, the category \mathcal{C} of sheaves E on \mathcal{S} such that $\mathbf{R}\pi_*(E) = 0$ satisfies $\mathcal{C} = \Phi^{-1}\tilde{\mathcal{C}}$, and is generated by the sheaves $\mathcal{O}_p \otimes \rho_i, i \neq 0$.

We finish this section by recalling a definition which will be useful later.

Definition 2.5. Let W be a quasi-projective variety, on which a finite group G is acting. A G -constellation on W is a G -equivariant sheaf E on W with finite support such that $H^0(E)$ is isomorphic to the regular representation of G , as G -representation. A G -cluster is the structure sheaf \mathcal{O}_Z of a subscheme $Z \subset W$ which is also a G -constellation.

If G is a finite subgroup of GL_2 acting on \mathbb{C}^2 , then the space of G -clusters, denoted $G\text{-Hilb}(\mathbb{C}^2)$, is the minimal resolution of \mathbb{C}^2/G [8]. The skyscraper sheaves of points in the exceptional locus correspond under Φ^{-1} to clusters supported at the origin in \mathbb{C}^2 .

2.3. Characteristic classes. From now on, we assume that \mathcal{S} is a projective Kleinian orbisurface with a unique stacky point $p \in \mathcal{S}$ and residual gerbe $BG = [*/G]$. Set $\iota: BG \hookrightarrow \mathcal{S}$ the corresponding closed substack.

We use Vistoli's intersection theory in what follows [27]. In particular, Chern classes and Todd classes are defined, as well as a degree map. The Hodge index theorem still holds, i.e. the intersection form on $\text{NS}(\mathcal{S}) \otimes \mathbb{R}$ is of signature $(1, r - 1)$:

Theorem 2.6 (Hodge Index Theorem). *Suppose H is an ample Cartier divisor on \mathcal{S} . If $D \neq 0$ is a divisor such that $D \cdot H = 0$ then $D^2 < 0$.*

For E a sheaf, and H an ample divisor class on \mathcal{S} , the slope of E with respect to H is

$$\mu(E) = \frac{H \cdot \text{ch}_1(E)}{\text{rk}(E)},$$

with the convention that $\mu(E) = +\infty$ if $\text{rk}(E) = 0$. We say that E is μ -(semi)stable if for all non-zero proper subsheaves $E' \subset E$ one has $\mu(E') < (\leq) \mu(E/E')$.

Define also the discriminant of E by

$$\Delta(E) = (\text{ch}_1(E))^2 - 2\text{rk}(E) \text{ch}_2(E).$$

The usual Bogomolov-Gieseker inequality still holds on \mathcal{S} :

Theorem 2.7 ([19, Prop. 4.2.4]). *If E is a μ -semistable sheaf, then $\Delta(E) \geq 0$, or equivalently*

$$(2) \quad \text{ch}_2(E) \leq \frac{(\text{ch}_1(E))^2}{\text{rk}(E)}.$$

The results above only involve a part of the Grothendieck group of \mathcal{S} , and ignore contributions from the residual gerbe BG . The Grothendieck group of BG is free, Abelian and generated by the irreducible representations of G $\{\rho_i \mid i = 0, \dots, M\}$. For any perfect complex of sheaves E on \mathcal{S} , we have

$$[\mathbf{L}\iota^*E] = \sum_{i=0}^M a_i \rho_i.$$

Definition 2.8. Given a perfect complex $E \in \mathcal{D}(\mathcal{S})$, we define the *orbifold Chern character*

$$\text{ch}_{orb}(E) = (\text{ch}(E), a_0, \dots, a_M).$$

2.4. The Toën–Hirzebruch–Riemann–Roch theorem. We use a version of the Hirzebruch-Riemann-Roch theorem for smooth projective Deligne-Mumford stacks due to Toën [23]. The formula is analogous to the usual Hirzebruch-Riemann-Roch theorem, but it presents a correction term. For the convenience of the reader, we give a brief description of the formula, following [25, Appendix A].

Let $I\mathcal{S}$ denote the inertia stack of \mathcal{S} , and define a map

$$\rho: K(I\mathcal{S}) \rightarrow K(I\mathcal{S}) \otimes \mathbb{Q}(\mu_\infty)$$

as follows: if E is a bundle on $I\mathcal{S}$ decomposing as a sum $\bigoplus_{\zeta} E^{(\zeta)}$ of eigenbundles with eigenvalue ζ , let

$$\rho(E) = \sum_{\zeta} \zeta E^{(\zeta)}.$$

One then defines the weighted Chern character as the composition

$$\tilde{\text{ch}}: K(\mathcal{S}) \xrightarrow{\sigma^*} K(IS) \xrightarrow{\rho} K(IS) \xrightarrow{\text{ch}} H^*(IS)$$

where $\sigma: IS \rightarrow \mathcal{S}$ is the projection and ch is the usual Chern character. The weighted Todd class $\widetilde{\text{Td}}_{\mathcal{S}}$ is defined in a similar way [25, Def. A.0.5]. Then we have

Theorem 2.9 (Toën-Hirzebruch-Riemann-Roch). *Let E be a perfect complex of sheaves on \mathcal{S} , then*

$$\chi(E) = \int_{IS} \tilde{\text{ch}}(E) \cdot \widetilde{\text{Td}}_{\mathcal{S}} = \int_{\mathcal{S}} \tilde{\text{ch}}(E) \cdot \widetilde{\text{Td}}_{\mathcal{S}} + \delta(E)$$

where $\delta(E) := \int_{IS \setminus \mathcal{S}} \tilde{\text{ch}}(E) \cdot \widetilde{\text{Td}}_{\mathcal{S}}$ is the aforementioned correction term.

Our short term goal is now to investigate the term $\delta(E)$ in the case of a Kleinian orbisurface, by computing the weighted Chern characters of $[\mathbf{L} \iota^* E]$. The inertia stack of \mathcal{S} is

$$IS = \mathcal{S} \sqcup (IBG \setminus BG)$$

where

$$IBG \setminus BG = \bigsqcup_{(g) \neq (1)} BC_G(g)$$

(here, the union is taken over all conjugacy classes (g) of non-trivial elements $g \in G$). The degree of the Todd class on $IBG \setminus BG$ is given by the formula

$$\int_{IBG \setminus BG} \widetilde{\text{Td}}_{\mathcal{S}} = \sum_{(g) \neq (1)} \frac{1}{|C_G(g)|} \cdot \frac{1}{2 - \xi_g - \xi_g^{-1}}$$

where ξ_g and ξ_g^{-1} are the eigenvalues of the action of g on the tangent space $T_p \mathcal{S}$ of the stacky point on \mathcal{S} . This number is computed in [9] to be

$$\delta(\mathcal{O}_{\mathcal{S}}) = \frac{1}{12} \left(\chi_{\text{top}}(C_{\text{red}}) - \frac{1}{|G|} \right),$$

where C is the fundamental cycle of the minimal resolution (see Section 2.2).

The fiber of a sheaf E at p decomposes as $[\mathbf{L} \iota^* E] = \sum_{i=0}^M a_i \rho_i$ where the sum runs over all irreducible representations ρ_i of G . On $BC_G(g)$, the element g acts on ρ_i with eigenvalues $\zeta_i^{(l)}$, to which correspond eigenspaces $\rho_i^{(l)}$. Therefore, $\mathbf{L} \iota^* E$ decomposes on $BC_G(g)$ into weighted eigenbundles as $\sum_{i=0}^M \sum_{l=1}^{r_i} a_i \zeta_i^{(l)} \rho_i^{(l)}$.

Then, the weighted Chern character of $\mathbf{L} \iota^* E|_{BC_G(g)}$ is given by

$$(3) \quad \tilde{\text{ch}}(\mathbf{L} \iota^* E|_{BC_G(g)}) = \sum_{i=0}^M \sum_{l=1}^{r_i} a_i \zeta_i^{(l)} = \sum_{i=0}^M a_i \chi_i(g),$$

where $\chi_i := \chi_{\rho_i} = \text{Tr} \circ \rho_i$ is the character of the representation ρ_i . Our main interest lies in the following computation:

Lemma 2.10. *Let ρ be an irreducible representation of G of dimension r . Then the second Chern character of $\mathcal{O}_p \otimes \rho$ is $\frac{r}{N}$, and*

$$(4) \quad \delta(\mathcal{O}_p \otimes \rho) = \begin{cases} 1 - \frac{1}{N} & \text{if } \rho = \mathbb{1}; \\ -\frac{r}{N} & \text{if } \rho \neq \mathbb{1}. \end{cases}$$

Proof. This is a local computation and so we can assume $\mathcal{S} = [U/G]$ where U is an open subset of \mathbb{A}^2 . In this case, we have the equivariant Koszul complex (write V to denote $T_p\mathcal{S}$ as a representation of G)

$$0 \rightarrow \mathcal{O}_U \otimes \Lambda^2 V \cong \mathcal{O}_U \rightarrow \mathcal{O}_U \otimes V \rightarrow \mathcal{O}_U \rightarrow \mathcal{O}_p,$$

which resolves \mathcal{O}_p . Hence,

$$[\mathbf{L} \iota^*(\mathcal{O}_p \otimes \rho)] = (2 \cdot \mathbb{1} - V) \otimes \rho.$$

By Theorem 2.9 and multiplicativity of characters, the correction term is

$$\begin{aligned} \delta(\mathcal{O}_p \otimes \rho) &= \sum_{(g) \neq (I)} \frac{1}{|C_G(g)|} \cdot \frac{\widetilde{\text{ch}}(\mathbf{L} \iota^* \mathcal{O}_{p|BC_G(g)})}{2 - \xi_g - \xi_g^{-1}} \\ &= \sum_{(g) \neq (I)} \frac{1}{|C_G(g)|} \cdot \frac{(2\chi_{\mathbb{1}}(g) - \chi_V(g))\chi_\rho(g)}{2 - \chi_V(g)} \\ &= \sum_{(g) \neq (I)} \frac{\chi_\rho(g)}{|C_G(g)|} \end{aligned}$$

Denote by N_g the cardinality of the conjugacy class of $g \in G$, and write the orthogonality relation between characters:

$$\delta_{\mathbb{1}\rho} = \frac{1}{N} \sum_{g \in G} \chi_\rho(g) \overline{\chi_{\mathbb{1}}(g)} = \frac{1}{N} \sum_{(g)} N_g \chi_\rho(g) = \frac{1}{N} \sum_{(g)} \frac{N}{|C_G(g)|} \chi_\rho(g) = \sum_{(g)} \frac{\chi_\rho(g)}{|C_G(g)|},$$

where $\delta_{\mathbb{1}\rho} = 1$ if $\rho = \mathbb{1}$ and 0 otherwise.

The summand corresponding to $(g) = (I)$ is $\frac{\chi_\rho(I)}{N} = \frac{r}{N}$. Isolating it, one obtains

$$\sum_{(g) \neq (I)} \frac{\chi_\rho(g)}{|C_G(g)|} = \begin{cases} 1 - \frac{1}{N} & \text{if } \rho = \mathbb{1}; \\ -\frac{r}{N} & \text{if } \rho \neq \mathbb{1}. \end{cases}$$

Since $\chi(\mathcal{S}, \mathcal{O}_p \otimes \rho) = \chi(\mathcal{S}, \pi_*(\mathcal{O}_p \otimes \rho)) = \delta_{\mathbb{1}\rho}$, the statement about second Chern characters follows. □

3. A Bogmolov-Gieseker-type inequality for slope semistable sheaves

Let \mathcal{S} be a projective Kleinian orbisurface, denote $\pi: \mathcal{S} \rightarrow S$ the structure morphism, and $f: \tilde{S} \rightarrow S$ the minimal resolution of S . We keep our standing assumption that \mathcal{S} has only one stacky point p with residual gerbe BG and G acts through SL_2 .

Recall that for a sheaf E on \mathcal{S} or $\tilde{\mathcal{S}}$, and H an ample (on \mathcal{S}) divisor class, the slope of E with respect to H is

$$\mu_H(E) = \frac{H \cdot \text{ch}_1(E)}{\text{ch}_0(E)}.$$

and its discriminant is $\Delta(E) = (\text{ch}_1(E))^2 - \text{ch}_2(E) \text{ch}_0(E)$. The discriminant is non-negative on μ_H -semistable sheaves by (2). We seek a form, analog to Δ , which involves the whole ch_{orb} and enjoys a similar positivity property. We use the notation of Section 2.2 throughout.

Let $E \in \text{Coh}(\mathcal{S})$, define its *orbifold discriminant* $\Delta_{orb}(E) := \Delta(\Phi(E))$. The goal of this section is to prove the following:

Theorem 3.1. *Let H be ample on \mathcal{S} . Let E be a μ_H -semistable sheaf on \mathcal{S} . Let $\tilde{E} = \Phi E$ be its image on $\tilde{\mathcal{S}}$. Then, $\Delta_{orb}(E) = \Delta(\tilde{E}) \geq 0$.*

First, observe the following lemma:

Lemma 3.2. *Let E be a torsion-free sheaf on \mathcal{S} , then \tilde{E} is a sheaf.*

Proof. If \tilde{E} is not a sheaf, then there is an exact sequence

$$H^{-1}(\tilde{E})[1] \rightarrow \tilde{E} \rightarrow H^0(\tilde{E})$$

in ${}^0\text{Per}(\tilde{\mathcal{S}}/S)$, where $H^{-1}(\tilde{E})$ is a torsion sheaf, since it lies in $\tilde{\mathcal{F}}_0$. Applying Φ^{-1} to the sequence above one obtains a short exact sequence in $\text{Coh}(\mathcal{S})$

$$B \rightarrow E \rightarrow A.$$

with $B \in \mathcal{F}_0$. We have $\pi_*(B) = f_*(H^{-1}(\tilde{E})) = 0$ by definition of $\tilde{\mathcal{F}}_0$, so B is a torsion sheaf and E is not torsion-free. \square

Definition 3.3. We say that a sheaf E on \mathcal{S} (resp. \tilde{E} on $\tilde{\mathcal{S}}$) *descends to S* if the natural map

$$\pi^* \pi_* E \rightarrow E$$

(resp. $f^* f_* E \rightarrow E$) is an isomorphism.

Lemma 3.4. *\tilde{E} is a torsion-free sheaf if and only if E is torsion-free and it descends to S .*

Proof. By the previous lemma, if E is torsion-free then \tilde{E} is a sheaf. Now we show that if, additionally, E descends to S , then \tilde{E} is torsion-free. Suppose \tilde{E} has a torsion subsheaf \tilde{F} , for sake of contradiction. Then, applying Φ^{-1} to the sequence $\tilde{F} \rightarrow \tilde{E} \rightarrow \tilde{E}'$ and taking the associated long exact sequence of sheaves, one gets

$$H^0(\Phi^{-1}\tilde{F}) \rightarrow E \rightarrow E' \rightarrow H^1(\Phi^{-1}\tilde{F})$$

(the other terms in the long exact sequence of cohomology sheaves vanish because E is a sheaf, and $H^{-1}(E') = 0$ since images of sheaves under Φ^{-1} may only have cohomologies in degrees 0,1). The sheaf $H^0(\Phi^{-1}\tilde{F})$ is torsion: this follows from the triangle

$$H^0(\Phi^{-1}\tilde{F}) \rightarrow \Phi^{-1}\tilde{F} \rightarrow H^1(\Phi^{-1}\tilde{F})[-1]$$

and the fact that $H^1(\Phi^{-1}\tilde{F}) \in \mathcal{F}_0$ is torsion, and $\Phi^{-1}\tilde{F}$ has rank 0 since Φ and Φ^{-1} preserve ranks. Therefore, either E has torsion, or fits in a short exact sequence

$$(5) \quad E \rightarrow E' \rightarrow F$$

where $F := \Phi^{-1}\tilde{F} \in \mathcal{F}_0[-1]$. Then, there is a diagram with exact rows and columns

$$\begin{array}{ccccc} \pi^*\pi_*E & \longrightarrow & E & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ \pi^*\pi_*E' & \longrightarrow & E' & \longrightarrow & K \\ \downarrow & & \downarrow & & \downarrow \\ M & \longrightarrow & F & \longrightarrow & K \end{array}$$

where the top right corner is zero since E descends to S , and M is a repeated extension of pull-backs of \mathcal{O}_p from S . The middle row of the diagram shows $\mathbf{R}\pi_*K = 0$. If $M \neq 0$, we have $\pi_*M \neq 0$, contradicting $\pi_*F = f_*\tilde{F} = 0$. If $M = 0$, then $F \simeq K$ but $K \in \tilde{\mathcal{T}}_0$ while $F \in \tilde{\mathcal{F}}_0[-1]$.

Conversely, if E has torsion then \tilde{E} is either not a sheaf, or it has torsion, so we may assume that E is torsion-free. Then E fits in a sequence (5), where $F[-1]$ is the image of the torsion of \tilde{E} . If E does not descend to S , we have $K \neq 0$ in the diagram

$$\begin{array}{ccccc} \pi^*\pi_*E & \longrightarrow & E & \longrightarrow & K \\ \downarrow & & \downarrow & & \downarrow^a \\ \pi^*\pi_*E' & \longrightarrow & E' & \longrightarrow & K' \\ \downarrow & & \downarrow & & \\ M & \longrightarrow & F & & \end{array}$$

If $F = 0$ we must have $\ker(a) \simeq M$, which is a contradiction because $\mathbf{R}\pi_*K = \mathbf{R}\pi_*K' = 0$ while $\mathbf{R}\pi_*M \neq 0$. We showed that \tilde{E} has torsion whenever E has torsion or does not descend. \square

Next, we show:

Lemma 3.5. *Suppose E descends to S and is μ_H -semistable, then \tilde{E} is μ_H -semistable of the same slope.*

Proof. Since E descends and is torsion-free, \tilde{E} is torsion-free by Lemma 3.4. Suppose \tilde{E} is destabilized by a sequence $\tilde{F} \rightarrow \tilde{E} \rightarrow \tilde{K}$ with \tilde{F} torsion-free. Apply Φ^{-1} to the sequence and get a triangle

$$F \rightarrow E \rightarrow K$$

where E is a sheaf, F is a sheaf since \tilde{F} is torsion-free, and K has cohomologies in degree 0 and -1. This implies immediately that K is just a sheaf, and $F \rightarrow E$ is still injective. However, $\mu_H(F) = \mu_H(\tilde{F})$ and $\mu_H(E) = \mu_H(\tilde{E})$, so E is also unstable. \square

Corollary 3.6. *Let $E \in \text{Coh}(S)$, and $\tilde{E} := \Phi(E)$. If E descends to S and is μ_H -semistable, then $\Delta(\tilde{E}) \geq 0$.*

Proof. By Lemma 3.5, \tilde{E} is slope semistable, so it satisfies the Bogomolov inequality on \tilde{S} . \square

We are finally ready to prove Theorem 3.1:

Proof of 3.1. If E descends to S , then \tilde{E} is slope-semistable, and Cor. 3.6 applies. Otherwise, the sheaf \tilde{E} may have a torsion subsheaf \tilde{T} , supported on the exceptional locus, and fit in a sequence

$$\tilde{T} \rightarrow \tilde{E} \rightarrow \tilde{E}'$$

where \tilde{E}' is torsion-free. In particular, $\mu_H(\tilde{E}) = \mu_H(\tilde{E}')$. Notice moreover that $\tilde{T} \in \tilde{\mathcal{F}}_0$, otherwise E would have torsion. Now consider

$$(6) \quad \Delta(\tilde{E}) = \Delta(\tilde{E}') + 2 \operatorname{ch}_1(\tilde{E}') \operatorname{ch}_1(\tilde{T}) + \operatorname{ch}_1(\tilde{T}^2) - 2 \operatorname{ch}_0(\tilde{E}') \operatorname{ch}_2(\tilde{T}).$$

First, we claim \tilde{E}' is μ_H -semistable. We show this by showing that E' is semistable, and applying Lemma 3.5. A destabilizer F' of E' either factors through the inclusion $E \rightarrow E'$, or it fits in a diagram

$$\begin{array}{ccccc} F & \longrightarrow & F' & \longrightarrow & T' \\ \downarrow & & \downarrow & & \downarrow \\ E & \longrightarrow & E' & \longrightarrow & T \end{array}$$

where $T = \Phi^{-1}\tilde{T}[1]$ is a sheaf, T' is the image of F' in T , and F is a subsheaf of E with $\mu_H(F) = \mu_H(F') > \mu_H(E') = \mu_H(E)$, which destabilizes E . Hence we have $\Delta(E') \geq 0$.

Observe that the summand $2 \operatorname{ch}_1(\tilde{E}') \operatorname{ch}_1(\tilde{T})$ in (6) vanishes. In fact, if \tilde{E}' descends to S , then $\operatorname{ch}_1(\tilde{E}')$ is orthogonal to the exceptional curve, and hence it's orthogonal to $\operatorname{ch}_1(\tilde{T})$.

It remains to understand the last two summands. By construction, the sheaf T is a repeated extension of at most $\operatorname{rk}(E')$ proper quotients of clusters (see Sec. 2.2). Lemma 3.8 below, applied with $M = \operatorname{rk}(E')$, shows that

$$(7) \quad \operatorname{ch}_1(\tilde{T}^2) - 2 \operatorname{ch}_0(\tilde{E}') \operatorname{ch}_2(\tilde{T}) \geq 0.$$

The conclusion is that $\Delta(\tilde{E}) \geq 0$. \square

To prove Lemma 3.8, we need to understand the structure of proper quotients of clusters:

Lemma 3.7. *Quotients of clusters correspond under the McKay functor Φ to sheaves $L_D \subseteq \omega_C$ whose Chern character satisfies $\operatorname{ch}(L_D) = (0, D, -1)$, where $D \leq C$ is an effective one dimensional cycle whose coefficients are those of a root in the root system associated to the singularity. In particular, if D_1 and D_2 are two such cycles, then $D_1 \cdot D_2 \geq -2$.*

Proof. Consider the proper quotient Q of a cluster M , and the diagram with exact rows

$$\begin{array}{ccccc} K' & \longrightarrow & M & \longrightarrow & Q \\ \downarrow & & \downarrow & & \downarrow \\ K & \longrightarrow & M & \longrightarrow & \mathcal{O}_p \end{array}$$

Denote by H , resp. H' , the images of K , resp. K' under the McKay functor. Since H' is built of repeated extensions of $\mathcal{O}_{C_i}(-1)$, we get immediately that $L := \Phi(Q)[-1]$ satisfies $[L] = [H'] - [C_p]$ and has Chern character $\operatorname{ch}(L) = (0, D, -1)$ where D is a positive linear combination of the C_i . Moreover, the image under Φ of the diagram above exhibits L as a subobject of $\Phi(\mathcal{O}_p)[-1] = \omega_C$.

What is left to argue is that subobjects of ω_C with Chern character $(0, D, -1)$ must satisfy that the coefficients of D are those of a positive root of the associated root system. Suppose L is as above. Since $\iota_* L \subseteq \iota_* \omega_C$, we must have

$$\operatorname{ch}_2 \iota_* L \leq \operatorname{ch}_2 \iota_*(\omega_C)|_D - (C - D) \cdot D$$

(its degree cannot exceed that of the restriction of $\iota_* \omega_C$, but its sections must additionally vanish along the intersection between D and its complement). Now $\operatorname{ch}_2 \iota_* L = -1$, and $\operatorname{ch}_2 \iota_*(\omega_C)|_D =$

$C.D - D^2/2$ by the Grothendieck–Riemann–Roch theorem. Then the inequality above reads

$$-2 \leq D^2.$$

Identify D with an element α of the root lattice. Then the inequality states

$$2 \geq \langle \alpha, \alpha \rangle$$

which can only happen (realizing equality) if α is a root. For the last statement, it suffices to observe that if D_1 and D_2 correspond to roots α and β , then $D_1.D_2 = -\langle \alpha, \beta \rangle \geq -2$. This follows from the theory of simply laced root systems. \square

We can then show:

Lemma 3.8. *Let T be a repeated extension of at most M proper quotients of clusters. Denote $\tilde{T} := \Phi(T)$, then*

$$(8) \quad \text{ch}_1(\tilde{T}^2) - 2M \text{ch}_2(\tilde{T}) \geq 0.$$

Proof. Quotients of clusters correspond to objects L_D as described in Lemma 3.7.

Since $[\tilde{T}] = \sum_{j=1}^m [L_{D_j}]$ in $K(S)$, with $m \leq M$, we have $\text{ch}(\tilde{T}) = (0, \sum_j D_j, -m)$. Then

$$\text{ch}_1(\tilde{T}^2) - 2M \text{ch}_2(\tilde{T}) = \left(\sum_{i=1}^m D_i \right)^2 + 2Mm \geq -2m^2 + 2Mm \geq 0,$$

where the first inequality is a consequence of the last statement of Lemma 3.7. \square

4. Stability conditions on $D^b(\mathcal{S})$

In this section we construct stability conditions on $D^b(\mathcal{S})$.

4.1. Background on Bridgeland stability. We recall some aspects of the theory of stability conditions in what follows, and direct the interested reader to the seminal works of Bridgeland [5], [6] and to the survey [21]. Let \mathcal{D} be a triangulated category.

Definition 4.1. A *heart of a bounded t-structure* in \mathcal{D} is a full additive subcategory $\mathcal{A} \subset \mathcal{D}$ satisfying the following properties:

- (i) $\text{Hom}^i(A, B) = 0$ for $i < 0$;
- (ii) every object in \mathcal{D} has a filtration by cohomology objects in \mathcal{A} . In other words, for all non-zero $E \in \mathcal{D}$ there are integers $k_1 > \dots > k_m$ and a collection of triangles

$$\begin{array}{ccccccccccc}
 0 = E_0 & \longrightarrow & E_1 & \longrightarrow & E_2 & \longrightarrow & \dots & \longrightarrow & E_{m-1} & \longrightarrow & E_m = E \\
 & & \swarrow & & \swarrow & & & & \swarrow & & \swarrow \\
 & & A_1 & & A_2 & & & & A_m & &
 \end{array}$$

where $A_i[-k_i] \in \mathcal{A}$.

One can check that if \mathcal{A} is the heart of a bounded t-structure in \mathcal{D} , then \mathcal{A} is abelian.

The definition of a stability condition also involves the choice of a finite rank lattice Λ and a surjective group homomorphism $v: K(\mathcal{D}) \twoheadrightarrow \Lambda$.

Definition 4.2. A *pre-stability condition* on \mathcal{D} is a pair $\sigma = (Z, \mathcal{A})$ where:

- (i) \mathcal{A} is the heart of a bounded t-structure in $D^b(\mathcal{S})$;
- (ii) $Z: \Lambda \rightarrow \mathbb{C}$ is an additive homomorphism called the *central charge*;

and they satisfy the following properties:

- (1) For any non-zero $E \in \mathcal{A}$,

$$Z(v(E)) \in \mathbb{R}_{>0} \cdot e^{i\pi\phi}$$

with $\phi \in (0, 1]$. Define the *phase* of $0 \neq E \in \mathcal{A}$ to be $\phi(E) := \phi$. We say that $E \in \mathcal{A}$ is σ -*semistable* if for all non-zero subobjects $F \in \mathcal{A}$ of E , $\phi(F) \leq \phi(E)$. E is σ -*stable* if for all non-zero proper subobjects $F \in \mathcal{A}$ of E , $\phi(F) < \phi(E)$. We denote by $\mathcal{P}(\phi)$ the category of semistable objects of phase ϕ .

- (2) (HN filtrations) The objects of \mathcal{A} have *Harder-Narasimhan filtrations* with respect to Z . In other words, for every $E \in \mathcal{A}$ there is a unique filtration

$$0 = E_0 \subset E_1 \subset \dots \subset E_{n-1} \subset E_n = E$$

such that the quotients $E_i/E_{i-1} \in \mathcal{P}(\phi_i)$ with $\phi_1 > \phi_2 > \dots > \phi_n$.

Definition 4.3. A pre-stability condition σ is a *stability condition* if it additionally satisfies the *support property*, i.e.

$$C_\sigma := \inf \left\{ \frac{|Z(v(E))|}{\|v(E)\|} : 0 \neq E \in \mathcal{P}(\phi), \phi \in \mathbb{R} \right\} > 0.$$

There is an alternative characterization of the support property, given by the following Proposition [16, Section 2.1]:

Proposition 4.4. A pre-stability condition $\sigma = (Z, \mathcal{A})$ satisfies the support property if and only if there exists a quadratic form Q such that Q is negative definite on the kernel of Z , and $Q(E) \geq 0$ for every σ -semistable object E in \mathcal{A} .

Let $\text{Stab}(\mathcal{D})$ denote the set of stability conditions on \mathcal{D} . In [5, Sec. 6], the author defines a generalized metric f on $\text{Stab}(\mathcal{D})$ which makes it into a topological space. Moreover, a deformation result holds:

Theorem 4.5 ([5, Thm. 7.1]). Let $\sigma = (Z, \mathcal{A}) \in \text{Stab}(\mathcal{D})$. Then for every $\epsilon > 0$ there exists a disc $\Delta \subset \text{Hom}(\Lambda, \mathbb{C})$, centered at Z , such that for every $W \in \Delta$ there exists a stability condition $\tau = (W, \mathcal{A}')$ with $f(\sigma, \tau) < \epsilon$.

In turn, this leads to the following:

Theorem 4.6 ([5, Thm. 1.2]). The central charge map $\varpi: \text{Stab}(\mathcal{D}) \rightarrow \text{Hom}(\Lambda, \mathbb{C})$ given by $(Z, \mathcal{A}) \mapsto Z$ is a local homeomorphism. In particular, $\text{Stab}(\mathcal{D})$ is a complex manifold of dimension $\text{rk}(\Lambda)$.

In what follows, we set $\mathcal{D} = D(\mathcal{S})$, we choose v to be the the orbifold Chern character ch_{orb} (Definition 2.8), and let Λ denote its image in $H_{orb}^*(\mathcal{S}) := H^*(I\mathcal{S})$.

4.2. Construction of a pre-stability condition. Again, \mathcal{S} denotes a projective Kleinian orbisurface with a unique stacky point $p \in \mathcal{S}$ with residual gerbe $BG = [*/G]$ and ample divisor H . Set $\iota: BG \hookrightarrow \mathcal{S}$ the corresponding closed substack.

Define subcategories of $\text{Coh}(\mathcal{S})$ by

$$\begin{aligned} T_{H,b} &= \{E \in \text{Coh}(\mathcal{S}) \mid \text{for all } \mu_H\text{-semistable factors } F \text{ of } E, \mu_H(F) > b\}; \\ F_{H,b} &= \{E \in \text{Coh}(\mathcal{S}) \mid \text{for all } \mu_H\text{-semistable factors } F \text{ of } E, \mu_H(F) \leq b\}. \end{aligned}$$

The existence of Harder-Narasimhan filtrations for slope stability in our context is proven exactly as in the case of schemes, which is detailed in [14, Sec. 1.6]. As a consequence, $(T_{H,b}, F_{H,b})$ is a torsion pair, so we can perform the usual tilt and obtain the heart of a bounded t-structure

$$\text{Coh}^b(\mathcal{S}) := (F_{H,b}[1], T_{H,b}).$$

Since $\delta(F)$ is a linear function of $\text{ch}_{orb}(F)$ (see Section 2.3), the function $Z_{w,\gamma}: \Lambda \rightarrow \mathbb{C}$ defined as

$$Z_{w,\gamma}(E) = Z(\text{ch}_{orb}(E)) = -\text{ch}_2(E) + w \text{ch}_0(E) + \gamma \cdot \delta(E) + iH \cdot \text{ch}_1(E)$$

is also linear (here, $w \in \mathbb{C}$ and $\gamma \in \mathbb{R}$). Note that we identify rk and ch_0 here, slightly deviating from the usual notation of having $H^2 \text{ch}_0$ as a summand in the central charge. The goal of this section is to prove the following theorem. We denote $N := |G|$ and $D := \delta(\mathcal{O}_{\mathcal{S}})$.

Theorem 4.7. *Choose parameters $\gamma \in (0, \frac{1}{N-1})$ and $w \in \mathbb{C}$ such that:*

- (i) $\text{Re } w > -\frac{(\text{Im } w)^2}{H^2} + (2 + \gamma)D - (1 + \gamma)^2$;
- (ii) $\text{Re } w > \frac{1}{2} \frac{(\text{Im } w)^2}{H^2} - \gamma(D - \frac{N-1}{N}) > 0$.

Then, the pair $(Z_{w,\gamma}, \text{Coh}^{-\text{Im } w}(\mathcal{S}))$ is a stability condition on $D(\mathcal{S})$.

We split the proof of the theorem in two sections: the present Section 4.2 contains the construction of a pre-stability condition, while Section 4.3 contains arguments about the support property and concludes the proof. First, we prove a preliminary lemma:

Lemma 4.8. *Let F be a torsion free sheaf on \mathcal{S} . Consider the sequence*

$$E := \pi^* \pi_* F \rightarrow F \rightarrow M$$

Then $\delta(F) = \delta(E) + \delta(M) \geq (\text{rk } F)(D - \frac{N-1}{N})$.

Proof. The sheaf M is torsion, supported on the stacky point, and obtained by repeated extensions of copies of $\mathcal{O}_p \otimes \rho_i$ (because it pushes forward to zero). Every $\mathcal{O}_p \otimes \rho_i$ appears in the composition series of M at most $\text{rk}(F) \cdot \dim \rho_i$ times. Every one of these copies contributes $\delta(\mathcal{O}_p \otimes \rho_i) = -\dim(\rho_i)/N$, so

$$\delta(M) \geq (\text{rk}(F)) \sum (r_i) \delta(\mathcal{O}_p \otimes \rho_i) \geq -(\text{rk}(F)) \frac{N-1}{N}.$$

Now, $[E]$ is in the span of $[\mathcal{O}]$, $[\mathcal{O}_q]$ (for $q \neq p$) and $NS(\mathcal{S})$, write

$$[E] = (\text{rk } F)[\mathcal{O}] + a[\mathcal{O}_q] + \phi$$

where $\phi \in NS(\mathcal{S})$ and observe that $\delta(\mathcal{O}_q) = 0$ and $\delta(\phi) = 0$: this can be checked by observing that that $[\mathbf{L} \iota^* \mathcal{O}_K] = 0$, where $[K] = \phi$ is the class of any curve on \mathcal{S} . So $\delta(E) = \text{rk}(F)D$. \square

Lemma 4.9. *Provided that $0 < \gamma < \frac{1}{N-1}$ and $\operatorname{Re} w - \frac{1}{2} \frac{(\operatorname{Im} w)^2}{H^2} + \gamma(D - \frac{N-1}{N}) > 0$, the group homomorphism $Z_{w,\gamma}$ is a stability function on $\mathcal{A} := \operatorname{Coh}^{-\operatorname{Im} w}(\mathcal{S})$.*

Proof. First, one observes that for a sheaf E we have

$$\frac{H \cdot \operatorname{ch}_1(E)}{\operatorname{ch}_0(E)} = \mu(E) > -\operatorname{Im} w$$

if and only if $E \in T_{H, -\operatorname{Im} w}$, so that $\operatorname{Im} Z(E) \geq 0$ for all $E \in \mathcal{A}$.

Then, we only need to check that $\operatorname{Re} Z(E) < 0$ whenever $\operatorname{Im} Z(E) = 0$. Assume then that $\operatorname{Im} Z(E) = 0$. If E is torsion, then $\operatorname{ch}_0(E) = 0$ and by $\operatorname{Im} Z(E) = 0$ we get $\operatorname{ch}_1(E) = 0$. So E is supported on points.

We have that $\operatorname{Re} Z(E) < 0$ for all E supported on points: $\operatorname{Re} Z(\mathcal{O}_p) = -1/N + \gamma(1 - 1/N)$ and $\operatorname{Re} Z(\mathcal{O}_p \otimes \rho) = -\dim(\rho)/N + \gamma(-\dim(\rho)/N)$ (see Lemma 2.10) are both negative by the assumptions on γ , and every E is an extension of these.

The other case to check is the following: $\operatorname{Re} Z(E) > 0$ (because $E[1] \in \mathcal{A}$) for all μ -semistable sheaves E with $\mu(E) = -\operatorname{Im} w$. In this case, the Bogomolov-Gieseker inequality (2) yields $-\operatorname{ch}_2 \geq -\frac{\operatorname{ch}_1^2}{2\operatorname{ch}_0(E)}$, since $\operatorname{ch}_0(E) > 0$. Hence

$$\begin{aligned} \operatorname{Re} Z(E) &= -\operatorname{ch}_2(E) + \operatorname{Re} w \operatorname{ch}_0(E) + \gamma \cdot \delta(E) \\ &\geq \operatorname{ch}_0(E) \left[\operatorname{Re} w - \frac{\operatorname{ch}_1(E)^2}{2\operatorname{ch}_0(E)^2} \right] + \gamma \cdot \delta(E) \end{aligned}$$

Now, since $\mu(E) = -\operatorname{Im} w$ we have

$$\left(\operatorname{ch}_1(E) + \frac{\operatorname{Im} w \operatorname{ch}_0(E)}{H^2} H \right)^2 \leq 0$$

by the Hodge index theorem. Expanding this and using $\mu(E) = -\operatorname{Im} w$ once more, one gets

$$(9) \quad -\operatorname{ch}_1(E)^2 \geq -\frac{(\operatorname{Im} w)^2 (\operatorname{ch}_0(E))^2}{H^2}$$

Combining the inequalities above, and using Lemma 4.8 to estimate $\delta(E)$, we get:

$$\begin{aligned} \operatorname{Re} Z(E) &\geq \operatorname{ch}_0(E) \left[\operatorname{Re} w - \frac{\operatorname{ch}_1(E)^2}{2\operatorname{ch}_0(E)^2} \right] + \gamma \delta(E) \\ &\geq \operatorname{ch}_0(E) \left[\operatorname{Re} w - \frac{1}{2} \frac{(\operatorname{Im} w)^2}{H^2} \right] + \gamma \delta(E) \\ &\geq \operatorname{ch}_0(E) \left[\operatorname{Re} w - \frac{1}{2} \frac{(\operatorname{Im} w)^2}{H^2} + \gamma \left(D - \frac{N-1}{N} \right) \right]. \end{aligned}$$

This last part is positive by the assumption on $\operatorname{Re} w$. □

Lemma 4.10. *If H is a rational class and $\operatorname{Im} w \in \mathbb{Q}$, then Z satisfies Harder-Narasimhan filtrations on \mathcal{A} .*

Proof. Under the rationality assumptions, it is easy to see that the image of $\operatorname{Im} Z_{w,\gamma}$ is discrete. Then, it is enough to show that \mathcal{A} is Noetherian [21, Prop. 4.10]. This is proven exactly in the same way as [21, Lemma 6.17]. □

Summarizing this discussion:

Proposition 4.11. *Let H be a rational class, and γ and w chosen so that $0 < \gamma < \frac{1}{N-1}$, $\operatorname{Re} w - \frac{1}{2} \frac{(\operatorname{Im} w)^2}{H^2} + \gamma(D - \frac{N-1}{N}) > 0$ and $\operatorname{Im} w \in \mathbb{Q}$. Then, the pair $\sigma_{w,\gamma} = (Z_{w,\gamma}, \mathcal{A})$ is a pre-stability condition on $D(\mathcal{S})$.*

Proof. The previous Lemmata 4.9 and 4.10 show that $(Z_{w,\gamma}, \mathcal{A})$ satisfies Definition 4.2. \square

4.3. Support property. We now define a quadratic form Q on $K(\mathcal{S})$ which is negative definite on $\ker Z_{w,\gamma}$ and $Q(E) \geq 0$ for all $E \in \mathcal{A}$ which are $\sigma_{w,\gamma}$ -semistable. To do so, we need to investigate objects which are semistable for a limiting value of $\operatorname{Re} w$:

Lemma 4.12. *Let \mathcal{R} be the set of objects $E \in \mathcal{A}$ that are $\sigma_{w,\gamma}$ -semistable for all $\alpha := \operatorname{Re} w \gg 0$. Then every $E \in \mathcal{R}$ has one of the following forms:*

- (1) E is a slope semistable sheaf.
- (2) $H^{-1}(E) = 0$ and $H^0(E)$ is torsion;
- (3) $H^{-1}(E)$ is a torsion free, slope semistable sheaf, and $H^0(E)$ is supported on points.

Proof. The proof of [21, Lemma 6.18] carries over *mutatis mutandis*. \square

Definition 4.13. Define a preliminary quadratic form

$$Q_0(E) := \Delta_{orb}(E).$$

The form Q_0 is the pull-back on the cohomology of the stack of $\Delta = \operatorname{ch}_1^2 - 2 \operatorname{ch}_0 \operatorname{ch}_2$ on the surface under the McKay correspondence. The image in $N(\mathcal{S})$ of a numerical class $\mathbf{w} := (r, \phi + \sum t_j C_j, d)$ on \tilde{S} is

$$(10) \quad \mathbf{v} := r[\mathcal{O}] + \phi + d[\mathcal{O}_{Np}] + \sum_i (t_i + dr_i)[\mathcal{O}_p \otimes \rho_i]$$

Applying Lemma 2.10 to (10) yields:

$$(11) \quad \operatorname{ch}_2(\mathbf{v}) = d + \frac{1}{N} \sum r_i t_i;$$

$$(12) \quad \delta(\mathbf{v}) = rD - \frac{1}{N} \sum r_i t_i;$$

Lemma 4.14. *The form Q_0 is negative definite on $\ker Z_{w,\gamma}$, as long as $\operatorname{Re} w > (2 + \gamma)D - (1 + \gamma)^2 - \frac{(\operatorname{Im} w)^2}{H^2}$.*

Proof. Keep the notation as above, and suppose \mathbf{v} belongs to $\ker Z$. The condition on the real part reads $\operatorname{ch}_2(\mathbf{v}) = \operatorname{Re} wr + \gamma \delta(\mathbf{v})$. One sees from (11) and (12) that $d = \operatorname{ch}_2(\mathbf{v}) + \delta(\mathbf{v}) - rD$. Then, $Q_0(\mathbf{v})$ rewrites as

$$\begin{aligned} Q_0(\mathbf{v}) &= \Delta(\mathbf{w}) = \phi^2 + \left(\sum t_j C_j \right)^2 - 2r(d) \\ &= \phi^2 + \left(\sum t_j C_j \right)^2 - 2r(\operatorname{ch}_2(\mathbf{v}) + \delta(\mathbf{v}) - rD) \\ &= \phi^2 + \left(\sum t_j C_j \right)^2 - 2r((\operatorname{Re} w - D)r + (1 + \gamma)\delta(\mathbf{v})). \end{aligned}$$

Now, (12) gives

(13)

$$\begin{aligned} Q_0(\mathbf{v}) &= \phi^2 - 2r^2(\operatorname{Re} w - D) + \left(\sum t_j C_j\right)^2 - 2r(1 + \gamma) \left(rD - \frac{1}{N} \sum r_i t_i\right) \\ &= \phi^2 - 2r^2(\operatorname{Re} w - D - (1 + \gamma)D) + \left(\sum t_j C_j\right)^2 + 2r(1 + \gamma) \left(\frac{1}{N} \sum r_i t_i\right). \end{aligned}$$

We concentrate now on the quantity

$$(14) \quad \left(\sum t_j C_j\right)^2 + 2r(1 + \gamma) \left(\frac{\sum r_i t_i}{N}\right).$$

By adding and subtracting $\left(\frac{\sum r_i t_i}{N}\right)^2$ and completing a square, we have

$$(15) \quad \begin{aligned} &\left(\sum t_j C_j\right)^2 + 2r(1 + \gamma) \left(\frac{\sum r_i t_i}{N}\right) = \\ &\left[\left(\sum t_i C_i\right)^2 + \left(\frac{\sum r_i t_i}{N}\right)^2\right] - \left[\frac{\sum r_i t_i}{N} - r(1 + \gamma)\right]^2 + r^2(1 + \gamma)^2. \end{aligned}$$

The condition on the imaginary part is $\operatorname{Im} Z(v) = H \cdot \phi + \operatorname{Im} wr = 0$. Hence the Hodge index theorem yields $\phi^2 \leq \frac{(\operatorname{Im} w)^2 r^2}{H^2}$ as in (9). Combine this with equations (13) and (15) to obtain

(16)

$$Q_0(\mathbf{v}) \leq -2r^2 \left[\operatorname{Re} w - (2 + \gamma)D + (1 + \gamma)^2 + \frac{(\operatorname{Im} w)^2}{H^2} \right] + \left[\left(\sum t_i C_i\right)^2 + \left(\frac{\sum r_i t_i}{N}\right)^2 \right].$$

The second summand is negative unless $t_i = 0$ for all i by Lemma 4.15 below. In this case, the first summand is negative unless $r = 0$. If $r = t_i = 0$, then we must have $\delta(\mathbf{v}) = 0$, $\operatorname{ch}_2(\mathbf{v}) = 0$, and $H \cdot \phi = 0$, which implies $Q_0(\mathbf{v}) = \phi^2$ is negative definite by the Hodge index theorem. This concludes the proof. \square

Lemma 4.15. *The quantity $\left[\left(\sum t_i C_i\right)^2 + \left(\frac{1}{N} \sum r_i t_i\right)^2\right]$ is non-positive, and it is zero only if all $t_i = 0$.*

Proof. Let H denote the intersection matrix of the exceptional curves. Its negative $-H$ is the Cartan matrix associated with the root system corresponding to the singularity. Let J denote the matrix associated with the symmetric bilinear form $(t_1, \dots, t_M) \mapsto \left(\sum r_i t_i\right)^2$. It is sufficient to prove that the matrix

$$A := H + \frac{1}{N^2} J$$

is negative definite. To do so, we study the eigenvalues of H and J . The entries of J are

$$(J)_{i,j} = r_i r_j,$$

and J can be written as $\mathbf{r}\mathbf{r}^T$ where $\mathbf{r} = (r_1, \dots, r_M)$. Then, J has rank 1 and an eigenvector is \mathbf{r} with eigenvalue $\sum_{i=1}^M r_i \leq N - 1$. All other eigenvalues are 0.

Remark 4.16. From the representation-theoretic viewpoint, the quantity

$$h := \sum_{i=0}^M r_i = \sum_{i=1}^M r_i + 1$$

coincides with the Coxeter number of the root system associated with the singularity, since the r_i are the coefficients of the longest root.

Write $\alpha_1 \geq \dots \geq \alpha_{N-1}$ for the eigenvalues of A . By Weyl's inequality on sums of symmetric matrices, we have

$$\alpha_1 \leq \eta_1 + \frac{h-1}{N^2},$$

where η_1 is the biggest eigenvalue of H . The eigenvalues of H are computed in [11], and we have that

$$(17) \quad \eta_1 = -2 + 2 \cos\left(\frac{\pi}{h}\right).$$

The Coxeter numbers and orders of the groups are

Root system	h	N
A_{n-1}	n	n
D_n	$2n$	$4n$
E_6	12	24
E_7	18	48
E_8	30	120

It is then straightforward to check that $\eta_1 + \frac{h-1}{N^2} < 0$ in all the cases listed, and A is therefore negative definite. \square

Notice that replacing Q_0 with a quadratic form

$$Q := Q_0 + S(\operatorname{Re} Z)^2 + T(\operatorname{Im} Z)^2$$

does not affect its signature on $\ker Z_{w,\gamma}$.

The following lemma holds on projective orbisurfaces as well (it can be proven on the coarse moduli space):

Lemma 4.17 ([21, Ex. 6.11]). *Let ω be an ample real divisor class. Then there exists a constant $C_\omega \geq 0$ such that, for every effective divisor class D , we have*

$$C_\omega(\omega \cdot D)^2 + D^2 \geq 0.$$

Let C_H the constant in the lemma corresponding to the class H , and define

$$(18) \quad Q_1 := Q_0 + C_H(\operatorname{Im} Z)^2.$$

Lemma 4.18. *The form Q_1 satisfies $Q_1(E) \geq 0$ if E a torsion-free slope-semistable sheaf, or if E is supported on a curve.*

Proof. If E is torsion supported on a curve then $\operatorname{ch}_1(E)$ is effective and

$$Q_1(E) = \operatorname{ch}_1(E)^2 + C_H(H \cdot \operatorname{ch}_1(E))^2 \geq 0$$

where the inequality is that of Lemma 4.17, and Q_1 reduces to that expression because $\operatorname{ch}_0(E) = 0$ and $\delta(E) = 0$ as argued in the proof of Lemma 4.8.

If E is a torsion-free, semistable sheaf, we have shown that

$$Q_0(E) = \Delta_{orb}(E) \geq 0$$

by Theorem 3.1. □

Now observe that if T is a sheaf supported on points, we may write

$$[T] = d[\mathcal{O}_{Np}] + \sum_i (dr_i + t_i)[\mathcal{O}_p \otimes \rho_i]$$

with $d \geq 0$ and $t_i + dr_i \geq 0$. Let

$$K := \max_{i=1, \dots, M} \{Z_{w, \gamma}(\mathcal{O}_p), Z_{w, \gamma}(\mathcal{O}_p \otimes \rho_i)\} < 0.$$

Then $\operatorname{Re} Z_{w, \gamma}(T)^2 \geq K^2(d + \sum(d + t_j))^2$. Now pick S such that $SK^2 > 2N$, and observe that

$$\begin{aligned} (19) \quad Q_1(T) + S(\operatorname{Re} Z(T))^2 &= \left(\sum t_i C_i \right)^2 + S(\operatorname{Re} Z(T))^2 \geq \\ &= \left(\sum t_i C_i \right)^2 + SK^2 \left(d + \sum (dr_i + t_i) \right)^2 \geq \\ &= -2 \sum t_j^2 + SK^2 \left(d^2 + \sum (dr_i + t_j)^2 \right), \\ &= -2 \sum t_j^2 + SK^2 \left(d^2 + \sum (d^2 r_i^2 + t_i^2 + 2dr_i t_i) \right), \end{aligned}$$

where we used that the eigenvalues of the intersection matrix of C are all ≥ -2 , and that the mixed products appearing in the second summand are all non-negative. Now write

$$\begin{aligned} (20) \quad & d^2 + \sum (d^2 r_i^2 + t_i^2 + 2dr_i t_i) = \\ & \frac{\sum t_i^2}{N} + d^2 - d^2 \frac{\sum r_i^2}{N-1} + \sum \left(\frac{N}{N-1} d^2 r_i^2 + 2dr_i t_i + \frac{N-1}{N} t_i^2 \right) = \\ & \frac{\sum t_i^2}{N} + \sum \left(\sqrt{\frac{Ndr_i}{N-1}} + \sqrt{\frac{(N-1)t_i}{N}} \right)^2 \geq \frac{\sum t_i^2}{N}. \end{aligned}$$

We may continue the chain of inequalities (19):

$$Q_1(T) + S(\operatorname{Re} Z(T))^2 \geq -2 \sum t_j^2 + \frac{SK^2}{N} \sum t_j^2 > 0$$

by our choice of S . We can finally define

$$Q(E) := Q_1(E) + S(\operatorname{Re} Z(E))^2$$

and observe that Q is negative definite on $\ker Z_{w, \gamma}$ and non-negative on objects of \mathcal{R} . Then we have:

Theorem 4.19 (Rational case). *The pair $\sigma_{w, \gamma} = (Z_{w, \gamma}, \operatorname{Coh}^{-\operatorname{Im} w}(\mathcal{S}))$ satisfies the support property with respect to the quadratic form Q , and it is then a stability condition on $D(\mathcal{S})$.*

Proof. This is a standard argument (see for example [24, Theor. 6.11] or [21, Theor. 6.13]). Suppose E is $\sigma_{w, \gamma}$ -semistable and $\operatorname{Im} Z_{w, \gamma}(E)$ is minimal (note that such a minimum exists only because we are tilting at a rational slope, and hence the image of $\operatorname{Im} Z_{w, \gamma}$ is discrete). Then E must be semistable for all $\alpha' > \alpha = \operatorname{Re} w$, and thus belong to \mathcal{R} , so $Q(E) \geq 0$.

We proceed now by induction on $\text{Im } Z_{w,\gamma}$: suppose there is a semistable object E for which $Q(E) < 0$, and that $Q(F) \geq 0$ for all semistable objects with $\text{Im } Z_{w,\gamma}(F) < \text{Im } Z_{w,\gamma}(E)$. The object $E \notin \mathcal{R}$, so there is some $\alpha' > \alpha$ such that E is strictly semistable with respect to $\sigma_{w',\gamma}$ (where $w' = \alpha' + i \text{Im } w$), with Jordan-Hölder factors E_1, \dots, E_m . The inductive hypothesis applies to E_i , hence $Q(E_i) \geq 0$ for all i . The images $Z_{w',\gamma}(E_i)$ all lie in the same ray in \mathbb{C} , so for any pair E_i, E_j there exists $a > 0$ such that $Z_{w',\gamma}(E_i) - aZ_{w',\gamma}(E_j) = 0$. Since Q is negative definite on $\ker Z_{w',\gamma}$, the class $[E_i] - a[E_j]$ belongs to the negative cone of Q in $K(\mathcal{S})$. This implies that any linear combination with positive coefficients of $[E_i]$ and $[E_j]$ lies in the positive cone of Q . Since this holds for any i, j , we must have $Q(E) \geq 0$.

This shows that the support property is satisfied for all semistable objects of positive imaginary charge. We checked above that the support property with respect to Q is satisfied for stable objects of phase 1 as well, which allows us to conclude. \square

Proof of Theorem 4.7. It remains to argue that one can drop the rationality assumptions on H and $\text{Im } w$. The argument is carried out in detail in [6] in the case of a K3 surface and follows from the discussion in [5, Sec. 6,7], but it requires an observation about the heart of a stability condition in the geometric chamber. The analog of this observation is the following Lemma, which can be proven exactly as [21, Lemma 6.20] \square

Lemma 4.20. *Let $(\mathcal{B}, Z_{w,\gamma})$ be a stability condition for which all skyscraper sheaves $\mathcal{O}_p \otimes \rho^i$, $j = 0, \dots, M$ and \mathcal{O}_q for $q \neq p$ are stable of phase 1. Then $\mathcal{B} = \text{Coh}^{-\text{Im } w}(\mathcal{S})$.*

5. Wall-crossing: clusters and constellations

Throughout this section, \mathcal{S} is an *ADE*-orbisurface with a single isolated stacky point p . We investigate wall-crossing for objects of class $v := [\mathcal{O}_x]$, where $x \in \mathcal{S}$ is a closed point with trivial stabilizer. Let $\sigma^* := \sigma_{w,\gamma}$ be one of the stability conditions of Theorem 4.7, and denote by \mathcal{A} its heart $\text{Coh}^{-\text{Im } w}(\mathcal{S})$.

Lemma 5.1. *Skyscraper sheaves \mathcal{O}_x for $x \neq p$, and sheaves $\mathcal{O}_p \otimes \rho$ are simple objects in \mathcal{A} . Therefore, they are σ^* -stable and all have phase 1.*

Proof. The long exact sequence of cohomology sheaves associated to a short exact sequence in \mathcal{A}

$$0 \rightarrow A \rightarrow \mathcal{O}_x \rightarrow B \rightarrow 0$$

shows that $H^0(B)$ is 0 or \mathcal{O}_x . If $H^0(B) \simeq \mathcal{O}_x$, then

$$H^{-1}(B) \simeq H^0(A) = A = 0.$$

If $H^0(B) = 0$, then $H^{-1}(B)$ and $H^0(A)$ have the same slope, which is a contradiction, unless $H^{-1}(B) = 0$ and $A \simeq \mathcal{O}_x$. This shows that \mathcal{O}_x is simple in \mathcal{A} . The argument for sheaves $\mathcal{O}_p \otimes \rho$ is identical. One then observes that $Z_{w,\gamma}$ maps these objects to the negative real axis to conclude they are stable of phase 1. \square

Lemma 5.2. *The objects of class $[\mathcal{O}_x]$ in \mathcal{A} are skyscraper sheaves \mathcal{O}_x , or they have a composition series whose factors are the $\mathcal{O}_p \otimes \rho_i$, repeated with multiplicity $r_i = \dim \rho_i$. The former are σ^* -stable, while the latter are σ^* -semistable and all share the same S -equivalence class.*

Proof. The statement about stability follows immediately from Lemma 5.1. What needs justification is the "only" part of the statement: let $A \in \mathcal{A}$ be a complex with $[\mathcal{O}_x] = [A] = [H^0(A)] - [H^{-1}(A)]$. Then $H^0(A)$ and $H^{-1}(A)$ have the same slope, since their classes differ by a codimension 2 summand. This is only possible if $H^{-1}(A) = 0$ and A is a sheaf of class $[\mathcal{O}_x]$. If A is supported away from p , then $A = \mathcal{O}_x$ for some x . If A is supported at p , then it has a composition series with factors $\mathcal{O}_p \otimes \rho_i$. The multiplicities must be r_i since $[\mathcal{O}_x] = \sum_i r_i [\mathcal{O}_p \otimes \rho_i]$. \square

Now apply Bridgeland's deformation result 4.5 to σ^* , and obtain a neighborhood $\Delta \subset \text{Stab}(\mathcal{S})$ and a stability condition $\sigma_0 = (Z_0, \mathcal{A}_0) \in \Delta$ such that $Z_0([\mathcal{O}_x]) = -1$ and $\phi_0(\mathcal{O}_p \otimes \rho) < 1$ for all $\rho \neq \mathbb{1}$. Denote by $\mathcal{V} \simeq [W/G]$, with $W = \mathbb{C}^2$, the chart of \mathcal{S} around the stacky point, and recall Definition 2.5.

Proposition 5.3. *The moduli space $M_{\sigma^*}(v)$ of σ^* -semistable objects is isomorphic to the coarse moduli space of \mathcal{S} , and $M_{\sigma_0}(v)$ is isomorphic to its minimal resolution. The wall-crossing morphism*

$$M_{\sigma_0}(v) \rightarrow M_{\sigma^*}(v)$$

sending σ_0 -semistable objects to their σ^ - S -equivalence class is the contraction of the exceptional divisors.*

Proof. The stability condition σ_0 ensures that no proper subsheaf of a σ_0 -stable object contains $\mathcal{O}_p \otimes \mathbb{1}$ in its composition series. Then, objects of $M_{\sigma_0}(v)$ supported at p are exactly G -clusters, and $M_{\sigma_0}(v)$ is locally isomorphic to $G\text{-Hilb}(W)$. The G -Hilbert scheme is the minimal resolution of W/G [8], with exceptional locus parameterizing clusters supported at p .

At σ^* , clusters become strictly semistable with the same S -equivalence class. In other words, the exceptional divisor of $G\text{-Hilb}(W)$ is contracted to a point, showing that $M_{\sigma^*}(v)$ is locally isomorphic to $V := W/G$. Away from this chart, $M_{\sigma^*}(v)$ is isomorphic to $S \setminus V$. This shows that $M_{\sigma^*}(v) \simeq S$ and concludes the proof of the proposition. \square

In a completely analogous way, we can define deformations $\sigma_i = (Z_i, \mathcal{A}_i)$ of σ^* such that $Z_i([\mathcal{O}_x]) = -1$ and $\phi_0(\mathcal{O}_p \otimes \rho) < 1$ for all $\rho \neq \rho_i$. One argues then as in Prop. 5.3, to show that the moduli spaces $M_{\sigma_i}(v)$ are moduli spaces of G -constellations, and are crepant resolutions of $M_{\sigma^*}(v)$. This is not surprising, as the stability conditions σ_i correspond to certain generic stability parameters on quiver representations, as illustrated in the next subsection. Proposition 5.3 is then an analog of a well-known result in King's theory of stability for quiver representations, see [15] and [10].

5.1. Comparison with [7]. Let \mathcal{B} be the finite length abelian subcategory of $\text{Coh}(\mathcal{S})$ generated by the simple sheaves $\mathcal{O}_p \otimes \rho_i$, with $i \neq 0$. Denote by \mathcal{T} the triangulated subcategory of $D(\mathcal{S})$ consisting of complexes whose cohomologies lie in \mathcal{B} , the main result of [7] is the description of a connected component of $\text{Stab}(\mathcal{T})$.

The Grothendieck group $K(\mathcal{T})$ endowed with the Euler pairing is a root lattice and the classes $\alpha_i := [\mathcal{O}_p \otimes \rho_i]$ are roots. Therefore, the space $\text{Hom}(K(\mathcal{T}), \mathbb{C})$ of central charges of $\text{Stab}(\mathcal{T})$ is identified with the Cartan algebra of the root system, and admits an action of the Weyl group which is free on the set of regular orbits

$$\mathfrak{h}^{\text{reg}} = \{Z \in \text{Hom}(K(\mathcal{T}), \mathbb{C}) \mid Z(\alpha) \neq 0 \text{ for all roots } \alpha \in K(\mathcal{T})\}.$$

By [7, Lemma 3.1], there is a region U in $\text{Stab}(\mathcal{T})$, homeomorphic to a complexified Weyl chamber in $\mathfrak{h}^{\text{reg}}$, containing stability conditions (Z, \mathcal{B}) with $\text{Im } Z(\mathcal{O}_p \otimes \rho_i) > 0$ for all $i \neq 0$. Moreover, the central charge map $\varpi: \text{Stab}(\mathcal{T}) \rightarrow \text{Hom}(K(\mathcal{T}), \mathbb{C})$ is a covering space over $\mathfrak{h}^{\text{reg}}$ [7, Prop. 3.3].

Since $Z_{0|K(\mathcal{T})}$ satisfies $\text{Im } Z_{0|K(\mathcal{T})}(\mathcal{O}_p \otimes \rho_i) > 0$ for all $i \neq 0$, the stability condition σ_0 gives rise to a stability condition $(Z_{0|K(\mathcal{T})}, \mathcal{B}) \in U$ by [7, Lemma 3.1]. More generally, stability conditions $\sigma = (Z_\sigma, \mathcal{A}_\sigma) \in \Delta \subset \text{Stab}(D(\mathcal{S}))$, satisfy $Z_{\sigma|K(\mathcal{T})} \in \mathfrak{h}^{\text{reg}}$.

Therefore, restriction of central charge defines a map $\Delta \rightarrow \mathfrak{h}^{\text{reg}}$, which lifts to a map

$$\delta: \Delta \rightarrow \text{Stab}(\mathcal{T}).$$

The boundary ∂U decomposes as $\partial U = \cup_i U_i$, where

$$U_i := \{\tau \in \partial U \mid \text{Im } Z_\tau(\mathcal{O}_p \otimes \rho_i) = 0\}.$$

This shows that $\delta(\sigma^*) \in \cap_i U_i$. Moreover, $\delta^{-1}(U_i)$ is a wall for class v , because $\sigma \in \delta^{-1}(U_i)$ satisfies $Z_\sigma(v) // Z_\sigma(\mathcal{O}_p \otimes \rho_i)$. Summarizing:

Proposition 5.4. *There exists a map $\delta: \Delta \rightarrow \text{Stab}(\mathcal{T})$. The preimages of the components of ∂U along δ are walls for class v in $\text{Stab}(\mathcal{S})$, and σ^* lies in their intersection.*

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