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Coupled electro-elastic deformation and instabilities of a toroidal membrane

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Abstract

We analyse here the problem of large deformation of dielectric elastomeric membranes under coupled electromechanical loading. Extremely large deformations (enclosed volume changes of 100 times and greater) of a toroidal membrane are studied by the use of a variational formulation that accounts for the total energy due to mechanical and electrical fields. A modified shooting method is adopted to solve the resulting system of coupled and highly nonlinear ordinary differential equations. We demonstrate the occurrence of limit point, wrinkling, and symmetry-breaking buckling instabilities in the solution of this problem. Onset of each of these “reversible” instabilities depends significantly on the ratio of the mechanical load to the electric load, thereby providing a control mechanism for state switching.

Keywords: Electroelastic membrane, Limit point, Wrinkling, Buckling, Stability analysis

1. Introduction

Thin electroelastic structures made from electroactive polymers [56] find wide use in engineering applications including artificial muscles [5], soft grippers [3, 31, 34], and energy generators [50]. In this work we use the theory of nonlinear electroelasticity [18] to analyse the large deformation of a toroidal electroelastic membrane inflated by a mechanical pressure and actuated by an electric potential difference applied across its thickness. Extreme deformations induce limit point, wrinkling, and symmetry-breaking buckling instabilities in the membrane. These are systematically studied in this contribution.

1.1. Nonlinear electroelasticity

Developments in the theory of electroelasticity date back to the classic work of Toupin [75]. By combining the theory of continuum mechanics and electrostatics, a framework was established for

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analysing the nonlinear response of isotropic dielectric materials. Toupin [76] extended his seminal work by deriving the governing equations for the dynamics of elastic dielectrics. Eringen [25] followed by formulating the governing equations in an alternative way and applied his method to the problem of an incompressible thick-walled cylindrical tube subjected to a radial electric field. Tiersten [73] simplified the formulations proposed by Toupin [75] and Eringen [25], thereby making a fundamental contribution to nonlinear electroelastic material modelling. He also made significant contributions to the theory of piezoelectricity [73, 74].

Interest in the development of nonlinear theories of electroelasticity was renewed owing to the development and industry adoption of dielectric elastomers that can undergo large deformations and nonlinear electroelastic coupling. McMeeking and Landis [44] used the principle of virtual work to derive simpler governing equations for quasi-electrostatics in Eulerian form. Concurrently, Dorfmann and Ogden [18, 19] developed a general Lagrangian formulation and constitutive relations within the framework of continuum mechanics to simulate the finite deformation of electroelastic materials coupled to electric fields. Rate-dependent theories to account for dissipation owing to viscoelasticity in dielectric elastomers were developed by Ask et al. [4] and Saxena et al. [65]. Variational formulations of electroelasticity to enable the development of computational methods were presented by Vu et al. [78], Liu [39], and have been applied to analyse stability by computation of higher variations by Bus-tamante et al. [9], and Saxena and Sharma [64]. A comprehensive review of the theory of nonlinear electroelasticity and its applications is presented in [23].

1.2. Dielectric elastomers

Electroactive polymers are materials that can undergo deformation due to an applied electric field. Dielectric elastomers are one of the most commonly used electroactive polymers. They are composed of a soft elastomer sandwiched between two compliant electrodes. Application of a potential difference between the two electrodes results in a large deformation in the elastomer due to the electrostatic forces generated by the opposite electric charges [56].

This principle has been widely used in the design of sensing and actuating systems. For example, Bar-Cohen et al. [5] explored their use as artificial muscles. Moretti et al. [50] developed wave energy generators based on the inflation of dielectric elastomers. Kofod et al. [30] presented the principle of self-organized dielectric elastomer minimum energy structures (DEMESs) and developed a gripper [31]. Araromi et al. [3] applied DEMESs as a gripper to capture debris in space. Lau et al. [34] developed a dielectric elastomer finger for grasping and pinching highly deformable objects. For a detailed review on grippers made of dielectric elastomers, see Shintake et al. [66]. Ozsecen et al. [55] developed haptic

interfaces, Michel et al. [47] performed a feasibility study for a bionic propulsion system, and O'Halloran et al. [54] explored sensing systems based on dielectric elastomers. We refer to the review papers by Suo [68] and Lu et al. [41] for further detailed discussion on dielectric elastomers.

1.3. *Biological and engineering applications with toroidal membranes*

Biological membranes with a toroidal shape also naturally exist in human body. One of the most important cells in the human body, Erythrocyte (red blood cell), is a small dielectrophoretic (DEP) electromagnetic field (EMF) driven cell [58]. It has a unique toroidal shape in order to increase the surface area-to-volume ratio which increases the diffusion of oxygen and carbon dioxide through their cell membrane. Toroidal structures are common in DNA-binding enzymes [27] and bacteriophages [35]. Further discussion on natural biological toroidal structures can be found in [17]. Topologically, a torus can be considered as the simplest example of a genus 1 orientable surface which appeals to mathematicians and engineers. Toroidal membranes and shells are widely applied in engineering such as tyres, air springs [13], soft grippers and inflatable actuators. Zang et al. [81] designed a bionic toroidal soft gripper in order to catch objects with arbitrary shapes and sizes. Adams et al. [1] designed an electro-pneumatic device using a string of inflatable toroidal membranes for water pipe inspection. Thus, analysing the mechanics of large deformation in toroidal membranes can not only aid the engineering design of such devices but also develop a better understanding of certain biological processes at the cellular level.

1.4. *Instabilities in nonlinear membranes*

Nonlinear membranes are widely applied in engineering structures and naturally appear in the form of biological tissues. Air bags, diaphragm valves, balloons, skin tissue, and cell walls are examples of nonlinear membranes.

Inflation can cause large deformation in the membranes resulting in instabilities. A well-known instability phenomenon of inflating membranes is the limit point. This is a critical point after which the membrane appears to lose stiffness to inflation and undergoes very large inflation with a small increase in pressure. This phenomenon is also called snap-through bifurcation and has been well studied [see e.g. 7, 10, 29, 52, 70]. Computation of accurate pressure-volume characteristics in this case require a path-following scheme due to the non-uniqueness of solution [59].

In-plane deformation of membrane can also result in wrinkling which is a form of localised buckling. An ideal membrane is a structure with negligible bending stiffness and can only sustain tensile loading. If any part of the membrane structure experiences compression, it undergoes local out-of-plane

deformation to avoid the in-plane compressive stresses. Tension field theory developed by Pipkin [57] and Steigmann [67] is a widely used tool to model wrinkles in nonlinear elastic membranes. It assumes zero bending stiffness and an infinitely continuous distribution of wrinkles orientated in the direction of the positive principal stress. To avoid a contribution to the energy by compressive stresses, a relaxed energy function is used that constrains the stress tensor to be positive semi-definite. As a result the amplitude and wavelength of wrinkles cannot be computed by using this theory. This theory has been applied to study wrinkles in axisymmetric hyperelastic membranes [37, 38] and to model wrinkles in skin during wound closure [69], to name a few applications. A generalisation of the tension field theory has been attempted (although without a rigorous mathematical proof) for the case of electroelasticity by De Tommasi et al. [14], De Tommasi et al. [15], Greaney et al. [26], and for the case of magnetoelasticity by Reddy and Saxena [59, 60], Saxena et al. [63]. Wong and Pellegrino [79] developed an analytical method to quantify the location, amplitude, and wavelengths of linear elastic membranes. Nayyar et al. [53] and Barsotti [6] developed a nonlinear finite element method for simulating stretch-induced wrinkling of hyperelastic thin sheets.

While wrinkling is a localised buckling, the membrane structure can also experience a global buckling on account of large deformations. In structures with a geometrical symmetry, this instability manifests as a bifurcation from the symmetric principal solution and leads to a loss of symmetry. The theory of elastic buckling, developed by Koiter [32] and Budiansky [8], provides methods to evaluate the critical point of such instability. This typically requires checking the sign of the second variation of the total potential energy to determine the stability state. Chaudhuri and Dasgupta [11] studied the perturbed deformations of inflated hyperelastic circular membranes, Venkata and Saxena [77] analysed buckling of hyperelastic toroidal membranes, Xie et al. [80] analysed the shape bifurcations of a dielectric elastomeric sphere through a direct perturbation approach, and Reddy and Saxena [59, 60], Saxena et al. [63] analysed shape bifurcations of magnetoelastic membranes.

1.5. *Electroelastic instability*

Experimental investigation of dielectric elastomeric membranes have revealed all the three instabilities discussed in Section 1.4. An experimental investigation of electroelastic membranes by Kollosche et al. [33] demonstrates an interplay between the limit point and wrinkling instabilities due to coupling effects. Li et al. [36] studied large voltage-induced deformation of dielectric elastomers. In addition to wrinkling and limit point, they also demonstrate symmetry-breaking and bulge formation in the inflation of a circular membrane. Careful experimental investigations on the rate-dependent behaviour of dielectric elastomers have shown a relation between the viscoelastic response and the breakdown limit

107 [2, 28, 45]. Zhang et al. [82] and Mao et al. [42] presented a controlled experimental procedure to
108 produce wrinkles in dielectric elastomer membranes.

109 Theoretical and computational procedures to model these electroelastic instabilities have been de-
110 veloped largely for bulk media with some recent works towards the analysis of membranes. Zhao and
111 Suo [83] analysed the instability of dielectric elastomers to guide the design of actuator configurations
112 and materials. Rudykh et al. [62] presented a method to use snap-through instability of thick-wall
113 electroactive balloons to design actuators. Miehe et al. [48] developed an algorithm for finite element
114 computations of both structural and material stability analysis in electroelasticity. Dorfmann and Og-
115 den [20] studied the critical stretch corresponding to loss of stability of a thick electroelastic plate by
116 perturbation of the equilibrium equations. Dorfmann and Ogden [21] also investigated radial deforma-
117 tions of a thick-walled spherical shell using the nonlinear electroelastic theory. Melnikov and Ogden [46]
118 presented a mathematical approach to study bifurcation of a finitely deformed thick-walled cylindrical
119 tube. A more complete set of references can be found in the comprehensive review on the instability of
120 soft dielectrics by Dorfmann and Ogden [24].

121 Xie et al. [80] undertook a bifurcation analysis of a spherical dielectric membrane under inflation and
122 also derived post-buckling solutions. Greaney et al. [26] used a modified tension field theory to analyse
123 wrinkling and pull-in instabilities in dielectric membranes. For the related problem of magnetoelastic
124 membranes, Reddy and Saxena [59] recently demonstrated new instability phenomena of an additional
125 limit point and reversal of wrinkling location in an inflating toroidal magnetoelastic membrane. In
126 addition to the engineering and biological applications discussed in Section 1.3, these results provided a
127 partial motivation to study instabilities in a toroidal electroelastic membranes. In this contribution we,
128 for the first time, present a novel analysis of the interaction between limit point, wrinkling, and buckling
129 instabilities in a toroidal membrane. For the wrinkling analysis we apply the tension field theory of
130 Steigmann [67] generalised to electroelasticity by De Tommasi et al. [14] and Greaney et al. [26]. We
131 extend the aforementioned theoretical works by providing the numerical details to apply the tension
132 field theory and thereby deliver physically meaningful predictions of wrinkled solutions post-instability.
133 These predictions are compared to primary solutions that ignore the inability of a membrane to sustain
134 compressive states.

135 1.6. Organisation of the manuscript

136 This contribution is organised as follows: Section 2 introduces the kinematics of the deformation.
137 Section 3 formulates the equilibrium equations using the first variation of the total potential energy func-
138 tional that is composed of mechanical and electrical contributions. The Mooney-Rivlin model [49, 61]

is adopted for the hyperelastic energy density and the electrical contribution is accounted for via the energy density function by coupling the electric displacement to the deformation tensor. Three instabilities (snap-through, wrinkling and loss of symmetry) are analysed and discussed in detail. Computations of wrinkling using the energy relaxation method are presented in Section 4. Section 5 describes the second variation analysis of the energy function to consider the loss of symmetry in the circumferential direction of the torus. Section 6 presents several numerical examples to elucidate the theory. Conclusions are presented in Section 7.

1.7. Notation

Brackets. Three types of brackets are used. Square brackets $[\]$ are used to clarify the order of operations in an algebraic expression. Curly brackets $\{ \ }$ define a set and circular brackets $(\)$ are used to define the parameters of a function. If brackets are used to denote an interval then $(\)$ stands for an open interval and $[\]$ a closed interval.

Symbols. A variable typeset in a normal weight font represents a scalar. A bold weight font denotes a vector or a second-order tensor. An upper-case bold letter denotes a vector or tensor in the reference configuration and a lowercase bold letter denotes a vector or tensor in the current (deformed) configuration. A tensor directly enclosed by square brackets, for example $[\mathbf{A}]$, denotes the matrix representation of the tensor in a selected coordinate system. A subscript denotes the partial derivative with respect to the field. For example, consider a function $A(a, b(a), c(a))$. A_b denotes the partial derivative of A with respect to b , which is equivalent to $\frac{\partial A}{\partial b}$. $\frac{dA}{da}$ is the full derivative with respect to a , given by

$$\frac{dA}{da} = \frac{\partial A}{\partial a} + \frac{\partial A}{\partial b} \frac{\partial b}{\partial a} + \frac{\partial A}{\partial c} \frac{\partial c}{\partial a}.$$

Functions. $\det(\mathbf{A})$ denotes the determinant of the second-order tensor $\mathbf{A}$. $\text{diag}(a, b, c)$ denotes a second-order tensor with only diagonal entries a, b and c .

2. Kinematics

Consider the toroidal membrane in Figure 1 with major and minor radii R_b and $R_s (< R_b)$, respectively. The initial thickness of the membrane H is assumed constant, where $H/R_s \ll 1$. The geometry and kinematics of this system are similar to previously studied problems [59, 77]. The torus is inflated by an internal pressure $\tilde{P}$ and an electric potential difference Φ_0 is applied across its thickness. The membrane is assumed to be incompressible.

Figure 1: (a) The reference configuration of a toroidal membrane with circular cross-section highlighted. (b) The reference and deformed configurations of a cross section of the toroidal membrane. (c) Close-up of part of the deformed membrane.

The position vector $\mathbf{X}$ of a point in the undeformed toroidal membrane is given by

$$\begin{aligned} \mathbf{X}(\theta, \phi, \xi) = & [R_b + [R_s + \xi] \cos \theta] \cos \phi \mathbf{E}_1 + [R_b + [R_s + \xi] \cos \theta] \sin \phi \mathbf{E}_2 \\ & + [R_s + \xi] \sin \theta \mathbf{E}_3, \end{aligned} \quad (1)$$

where ξ is the distance of the point from the mid-surface (defined by $\xi = 0$) of the membrane along the radius. The set $\{\mathbf{E}_i\}$ of orthonormal vectors is the basis corresponding to the (Y^1, Y^2, Y^3) coordinate system with origin O . The components of the covariant metric tensor with respect to the local (curvilinear) system (θ, ϕ, ξ) (see Figure 1) for the membrane in its reference (perfect torus) configuration, are given by

$$[\mathbf{G}] = \begin{bmatrix} R_s^2 & 0 & 0 \\ 0 & [R_b + R_s \cos \theta]^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (2)$$

with determinant $G = \det \mathbf{G} = R_s^2 [R_b + R_s \cos \theta]^2$. Here, we have used the small thickness assumption ($H \ll R_s < R_b$).

2.2. Deformed configuration

Let the position vector of the deformed mid-surface be given by $\mathbf{x}$ (corresponding to $\mathbf{X}$ in the reference configuration) with the unit outward normal vector $\mathbf{n}$. It can be shown that

$$\mathbf{x} = \tilde{\varrho} \cos \phi \mathbf{E}_1 + \tilde{\varrho} \sin \phi \mathbf{E}_2 + \tilde{\eta} \mathbf{E}_3, \quad (3)$$

where $\tilde{\varrho} : [0, 2\pi) \times [0, 2\pi) \rightarrow \mathbb{R}$ and $\tilde{\eta} : [0, 2\pi) \times [0, 2\pi) \rightarrow \mathbb{R}$ are functions depending on θ and ϕ that describe the position of points on the mid-surface as shown in Figure 1c. Components of the three-dimensional covariant metric tensor for the deformed configuration are given by

$$[\tilde{\mathbf{g}}] = \begin{bmatrix} \tilde{\varrho}_\theta^2 + \tilde{\eta}_\theta^2 & \tilde{\varrho}_\theta \tilde{\varrho}_\phi + \tilde{\eta}_\theta \tilde{\eta}_\phi & 0 \\ \tilde{\varrho}_\theta \tilde{\varrho}_\phi + \tilde{\eta}_\theta \tilde{\eta}_\phi & \tilde{\varrho}_\phi^2 + \tilde{\varrho}^2 + \tilde{\eta}_\phi^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix}, \quad (4)$$

where $\lambda_3 = h/H$ with h being the thickness of the torus in the deformed configuration. We further introduce the dimensionless quantities

$$\gamma = \frac{R_s}{R_b}, \quad \varrho = \frac{\tilde{\varrho}}{R_b}, \quad \eta = \frac{\tilde{\eta}}{R_b}, \quad (5)$$

where the parameter γ describes the aspect ratio of the undeformed torus. The two principle stretches λ_1 and λ_2 can be expressed using the reference and deformed covariant metric tensors as

$$\begin{aligned}\lambda_1^2 &= \mathcal{P} + \mathcal{Q}, \\ \lambda_2^2 &= \mathcal{P} - \mathcal{Q},\end{aligned}\tag{6}$$

where

$$\begin{aligned}\mathcal{P} &= \frac{1}{2} \left[\frac{\varrho_\theta^2 + \eta_\theta^2}{\gamma^2} + \frac{\varrho_\phi^2 + \eta_\phi^2 + \varrho^2}{[1 + \gamma \cos \theta]^2} \right], \\ \mathcal{Q} &= \frac{1}{2} \sqrt{\left[\frac{\varrho_\theta^2 + \eta_\theta^2}{\gamma^2} - \frac{\varrho_\phi^2 + \eta_\phi^2 + \varrho^2}{[1 + \gamma \cos \theta]^2} \right]^2 + 4 \left[\frac{\varrho_\theta \varrho_\phi + \eta_\theta \eta_\phi}{\gamma[1 + \gamma \cos \theta]} \right]^2}.\end{aligned}$$

The deformation map $\mathbf{X}$, from the undeformed configuration to the deformed configuration, is defined via

$$\mathbf{x} = \mathbf{X}(\mathbf{X}),$$

and the corresponding deformation gradient is defined by

$$\mathbf{F}(\mathbf{X}) := \text{Grad } \mathbf{X}(\mathbf{X}).$$

Henceforth, for notational convenience, it is assumed that $\mathbf{X}$ and $\mathbf{x}$ are related as above to map the evaluation at $\mathbf{x}$ of quantities defined on the deformed configuration to their counterparts at $\mathbf{X}$ on the undeformed configuration. The incompressibility constraint is given by

$$J = \det(\mathbf{F}) = 1,$$

as a consequence of which, the third principle stretch follows as

$$\lambda_3^2 = \frac{1}{\lambda_1^2 \lambda_2^2} = \frac{\gamma^2 [1 + \gamma \cos \theta]^2}{[\varrho_\theta \eta_\phi - \varrho_\phi \eta_\theta]^2 + \varrho^2 [\varrho_\theta^2 + \eta_\theta^2]}.\tag{7}$$

The right Cauchy–Green deformation tensor $\mathbf{C} = \mathbf{F}^\top \mathbf{F}$ is given in the local coordinate system of the torus by

$$[\mathbf{C}] = \text{diag}(\lambda_1^2, \lambda_2^2, \lambda_3^2).\tag{8}$$

Remark 1. *If the solution is symmetric with respect to $\mathbf{E}_3$ along the ϕ (azimuthal) direction (which happens to be the case for the principal solution as shown in Section 3.3.), the first two principle stretches can be simplified as*

$$\lambda_1 = \frac{1}{\gamma} [\varrho_\theta^2 + \eta_\theta^2]^{1/2}, \quad \lambda_2 = \frac{\varrho}{1 + \gamma \cos \theta}.\tag{9}$$

163 3. Electroelastic energy based variational formulation and equations of equilibrium

164 This section formulates the equations of electroelastic equilibrium using the first variation of the total
 165 potential energy functional. Section 3.1 briefly presents the equations for electrostatics. Thereafter the
 166 total potential energy of the system under an applied pressure and an electric field is given. Section 3.3
 167 considers the first variation of the total potential energy and the resulting three governing equations.
 168 Section 3.4 decomposes the energy density function into an elastic energy density function and an
 169 electric contribution. A Mooney–Rivlin constitutive model is employed for the elastic energy density.
 170 This yields the governing equations presented in Section 3.5. A numerical method is proposed in
 171 Section 3.6 to solve the resulting system of nonlinear ordinary differential equations (ODEs).

172 3.1. Electrostatics

Maxwell’s equations for electrostatics are given by

$$\text{Curl } \mathbb{E} = \mathbf{0}, \quad \text{and} \quad \text{Div } \mathbb{D} = 0, \quad (10)$$

where $\mathbb{E}$ is the electric field in the reference configuration and $\mathbb{D}$ is the electric displacement in the reference configuration assuming the free charge density in the volume is zero. Equation (10)₂ motivates the introduction of an electric vector potential $\mathbb{A}$ defined as

$$\mathbb{D} = \text{Curl } \mathbb{A}. \quad (11)$$

The referential vectors $\mathbb{E}$ and $\mathbb{D}$ can be expressed as the pull-backs of the electric field and displacement in the current (deformed) configuration, $\mathfrak{e}$ and $\mathfrak{d}$, as [22]

$$\mathbb{E} = \mathbf{F}^\top \mathfrak{e} \quad \text{and} \quad \mathbb{D} = J \mathbf{F}^{-1} \mathfrak{d}. \quad (12)$$

Within the electroelastic solid, the constitutive relation

$$\mathbb{E} = \Omega_{\mathbb{D}} \quad (13)$$

relates $\mathbb{E}$ and $\mathbb{D}$ via the total energy density Ω of the material. In free space, $\mathfrak{e}$ and $\mathfrak{d}$ are related through the electric permittivity of vacuum ε_0 as

$$\mathfrak{e} = \varepsilon_0^{-1} \mathfrak{d}. \quad (14)$$

174 3.2. Potential energy functional

The toroidal membrane occupies the region $\mathcal{B}_0$ in the reference configuration and its total internal energy density per unit volume Ω is parameterised by the deformation gradient $\mathbf{F}$ and the referential electric displacement $\mathbb{D}$. Under an applied pressure $\tilde{P}$, the total potential energy of the system can be written as

$$\mathfrak{E}(\boldsymbol{\chi}, \mathbf{A}) = \int_{\mathcal{B}_0} \Omega(\mathbf{F}, \mathbb{D}) dv_0 - \int_{V_0}^{V_0+\Delta V} \tilde{P} dV. \quad (15)$$

175 We note here that the first integral is over the region occupied by the solid membrane while the second
 176 integral of pressure work is over the volume of fluid enclosed by the inflated membrane (that is the
 177 region lying in the interior of the torus). Hence the differentials are distinguished as dv_0 and dV . Since
 178 the electric potential is specified on the inner and outer surfaces of the torus, we assume that the electric
 179 field does not leak out and therefore there is no contribution to the energy in the region outside the
 180 membrane given by $\mathbb{R}^3 \setminus \mathcal{B}_0$.

As the membrane is considered to be thin ($H/R_s \ll 1$) with negligible bending stiffness, the deformation field $\boldsymbol{\chi}$ is determined by the functions $\tilde{\varrho}$ and $\tilde{\eta}$ which describe the geometry of the mid-surface of the membrane (see Figure 1b). The potential energy functional $\mathfrak{E}$ (15) can therefore be reparametrised as

$$E(\tilde{\varrho}, \tilde{\eta}, \mathbf{A}) = H \int_0^{2\pi} \int_0^{2\pi} \Omega(\mathbf{F}, \mathbb{D}) \sqrt{G} d\theta d\phi - \frac{1}{2} \int_0^{2\pi} \int_0^{2\pi} \tilde{P} \tilde{\varrho}^2 \tilde{\eta}_\theta d\theta d\phi. \quad (16)$$

181 The modification of the second term for the pressure work is based on the calculations provided in
 182 Appendix A (supporting file).

183 3.3. First variation and equations of equilibrium

At equilibrium, the total potential energy of the inflated toroidal membrane will be stationary, that is

$$\delta E \equiv \delta E((\tilde{\varrho}, \tilde{\eta}, \mathbf{A}); (\delta \tilde{\varrho}, \delta \tilde{\eta}, \delta \mathbf{A})) = 0, \quad (17)$$

$$\begin{aligned} \text{with } \delta E = H \int_0^{2\pi} \int_0^{2\pi} & \left[\Omega_{\tilde{\varrho}} \delta \tilde{\varrho} + \Omega_{\tilde{\varrho}_\theta} \delta \tilde{\varrho}_\theta + \Omega_{\tilde{\eta}_\theta} \delta \tilde{\eta}_\theta + \Omega_{\tilde{\varrho}_\phi} \delta \tilde{\varrho}_\phi + \Omega_{\tilde{\eta}_\phi} \delta \tilde{\eta}_\phi + \Omega_{\mathbb{D}} \cdot \delta \mathbb{D} \right] \sqrt{G} \\ & - \frac{1}{2H} \tilde{P} [2 \tilde{\varrho} \tilde{\eta}_\theta \delta \tilde{\varrho} + \tilde{\varrho}^2 \delta \tilde{\eta}_\theta] d\theta d\phi = 0. \end{aligned} \quad (18)$$

As shown in Figure 1a, the geometry of the toroidal membrane is axisymmetric with respect to $\mathbf{E}_3$. Thus, the principle solutions $\tilde{\varrho}$ and $\tilde{\eta}$ should be constant along the ϕ direction, which implies

$\tilde{\varrho}_\phi = \tilde{\eta}_\phi = 0$. Thus Equation (18) can be simplified as

$$\delta E = 2\pi H \int_0^{2\pi} \left[\left[\Omega_{\tilde{\varrho}} \delta \tilde{\varrho} + \Omega_{\tilde{\varrho}_\theta} \delta \tilde{\varrho}_\theta + \Omega_{\tilde{\eta}_\theta} \delta \tilde{\eta}_\theta + \Omega_{\mathbf{D}} \cdot \delta \mathbf{D} \right] \sqrt{G} - \frac{1}{2H} \tilde{P} [2 \tilde{\varrho} \tilde{\eta}_\theta \delta \tilde{\varrho} + \tilde{\varrho}^2 \delta \tilde{\eta}_\theta] \right] d\theta = 0. \quad (19)$$

184

The rotational symmetry of the torus leads to the periodic boundary conditions

$$\delta \tilde{\varrho}|_{\theta=0} = \delta \tilde{\varrho}|_{\theta=2\pi}, \quad \delta \tilde{\eta}|_{\theta=0} = \delta \tilde{\eta}|_{\theta=2\pi}, \quad (20)$$

while a perturbation of equation (11) gives $\delta \mathbf{D} = \text{Curl } \delta \mathbf{A}$. Thus we can express the variations of all the quantities in equation (19) in terms of the variations $\delta \tilde{\varrho}$, $\delta \tilde{\eta}$ and $\delta \mathbf{A}$, as desired. Using integration by parts on the terms containing $\delta \tilde{\varrho}_\theta$ and $\delta \tilde{\eta}_\theta$, the vector identity

$$[\nabla \times \mathbf{u}] \cdot \mathbf{v} = \nabla \cdot [\mathbf{u} \times \mathbf{v}] + [\nabla \times \mathbf{v}] \cdot \mathbf{u}, \quad \mathbf{u}, \mathbf{v} \in \mathbb{R}^3 \quad (21)$$

for the terms containing $\text{Curl } \delta \mathbf{A}$, and the arbitrariness of the variations results in the following Euler-Lagrange equations for this system

$$\sqrt{G} \Omega_{\tilde{\varrho}} - \frac{d}{d\theta} \left(\sqrt{G} \Omega_{\tilde{\varrho}_\theta} \right) - \frac{\tilde{P} \tilde{\varrho} \tilde{\eta}_\theta}{H} = 0, \quad (22a)$$

$$\frac{d}{d\theta} \left(\sqrt{G} \Omega_{\tilde{\eta}_\theta} - \frac{\tilde{P} \tilde{\varrho}^2}{2H} \right) = 0, \quad (22b)$$

$$\text{Curl } \mathbb{E} = \mathbf{0}, \quad (22c)$$

185 for $\theta \in (0, 2\pi]$ along with the boundary conditions (20) and a specified potential difference Φ_0 across
186 the torus thickness.

For the thin-walled membrane considered, the electric field can be reasonably approximated as [80]

$$\mathbf{e} = \frac{\Phi_0}{h} \mathbf{n}. \quad (23)$$

Thus, from (13) the electric field in reference configuration is given by

$$\mathbb{E} = \frac{\Phi_0}{\lambda_3 H} \mathbf{F}^\top \mathbf{n}, \quad (24)$$

187 which also satisfies the governing equation (22c).

The total energy density Ω can be further decomposed into an elastic energy density $\widehat{W}$ and an electric contribution as follows, see [19]

$$\Omega(\mathbf{F}, \mathbb{D}) = \widehat{W}(\mathbf{F}) + \beta[\mathbf{C} \mathbb{D}] \cdot \mathbb{D}, \quad (25)$$

where β is a positive material constant representing electroelastic coupling and the term $[\mathbf{C} \mathbb{D}] \cdot \mathbb{D}$ is a scalar invariant. This allows the electric field in the reference configuration to be expressed as

$$\mathbb{E} = \Omega_{\mathbb{D}} = 2\beta \mathbf{C} \mathbb{D}. \quad (26)$$

On comparing equations (23) and (26), the electric displacement in the reference configuration can be expressed as

$$\mathbb{D} = \frac{\Phi_0}{2\beta\lambda_3 H} \mathbf{F}^{-1} \mathbf{n} = \frac{\Phi_0}{2\beta H} \mathbf{C}^{-1} \mathbf{N}, \quad (27)$$

where $\mathbf{n}$ and $\mathbf{N}$ are unit vectors along the outward normal in current (deformed) and reference configurations, respectively. Here Nanson's relation is employed to relate the normal vectors as $\mathbf{n} da = J \mathbf{F}^{-\top} \mathbf{N} dA$ with $da = \lambda_1 \lambda_2 dA$.

We consider an incompressible Mooney–Rivlin constitutive model for the elastic energy density $\widehat{W}$ given in terms of two material constants C_1 and C_2 where

$$\widehat{W}(I_1, I_2) = C_1[I_1 - 3] + C_2[I_2 - 3], \quad (28a)$$

$$\text{with } I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \quad \text{and} \quad I_2 = \lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2}. \quad (28b)$$

The energy density Ω in Equation (25) can therefore be expressed as

$$\Omega = C_1[I_1 - 3] + C_2[I_2 - 3] + \beta[\mathbf{C} \mathbb{D}] \cdot \mathbb{D}. \quad (29)$$

The derivatives of Ω required for the subsequent derivations and computations are tabulated in Appendix B (supporting file).

3.5. Governing equations

We introduce the following dimensionless versions of pressure P and electric loading $\mathcal{E}$ to simplify our calculations

$$P = \frac{\widetilde{P} R_b}{C_1 H}, \quad \mathcal{E} = \frac{\Phi_0^2}{C_1 \beta H^2}. \quad (30)$$

Further introducing a dimensionless parameter $\alpha = C_2/C_1$, the ordinary differential equations (22a) and (22b) can be written in dimensionless form for $\theta \in [0, \pi]$ as

$$\begin{aligned} \frac{d}{d\theta} \left([1 + \gamma \cos \theta] \left[\frac{2\varrho_\theta}{\gamma^2} [1 + \alpha\lambda_2^2] \left[1 - \frac{1}{\lambda_1^4 \lambda_2^2} \right] - \frac{\rho_\theta \mathcal{E} \lambda_2^2}{2\gamma^2} \right] \right. \\ \left. - 2\lambda_2 [1 + \alpha\lambda_1^2] \left[1 - \frac{1}{\lambda_1^2 \lambda_2^4} \right] + \frac{\mathcal{E} \lambda_2 \lambda_1^2}{2} + \frac{P \varrho \eta_\theta}{\gamma} \right) = 0, \end{aligned} \quad (31a)$$

$$\frac{d}{d\theta} \left([1 + \gamma \cos \theta] \left[\frac{2\eta_\theta}{\gamma^2} [1 + \alpha\lambda_2^2] \left[1 - \frac{1}{\lambda_1^4 \lambda_2^2} \right] - \frac{\eta_\theta \mathcal{E} \lambda_2^2}{2\gamma^2} \right] - \frac{P \varrho^2}{2\gamma} \right) = 0, \quad (31b)$$

along with the boundary conditions, accounting for the additional assumption of a reflection symmetry of the system with respect to the $Y^1 - Y^2$ plane, given by

$$\varrho_\theta = 0, \quad \eta = 0 \quad \text{at} \quad \theta = 0 \quad (32a)$$

$$\text{and } \varrho_\theta = 0, \quad \eta = 0 \quad \text{at} \quad \theta = \pi. \quad (32b)$$

195

196 3.6. Numerical solution procedure

The governing equations (31a) and (31b) are coupled nonlinear second-order ordinary differential equations. These can be converted to a system of first-order ODEs by defining

$$y_1 = \varrho, \quad y_2 = \varrho_\theta, \quad y_3 = \eta, \quad y_4 = \eta_\theta, \quad (33)$$

and rewriting the system as

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & A_1 & 0 & A_2 \\ 0 & 0 & 1 & 0 \\ 0 & B_1 & 0 & B_2 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} y'_1 \\ y'_2 \\ y'_3 \\ y'_4 \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} y_2 \\ -A_3 \\ y_4 \\ -B_3 \end{bmatrix}}_{\mathbf{b}}, \quad (34)$$

where $(\bullet)'$ denotes the derivative with respect to θ , together with boundary conditions

$$y_2 = 0, \quad y_3 = 0 \quad \text{at} \quad \theta = 0 \quad \text{and} \quad \theta = \pi. \quad (35)$$

197 The components of matrices $\mathbf{A}$ and $\mathbf{b}$ are derived in Appendix D (supporting file).

198 We discretise the reference configuration of the membrane into n_θ segments as shown in Figure 2. The
199 $n_\theta + 1$ discretised points in the deformed configuration are denoted by $\mathbf{x}^i(\theta)$, where $i = 0, 1, 2, \dots, n_\theta$.
200 For each θ^i , the matrix $\mathbf{A}$ and the right hand vector $\mathbf{b}$ can be computed given the mechanical and
201 electrical loads P and $\mathcal{E}$ that are independent of θ . P is only present in the vector $\mathbf{b}$ as the mechanical

Figure 2: Half of the cross-section is discretised into n_θ segments with respect to θ and the boundary conditions (32) are applied at $\theta = 0$ and π .

load does not affect the material “stiffness”. However, $\mathcal{E}$ appears in both the matrix $\mathbf{A}$ and the vector $\mathbf{b}$ since the electrical load affects not only the boundary conditions but also changes the material behaviour.

The system (34) and (35) is solved using a variation of the shooting method [51] employing arc-length control [59]. We use the `ode45` solver in `Matlab` [43] for the numerical approximation of the ODEs. y_2^0 and y_3^0 are zero as per the boundary conditions (35) and a value of y_1^0 is chosen. To solve the system of equation, a reasonable initial guess for the variables y_4^0 and P is required. Then the function `ode45` can be used to solve for the unknowns $y_1^i, y_2^i, y_3^i, y_4^i$ for any $i = 0, 1, 2, 3, \dots, n_\theta$. A cost function $f_c = \sqrt{[y_2^{n_\theta}]^2 + [y_3^{n_\theta}]^2}$ is introduced to ensure satisfaction of the boundary conditions; this is required to be minimised as part of the shooting method. Then the function `fminsearchbnd` [16] is used to search for a pair of y_4^0 and P such that the cost function f_c is less than a tolerance $\epsilon = 10^{-6}$. Once $f_c < \epsilon$, the values of y_4^0 and P are assumed sufficiently converged and the values of $y_1^i, y_2^i, y_3^i, y_4^i$ for all $i = 0, 1, 2, 3, \dots, n_\theta$ are computed. The arc-length control (by specifying y_1^0 and computing P) helps in the evaluation of solutions between the snap-through path, see Figure 3.

Since the solutions are computed using a minimisation procedure, convergence is dependent upon the initial guess of P and y_4^0 . In general, the values of y_4^0 and P from the previous load step can be used as initial guesses for the updated deformed profile, thereby allowing the inflation of the toroidal membrane to be simulated. However, in the initial stages of the inflation of the membrane, a relatively large pressure increase results in a very small deformation. This is due to the high initial stiffness as shown in Figure 3. In this initial region, the gradient of P with respect to volume change is very high

Figure 3: A sketch of the initial stiffness and snap-through path for an ideal electroelastic toroidal membrane under large inflation. For a membrane with imperfections, snap-through may occur prior to reaching the limit point.

and convergence of the method using `fminsearchbnd` is problematic. To simulate the inflation in this region, one should seek solutions close to the limit point and then solve for a less deformed membrane profile by decreasing dy_1^0 with a small decrement. Convergence is also improved by introducing a scaling coefficient $\kappa = |dP/dy_1^0|$, where the change of pressure dP is estimated based on the previous solutions. Then, instead of searching for y_4^0 and P to achieve $f_c = 0$, one searches for P and κy_4^0 . Following this approach, the convergence of the `fminsearchbnd` is greatly improved and the method is robust. A computer code employing the above-described scheme is available at [40].

4. Wrinkling instability analysis

A membrane structure can only sustain tensile loading and has no resistance to compressive stress. When an in-plane compressive stress is about to occur in a membrane structure, the membrane tends to develop localised out-of-plane deformation to lower the energy. This phenomenon is known as wrinkling. For the problem considered here, when the toroidal membrane inflates and gradually undergoes increasing deformation, the tensile stresses in inner regions ($\theta \approx \pi$) are gradually reduced. Eventually, this will lead to wrinkling in the inner regions of the toroidal membrane. Post wrinkle formation, the principle governing equations (31a) and (31b) are no longer descriptors of the state of the system. An

energy relaxation method is then adopted to modify the governing equations and compute new solutions valid for the post-wrinkling regime.

4.1. In-plane stress components

The total Piola stress tensor $\mathbf{P}$ for the incompressible case is given by [19]

$$\mathbf{P} = \Omega_{\mathbf{F}} - p \mathbf{F}^{-\top}, \quad (36)$$

where p is a Lagrange multiplier associated with the constraint of incompressibility. The total Cauchy stress is (as $J = 1$)

$$\boldsymbol{\sigma} = \mathbf{P} \mathbf{F}^\top = \Omega_{\mathbf{F}} \mathbf{F}^\top - p \mathbf{i}, \quad (37)$$

where $\mathbf{i}$ is the second-order unit tensor in the current configuration. Let the local orthonormal coordinate system in the membrane mid-surface be given by $\{\mathbf{n}, \mathbf{t}_1, \mathbf{t}_2\}$, where the unit normal $\mathbf{n}$ is defined in Appendix A (supporting file) and $\mathbf{t}_1$ and $\mathbf{t}_2$ are unit vectors in the tangent plane of the membrane. Following the discussion in Section 2, the tensor $\mathbf{F}$ and subsequently the Cauchy stress $\boldsymbol{\sigma}$ are represented by diagonal matrices in this local basis. We can compute the in-plane stress components as

$$s_{11} = [\boldsymbol{\sigma} \mathbf{t}_1] \cdot \mathbf{t}_1, \quad s_{22} = [\boldsymbol{\sigma} \mathbf{t}_2] \cdot \mathbf{t}_2, \quad s_{12} = s_{21} = [\boldsymbol{\sigma} \mathbf{t}_2] \cdot \mathbf{t}_1 = 0, \quad (38)$$

Using the balance of traction at the inner boundary,

$$\boldsymbol{\sigma} \mathbf{n} + \tilde{P} \mathbf{n} = \mathbf{0}, \quad (39)$$

One determines the Lagrange multiplier p as

$$p = \tilde{P} + [\Omega_{\mathbf{F}} \mathbf{F}^\top \mathbf{n}] \cdot \mathbf{n}, \quad (40)$$

using which one can write the two in-plane stress components as

$$\begin{aligned} s_{11} &= [\Omega_{\mathbf{F}} \mathbf{F}^\top \mathbf{t}_1] \cdot \mathbf{t}_1 - \tilde{P} - [\Omega_{\mathbf{F}} \mathbf{F}^\top \mathbf{n}] \cdot \mathbf{n}, \\ s_{22} &= [\Omega_{\mathbf{F}} \mathbf{F}^\top \mathbf{t}_2] \cdot \mathbf{t}_2 - \tilde{P} - [\Omega_{\mathbf{F}} \mathbf{F}^\top \mathbf{n}] \cdot \mathbf{n}. \end{aligned} \quad (41)$$

Substituting in the specific form of Ω from equations (25) and (28a), and making use of (27), yields

$$\Omega_{\mathbf{F}} \mathbf{F}^\top = 2C_1 \mathbf{b} + 2C_2 [I_1 \mathbf{b} - \mathbf{b}^2] + \frac{\Phi_0^2}{2\beta\lambda_3^2 H^2} \mathbf{n} \otimes \mathbf{n}, \quad (42)$$

allowing one to rewrite the in-plane principal stress components as

$$s_{11} = C_1 \left[-\frac{PH}{R_b} + 2\lambda_1^2 + 2\alpha\lambda_1^2 \left[\lambda_2^2 + \frac{1}{\lambda_1^2\lambda_2^2} \right] - \frac{2}{\lambda_1^2\lambda_2^2} - 2\alpha \left[\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} \right] - \frac{\mathcal{E}\lambda_1^2\lambda_2^2}{2} \right], \quad (43a)$$

$$s_{22} = C_1 \left[-\frac{PH}{R_b} + 2\lambda_2^2 + 2\alpha\lambda_2^2 \left[\lambda_1^2 + \frac{1}{\lambda_1^2\lambda_2^2} \right] - \frac{2}{\lambda_1^2\lambda_2^2} - 2\alpha \left[\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} \right] - \frac{\mathcal{E}\lambda_1^2\lambda_2^2}{2} \right]. \quad (43b)$$

240 Note that the thinness assumption of the membrane, $H/R_s \ll 1$, nullifies the influence of the pressure
 241 term on the in-plane membrane stresses [12]. A constant value $H/R_s = \gamma^{-1}10^{-4}$ is thus adopted in all
 242 numerical examples.

243 4.2. Energy Relaxation

244 Adopting the tension field theory developed by [57, 67] for hyperelastic membranes and later ex-
 245 tended to electroelastic membranes by [15, 26], we introduce the concept of a *generalised natural state*.
 246 In the current problem, the key kinematic variables in the total energy density function $\Omega(\mathbf{F}, \mathbb{D})$ are
 247 $\{\lambda_1, \lambda_2, \mathcal{E}\}$ and hence $\Omega = \widehat{\Omega}(\lambda_1, \lambda_2, \mathcal{E})$.

If the membrane is mechanically stretched along one direction with stretch $\lambda_1 > 1$ (resp. $\lambda_2 > 1$)
 and an electric load $\mathcal{E}$ is applied, then the value of λ_2 (resp. λ_1) that sets the principal stress component
 s_{22} (resp. s_{11}) to zero is denoted as the *natural width* w_1 . This natural width is given by

$$w_1(\lambda_1, \mathcal{E}) := \lambda_2^* \quad \text{such that} \quad s_{22}(\lambda_1, \lambda_2^*, \mathcal{E}) = 0, \quad (44)$$

where λ_2^{*2} can be expressed as a function of λ_1 and $\mathcal{E}$ as

$$\lambda_2^{*2}(\lambda_1, \mathcal{E}) = \frac{\frac{PH}{R_b} \lambda_1 + \sqrt{\frac{P^2 H^2}{R_b^2} \lambda_1^2 - 4\alpha \mathcal{E} \lambda_1^4 + 16\alpha^2 \lambda_1^4 + 32\alpha \lambda_1^2 - 4\mathcal{E} \lambda_1^2 + 16}}{4\lambda_1 + 4\alpha \lambda_1^3 - \mathcal{E} \lambda_1^3}. \quad (45)$$

The relaxed energy is then given by

$$\Omega^*(\lambda_1, \lambda_2^*) = C_1[I_1^* - 3] + C_2[I_2^* - 3] + \beta[\mathbf{C} \mathbb{D}] \cdot \mathbb{D}, \quad (46)$$

where

$$I_1^*(\lambda_1, \lambda_2^*) = \lambda_1^2 + \lambda_2^{*2} + \frac{1}{\lambda_1^2 \lambda_2^{*2}}, \quad I_2^*(\lambda_1, \lambda_2^*) = \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^{*2}} + \lambda_1^2 \lambda_2^{*2}, \quad (47)$$

and

$$\beta[\mathbf{C} \mathbb{D}] \cdot \mathbb{D} = \frac{C_1 \mathcal{E}}{4} \lambda_1^2 \lambda_2^{*2}. \quad (48)$$

Since λ_2^* is a function of λ_1 and $\mathcal{E}$ as shown in (45), I_1^* and I_2^* are only functions of λ_1 and $\mathcal{E}$.
 Similarly, $\beta[\mathbf{C} \mathbb{D}] \cdot \mathbb{D}$ is a function of λ_1 and $\mathcal{E}$. Hence

$$\Omega^*(\lambda_1, \mathcal{E}) = C_1[I_1^* - 3] + C_2[I_2^* - 3] + \beta[\mathbf{C} \mathbb{D}] \cdot \mathbb{D}. \quad (49)$$

The governing equations (31) thus become

$$\begin{aligned} [1 + \gamma \cos \theta] \frac{d}{d\theta} \left(\frac{\partial \Omega^*}{\partial \varrho_\theta} \right) - \gamma \sin \theta \frac{\partial \Omega^*}{\partial \varrho_\theta} - \frac{\partial \Omega^*}{\partial \varrho} [1 + \gamma \cos \theta] + \frac{\tilde{P} R_b \varrho \eta_\theta}{\gamma H} &= 0, \\ [1 + \gamma \cos \theta] \frac{d}{d\theta} \left(\frac{\partial \Omega^*}{\partial \eta_\theta} \right) - \gamma \sin \theta \frac{\partial \Omega^*}{\partial \eta_\theta} - \frac{d}{d\theta} \left(\frac{\tilde{P} R_b \varrho^2}{2\gamma H} \right) &= 0. \end{aligned} \quad (50)$$

These can be rearranged into a matrix form as in (34). The reformulation is given in Appendix E (supporting file). The same shooting method described in Section 3.6 is used to solve these modified equations.

5. Loss of symmetry in the ϕ direction

This section considers the loss of symmetry of the principal solution obtained in Section 3 due to a perturbation along the outer equator which is an instability of the toroidal membrane. If the second variation of the potential energy functional vanishes, the solution bifurcates to a lower energy branch that is no longer symmetric.

5.1. Second variation of the potential energy functional

For the analysis of the critical point $(\mathbf{X}, \mathbf{A})$ of instability, one seeks $\Delta \mathbf{X} := (\Delta \tilde{\varrho}, \Delta \tilde{\eta})$ and $\Delta \mathbf{A}$ such that the following bilinear functional vanishes

$$\delta^2 E[\mathbf{X}, \mathbf{A}; (\delta \mathbf{X}, \delta \mathbf{A}), (\Delta \mathbf{X}, \Delta \mathbf{A})] = 0. \quad (51)$$

The principal solution has no dependence on the ϕ coordinate, but one might be interested in perturbations along the ϕ direction. Hence, we derive a more general Taylor expansion of the functional E in (16) considering $\tilde{\varrho}$ and $\tilde{\eta}$ to have dependence on both θ and ϕ , i.e. $\tilde{\varrho}(\theta, \phi)$ and $\tilde{\eta}(\theta, \phi)$. This yields

$$\begin{aligned} E[\tilde{\varrho} + \delta \tilde{\varrho}, \tilde{\eta} + \delta \tilde{\eta}, \mathbf{A} + \delta \mathbf{A}] &= H \int_0^{2\pi} \int_0^{2\pi} \left[\Omega + \Omega_{\tilde{\varrho}} \delta \tilde{\varrho} + \Omega_{\tilde{\varrho}\theta} \delta \tilde{\varrho}_\theta + \Omega_{\tilde{\varrho}\phi} \delta \tilde{\varrho}_\phi + \Omega_{\tilde{\eta}} \delta \tilde{\eta}_\theta \right. \\ &\quad + \Omega_{\tilde{\eta}\phi} \delta \tilde{\eta}_\phi + \Omega_{\mathbb{D}} \cdot \delta \mathbb{D} + \frac{1}{2} \left[\Omega_{\tilde{\varrho}\tilde{\varrho}} \delta \tilde{\varrho} \delta \tilde{\varrho} + \Omega_{\tilde{\varrho}\theta\tilde{\varrho}} \delta \tilde{\varrho}_\theta \delta \tilde{\varrho}_\theta + \Omega_{\tilde{\varrho}\phi\tilde{\varrho}} \delta \tilde{\varrho}_\phi \delta \tilde{\varrho}_\phi \right. \\ &\quad + \Omega_{\tilde{\eta}\theta\tilde{\eta}} \delta \tilde{\eta}_\theta \delta \tilde{\eta}_\theta + \Omega_{\tilde{\eta}\phi\tilde{\eta}} \delta \tilde{\eta}_\phi \delta \tilde{\eta}_\phi + \delta \mathbb{D} \cdot [\Omega_{\mathbb{D}\mathbb{D}} \delta \mathbb{D}] + 2\Omega_{\tilde{\varrho}\tilde{\varrho}\theta} \delta \tilde{\varrho} \delta \tilde{\varrho}_\theta + 2\Omega_{\tilde{\varrho}\tilde{\varrho}\phi} \delta \tilde{\varrho} \delta \tilde{\varrho}_\phi \\ &\quad + 2\Omega_{\tilde{\varrho}\tilde{\eta}\theta} \delta \tilde{\varrho} \delta \tilde{\eta}_\theta + 2\Omega_{\tilde{\varrho}\tilde{\eta}\phi} \delta \tilde{\varrho} \delta \tilde{\eta}_\phi + 2\delta \mathbb{D} \cdot \Omega_{\tilde{\varrho}\mathbb{D}} \delta \tilde{\varrho} + 2\Omega_{\tilde{\varrho}\theta\mathbb{D}} \delta \tilde{\varrho}_\theta \delta \tilde{\varrho} + 2\Omega_{\tilde{\varrho}\phi\mathbb{D}} \delta \tilde{\varrho}_\phi \delta \tilde{\varrho} \\ &\quad + 2\Omega_{\tilde{\eta}\theta\mathbb{D}} \delta \tilde{\eta}_\theta \delta \tilde{\eta} + 2\Omega_{\tilde{\eta}\phi\mathbb{D}} \delta \tilde{\eta}_\phi \delta \tilde{\eta} + 2\delta \mathbb{D} \cdot \Omega_{\tilde{\eta}\mathbb{D}} \delta \tilde{\eta} + 2\delta \mathbb{D} \cdot \Omega_{\tilde{\eta}\theta\mathbb{D}} \delta \tilde{\eta}_\theta + 2\delta \mathbb{D} \cdot \Omega_{\tilde{\eta}\phi\mathbb{D}} \delta \tilde{\eta}_\phi \left. \right] \sqrt{G} d\theta d\phi \\ &\quad - \frac{1}{2} \int_0^{2\pi} \int_0^{2\pi} \tilde{P} [[\tilde{\varrho} + \delta \tilde{\varrho}]^2 [\tilde{\eta}_\theta + \delta \tilde{\eta}_\theta]] d\theta d\phi. \end{aligned} \quad (52)$$

Application of integration by parts and making use of the vector identity (21), the second variation is

expressed as

$$\begin{aligned}
\delta^2 E[\mathbf{X}, \mathbf{A}; (\delta \mathbf{X}, \delta \mathbf{A}), (\Delta \mathbf{X}, \Delta \mathbf{A})] = & -H \int_0^{2\pi} \int_0^{2\pi} \left[\right. \\
& \left[\sqrt{G} \left[-\Omega_{\tilde{\varrho}} \tilde{\varrho} \Delta \tilde{\varrho} - \Omega_{\tilde{\varrho}} \tilde{\varrho}_\theta \Delta \tilde{\varrho}_\theta - \Omega_{\tilde{\varrho}} \tilde{\varrho}_\phi \Delta \tilde{\varrho}_\phi - \Omega_{\tilde{\varrho}} \tilde{\eta}_\theta \Delta \tilde{\eta}_\theta - \Omega_{\tilde{\varrho}} \tilde{\eta}_\phi \Delta \tilde{\eta}_\phi - \Omega_{\tilde{\varrho}} \mathbb{D} \cdot \Delta \mathbb{D} \right] \right. \\
& + \frac{d}{d\theta} \left(\left[\Omega_{\tilde{\varrho}_\theta} \tilde{\varrho}_\theta \Delta \tilde{\varrho}_\theta + \Omega_{\tilde{\varrho}} \tilde{\varrho}_\theta \Delta \tilde{\varrho} + \Omega_{\tilde{\varrho}_\theta} \tilde{\varrho}_\phi \Delta \tilde{\varrho}_\phi + \Omega_{\tilde{\varrho}_\theta} \tilde{\eta}_\theta \Delta \tilde{\eta}_\theta + \Omega_{\tilde{\varrho}_\theta} \tilde{\eta}_\phi \Delta \tilde{\eta}_\phi + \Omega_{\tilde{\varrho}_\theta} \mathbb{D} \cdot \Delta \mathbb{D} \right] \sqrt{G} \right) \\
& + \frac{d}{d\phi} \left(\left[\Omega_{\tilde{\varrho}_\phi} \tilde{\varrho}_\phi \Delta \tilde{\varrho}_\phi + \Omega_{\tilde{\varrho}} \tilde{\varrho}_\phi \Delta \tilde{\varrho} + \Omega_{\tilde{\varrho}_\phi} \tilde{\varrho}_\theta \Delta \tilde{\varrho}_\theta + \Omega_{\tilde{\varrho}_\phi} \tilde{\eta}_\theta \Delta \tilde{\eta}_\theta + \Omega_{\tilde{\varrho}_\phi} \tilde{\eta}_\phi \Delta \tilde{\eta}_\phi + \Omega_{\tilde{\varrho}_\phi} \mathbb{D} \cdot \Delta \mathbb{D} \right] \sqrt{G} \right) \Big] \delta \tilde{\varrho} \\
& + \left[\frac{d}{d\theta} \left(\left[\Omega_{\tilde{\eta}_\theta} \tilde{\eta}_\theta \Delta \tilde{\eta}_\theta + \Omega_{\tilde{\varrho}} \tilde{\eta}_\theta \Delta \tilde{\varrho} + \Omega_{\tilde{\varrho}_\theta} \tilde{\eta}_\theta \Delta \tilde{\varrho}_\theta + \Omega_{\tilde{\varrho}_\phi} \tilde{\eta}_\theta \Delta \tilde{\varrho}_\phi + \Omega_{\tilde{\eta}_\theta} \tilde{\eta}_\phi \Delta \tilde{\eta}_\phi + \Omega_{\tilde{\eta}_\theta} \mathbb{D} \cdot \Delta \mathbb{D} \right] \sqrt{G} \right) \right. \\
& + \frac{d}{d\phi} \left(\left[\Omega_{\tilde{\eta}_\phi} \tilde{\eta}_\phi \Delta \tilde{\eta}_\phi + \Omega_{\tilde{\varrho}} \tilde{\eta}_\phi \Delta \tilde{\varrho} + \Omega_{\tilde{\varrho}_\theta} \tilde{\eta}_\phi \Delta \tilde{\varrho}_\theta + \Omega_{\tilde{\varrho}_\phi} \tilde{\eta}_\phi \Delta \tilde{\varrho}_\phi + \Omega_{\tilde{\eta}_\phi} \tilde{\eta}_\theta \Delta \tilde{\eta}_\theta + \Omega_{\tilde{\eta}_\phi} \mathbb{D} \cdot \Delta \mathbb{D} \right] \sqrt{G} \right) \Big] \delta \tilde{\eta} \\
& - \text{Curl} \left[\Omega_{\mathbb{D}} \mathbb{D} \Delta \mathbb{D} + \Omega_{\tilde{\varrho}} \mathbb{D} \Delta \tilde{\varrho} + \Omega_{\tilde{\varrho}_\theta} \mathbb{D} \Delta \tilde{\varrho}_\theta + \Omega_{\tilde{\varrho}_\phi} \mathbb{D} \Delta \tilde{\varrho}_\phi + \Omega_{\tilde{\eta}_\theta} \mathbb{D} \Delta \tilde{\eta}_\theta + \Omega_{\tilde{\eta}_\phi} \mathbb{D} \Delta \tilde{\eta}_\phi \right] \cdot \delta \mathbf{A} \sqrt{G} \Big] d\theta d\phi \\
& - \int_0^{2\pi} \int_0^{2\pi} \tilde{P} \left[[\tilde{\varrho} \Delta \tilde{\eta}_\theta + \tilde{\eta}_\theta \Delta \tilde{\varrho}] \delta \tilde{\varrho} - [\tilde{\varrho}_\theta \Delta \tilde{\varrho} + \tilde{\varrho} \Delta \tilde{\varrho}_\theta] \delta \tilde{\eta} \right] d\theta d\phi. \tag{53}
\end{aligned}$$

257 The Euler-Lagrange equations corresponding to (51) are derived in dimensionless form as

$$\begin{aligned}
& \left[-\Omega_{\varrho} \varrho \Delta \varrho - \Omega_{\varrho} \varrho_\theta \Delta \varrho_\theta - \Omega_{\varrho} \varrho_\phi \Delta \varrho_\phi - \Omega_{\varrho} \eta_\theta \Delta \eta_\theta - \Omega_{\varrho} \eta_\phi \Delta \eta_\phi - \Omega_{\varrho} \mathbb{D} \cdot \Delta \mathbb{D} \right] [1 + \gamma \cos \theta] \\
& + \frac{d}{d\theta} \left(\left[\Omega_{\varrho_\theta} \varrho_\theta \Delta \varrho_\theta + \Omega_{\varrho} \varrho_\theta \Delta \varrho + \Omega_{\varrho_\theta} \varrho_\phi \Delta \varrho_\phi + \Omega_{\varrho_\theta} \eta_\theta \Delta \eta_\theta + \Omega_{\varrho_\theta} \eta_\phi \Delta \eta_\phi + \Omega_{\varrho_\theta} \mathbb{D} \cdot \Delta \mathbb{D} \right] [1 + \gamma \cos \theta] \right) \\
& + \frac{d}{d\phi} \left(\left[\Omega_{\varrho_\phi} \varrho_\phi \Delta \varrho_\phi + \Omega_{\varrho} \varrho_\phi \Delta \varrho + \Omega_{\varrho_\phi} \varrho_\theta \Delta \varrho_\theta + \Omega_{\varrho_\phi} \eta_\theta \Delta \eta_\theta + \Omega_{\varrho_\phi} \eta_\phi \Delta \eta_\phi + \Omega_{\varrho_\phi} \mathbb{D} \cdot \Delta \mathbb{D} \right] [1 + \gamma \cos \theta] \right) \\
& + \frac{C_1 P}{\gamma} \left[\varrho \Delta \eta_\theta + \eta_\theta \Delta \varrho \right] = 0, \tag{54}
\end{aligned}$$

$$\begin{aligned}
& \frac{d}{d\theta} \left(\left[\Omega_{\eta_\theta} \eta_\theta \Delta \eta_\theta + \Omega_{\varrho} \eta_\theta \Delta \varrho + \Omega_{\varrho_\theta} \eta_\theta \Delta \varrho_\theta + \Omega_{\varrho_\phi} \eta_\theta \Delta \varrho_\phi + \Omega_{\eta_\theta} \eta_\phi \Delta \eta_\phi + \Omega_{\eta_\theta} \mathbb{D} \cdot \Delta \mathbb{D} \right] [1 + \gamma \cos \theta] \right) \\
& + \frac{d}{d\phi} \left(\left[\Omega_{\eta_\phi} \eta_\phi \Delta \eta_\phi + \Omega_{\varrho} \eta_\phi \Delta \varrho + \Omega_{\varrho_\theta} \eta_\phi \Delta \varrho_\theta + \Omega_{\varrho_\phi} \eta_\phi \Delta \varrho_\phi + \Omega_{\eta_\theta} \eta_\phi \Delta \eta_\theta + \Omega_{\eta_\phi} \mathbb{D} \cdot \Delta \mathbb{D} \right] [1 + \gamma \cos \theta] \right) \\
& - \frac{C_1 P}{\gamma} \left[\varrho \Delta \varrho_\theta + \varrho_\theta \Delta \varrho \right] = 0, \tag{55}
\end{aligned}$$

$$\text{Curl} \left[\Omega_{\mathbb{D}} \mathbb{D} \Delta \mathbb{D} + \Omega_{\varrho} \mathbb{D} \Delta \varrho + \Omega_{\varrho_\theta} \mathbb{D} \Delta \varrho_\theta + \Omega_{\varrho_\phi} \mathbb{D} \Delta \varrho_\phi + \Omega_{\eta_\theta} \mathbb{D} \Delta \eta_\theta + \Omega_{\eta_\phi} \mathbb{D} \Delta \eta_\phi \right] = \mathbf{0}, \tag{56}$$

with periodic boundary conditions along θ and ϕ given by

$$\{\Delta \varrho, \Delta \eta, \Delta \mathbb{D}, \Delta \varrho_\theta, \Delta \varrho_\phi, \Delta \eta_\theta, \Delta \eta_\phi\}|_{\phi=0} = \{\Delta \varrho, \Delta \eta, \Delta \mathbb{D}, \Delta \varrho_\theta, \Delta \varrho_\phi, \Delta \eta_\theta, \Delta \eta_\phi\}|_{\phi=2\pi}, \quad (57)$$

$$\{\Delta \varrho, \Delta \eta, \Delta \mathbb{D}, \Delta \varrho_\theta, \Delta \varrho_\phi, \Delta \eta_\theta, \Delta \eta_\phi\}|_{\theta=0} = \{\Delta \varrho, \Delta \eta, \Delta \mathbb{D}, \Delta \varrho_\theta, \Delta \varrho_\phi, \Delta \eta_\theta, \Delta \eta_\phi\}|_{\theta=2\pi}. \quad (58)$$

Since ϱ_ϕ and η_ϕ vanish at the bifurcation point, $\Omega_{\varrho\varrho_\phi}$, $\Omega_{\varrho\eta_\phi}$, $\Omega_{\varrho_\theta\varrho_\phi}$, $\Omega_{\varrho_\theta\eta_\phi}$, $\Omega_{\varrho_\phi\mathbb{D}}$ and $\Omega_{\eta_\phi\mathbb{D}}$ are also zero as shown in Appendix C (supporting file). Rearranging equations (54) and (55) gives

$$\begin{aligned} & \left[[1 + \gamma \cos \theta] \left[-\Omega_{\varrho\varrho} + \frac{d\Omega_{\varrho\varrho\theta}}{d\theta} \right] - \gamma \sin \theta \Omega_{\varrho\varrho\theta} + \frac{C_1 P}{\gamma} \eta_\theta \right] \Delta \varrho \\ & + \left[[1 + \gamma \cos \theta] \left[-\Omega_{\varrho\varrho_\theta} + \frac{d\Omega_{\varrho\varrho_\theta\theta}}{d\theta} \right] - \gamma \sin \theta \Omega_{\varrho\varrho_\theta\theta} \right] \Delta \varrho_\theta \\ & + [1 + \gamma \cos \theta] \Omega_{\varrho_\theta\varrho_\theta} \Delta \varrho_{\theta\theta} + \left[[1 + \gamma \cos \theta] \left[\frac{d\Omega_{\varrho_\phi\varrho_\phi}}{d\phi} \right] \right] \Delta \varrho_\phi \\ & + \left[[1 + \gamma \cos \theta] \Omega_{\varrho_\phi\varrho_\phi} \right] \Delta \varrho_{\phi\phi} + \left[[1 + \gamma \cos \theta] \left[-\Omega_{\varrho\eta_\theta} + \frac{d\Omega_{\varrho\eta_\theta\theta}}{d\theta} \right] - \gamma \sin \theta \Omega_{\varrho\eta_\theta\theta} + \frac{C_1 P}{\gamma} \varrho \right] \Delta \eta_\theta \\ & + [1 + \gamma \cos \theta] \Omega_{\varrho_\theta\eta_\theta} \Delta \eta_{\theta\theta} + \left[[1 + \gamma \cos \theta] \left[\frac{d\Omega_{\varrho_\phi\eta_\phi}}{d\phi} \right] \right] \Delta \eta_\phi + [1 + \gamma \cos \theta] \Omega_{\varrho_\phi\eta_\phi} \Delta \eta_{\phi\phi} \\ & + \left[[1 + \gamma \cos \theta] \left[-\Omega_{\varrho\mathbb{D}} + \frac{d\Omega_{\varrho\mathbb{D}\theta}}{d\theta} \right] - \gamma \sin \theta \Omega_{\varrho\mathbb{D}\theta} \right] \cdot \Delta \mathbb{D} + [1 + \gamma \cos \theta] \Omega_{\varrho_\theta\mathbb{D}} \cdot \frac{d\Delta \mathbb{D}}{d\theta} = 0 \end{aligned} \quad (59)$$

$$\begin{aligned} & \left[[1 + \gamma \cos \theta] \frac{d\Omega_{\varrho\eta_\theta}}{d\theta} - \gamma \sin \theta \Omega_{\varrho\eta_\theta} - \frac{C_1 P}{\gamma} \varrho_\theta \right] \Delta \varrho \\ & + \left[[1 + \gamma \cos \theta] \left[\Omega_{\varrho\eta_\theta} + \frac{d\Omega_{\varrho_\theta\eta_\theta}}{d\theta} \right] - \gamma \sin \theta \Omega_{\varrho_\theta\eta_\theta} - \frac{C_1 P}{\gamma} \varrho \right] \Delta \varrho_\theta \\ & + [1 + \gamma \cos \theta] \Omega_{\varrho_\theta\eta_\theta} \Delta \varrho_{\theta\theta} + \left[[1 + \gamma \cos \theta] \frac{d\Omega_{\varrho_\phi\eta_\phi}}{d\phi} \right] \Delta \varrho_\phi \\ & + \left[[1 + \gamma \cos \theta] \Omega_{\varrho_\phi\eta_\phi} \right] \Delta \varrho_{\phi\phi} + \left[[1 + \gamma \cos \theta] \frac{d\Omega_{\eta_\theta\eta_\theta}}{d\theta} - \gamma \sin \theta \Omega_{\eta_\theta\eta_\theta} \right] \Delta \eta_\theta \\ & + [[1 + \gamma \cos \theta] \Omega_{\eta_\theta\eta_\theta} \Delta \eta_{\theta\theta} + \left[[1 + \gamma \cos \theta] \frac{d\Omega_{\eta_\phi\eta_\phi}}{d\phi} \right] \Delta \eta_\phi \\ & + [1 + \gamma \cos \theta] \Omega_{\eta_\phi\eta_\phi} \Delta \eta_{\phi\phi} + \left[[1 + \gamma \cos \theta] \frac{d\Omega_{\eta_\theta\mathbb{D}}}{d\theta} - \gamma \sin \theta \Omega_{\eta_\theta\mathbb{D}} \right] \cdot \Delta \mathbb{D} \\ & + [[1 + \gamma \cos \theta] \Omega_{\eta_\theta\mathbb{D}}] \cdot \frac{d\Delta \mathbb{D}}{d\theta} = 0 \end{aligned} \quad (60)$$

The third governing equation (56) physically means that the curl of the perturbed electrical field is zero, i.e. $\text{Curl}(\Delta \mathbb{E}) = \mathbf{0}$. As the electrical field does not change in the normal directions, i.e. $\frac{d}{dN}(\Delta \mathbb{E}) = \mathbf{0}$, (56) reduces to

$$[\Omega_{\mathbb{D}\mathbb{D}} \Delta \mathbb{D} + \Omega_{\varrho\mathbb{D}} \Delta \varrho + \Omega_{\varrho_\theta\mathbb{D}} \Delta \varrho_\theta + \Omega_{\varrho_\phi\mathbb{D}} \Delta \varrho_\phi + \Omega_{\eta_\theta\mathbb{D}} \Delta \eta_\theta + \Omega_{\eta_\phi\mathbb{D}} \Delta \eta_\phi] \cdot \mathbf{N} = 0. \quad (61)$$

To study prismatic-type bifurcation of the solution along the ϕ coordinate, we consider the following perturbations of the fields superimposed upon the principal solution

$$\Delta \varrho = \mathcal{F} \exp(im\phi), \quad \Delta \eta = \mathcal{G} \exp(im\phi), \quad \Delta \mathbb{D} = \frac{\Phi_0}{\beta H} \mathcal{H} \exp(im\phi) \mathbf{N}, \quad m \in \mathbb{Z}^+. \quad (62)$$

Here we have assumed $\mathcal{F}, \mathcal{G}$ and $\mathcal{H}$ are constants and, invoking the thin-membrane assumption, we restrict the solution $\Delta \mathbb{D}$ to along the thickness direction $\mathbf{N}$. Upon substitution of (62), the equations (59), (60) and (61) can be expressed in terms of three unknown variables $\mathcal{F}, \mathcal{G}$ and $\mathcal{H}$ as

$$\underbrace{\left[[1 + \gamma \cos \theta] \left[-\Omega_{\varrho\varrho} + \frac{d\Omega_{\varrho\varrho}}{d\theta} \right] - \gamma \sin \theta \Omega_{\varrho\varrho\theta} + \frac{C_1 P}{\gamma} \eta_\theta - m^2 [1 + \gamma \cos \theta] \Omega_{\varrho\phi\varrho\phi} \right]}_{M_{11}} \mathcal{F} \quad (63)$$

$$\underbrace{-m^2 [1 + \gamma \cos \theta] \Omega_{\varrho\phi\eta\phi}}_{M_{12}} \mathcal{G} + \underbrace{\frac{\Phi_0}{\beta H} \left[[1 + \gamma \cos \theta] \left[-\Omega_{\varrho\mathbb{D}} + \frac{d\Omega_{\varrho\mathbb{D}}}{d\theta} \right] - \gamma \sin \theta \Omega_{\varrho\theta\mathbb{D}} \right] \cdot \mathbf{N}}_{M_{13}} \mathcal{H} = 0, \quad (64)$$

$$\underbrace{\left[[1 + \gamma \cos \theta] \frac{d\Omega_{\varrho\eta\theta}}{d\theta} - \gamma \sin \theta \Omega_{\varrho\eta\theta} - \frac{C_1 P}{\gamma} \varrho_\theta - m^2 [1 + \gamma \cos \theta] \Omega_{\varrho\phi\eta\phi} \right]}_{M_{21}} \mathcal{F} \\ \underbrace{-m^2 [1 + \gamma \cos \theta] \Omega_{\eta\phi\eta\phi}}_{M_{22}} \mathcal{G} + \underbrace{\frac{\Phi_0}{\beta H} \left[[1 + \gamma \cos \theta] \frac{d\Omega_{\eta\theta\mathbb{D}}}{d\theta} - \gamma \sin \theta \Omega_{\eta\theta\mathbb{D}} \right] \cdot \mathbf{N}}_{M_{23}} \mathcal{H} = 0, \quad (65)$$

$$\underbrace{\Omega_{\varrho\mathbb{D}} \cdot \mathbf{N}}_{M_{31}} \mathcal{F} + \underbrace{\left[\frac{\Phi_0}{\beta H} \Omega_{\mathbb{D}\mathbb{D}} \right]}_{M_{33}} \mathcal{H} = 0. \quad (66)$$

Finally the governing system of equations (64), (65) and (66) can be expressed in a matrix form as

$$\underbrace{\begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & 0 & M_{33} \end{bmatrix}}_{\mathbf{M}} \begin{bmatrix} \mathcal{F} \\ \mathcal{G} \\ \mathcal{H} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \quad (67)$$

259 If $\det(\mathbf{M}) = 0$, the system of equations has a solution, which means the toroidal membrane could lose
260 stability due to a bifurcation in the ϕ direction.

261 6. Numerical examples

262 Section 6.1 presents numerical results obtained by solving the governing equations (31a) and (31b)
263 with associated boundary conditions (32). These results are only valid before the torus loses stability.

Recall that as the membrane inflates, there are three kinds of possible instabilities considered here. They are snap-through, wrinkling, and bifurcation in the ϕ direction. If wrinkling happens first, the energy relaxation method is adopted and the governing equation are modified as described in Section 4.2. The new solutions obtained with the energy relaxation method are computed in Section 6.2 and compared with the principle solutions. If the bifurcation happens first, the toroidal membrane loses its symmetry along the ϕ direction at the critical point as demonstrated in Section 6.3. The principle solution becomes unstable after the bifurcation point.

6.1. Principal solution and limit point instability

We first consider the behaviour of an inflating electro-elastic toroidal membrane under different electrical loads. Figure 4 shows the plots of pressure against volume change for a toroidal membrane with aspect ratio $\gamma = 0.6$, material coefficient $\alpha = 0.2$ and an electrical load $\mathcal{E}$ increasing from 0 to 0.3. The profiles of these plots are similar. The solution without electrical load is verified using the results in [59]. At small deformation, the membrane has a large stiffness against inflation. Thus the ratio of pressure to volume change is large at the beginning. The ratio stays the same under different electrical loads. As the electrical load increases the limit point also occurs at a similar volume change, but the limit point pressure is significantly decreased. After the limit point for different electrical loads, the pressures all decrease with increasing volume. After a finite volume change, the membrane regains stiffness and the pressures start to increase as the volume increases. The same trend can be seen in Figures 8 and 9 for a toroidal membrane with $\gamma = 0.4$ and $\alpha = 0.1, 0.2$ and 0.3 .

Figure 5 compares and visualises the inflation of purely elastic membranes and eletroelastic membranes with $\mathcal{E} = 0.3$. The principle solutions are solved for increasing ϱ at $\theta = 0$ and the deformed profiles for both cases are coloured by the pressure P . The electroelastic membrane can achieve the same volume change as the elastic membrane under considerably less pressure. We also note the significant difference in inflation profiles of stout ($\gamma = 0.4$) and slender ($\gamma = 0.1$) tori. For large γ values shown in Figure 5a, the inflation (prescribed by position of outer end ρ at $\theta = 0$) causes the inner end ($\theta = \pi$) to move slightly inwards. However, for small γ values the inner end ($\theta = \pi$) first moves outward and then inwards as shown in Figure 5b. The level of inward movement is significantly accentuated in the presence of the electric field. This dependence of $\rho(\pi)$ and $\rho(0)$ on the aspect ratio and material properties is of significance while designing actuator mechanisms from EAPs as detailed in [77].

In an experimental setting, inflation of the membrane can be accomplished using either a pressure controlled or a volume controlled process. During the inflation of the membrane, one can control

Figure 4: Plot of pressure against volume change for a toroidal membrane with geometry parameter $\gamma = 0.6$ and material parameter $\alpha = 0.2$ for different electrical loads $\mathcal{E}$.

the pressure inside the membrane. When the pressure reach the limit point in a pressure controlled experiment, there will be a sudden volume change of the membrane. This is known as a snap-through buckling of the hyperelastic membrane as described in Figures 3 and 5a. We also note that it would be possible for snap-through to occur prior to reaching the limit point if there are imperfections. The post bifurcation state can be analysed using a range of techniques including the Maxwell criterion [72, 71].

Figure 5b shows the inflation process for a toroidal membrane with a smaller aspect ratio, $\gamma = 0.1$ and $\alpha = 0.2$. The pressure to reach the limit point is much higher than a toroidal membrane with $\gamma = 0.4$. For such tori, the snap-through path is significantly longer and cannot be shown in Figure 5b.

The snap-through phenomenon can be eliminated if the torus is inflated in a volume-controlled experiment. Our calculations show that the snap-through phenomenon can also be eliminated in a mass-controlled experiment (assuming ideal gas law under constant temperature, mass is proportional to PV) since mass and volume increase monotonically for an electroelastic torus.

6.2. Computation of wrinkling instability

Wrinkling occurs during inflation when the in-plane stress in any direction becomes zero. The two in-plane principal stress components are calculated using the principal solutions obtained from (41). For the toroidal membrane, wrinkling first occurs at the inner region where $\theta \approx \pi$. Figures 6a and 6b are two examples of membranes with negative stresses calculated from the principal solutions. Figure 6a shows a half cross section of the membrane with aspect ratio $\gamma = 0.4$, material parameter $\alpha = 0.3$, and the electrical load $\mathcal{E} = 0.1$. Wrinkles start to form when $\varrho \approx 5.16$ at $\theta = 0$. When $\varrho \approx 5.68$ at $\theta = 0$,

Figure 5: Cross-section profiles during inflation for toroidal membrane with (a) $\gamma = 0.4$ and (b) $\gamma = 0.1$. The material parameter $\alpha = 0.2$ for both cases. The upper halves represent the deformation under pure elastic loads. The lower halves represent the deformation with an electrical load $\mathcal{E} = 0.3$.

the wrinkling region is between $\theta = (29/30)\pi$ to π . Keeping the material parameter and electrical load the same, a torus with large aspect ratio $\gamma = 0.6$ starts to wrinkle when $\rho \approx 2.78$ at $\theta = 0$. When $\rho \approx 3.38$, the wrinkling region is between $\theta = (29/30)\pi$ to π .

The relaxed form of the energy density (46) is adopted when negative stresses are detected. The modified ODE system (50) is used for computing the profile of the wrinkled membrane. Figures 7a and 7b compare the membrane profiles of the principal solution with the solution obtained using the relaxed energy density function for tori with two different aspect ratios ($\gamma = 0.4$ and $\gamma = 0.6$). Figures 8a and 8b compare the plots of the pressure against the volume change with different electrical loads. Two toroidal membranes with same aspect ratio $\gamma = 0.4$ but different material parameters, $\alpha = 0.2$ and 0.3 , are investigated. The red dots indicate the first appearance of wrinkling during inflation. The loading curve post the occurrence of wrinkling is dashed to demonstrate the unstable region. As the electrical loads increase, the wrinkles occur even at low values of pressure and inflation. However, we note that the volume change for the first appearance of wrinkling does not decrease monotonically with increasing $\mathcal{E}$. Significant variable changes in the onset points of wrinkling are observed in both cases when $\mathcal{E} < 0.1$.

Figure 6: Membrane profiles showing wrinkling regions. The black profile shows the wrinkling starting to form and the blue profile indicate the wrinkling region is $\theta = (29/30)\pi$ to π . Both cases have the same material property $\alpha = 0.3$, Electrical load is $\mathcal{E} = 0.1$. A toroidal membrane with larger aspect ratio wrinkles with less inflation.

6.3. Loss of symmetry

The loss of symmetry along the ϕ direction during inflation is examined. This bifurcation occurs when the second variation of the energy density function is zero as computed by checking the determinant of matrix $\mathbf{M}$ in Equation (67). Three toroidal membranes with the same aspect ratio $\gamma = 0.4$ but different material parameter $\alpha = 0.1, 0.2$ and 0.3 are compared. Figure 9a shows the plot of pressure against volume change for increasing electrical load $\mathcal{E} = 0, 0.1, 0.2$ and 0.3 and the material parameter $\alpha = 0.1$. The bifurcation occurs close to the limit points which is similar to the neo-Hookean material investigated in [77]. Figures 9b and 9c show the same plots for membranes with $\alpha = 0.2$ and 0.3 . In this case bifurcation occurs at the strain hardening stage. All the bifurcation points are found at buckling mode $m = 1$. However, the electrical load significantly influences the onset of bifurcation. When the electrical load remain small, it delays the appearance of bifurcation. After the electrical load exceeds a certain value, bifurcation happens with a smaller volume change. For very large electrical loads, bifurcation occurs close to the limit point as shown for $\mathcal{E} = 0.4$ in Figure 9b.

Finally, we investigate the combined effect of both wrinkling and loss of symmetry. We compare the response of membranes with same geometry ($\gamma = 0.4$) and different material parameter ($\alpha = 0.2, 0.3$) in Figures 10a and 10b, and membranes with same material parameter ($\alpha = 0.3$) and different geometry ($\gamma = 0.4, 0.5$) in Figures 10b and 10c. Wrinkling is the dominant instability mode for membranes with a large aspect ratio $\gamma = 0.5$ irrespective as to the electrical load. For membranes with a lower aspect ratio $\gamma = 0.4$, the combined effect of the material parameter α and the electrical load $\mathcal{E}$ dictates the instability mode. Typically loss of symmetry occurs before wrinkling for very small and very large electrical loads,

(a)

(b)

Figure 7: Comparison of the membrane profiles computed using the principal solution and the relaxed energy solution.

(a) $\gamma = 0.4$, $\alpha = 0.2$

(b) $\gamma = 0.4$, $\alpha = 0.3$

Figure 8: Plots of the pressure against the volume change with different electrical loads. Red dots indicate where the first wrinkles occur. The dashed portions of the loading curves denote the unstable region post-wrinkling.

(a) $\gamma = 0.4$, $\alpha = 0.1$

(b) $\gamma = 0.4$, $\alpha = 0.2$

(c) $\gamma = 0.4$, $\alpha = 0.3$

Figure 9: Plots of the pressure against the volume change with different electrical loads. Blue diamonds indicate the onset of bifurcation that causes loss of symmetry. The dashed portions of the loading curves denote the unstable region after loss of symmetry

Figure 10: Plots of the pressure against the volume change with different electrical loads. Red dots indicate where the first wrinkle occurs. Blue diamonds indicate where the bifurcation points. As the electric load increases, the bifurcation occurs at a larger volume change. However, after a certain value of $\mathcal{E}$, the bifurcation occurs earlier.

and wrinkling is preferred for moderate $\mathcal{E}$ values. A larger value of the material parameter α can increase the zone in which wrinkling is preferred.

This interesting interaction between the various modes of instabilities can be used as an actuation mechanism. For example, consider the idealised phase-space diagram depicted in Figure 11 based on the response in Figure 10. An initial loading can be performed in the presence of electric field along the path $A \rightarrow B \rightarrow C$. Moving from B to C induces wrinkles in the torus. Upon reducing the intensity of the electric load while keeping the volume constant, one can remove wrinkles by moving to the state D. Upon further reducing the electric load while keeping the volume constant, one can move to the state E where the torus is no longer symmetric.

Figure 11: Instability interaction diagram. One can induce (or eliminate) wrinkles and loss of symmetry in a reversible manner by controlling the applied electric load $\mathcal{E}$ and the volume of fluid in the membrane; for example by traversing the path $A \rightarrow B \rightarrow C \rightarrow D \rightarrow E$.

7. Conclusions

A study of the mechanics of inflation of a toroidal membrane under large coupled electromechanical loading has been presented. A numerical scheme has been developed to solve the highly nonlinear coupled ODEs to capture the rapidly changing solution for small inflation values together with the aid of an arc-length method. The classical limit point instability for inflated membranes is recovered and the limit point pressure can be significantly reduced upon the application of a potential difference across the thickness of the torus. Wrinkling instability has been modelled using an extension of the tension field theory to electroelasticity by employing a relaxed energy approach. The torus loses its rotational symmetry at large values of inflation and this instability has been computed using a second variation based analysis. This critical bifurcation point varies nonlinearly with the electrical load - it is delayed for moderate electric loads and is then favoured for large electric loads.

This interaction between the various instability modes can be exploited for the development of actuation mechanisms. One such example has been demonstrated in Figure 11. The phase space of this problem can be traversed by controlling the electric load and either one of the pressure or the volume thereby providing a great deal of flexibility in engineering design. Such features will be exploited in future works.

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