

# HOMOTOPY THEORY OF MODULES OVER A COMMUTATIVE S-ALGEBRA: SOME TOOLS AND EXAMPLES

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**ABSTRACT.** Modern categories of spectra such as that of Elmendorf et al equipped with strictly symmetric monoidal smash products allows the introduction of symmetric monoids providing a new way to study highly coherent commutative ring spectra. These have categories of modules which are generalisations of the classical categories of spectra that correspond to modules over the sphere spectrum; passing to their derived or homotopy categories leads to new contexts in which homotopy theory can be explored.

In this paper we describe some of the tools available for studying these ‘brave new homotopy theories’ and demonstrate them by considering modules over the  $K$ -theory spectrum, closely related to Mahowald’s theory of  $bo$ -resolutions.

In a planned sequel we will apply these techniques to the much less familiar context of modules over the 2-local connective spectrum of topological modular forms.

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## INTRODUCTION

Modern categories of spectra such as that of Elmendorf et al [EKMM97] equipped with strictly symmetric monoidal smash products allow for the introduction of symmetric monoids giving a new way to study highly coherent commutative ring spectra. In turn these have categories of modules which are generalisations of the classical categories of spectra (corresponding to modules over the sphere spectrum). For example, such categories have Quillen model structures and so homotopy (or derived) categories, thus allowing the study of ‘brave new homotopy theories’. This paper provides an introduction to some of the machinery available for engaging in this version of homotopy theory and Sections 1, 2 and 3 provide an overview on homotopy theory for  $R$ -modules over a commutative  $S$ -algebra  $R$  which should be sufficient for reading the present work. Although we only discuss connective spectra, many aspects also apply to non-connective settings with suitable modifications.

As an example we consider the important case of  $kO$  (the 2-local connective real  $K$ -theory spectrum) which is related to Mahowald’s theory of bo-resolutions and we review some aspects of this from the present perspective. The case of  $tmf$  (the 2-local connective spectrum of topological modular forms) is largely waiting to be developed in the spirit of Mahowald’s work and in the planned sequel we will discuss this, focusing especially on examples associated with  $kO$  considered as a  $tmf$ -module. Throughout our aim will be to exhibit interesting phenomena with connections to classical homotopy theory.

Since we use make use of modules over the mod 2 Steenrod algebra  $\mathcal{A}^*$  and its finite subHopf algebras  $\mathcal{A}(n)^*$ , we give some algebraic background in Section 4.

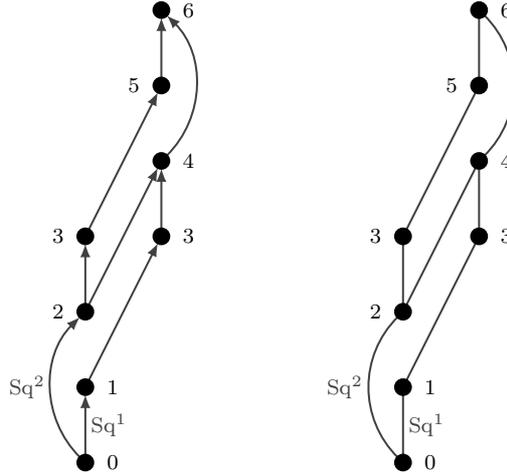
**Conventions & notations.** In this paper we will mainly work locally at the prime 2, so in that context  $H$  will denote the mod 2 Eilenberg-Mac Lane spectrum  $H\mathbb{F}_2$  and  $\mathcal{A}^*$  the mod 2 Steenrod algebra.

To avoid excessive display of gradings we will usually suppress cohomological degrees and write  $V$  for a cohomologically graded vector space  $V^*$ ; in particular we will often write  $\mathcal{A}$  for the Steenrod algebra. The linear dual of  $V$  is  $DV$  where  $(DV)^k = \text{Hom}_{\mathbb{F}_2}(V^{-k}, \mathbb{F}_2)$ , and we write  $V[m]$  for graded vector space with  $(V[m])^k = V^{k-m}$ , so for the cohomology of a spectrum  $X$ ,  $H^*(\Sigma^m X) = H^*(X)[m]$ . For a connected graded algebra  $\mathcal{B}^*$  we will often just write  $\mathcal{B}$ , and denote its positive degree part by  $\mathcal{B}^+$ .

When working with left modules over a Hopf algebra  $\mathcal{B}$  over a field we will write  $M \odot N$  to denote the vector space tensor product of two  $\mathcal{B}$ -modules  $M$  and  $N$  equipped with the diagonal action defined using the coproduct in  $\mathcal{B}$ ; for a vector space  $V$ ,  $M \otimes V$  will denote the left module with action obtained from the action on  $M$ . The product  $\odot$  defines the monoidal structure on the category of left  $\mathcal{B}$ -modules. It is well known that  $\mathcal{B} \odot N \cong \mathcal{B} \otimes N$  as left  $\mathcal{B}$ -modules, so  $\odot$  descends to the stable module category  $\mathbf{StMod}_{\mathcal{B}}$ .

When discussing modules we will often follow well established precedent and use diagrams such as those in the figure below which both represent  $\mathcal{A}(1)$  as a left  $\mathcal{A}(1)$ -module. We will usually interpret degrees as cohomological so that Steenrod actions are displayed pointing upwards

and we usually suppress arrow heads; labels on vertices denote degrees but are often omitted.



## 1. HOMOTOPY THEORY OF $R$ -MODULES

We adopt the terminology and notation of [EKMM97]. Initially we do not necessarily assume spectra are localised (or completed) at some prime although this possibility would not affect the generalities described here. Later we do focus on some specifically local aspects in order to incorporate notions from [BM04].

Let  $R$  be a connective commutative  $S$ -algebra. In the category of  $R$ -modules  $\mathcal{M}_R$  let  $S_R^n$  denote the functorial cofibrant replacement of the suspension  $\Sigma^n R$ . When  $R = S$  we will often suppress  $S$  from notation and for an  $R$ -module  $M$  also write  $M$  for the underlying  $S$ -module.

A morphism  $f: X \rightarrow Y$  in  $\mathcal{M} = \mathcal{M}_S$  gives rise to a morphism  $R \wedge X \rightarrow R \wedge Y$  in  $\mathcal{M}_R$ , namely  $1_r \wedge f$ . If  $M$  is an  $R$ -module, a morphism  $g: X \rightarrow M$  in  $\mathcal{M}$  gives rise to a morphism  $R \wedge X \rightarrow M$ .

$$R \wedge X \xrightarrow{1_r \wedge g} R \wedge M \longrightarrow M$$

For an  $R$ -module  $M$ ,

$$\pi_n(M) \cong \mathcal{D}_S(S^n, M) \cong \mathcal{D}_R(S_R^n, M).$$

Now suppose that  $R$  admits a morphism of commutative  $S$ -algebras  $R \rightarrow H = H\mathbb{F}_p$ ; for example, we might have  $\pi_0 R = \mathbb{Z}$  or  $\pi_0 R = \mathbb{Z}_{(p)}$ . Then we can define a homology theory  $H_*^R(-)$  on  $\mathcal{D}_R$  by setting

$$H_*^R M = \pi_*(H \wedge_R M)$$

when  $M$  is cofibrant. This theory has a dual cohomology theory  $H_R^*(-)$  defined by

$$H_R^* M = \mathcal{D}_R(M, H).$$

and which satisfies strict duality

$$H_R^n M \cong \text{Hom}_{\mathbb{F}_p}(H_n^R M, \mathbb{F}_p).$$

One approach to calculating is by using the Künneth spectral sequence of [EKMM97],

$$(1.1) \quad E_{s,t}^2 = \text{Tor}_{s,t}^{H_*^R}(\mathbb{F}_p, H_* M) \implies H_*^R M$$

which results from the isomorphism of  $H$ -modules

$$H \wedge_R M \cong H \wedge_{H \wedge_R} (H \wedge M).$$

Taking  $M = H$  we obtain a spectral sequence

$$\text{Tor}_{s,t}^{H_*^R}(\mathbb{F}_p, H_* H) \implies H_*^R H$$

which is known to be multiplicative. In a case where the induced homomorphism  $H_*R \rightarrow H_*H = \mathcal{A}_*$  is a monomorphism, this is especially useful. For  $p = 2$ , each of the cases  $R = H\mathbb{Z}, kO, kU, \text{tmf}, \text{tmf}_1(3)$  has this property. In such cases the dual Steenrod algebra  $\mathcal{A}_*$  is a free module over  $H_*R$  and we obtain

$$H_*^R H \cong \mathcal{A}_* \otimes_{H_*R} \mathbb{F}_p = \mathcal{A}_* // H_*R.$$

Dually we have

$$H_R^* H = \mathcal{D}_R(H, H)$$

and a spectral sequence

$$E_2^{s,t} = \text{Ext}_{H_*^R}^{s,t}(H_*H, \mathbb{F}_p) \implies H_R^* H.$$

In the situation where  $\mathcal{A}(p)_*$  is a free module over  $H_*R$ , this gives

$$H_R^n H \cong \text{Hom}_{\mathbb{F}_p}(H_n^R H, \mathbb{F}_p).$$

We also have

$$H^*R \cong \mathcal{A}(p)^* \otimes_{H_R^*H} \mathbb{F}_p.$$

As in the case  $R = S$ ,  $H_R^*H$  is a cocommutative Hopf algebra and its dual is a commutative Hopf algebra. Furthermore,  $H_R^*H$  has a natural left coaction on  $H_*^R M$  which induces a right action of  $H_*^R H$  on  $H_*^R M$  and a left action on  $H_R^* M$ . Using the algebra isomorphism

$$\mathcal{A} \xrightarrow{\cong} \mathcal{A}^\circ; \quad \theta \leftrightarrow (\chi\theta)^\circ$$

we can convert  $H_*^R M$  into a left  $H_R^*H$ -module with the grading convention that  $H_n^R M$  is in cohomological degree  $-n$  (so positive degree elements of  $H_R^*H$  act on  $H_*^R M$  by lowering degrees).

By using a geometric resolution of  $\mathbb{F}_p$  over  $H_*R$  to compute the  $E^2$ -term, it can be shown that (1.1) is a spectral sequence of left  $H_R^*H$ -comodules and right  $H_R^*H$ -modules.

In [BL01] we showed that there is an Adams spectral sequence for computing  $\mathcal{D}_R(L, M)$  for  $R$ -modules  $L, M$  with  $L$  strongly dualisable. Dualising from  $H_R^*H$ -comodules to  $H_R^*H$ -modules this has the form

$$(1.2) \quad E_2^{s,t}(L, M) = \text{Ext}_{H_R^*H}^{s,t}(H_*^R L, H_*^R M) \implies \mathcal{D}_R(\Sigma^{t-s} L, M)^\wedge,$$

where  $(-)\wedge$  denotes  $p$ -adic completion. When  $L = S_R^0$  we will set  $E_r^{*,*}(M) = E_r^{*,*}(L, M)$ . Assuming appropriate finiteness conditions this  $E_2$ -term can be rewritten to give

$$(1.3) \quad E_2^{s,t} = \text{Ext}_{H_R^*H}^{s,t}(H_R^* M, H_R^* L) \implies \mathcal{D}_R(\Sigma^{t-s} L, M)^\wedge.$$

When  $M = R \wedge Z$ , this gives

$$E_2^{s,t}(L, R \wedge Z) = \text{Ext}_{H_R^*H}^{s,t}(H_*^R L, H_*Z) \cong \text{Ext}_{H_R^*H}^{s,t}(H^*Z, H_R^*L).$$

Finally, if  $L = S_R^0$ ,

$$E_2^{s,t}(R \wedge Z) = \text{Ext}_{H_R^*H}^{s,t}(\mathbb{F}_p, H_*Z) \cong \text{Ext}_{H_R^*H}^{s,t}(H^*Z, \mathbb{F}_p).$$

This agrees with the classical Adams  $E_2$ -term

$$\text{Ext}_{\mathcal{A}^*}^{s,t}(H^*(R \wedge Z), \mathbb{F}_p) \implies \pi_{t-s}(R \wedge Z) \cong \mathcal{D}_R(S_R^{t-s}, R \wedge Z).$$

Notice that  $E_r^{*,*}(R)$  is a bi-graded commutative algebra and for any  $R$ -module  $M$ ,  $E_r^{*,*}(M)$  is a spectral sequence over it. When  $A$  is an  $R$ -ring spectrum,  $E_r^{*,*}(A)$  is a spectral sequence of  $E_r^{*,*}(R)$ -algebras.

For examples such as  $R = kO, kU, \text{tmf}, \text{tmf}_1(3)$  with  $p = 2$ , we know that  $H_*R$  is isomorphic to a left  $\mathcal{A}_* = H_*H$ -comodule algebra and then  $H_R^*H$  is a subHopf algebra of the Steenrod algebra  $\mathcal{A} = H^*H$ . In these cases  $\mathcal{A}$  is a free  $H_R^*H$ -module.

Now recall that for any ring homomorphism  $A \rightarrow B$ , left  $A$ -module  $U$  and left  $B$ -module  $V$ , there is a Cartan-Eilenberg change of rings spectral sequence of the form

$$E_2^{p,q} = \text{Ext}_B^q(\text{Tor}_p^A(B, U), V) \implies \text{Ext}_A^{q-p}(U, V).$$

If  $B$  is  $A$ -flat this collapses to give

$$\mathrm{Ext}_A^*(U, V) \cong \mathrm{Ext}_B^*(B \otimes_A U, V).$$

When  $A = H_R^*H$  and  $B = \mathcal{A}$  we obtain

$$\mathrm{Ext}_{H_R^*H}^*(\mathbb{F}_2, \mathbb{F}_2) \cong \mathrm{Ext}_{\mathcal{A}}^*(\mathcal{A} \otimes_{H_R^*H} \mathbb{F}_2, \mathbb{F}_2) \cong \mathrm{Ext}_{\mathcal{A}}^*(H^*R, \mathbb{F}_2)$$

which is the  $E_2$ -term of the Adams spectral sequence for computing  $\pi_*R$ ; of course, this is a standard change of rings isomorphism.

## 2. CELL AND CW $R$ -MODULES

Now we assume that  $R$  is  $p$ -local for some prime  $p$  and also  $(-1)$ -connected with  $R_0 = \pi_0R$  a acyclic  $\mathbb{Z}_{(p)}$ -module. This means that is a local graded ring whose maximal ideal  $\mathfrak{m} \triangleleft \pi_*R$  consists of the ideal  $(p) \triangleleft R_0$  together with all positive degree elements.

The notions of cell and CW  $R$ -modules have the usual forms described in [EKMM97]. The  $n$ -skeleton  $X^{[n]}$  of such a cell or CW  $R$ -module  $X$  is obtained from the  $(n-1)$ -skeleton  $X^{[n-1]}$  by forming a cofibre sequence of form

$$(2.1) \quad \bigvee_i S_R^{n-1} \xrightarrow{j^{n-1}} X^{[n-1]} \rightarrow X^{[n]}.$$

Here we take  $S_R^k$  to be the functorial cofibrant replacement of  $\Sigma^k R$  (recall that  $R$  is not cofibrant in the model category  $\mathcal{M}_R$  of [EKMM97]). Associated with such a cell  $R$ -module  $X$  there is a cellular chain complex of  $R_0$ -modules  $C_*^{\mathrm{CW}}(X)$  satisfying

$$(2.2) \quad H_*(C_*^{\mathrm{CW}}(X) \otimes_{R_0} \mathbb{F}_p) \cong H_*^R X.$$

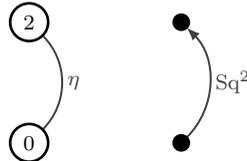
It is often useful to work with a minimal cell structure; we adapt the notion of minimal cell structure from [BM04] to the present context. A CW  $R$ -module  $X$  is *minimal* if for every  $n$ , the attaching map  $j^{n-1}$  of (2.1) satisfies

$$\mathrm{im}[j_*^{n-1}: \pi_*(\bigvee_i S_R^{n-1}) \rightarrow \pi_*X^{[n-1]}] \subseteq \mathfrak{m}\pi_*X^{[n-1]}.$$

It is easy to see that every connective  $R$ -module with finite type homotopy can be realised by a finite type connective minimal CW  $R$ -module. Furthermore, by (2.2) the cells of this give an  $\mathbb{F}_p$ -basis for  $H_*^R X$ , and the inclusion map induces a monomorphism

$$H_{n-1}^R X^{[n-1]} \rightarrow H_{n-1}^R X^{[n]}.$$

In order to describe CW modules or their (co)homology, we will often use cell diagrams or diagrams showing bases with action of  $H_R^*H$ . Usually we will assume a minimal cell structure has been used so cells will correspond to basis elements. For example, when  $R = S$ , the mapping cone of the Hopf map  $\eta: S^1 \rightarrow S^0$ ,  $X = S^0 \cup_{\eta} e^2$ , and its mod 2 cohomology,  $H^*X$ , can be represented by diagrams such as the following.



Such diagrams are standard in homotopy theory, for example they are discussed by Barratt, Jones & Mahowald [BJM84].

Occasionally we will require more complicated diagrams which involve maps between more general objects than just spheres. For example, given maps  $g: X \rightarrow Y$  and  $h: Y \rightarrow Z$  with  $hg$  null homotopic, we can factor  $\Sigma g: \Sigma X \rightarrow \Sigma Y$  through the mapping cone of  $h$  and so define

an object which is the mapping cone of a factor  $\Sigma X \rightarrow Z \cup_h CY$  represented by the following diagram.

$$\begin{array}{c} \Sigma^2 X \\ \Sigma g \downarrow \\ \Sigma Y \\ h \downarrow \\ Z \end{array}$$

Here is a useful result on building objects realising such diagrams; it generalises a well known result for maps between spheres.

**Lemma 2.1.** *Let  $f: W \rightarrow X$ ,  $g: X \rightarrow Y$  and  $h: Y \rightarrow Z$  where  $gf$  and  $hg$  are null homotopic. Then it is possible to realise the diagram*

$$\begin{array}{c} \Sigma^3 W \\ \Sigma^2 f \downarrow \\ \Sigma^2 X \\ \Sigma g \downarrow \\ \Sigma Y \\ h \downarrow \\ Z \end{array}$$

if and only if zero is contained in the Toda bracket  $\langle h, g, f \rangle_R \subseteq \mathcal{D}_R(\Sigma W, Z)$ .

Here  $\langle -, -, - \rangle_R$  denotes the Toda bracket calculated in the homotopy category of  $R$ -modules.

**Remark 2.2.** There is a different kind of Toda bracket that we might consider for an  $R$  ring spectrum  $A$  and a left  $A$ -module  $X$  spectrum (both being  $R$ -modules of course). For example we can consider  $\langle u, v, w \rangle_{R, X}$  for  $u \in \pi_r(A)$ ,  $v \in \pi_s(A)$  and  $w \in \pi_t(X)$  where  $uv = 0$  in  $\pi_*(A)$  and  $vw = 0$  in  $\pi_*(X)$  where these products are defined using the evident ring structure on  $\pi_*(A)$  and the  $\pi_*(A)$ -module structure of  $\pi_*(X)$ . The resulting bracket is a subset of  $\pi_{r+s+t+1}(X)$  with indeterminacy  $u\pi_{s+t+1}(X) + \pi_{r+s+1}(A)w$ . A version of this theory was described by G. Whitehead [Whi70] well before modern categories of spectra were developed but the essential ideas can be found in his work. Some applications of these brackets can be found in [BM04] using examples given in [Whi70]; in these we have  $R = A = S$  and  $X = k\mathbb{O}$  or  $k\mathbb{U}$ .

### 3. DUALITY

Various sorts of duality occur in the categories  $\mathcal{M}_R$  and  $\mathcal{D}_R$ , generalising classical cases.

**Spanier-Whitehead duality.** For an account of duality from a categorical viewpoint, we recommend the article of Dold & Puppe [DP80].

Following [DP80, EKMM97], the symmetric monoidal category  $\mathcal{D}_R$  has strongly dualisable objects and so there is a version of Spanier-Whitehead duality; we will denote the Spanier-Whitehead of an  $R$ -module  $X$  by  $D_R X = F_R(X, S_R^0)$ . As usual, when  $X$  is a finite CW module we can replace it by a weakly equivalent CW module; it is well known that an  $R$ -module is strongly dualisable if it is equivalent to a retract of a finite CW module. Of course, if  $Z$  is a strongly dualisable  $S$ -module (i.e., a spectrum) then  $R \wedge Z$  is a strongly dualisable  $R$ -module; more generally, if  $Z$  is a strongly dualisable  $R'$ -module where  $R$  is a commutative  $R'$ -algebra, then  $R \wedge_{R'} Z$  is strongly dualisable.

When  $L$  is strongly dualisable, the Adams spectral sequence of (1.3) can be expressed as

$$(3.1) \quad E_2^{s,t} = \text{Ext}_{H_R^*}^{s,t}(H_R^*(D_R L \wedge_R M), \mathbb{F}_p) \implies \mathcal{D}_R(\Sigma^{t-s}, D_R L \wedge_R M)^\wedge \cong \mathcal{D}_R(S_R^{t-s} L, M)^\wedge$$

since there are natural isomorphisms of functors

$$\mathrm{Hom}_{H_R^*H}(H_R^*(D_R L) \otimes (-), \mathbb{F}_p) \cong \mathrm{Hom}_{H_R^*H}(D(H_R^* L) \otimes (-), \mathbb{F}_p) \cong \mathrm{Hom}_{H_R^*H}(-, H_R^* L)$$

on left  $H_R^*H$ -modules extending to right derived functors. This is useful for computational purposes as it allows us to work consistently with projective resolutions and calculations with right derived functors of form  $\mathrm{Hom}_{H_R^*H}(-, \mathbb{F}_p)$ .

**Poincaré duality and Spanier-Whitehead duality.** We begin with some algebra. Let  $\mathbb{k}$  be a field and  $K^*$  a connected graded cocommutative Hopf algebra of finite type. We will indicate the coproduct  $\psi$  using the notation

$$\psi\theta = \theta \otimes 1 + 1 \otimes \theta + \sum_i \theta'_i \otimes \theta''_i = \theta \otimes 1 + 1 \otimes \theta + \sum_i \theta''_i \otimes \theta'_i$$

where the degrees of  $\theta', \theta''_i$  are positive and smaller than the degree of  $\theta$ . The action of the antipode  $\chi$  will often be indicated by writing  $\chi\theta = \bar{\theta}$ , so

$$\bar{\theta} = -\theta - \sum_i \bar{\theta}'_i \theta''_i = -\theta - \sum_i \theta'_i \bar{\theta}''_i.$$

Now let  $P_*$  be a local Poincaré duality algebra of degree  $d$  and let its graded dual be  $P^*$  where  $P^n = \mathrm{Hom}_{\mathbb{k}}(P_n, \mathbb{k})$ . This means that  $P_n = 0$  except when  $0 \leq n \leq d$ ,  $P_0 = \mathbb{k}$  and there is a  $\mathbb{k}$ -linear isomorphism  $P_0 \xrightarrow{\cong} P^d$  with  $1 \leftrightarrow \lambda$  which induces isomorphisms

$$P_n \xrightarrow{\cong} P^{d-n}; \quad x \mapsto x\lambda$$

where

$$(x\lambda)(y) = \lambda(yx)$$

for all  $y \in P_{d-n}$ . The pairing

$$P_* \otimes P^* \rightarrow P^*; \quad x \otimes \gamma \mapsto x\gamma$$

makes  $P^*$  a left  $P_*$ -module and the above duality isomorphism can be interpreted as defining an isomorphism of left  $P_*$ -modules

$$P_* \xrightarrow{\cong} P^*[-d]; \quad x \mapsto x\lambda.$$

Now suppose that  $P_*$  is a left  $K^*$ -module algebra. This means that there are pairings

$$K^r \otimes P_s \xrightarrow{\cong} P_{s-r}; \quad \theta \otimes x \mapsto \theta x$$

so that the Cartan formula holds for all  $x, y \in P_*$ :

$$\theta(xy) = (\theta x)y + x(\theta y) + \sum_i (\theta'_i x)(\theta''_i y).$$

There is also an action of  $K^*$  on  $P^*$  given by pairings

$$K^r \otimes P^s \xrightarrow{\cong} P^{r+s}; \quad \theta \otimes \gamma \mapsto \theta\gamma$$

where

$$(\theta\gamma)(z) = \gamma(\bar{\theta}z)$$

and this makes  $P^*$  a left  $H^*$ -module.

**Lemma 3.1.** *The duality isomorphism of left  $P_*$ -modules  $P_* \xrightarrow{\cong} P^*[-d]$  is also an isomorphism of left  $H^*$ -modules.*

*Proof.* We have to show that for all homogeneous elements  $x, y \in P_*$  and  $\theta \in K^*$ ,

$$\lambda(y(\theta x)) = ((\theta x)\lambda)(y) = \lambda(\bar{\theta}y)x.$$

We will prove this by induction on the degree of  $\theta$ . It is clearly true when  $\theta$  has degree 0. So assume it holds whenever  $\theta$  has degree less than  $n > 0$ .

Suppose that  $\theta$  has degree  $n$ . Consider  $\theta(yx)$ ; in order for this element to have degree  $d$ ,  $yx$  has to be of degree  $n + d > 0$ , hence  $yx = 0$ . So

$$\begin{aligned}
0 &= ((\theta(yx))\lambda)(1) = \lambda(\theta(yx)) \\
&= \lambda((\theta y)x) + \lambda(y\theta x) + \sum_i \lambda((\theta'_i y)(\theta''_i x)) \\
&= \lambda((\theta y)x) + \lambda(y\theta x) + \sum_i \lambda((\bar{\theta}'_i \theta''_i y)x) \\
&= \lambda(y\theta x) - \lambda((\bar{\theta} y)x) + \lambda\left((\theta y)x + \sum_i \lambda((\bar{\theta}'_i \theta''_i y)x) + (\bar{\theta} y)x\right) \\
&= \lambda(y\theta x) - \lambda((\bar{\theta} y)x).
\end{aligned}$$

Thus  $\lambda(y\theta x) = \lambda((\bar{\theta} y)x)$ , so the result holds for all  $n$ .  $\square$

Now let  $R$  be a commutative  $S$ -algebra satisfying the conditions assumed earlier. In particular, suppose that  $R_0$  is a cyclic  $\mathbb{Z}_{(p)}$ -module for some prime  $p$ .

Suppose that  $E$  is an  $R$  ring spectrum for which  $P_* = H_*^R E$  is a local Poincaré duality algebra over  $\mathbb{F}_p$ . Taking  $K^* = H_R^* H$ , the Spanier-Whitehead dual of  $E$  satisfies

$$H_*^R D_R E \cong H_*^R E \cong H_*^R E[d]$$

as  $H_R^* H$ -modules.

The next result is very useful for identifying Spanier-Whitehead stably self dual objects.

**Proposition 3.2.** *There is a morphism of  $R$ -modules  $E \rightarrow \Sigma^d R$  inducing a non-trivial homomorphism*

$$H_*^R E \rightarrow H_*^R R[-d] = \mathbb{F}_p.$$

*The multiplication map  $E \wedge_R E \rightarrow E$  composed with this map define a duality pairing  $E \wedge_R E \rightarrow \Sigma^d R$ . Hence  $E$  is a Spanier-Whitehead stably self dual  $R$ -module with  $D_R E \sim \Sigma^{-d} E$ .*

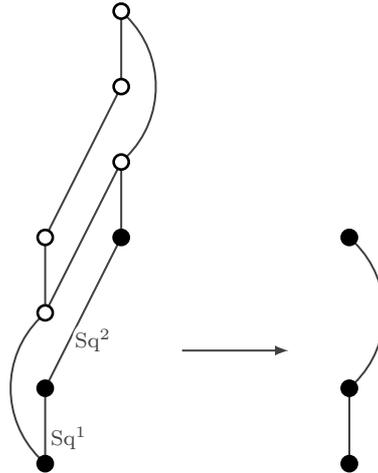
*Proof.* Choose a minimal CW  $R$ -module realisation of  $E$ ; this will have a single cell in each of the degrees 0 and  $d$ . Inclusion of the bottom cell induces the unit  $\mathbb{F}_p \rightarrow H_0^R E$ , while collapse onto the top cell induces a non-trivial linear mapping  $H_d^R E \rightarrow \mathbb{F}_p$  and this gives a basis element of  $H_R^d E$ ; by composing with a self map of  $S_R^d \sim \Sigma^d R$  we can assume this corresponds to  $1 \in H_0^R E$  under the Poincaré duality isomorphism  $H_0^R E \xrightarrow{\cong} H_R^d E$ . The product  $E \wedge_R E \rightarrow E$  composed with the projection  $E \rightarrow \Sigma^d R$  gives rise to a morphism  $f: E \rightarrow \Sigma^d D_R E$  and induces a non-degenerate pairing  $H_*^R E \otimes H_*^R E \rightarrow \mathbb{F}_p[d]$ . It follows that  $f_*: H_*^R E \xrightarrow{\cong} H_*^R D_R E[d]$  and so  $f: E \xrightarrow{\sim} \Sigma^d D_R E$ . Notice that all the algebraic maps here are compatible with the actions of  $H_R^* H$ .  $\square$

For any compact Lie group  $G$ ,  $E = R \wedge (G_+)$  provides an example of such a stably self-dual  $R$  ring spectrum. A generalisation to finite  $H$ -spaces was proved by Browder & Spanier [BS62], and some exotic examples can be found in the papers of Bauer, Pedersen and Rognes [Bau04, BP06, Rog08]; in all these classical cases,  $R = S$  is the base spectrum but the ideas work more generally.

**Cyclic modules.** It is common to encounter cyclic modules of the form  $\mathcal{A}(n) \otimes_{\mathcal{B}} \mathbb{F}_2$  arising as cohomology of  $R$ -modules for examples such as  $R = kO$  and  $R = \text{tmf}$ . Proposition 3.2 sometimes allows us to identify the underlying  $R$ -module as stably Spanier-Whitehead self dual, but this seems not to be a purely algebraic result. For example, consider the  $\mathcal{A}(1)$ -module

$$\mathcal{A}(1) \otimes_{\mathbb{F}_2(\text{Sq}^2)} \mathbb{F}_2 = \mathcal{A}(1)/\mathcal{A}(1)\{\text{Sq}^2\}.$$

Viewing this as the quotient of  $\mathcal{A}(1)$  obtained by killing the white circles in the diagram below, we see that this is the question mark module which is clearly not self dual.



The situation when  $\mathcal{B}$  is a subHopf algebra is more interesting. We will encounter many examples of this kind later arising as cohomology of  $kO$  or  $tmf$ -modules.

Next we give some algebraic results that we will use. We will assume the reader is aware of basic results such as the freeness of a Hopf algebra over a subHopf algebra (Milnor-Moore or Nichols-Zoeller for the ungraded case), and Poincaré duality/Frobenius property and self-injectivity for finite dimensional Hopf algebras (Browder-Spanier or Larson-Sweedler for the ungraded case).

First we give a statement and proof of a result for non-graded Hopf algebras which ought to be standard but we have been unable to locate a reference. This version incorporates suggestions of Ken Brown which led to a substantial improvement of our original attempt. Of course, if  $K$  is a *normal* subalgebra (i.e., if  $HK^+ = K^+H$ ) then  $H//K$  is a Hopf algebra and this result is immediate.

**Proposition 3.3.** *Let  $H$  be a finite dimensional commutative or cocommutative Hopf algebra over a field  $\mathbb{k}$  and let  $K$  be a subHopf algebra which is also unimodular. Then*

$$H//K = H \otimes_K \mathbb{k} \cong H/HK^+$$

*is a self-dual left  $H$ -module.*

*Proof.* For general results on Hopf algebras see Larson & Sweedler [LS69], Humphreys [Hum78, theorem 1] and Montgomery [Mon93]. When  $H$  is commutative or cocommutative its antipode is self inverse. i.e.,  $\chi \circ \chi = \text{Id}$ .

Let  $\lambda$  be a Frobenius form for  $H$  which we take to be a left and right integral in the dual Hopf algebra  $DH = \text{Hom}_{\mathbb{k}}(H, \mathbb{k})$ . The *Nakayama algebra automorphism*  $\nu: H \rightarrow H$  is characterised by the identity

$$(3.2) \quad \lambda(xy) = \lambda(y\nu(x))$$

for all  $x, y \in H$  (see [Lam99, section §16E]). If the associated bilinear pairing  $H \otimes_{\mathbb{k}} H \rightarrow \mathbb{k}$  is symmetric (which is true when  $H$  is unimodular) then  $\nu = \text{Id}_H$ .

As  $K$  is also unimodular, its (1-dimensional) vector spaces of left and right integrals coincide, i.e.,  $\int_K^l = \int_K^r$ , and we will just write  $\int_K$  for this subspace. By definition, the left and right annihilators of  $\int_K$  and any non-zero element  $s \in \int_K$  satisfy

$$\text{ann}_K^l \int_K = \text{ann}_K^l(s) = \text{ann}_K^r \int_K = \text{ann}_K^r(s) = K^+,$$

the kernel of the counit  $K \rightarrow \mathbb{k}$  which is a maximal ideal. By the Nichols-Zoeller theorem [NZ89],  $H$  is free as a left or right  $K$ -module, so the left annihilators in  $H$  satisfy

$$(3.3) \quad \text{ann}_H^l \int_K = \text{ann}_H^l(s) = HK^+,$$

the left ideal of  $H$  generated by  $K^+$ . Similarly the right annihilators satisfy

$$(3.4) \quad \text{ann}_H^r \int_K = \text{ann}_H^r(s) = K^+H = \chi(HK^+).$$

Now choose a non-zero element  $s_0 \in \int_K$ . The  $\mathbb{k}$ -linear mapping

$$\lambda': H \rightarrow \mathbb{k}; \quad x \mapsto \lambda(xs_0)$$

satisfies

$$\lambda'(xz) = 0$$

whenever  $x \in H$  and  $z \in K^+$ , so it factors through a linear mapping  $\lambda'': H//K \rightarrow \mathbb{k}$ .

It follows that the left  $H$ -module homomorphism

$$H \rightarrow H; \quad x \mapsto xs_0$$

has kernel  $HK^+$  and so induces an isomorphism  $H//K \xrightarrow{\cong} Hs_0$ .

Define a left  $H$ -module structure on  $\text{Hom}_{\mathbb{k}}(Hs_0, \mathbb{k})$  by setting

$$(x \cdot \alpha)(z) = \alpha(\chi(x)z)$$

for  $\alpha \in \text{Hom}_{\mathbb{k}}(Hs_0, \mathbb{k})$  and  $x \in H$ . Now define a left  $H$ -module homomorphism

$$H \rightarrow \text{Hom}_{\mathbb{k}}(Hs_0, \mathbb{k}); \quad x \mapsto x \cdot \lambda'$$

Using the Nakayama automorphism characterised in (3.2),

$$(x \cdot \lambda')(z) = \lambda(\chi(x)z) = \lambda(z\nu(\chi(x)))$$

so if  $x \in HK^+ = \chi(K^+H)$ ,  $(x \cdot \lambda')(z) = 0$ , showing that we can factor our homomorphism through a left  $H$ -module homomorphism  $H//K \rightarrow \text{Hom}_{\mathbb{k}}(Hs_0, \mathbb{k})$ . When  $x \notin HK^+$ , the kernel of the functional  $\lambda((-\)\nu(\chi(x))): H \rightarrow \mathbb{k}$  cannot contain a left ideal, therefore the functional  $x \cdot \lambda: Hs_0 \rightarrow \mathbb{k}$  must be non-trivial. This shows that we have an injection

$$H//K \rightarrow \text{Hom}_{\mathbb{k}}(Hs_0, \mathbb{k}) \cong \text{Hom}_{\mathbb{k}}(H//K, \mathbb{k})$$

which for dimensional reasons must be an isomorphism.  $\square$

Recall that a connected cohomologically graded  $\mathbb{k}$ -algebra  $A$  has  $A^i = 0$  when  $i < 0$  and  $A^0 \cong \mathbb{k}$ . A finite dimensional connected graded Hopf algebra over a field  $\mathbb{k}$  is a Poincaré duality algebra of some dimension  $d$ , and a basis element of the 1-dimensional  $\mathbb{k}$ -vector space  $\text{Hom}_{\mathbb{k}}(A^d, \mathbb{k})$  provides a ‘Frobenius form’ with similar properties to the ungraded case.

The proof of the following involves a modification of that for Proposition 3.3 with suitable allowances for gradings.

**Proposition 3.4.** *Let  $H$  be a finite dimensional commutative or cocommutative unimodular connected graded Hopf algebra over a field  $\mathbb{k}$  and let  $K$  be a subHopf algebra. Then*

$$H//K = H \otimes_K \mathbb{k} \cong H/HK^+$$

and there is an isomorphism of left  $H$ -modules

$$\text{Hom}_{\mathbb{k}}(H//K, \mathbb{k}) \xrightarrow{\cong} H//K[-d],$$

hence  $H//K$  is a stably self-dual left  $H$ -module.

**Remark 3.5.** When  $\text{char } \mathbb{k} = 2$ , the antipode  $\chi$  acts on the (necessarily 1-dimensional) top degree part of  $H$  as the identity, hence the dual Hopf algebra  $DH$  is unimodular. Hence we can apply Proposition 3.4 to pairs of finite dimensional Hopf subalgebras of the Steenrod algebra  $\mathcal{A}$ .

In the next result we give some useful consequences of Propositions 3.3 and 3.4 (in the latter case we need to interpret modules and morphisms as being suitably graded). First we need to set up some notation.

- We will set  $\otimes = \otimes_{\mathbb{k}}$  and  $\text{Hom} = \text{Hom}_{\mathbb{k}}$ .

- For a right  $K$ -module  $N$  (i.e., a left  $K^\circ$ -module), make  $\text{Hom}_K(H, N)$  a left  $H$ -module with action given by

$$(h \cdot \varphi)(x) = \varphi(\chi(h)x)$$

for  $h, x \in H$  and  $\varphi \in \text{Hom}_{K^\circ}(H, N)$  where  $H$  is regarded as a left  $K^\circ$ -module through right multiplication of  $K$ . Also make  $H//K \otimes N$  and  $\text{Hom}_{\mathbb{k}}(H//K, N)$  left  $K^\circ$ -modules by letting  $K^\circ$  act through its action on  $N$ ; this makes them both  $H \otimes K^\circ$ -modules.

- When  $L$  and  $M$  are left  $H$ -modules make  $\text{Hom}(L, M)$  a left  $H$ -module with action given by

$$(h \cdot \theta)(x) = \sum_i h_i'' \theta(\chi(h_i' x))$$

for  $h \in H$ ,  $x \in L$  and  $\theta \in \text{Hom}(L, M)$  and the coproduct on  $h$  being

$$\psi(h) = \sum_i h_i' \otimes h_i''.$$

It is well known that  $\theta$  satisfies  $h \cdot \theta = \varepsilon(h)\theta$  for all  $h \in H$  if and only if  $\theta \in \text{Hom}_H(L, M) \subseteq \text{Hom}(L, M)$ .

- Viewing  $H$  and  $M$  as left  $K$ -modules, make  $\text{Hom}_K(H, M)$  a left  $H$ -module by setting

$$(h \cdot \rho)(x) = \rho(xh).$$

**Proposition 3.6.** *Let  $H$  and  $K$  be as in Proposition 3.3 or 3.4. Let  $L$  and  $M$  be left  $H$ -modules and let  $N$  be a right  $K$ -module.*

- (a) *There are natural isomorphisms of left  $H \otimes K^\circ$ -modules*

$$H//K \otimes N \xrightarrow{\cong} \text{Hom}(H//K, \mathbb{k}) \otimes N \xrightarrow{\cong} \text{Hom}(H//K, N).$$

- (b) *There are natural isomorphisms of left  $H$ -modules*

$$H//K \odot M \xrightarrow{\cong} \text{Hom}(H//K, M) \xrightarrow{\cong} \text{Hom}_K(H, M).$$

- (c) *There is a natural isomorphism of  $\mathbb{k}$ -vector spaces*

$$\text{Hom}_H(L, H//K \odot M) \xrightarrow{\cong} \text{Hom}_K(L, M).$$

*Proof.* (a) The first isomorphism of vector spaces uses Proposition 3.3, the second is standard; these clearly respect the  $H \otimes K^\circ$ -module structures.

- (b) As for (a), there are isomorphisms of  $\mathbb{k}$ -vector spaces

$$H//K \otimes M \xrightarrow{\cong} \text{Hom}(H//K, \mathbb{k}) \otimes M \xrightarrow{\cong} \text{Hom}(H//K, M)$$

which are both  $H$ -linear.

Define a map

$$\text{Hom}(H//K, M) \rightarrow \text{Hom}_K(H, M); \quad \varphi \mapsto \tilde{\varphi}$$

where

$$\tilde{\varphi}(x) = \sum_i x_i' \varphi(\chi(x_i'')) + HK^+.$$

This has inverse

$$\text{Hom}_K(H, M) \rightarrow \text{Hom}(H//K, M); \quad \theta \mapsto \tilde{\theta}$$

where

$$\tilde{\theta}(x + HK^+) = \sum_i \chi(x_i'') \theta(x_i').$$

These are both  $H$ -linear.

- (c) Using (b) and a standard adjunction result we obtain

$$\begin{aligned} \text{Hom}_H(L, H//K \odot M) &\xrightarrow{\cong} \text{Hom}_H(L, \text{Hom}_K(H, M)) \\ &\xrightarrow{\cong} \text{Hom}_K(H \otimes_H L, M) \\ &\xrightarrow{\cong} \text{Hom}_K(L, M). \end{aligned}$$

□

These identifications can be used to deduce homological results. Here is one that is useful in our work.

**Proposition 3.7.** *Let  $H, K, L$  and  $M$  be as in Proposition 3.6. Then*

$$\mathrm{Ext}_H^*(L, H//K \odot M) \cong \mathrm{Ext}_K^*(L, M).$$

*Proof.* Take a projective resolution  $P_* \rightarrow L$  of the  $H$ -module  $L$ . Then for each  $s \geq 0$ ,

$$P_s \cong H \otimes_H P_s$$

is both a projective  $H$ -module and a projective  $K$ -module since  $H$  is a free left  $K$ -module. Therefore  $P_* \rightarrow L$  is also a projective resolution for  $L$  as a  $K$ -module. Also

$$\begin{aligned} \mathrm{Hom}_H(P_*, \mathrm{Hom}_K(H, M)) &\cong \mathrm{Hom}_K(H \otimes_H P_*, M) \\ &\cong \mathrm{Hom}_K(P_*, M), \end{aligned}$$

so on taking cohomology we obtain

$$\mathrm{Ext}_H^*(L, \mathrm{Hom}_K(H, M)) \cong \mathrm{Ext}_K^*(L, M).$$

The result now follows using Proposition 3.6(b).  $\square$

In particular, the case where  $L = H//K$  and  $M = \mathbb{k}$  is useful for some of the topological examples we will see later.

In the graded case, suppose that the top degrees of  $H$  and  $K$  are  $d$  and  $e$  respectively. Then the top degree of  $H//K$  is  $d - e$  and

$$D(H//K) \cong H//K[e - d].$$

**Poincaré duality for manifolds.** For a commutative ring spectrum  $E$ , classical Poincaré duality in  $E$ -theory is defined using the slant product for a space  $X_+$  with disjoint base point

$$E^*(X_+) \otimes_{E_*} E_*(X_+) \rightarrow E_*(X_+)$$

and the augmentation induced by collapse on the base point  $E_*(X_+) \rightarrow E_*(S^0) = E_*$ . Underlying this are compositions of the form

$$\Sigma^n E \wedge X_+ \xrightarrow{\mathrm{Id} \wedge \Delta} E \wedge X_+ \wedge X_+ \longrightarrow \Sigma^n E \wedge \Sigma^r E \wedge X_+ \xrightarrow{\mathrm{mult} \wedge \mathrm{Id}} \Sigma^{n+r} E \wedge X_+ \longrightarrow \Sigma^{n+r} E$$

and when  $E$  is a commutative  $S$ -algebra, the composition is a morphism of left  $E$ -modules. Of course this can also be formulated in terms of Spanier-Whitehead duality using Atiyah duality. This leads to the following result which gives a rich source of stably self dual  $R$ -modules.

**Proposition 3.8.** *Let  $R$  be a commutative  $S$ -algebra. Suppose that  $M$  is a compact closed smooth  $n$ -manifold whose normal bundle admits an orientation in  $R$ -cohomology. Then  $R \wedge (M_+)$  is a stably Spanier-Whitehead self  $R$ -module and*

$$D_R(R \wedge (M_+)) \sim R \wedge \Sigma^{-n}(M_+).$$

*Proof.* On choosing an  $R$ -orientation, Poincaré duality for  $R_*(M_+)$  gives an explicit isomorphism

$$R_*(M_+) \xrightarrow{\cong} R^{n-*}(M_+).$$

By the above remarks, this is induced by a weak equivalence of  $R$ -modules  $D_R(M_+) \xrightarrow{\sim} \Sigma^{-n}(M_+)$ .  $\square$

Since Spin manifolds are  $kO$ -orientable they satisfy the conditions; similarly when  $R = \mathrm{tmf}$ , String manifolds are  $\mathrm{tmf}$ -orientable. As examples of these, recall that  $\mathbb{R}P^n$  is a Spin manifold if and only if  $n \equiv 3 \pmod{4}$ , and it is a String manifold if  $n \equiv 7 \pmod{8}$ . So  $kO \wedge (\mathbb{R}P_+^{4k-1})$  is a stably self dual  $kO$ -module with

$$D_{kO}(kO \wedge (\mathbb{R}P_+^{4k-1})) \sim kO \wedge \Sigma^{-4k+1}(\mathbb{R}P_+^{4k-1})$$

and  $\mathrm{tmf} \wedge (\mathbb{R}P_+^{4k-1})$  is a stably self dual  $\mathrm{tmf}$ -module with

$$D_{\mathrm{tmf}}(\mathrm{tmf} \wedge (\mathbb{R}P_+^{8\ell-1})) \sim \mathrm{tmf} \wedge \Sigma^{-8\ell+1}(\mathbb{R}P_+^{8\ell-1}).$$

#### 4. RECOLLECTIONS ON THE STEENROD ALGEBRA AND FINITE SUBHOPF ALGEBRAS

The book of Margolis [Mar83] provides a thorough treatment of the Steenrod algebra  $\mathcal{A}$  and its finite subHopf algebras such as the  $\mathcal{A}(n)$  family. For more on bases of  $\mathcal{A}$  and its subalgebras see the survey article of Wood [Woo98].

In this appendix we highlight some important ideas that are useful for the present work.

**The Wall relations for  $\mathcal{A}(n)$ .** When working with  $\mathcal{A}$  or  $\mathcal{A}(n)$  we often describe elements by dualising the monomial basis in the dual. Recall that

$$\begin{aligned}\mathcal{A}_* &= \mathbb{F}_2[\xi_i : i \geq 1], \\ \mathcal{A}(n)_* &= \mathbb{F}_2[\xi_i : i \geq 1] / (\xi_1^{2^{n+1}}, \xi_2^{2^n}, \xi_3^{2^{n-1}}, \dots, \xi_{n+1}^2, \xi_{n+2}, \dots).\end{aligned}$$

Then the basis of (residue classes of) monomials  $\xi_1^{r_1} \xi_2^{r_2} \dots \xi_\ell^{r_\ell}$  defines a basis consisting of the elements  $\text{Sq}(r_1, \dots, r_\ell)$  in  $\mathcal{A}$  or  $\mathcal{A}(n)$ . We remark that in the computer algebra system **Sage**, only this basis is available for working in  $\mathcal{A}(n)$  making it the natural basis for expressing results found with its aid. Of course these basis elements can also be expressed in terms of monomials in  $\text{Sq}^r$  or indeed the indecomposables  $\text{Sq}^{2^s}$ .

The usual Adem relations in  $\mathcal{A}$  do not always restrict to a finite subalgebra  $\mathcal{A}(n)$ . For example, the following consequences of Adem relations

$$\text{Sq}^2 \text{Sq}^3 = \text{Sq}^4 \text{Sq}^1 + \text{Sq}^5 = \text{Sq}^4 \text{Sq}^1 + \text{Sq}^1 \text{Sq}^4$$

are not meaningful in  $\mathcal{A}(1)$  since  $\text{Sq}^4 \notin \mathcal{A}(1)$ .

A minimal set of relations between indecomposable generators  $\text{Sq}^{2^s}$  of  $\mathcal{A}$  was determined by Wall [Wal60] and these do restrict to defining relations for each  $\mathcal{A}(n)$ . Incidentally, these relations can be interpreted in either the sense of algebra relations or module relations for the augmentation ideal considered as a left or right module.

Consider the following elements of  $\mathcal{A}^*$ : for  $0 \leq s \leq r-2$  and  $1 \leq t$ ,

$$\begin{aligned}\text{(A)} \quad \Theta(r, s) &= \text{Sq}^{2^r} \text{Sq}^{2^s} + \text{Sq}^{2^s} \text{Sq}^{2^r}, \\ \text{(B)} \quad \Phi(t) &= \text{Sq}^{2^t} \text{Sq}^{2^t} + \text{Sq}^{2^{t-1}} \text{Sq}^{2^t} \text{Sq}^{2^{t-1}} + \text{Sq}^{2^{t-1}} \text{Sq}^{2^{t-1}} \text{Sq}^{2^t}.\end{aligned}$$

Then  $\Theta(r, s) \in \mathcal{A}(r-1)$  and  $\Phi(t) \in \mathcal{A}(r-1)^*$ , so these can be expressed as polynomial expressions in the  $\text{Sq}^{2^k}$  for  $0 \leq k \leq r-1$ . The elements of form

$$\text{Sq}^{2^r} \text{Sq}^{2^s} + \text{Sq}^{2^s} \text{Sq}^{2^r} + \Theta(r, s), \quad \text{Sq}^{2^t} \text{Sq}^{2^t} + \text{Sq}^{2^{t-1}} \text{Sq}^{2^t} \text{Sq}^{2^{t-1}} + \text{Sq}^{2^{t-1}} \text{Sq}^{2^{t-1}} \text{Sq}^{2^t} + \Phi(t)$$

give a minimal set of relations for  $\mathcal{A}$ . In particular, such elements with  $r, t \leq n$  form a minimal set of relations for  $\mathcal{A}(n)$ . In the first few cases the Wall relations are

$$\begin{aligned}\mathcal{A}(0) : \quad & \text{Sq}^1 \text{Sq}^1 = 0, \\ \mathcal{A}(1) : \quad & \text{Sq}^1 \text{Sq}^1 = \text{Sq}^2 \text{Sq}^2 + \text{Sq}^1 \text{Sq}^2 \text{Sq}^1 = 0 \\ \mathcal{A}(2) : \quad & \text{Sq}^1 \text{Sq}^1 = \text{Sq}^2 \text{Sq}^2 + \text{Sq}^1 \text{Sq}^2 \text{Sq}^1 \\ & = \text{Sq}^4 \text{Sq}^4 + \text{Sq}^2 \text{Sq}^4 \text{Sq}^2 + \text{Sq}^2 \text{Sq}^2 \text{Sq}^4 \\ & = \text{Sq}^1 \text{Sq}^4 + \text{Sq}^4 \text{Sq}^1 + \text{Sq}^2 \text{Sq}^1 \text{Sq}^2 = 0.\end{aligned}$$

We will sometimes use the Milnor primitives  $P_1^s$  ( $s \geq 0$ ) defined recursively by

$$P_1^0 = \text{Sq}(1), \quad P_1^s = \text{Sq}^{2^s} P_1^{s-1} + P_1^{s-1} \text{Sq}^{2^s} \quad (s \geq 1).$$

More generally, Margolis [Mar83] uses the notation  $\text{Sq}(r_1, \dots, r_\ell)$  for the element dual to the monomial  $\xi_1^{r_1} \dots \xi_\ell^{r_\ell}$  and sets  $P_t^s = \text{Sq}(0, \dots, 0, 2^s)$  where  $2^s$  occurs in the  $t$ -th place.

**Doubling.** Doubling is discussed by Margolis [Mar83, section 15.3] and also by the present author [Bak18, section 2]. The main thing to observe is that for any  $n \geq 1$  there is a grade halving homomorphism of Hopf algebras

$$\mathcal{A}(n) \xrightarrow{\cong} \mathcal{A}(n-1)$$

which induces a grade halving isomorphism

$$\mathcal{A}(n)//\mathcal{E}(n) \xrightarrow{\cong} \mathcal{A}(n-1)$$

under which the residue class of  $Sq^k$  maps to  $Sq^{k/2}$  if  $k$  is even and 0 otherwise.

If  $M$  is a left  $\mathcal{A}(n-1)$ -module then it becomes  $\mathcal{A}(n)//\mathcal{E}(n)$ -module  ${}^{(1)}M$  with all gradings doubled, hence it is also a left  $\mathcal{A}(n)$ -module. For example, when  $n = 1$  the  $\mathcal{A}(1)$ -module  ${}^{(1)}\mathcal{A}(0) \cong \mathcal{A}(1)//\mathcal{E}(1)$  is realised by the cohomology of the  $kO$ -module  $kU$  shown in (5.12), and when  $n = 2$ , the  $\mathcal{A}(2)$ -module  ${}^{(1)}\mathcal{A}(1) \cong \mathcal{A}(2)//\mathcal{E}(2)$  (the double of  $\mathcal{A}(1)$ ) is realised by the cohomology of the tmf-module  $\mathrm{tmf}_1(3) \sim BP(2)$ .

## 5. $kO$ -MODULES AND $\mathcal{A}(1)$ -MODULES

In this section we review some well known results on the (co)homology of  $kO$ -modules localised at the prime 2. Such results were originally due to Adams & Priddy, Mahowald and Milgram, see [AP76, Mah81, MM76]; the book of Margolis [Mar83] also provides a thorough treatment of stable module categories for finite subHopf algebras of  $\mathcal{A}$ .

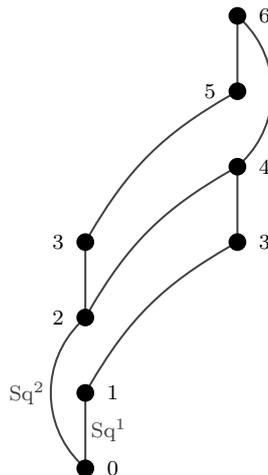
We begin by summarising some relationships between relative Steenrod algebras and their duals that involve connective  $K$ -theory; these are obtained using the ideas discussed in Section 1. These will be used heavily in what follows.

**Theorem 5.1.** *For  $H = H\mathbb{F}_2$  we have the following identifications of Hopf algebras:*

$$\begin{aligned} H_{kO}^* H &\cong \mathcal{A}(1)^* = \mathcal{A}(1), & H_*^{kO} H &\cong \mathcal{A}(1)_*, \\ H_{kU}^* H &\cong \mathcal{E}(1)^* = \mathcal{E}(1), & H_*^{kU} H &\cong \mathcal{E}(1)_*. \end{aligned}$$

Of course this means that the homology and cohomology of a  $kO$ -module are naturally  $\mathcal{A}(1)$ -modules and those of a  $kU$ -module are  $\mathcal{E}(1)$ -modules.

The  $kO$ -module  $H = H\mathbb{F}_2$  can be given a minimal cell structure with 8 cells corresponding to the obvious basis of  $\mathcal{A}(1)$ -module  $\mathcal{A}(1)$ .



Since  $H$  is a  $kO$  ring spectrum, by Proposition 3.2 it is stably self-dual and  $D_{kO}H \sim \Sigma^{-6}H$ .

When working in the stable module category  $\mathbf{StMod}_{\mathcal{A}(1)}$  we will often blur the distinction between a module and its stable equivalence class. We denote the kernel of the counit  $\varepsilon: \mathcal{A}(1) \rightarrow \mathbb{F}_2$  by  $\mathcal{I}(1)$ ; this module is stably invertible with stable inverse  $\mathcal{I}(1)^{-1} = D\mathcal{I}(1)$ , represented by the linear dual of  $\mathcal{I}(1)$ .

(5.1)

Recall that there are two mutually inverse endofunctors  $\Omega, \Omega^{-1}$  of  $\mathbf{StMod}_{\mathcal{A}(1)}$ , where

$$\Omega M = \mathcal{I}(1) \odot M, \quad \Omega^{-1} M = \mathcal{I}(1)^{-1} \odot M.$$

These commute with products and preserve duals, i.e.,

$$\begin{aligned} (\Omega M) \odot N &= \Omega(M \odot N), & (\Omega^{-1} M) \odot N &= \Omega^{-1}(M \odot N), \\ \Omega DM &= D\Omega M, & \Omega^{-1} DM &= D\Omega^{-1} M. \end{aligned}$$

Amongst the many cyclic quotient  $\mathcal{A}(1)$ -modules is the Joker module

(5.2)

$\mathcal{J} = \mathcal{A}(1)/\mathcal{A}(1)\{\mathrm{Sq}^3\}[-2]$

whose grading is arranged so that it is a self-dual  $\mathcal{A}(1)$ -module, indeed it represents the unique element of order 2 in the Picard group of  $\mathbf{StMod}_{\mathcal{A}(1)}$ ,  $\mathrm{Pic}_{\mathcal{A}(1)}$ .

In  $\mathbf{StMod}_{\mathcal{A}(1)}$  we have the following presentations as quotients of free modules:

$$\begin{aligned} \Omega \mathcal{J} &= \mathcal{A}(1)/\mathcal{A}(1)\{\mathrm{Sq}^1, \mathrm{Sq}^2 \mathrm{Sq}^1 \mathrm{Sq}^2\}[1] = \mathcal{A}(1)/\mathcal{A}(1)\{\mathrm{Sq}^1, \mathrm{Sq}^2 \mathrm{Sq}^3\}[1], \\ \Omega^{-1} \mathcal{J} &= \mathcal{A}(1)/\mathcal{A}(1)\{\mathrm{Sq}^2\}[-4], \\ \Omega^2 \mathcal{J} &= (\mathcal{A}(1)/\mathcal{A}(1)\{\mathrm{Sq}^1\}[2] \oplus \mathcal{A}(1)/\mathcal{A}(1)\{\mathrm{Sq}^1, \mathrm{Sq}^2 \mathrm{Sq}^1 \mathrm{Sq}^2\}[6]) / \mathcal{A}(1)\{(\mathrm{Sq}^2 \mathrm{Sq}^1 \mathrm{Sq}^2, \mathrm{Sq}^1)\}, \\ \Omega^{-2} \mathcal{J} &= \mathcal{A}(1)/\mathcal{A}(1)\{\mathrm{Sq}^2 \mathrm{Sq}^1\}[-7]. \end{aligned}$$

These modules have the following diagrams (the first two are often called the *question mark* and *upside down question mark* modules).

$$(5.3) \quad \begin{array}{cccc} \Omega \mathcal{J} & \Omega^{-1} \mathcal{J} & \Omega^2 \mathcal{J} & \Omega^{-2} \mathcal{J} \\ \begin{array}{c} \bullet 4 \\ | \\ \bullet 3 \\ | \\ \bullet 1 \end{array} & \begin{array}{c} \bullet -1 \\ | \\ \bullet -3 \\ | \\ \bullet -4 \end{array} & \begin{array}{c} \bullet 7 \\ | \\ \bullet 6 \\ | \\ \bullet 5 \\ | \\ \bullet 4 \\ | \\ \bullet 2 \end{array} & \begin{array}{c} \bullet -2 \\ | \\ \bullet -4 \\ | \\ \bullet -5 \\ | \\ \bullet -6 \\ | \\ \bullet -7 \end{array} \end{array}$$

We recall the structure of  $\text{Ext}_{\mathcal{A}(1)}(\mathbb{F}_2, \mathbb{F}_2)$ , the Adams  $E_2$ -term for  $\pi_* k\mathcal{O}$ , part of which appears in Figure 1. We also have

$$(5.4) \quad \text{Ext}_{\mathcal{A}(1)}^{s,*}(\mathcal{I}(1), \mathbb{F}_2) \cong \text{Ext}_{\mathcal{A}(1)}^{s+1,*}(\mathbb{F}_2, \mathbb{F}_2),$$

whose chart is a shifted version of Figure 1 with the  $\bullet$  at  $(0, 0)$  removed. For the dual  $D\mathcal{I}(1)$ ,

$$(5.5) \quad \text{Ext}_{\mathcal{A}(1)}^{s,*}(D\mathcal{I}(1), \mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2 & \text{if } s = 0, \\ \text{Ext}_{\mathcal{A}(1)}^{s-1,*}(\mathbb{F}_2, \mathbb{F}_2) & \text{if } s > 0. \end{cases}$$

Adams & Priddy [AP76, tables 3.10 & 3.11] give the Ext groups of the two question mark modules  $\Omega \mathcal{J}$  and  $\Omega^{-1} \mathcal{J}$ .

Such  $\mathcal{A}(1)$ -modules can often be realised as cohomology of  $k\mathcal{O}$ -modules, i.e., as  $H_{k\mathcal{O}}^* X$  for some  $k\mathcal{O}$ -module  $X$ . Indeed, in some cases  $X = k\mathcal{O} \wedge Z$  for an  $S$ -module  $Z$ . For example, there are  $S$ -modules  $S^0 \cup_\eta e^2 \cup_2 e^3$  and  $S^0 \cup_2 e^1 \cup_\eta e^3$  so that as  $\mathcal{A}(1)$ -modules,

$$\begin{aligned} H_{k\mathcal{O}}^*(k\mathcal{O} \wedge S^0 \cup_\eta e^2 \cup_2 e^3) &\cong H^*(S^0 \cup_\eta e^2 \cup_2 e^3) \cong \Omega \mathcal{J}[-1], \\ H_{k\mathcal{O}}^*(k\mathcal{O} \wedge S^0 \cup_2 e^1 \cup_\eta e^3) &\cong H^*(S^0 \cup_2 e^1 \cup_\eta e^3) \cong \Omega^{-1} \mathcal{J}[4]. \end{aligned}$$

In [Bak18], the Joker was shown to be realisable as  $H^* J$  for an  $S$ -module  $J$ , so as  $\mathcal{A}(1)$ -modules,  $H_{k\mathcal{O}}^*(k\mathcal{O} \wedge J) \cong H^* J$ . The realisability of the self-dual cyclic  $\mathcal{A}(1)$ -module  $\mathcal{A}(1)//\mathcal{A}(0)$  provides a more interesting question.

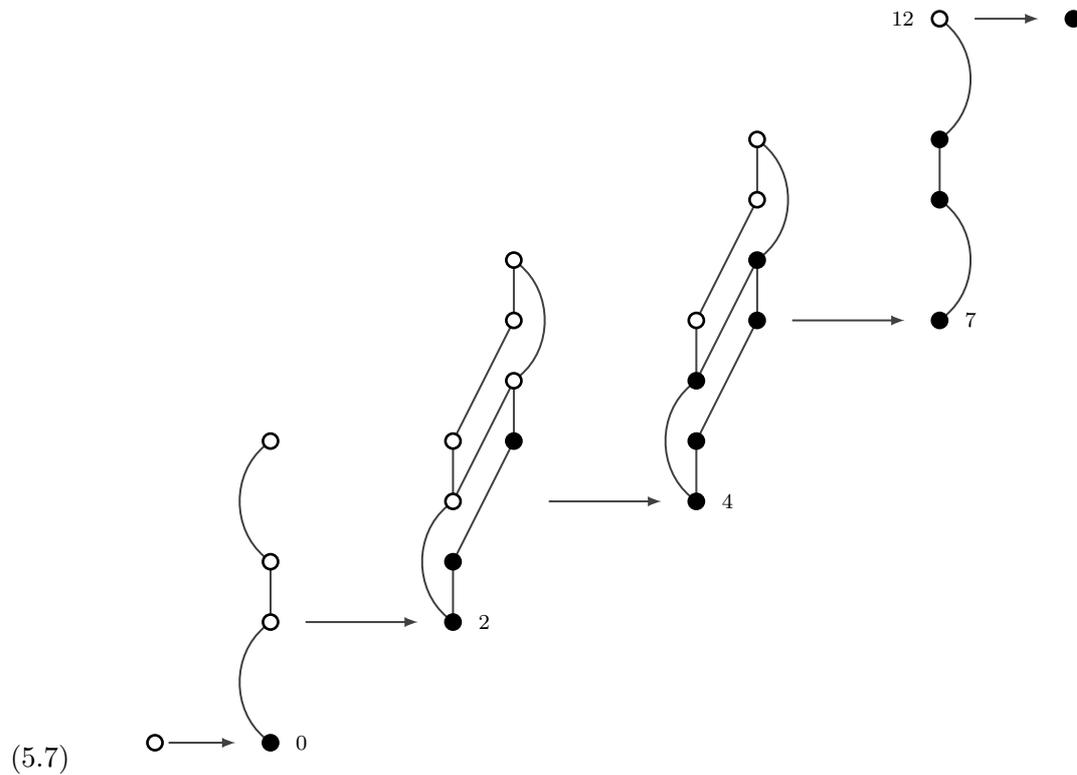
$$(5.6) \quad \mathcal{A}(1)//\mathcal{A}(0) = \mathcal{A}(1)/\mathcal{A}(1)\{\text{Sq}^1\} \quad \begin{array}{c} \bullet 5 \\ | \\ \bullet \\ | \\ \bullet \\ | \\ \bullet \\ | \\ \bullet 0 \end{array}$$

$\text{Sq}^1$   
 $\text{Sq}^2$

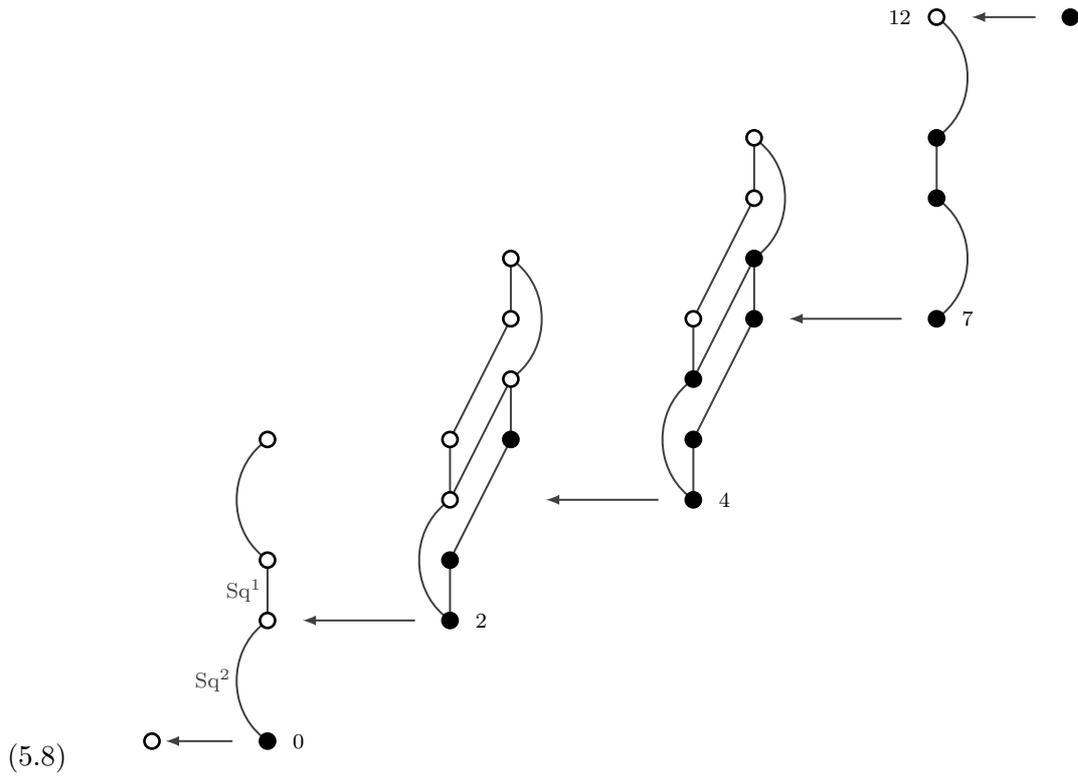
In  $\mathcal{M}_S$ , no CW complex of the form  $S^0 \cup_\eta e^2 \cup_2 e^3 \cup_\eta e^5$  can exist since the Toda bracket  $\langle \eta, 2, \eta \rangle = \{2\nu, -2\nu\} \subseteq \pi_3 S^0$  does not contain 0; alternatively, its existence is contradicted by the relation  $\text{Sq}^1 \text{Sq}^2 \text{Sq}^1 = \text{Sq}^4 \text{Sq}^1 + \text{Sq}^1 \text{Sq}^4$  in the Steenrod algebra  $\mathcal{A}$ . However, taken in  $\pi_* k\mathcal{O}$ , the Toda bracket  $\langle \eta, 2, \eta \rangle \subseteq \pi_3 k\mathcal{O}$  contains the images of  $\pm 2\nu$  which are both zero, hence there is a CW  $k\mathcal{O}$ -module with this cohomology. Of course  $H\mathbb{Z}$  is a  $k\mathcal{O}$ -module whose cohomology agrees with this  $\mathcal{A}(1)$ -module, so we are describing a minimal CW structure for it.

Since  $H\mathbb{Z}$  is a  $k\mathbb{O}$  ring spectrum and  $H_*^{k\mathbb{O}}H\mathbb{Z}$  satisfies Poincaré duality, Proposition 3.2 applies. Hence the  $k\mathbb{O}$ -module  $H\mathbb{Z}$  is stably self-dual with  $D_{k\mathbb{O}}H\mathbb{Z} \sim \Sigma^{-5}H\mathbb{Z}$ .

**The Whitehead tower for  $k\mathbb{O}$ .** The  $k\mathbb{O}$ -modules discussed above fit into the Whitehead tower of  $k\mathbb{O}$  shown in (5.7) as CW  $k\mathbb{O}$ -modules, where the numbers indicate degrees of cells in a minimal CW structure.



On applying  $H_{k\mathbb{O}}^*(-)$  we obtain the  $\mathcal{A}(1)$ -module extension of (5.8) and (5.9) which represents an element of  $\text{Ext}_{\mathcal{A}(1)}^{4,12}(\mathbb{F}_2, \mathbb{F}_2)$ , and in turn this represents the Bott periodicity element in  $\pi_8 k\mathbb{O}$  in the Adams spectral sequence. Of course this periodicity is also visible in Figure 1 which shows part of  $\text{Ext}_{\mathcal{A}(1)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$ . The periodicity is given by Yoneda product with the basis element of  $\text{Ext}_{\mathcal{A}(1)}^{8,4}(\mathbb{F}_2, \mathbb{F}_2)$ .



(5.9)  $0 \leftarrow \mathbb{F}_2 \leftarrow \mathcal{A}(1)/\mathcal{A}(1)\{Sq^1\} \leftarrow \mathcal{A}(1)[2] \leftarrow \mathcal{A}(1)[4] \leftarrow \mathcal{A}(1)/\mathcal{A}(1)\{Sq^1\}[7] \leftarrow \mathbb{F}_2[12] \leftarrow 0$

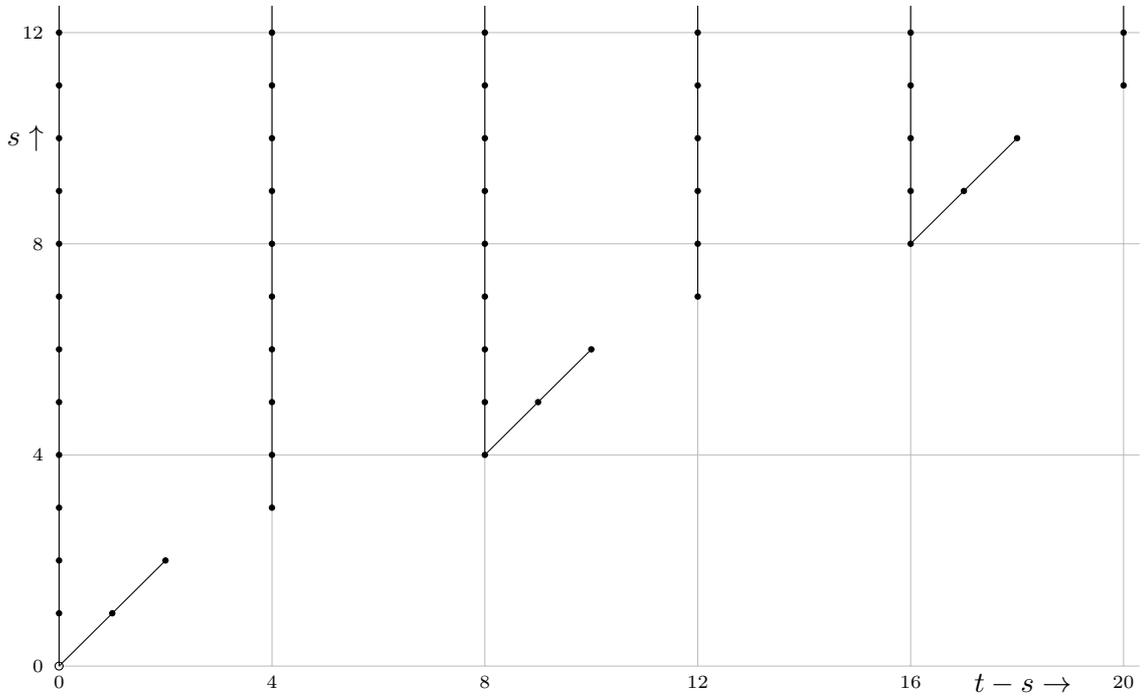


FIGURE 1.  $\text{Ext}_{\mathcal{A}(1)}^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$ : for  $0 \leq t - s \leq 20$ ,  $0 \leq s \leq 12$ . Removing the generator at  $(0,0)$  and shifting down and left gives  $\text{Ext}_{\mathcal{A}(1)}(\mathcal{I}(1), \mathbb{F}_2)$ .

The cofibres of the maps in (5.7) have the following cell structures

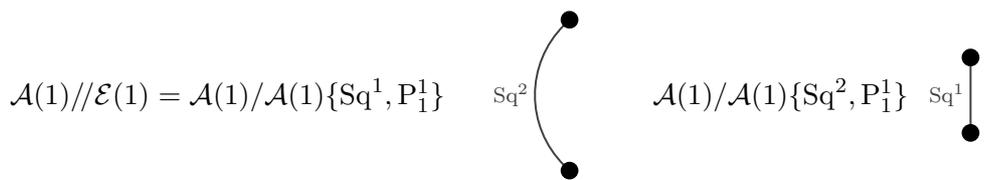
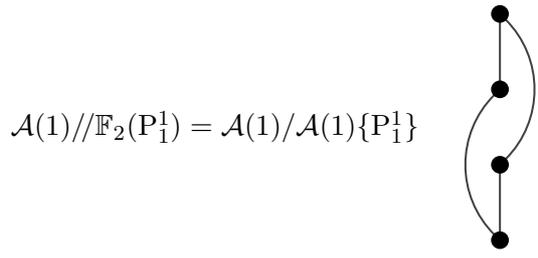
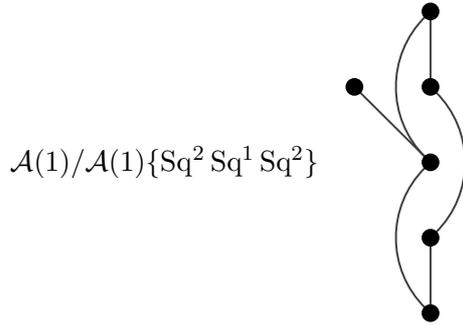
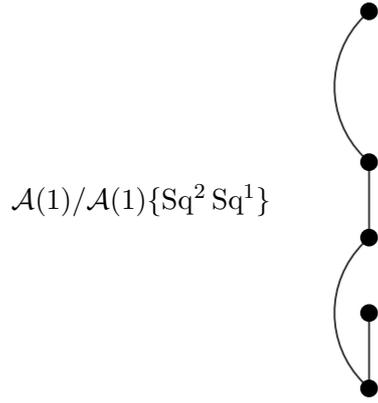
(5.10)

and their cohomologies are the following cyclic  $\mathcal{A}(1)$ -modules.

(5.11)

In fact the whole sequence (5.7) can be made to be Spanier-Whitehead self-dual in the sense that applying  $\Sigma^{12}D_{kO}$  gives an equivalent sequence.

**Realisation of cyclic  $\mathcal{A}(1)$ -modules.** There are of course other cyclic quotients of  $\mathcal{A}(1)$  which can be realised as cohomology of  $kO$ -modules. These fall into two groups, those containing a submodule isomorphic to the quotient  $\mathcal{A}(1)//\mathcal{A}(0)$  and those which do not. See Example 5.3 for the first one, the others involving  $\mathcal{A}(1)//\mathcal{A}(0)$  are easily realised. The others can all be constructed as  $kO$ -modules of the form  $kO \wedge Z$ .



Each of the last three has the form  $\mathcal{A}(1) \otimes_{\mathcal{B}} \mathbb{F}_2$  for some subalgebra  $\mathcal{B} \subseteq \mathcal{A}(1)$ . For each of the first two,  $\mathcal{B}$  is a subHopf algebra which explains the self-duality. In the last case,

$$\mathcal{B} = \mathbb{F}_2(\text{Sq}^2, P_1^1) = \mathbb{F}_2(\text{Sq}^2, \text{Sq}^1 \text{Sq}^2 + \text{Sq}^2 \text{Sq}^1)$$

is a 5-dimensional commutative subalgebra, but even so  $\mathcal{A}(1) \otimes_{\mathcal{B}} \mathbb{F}_2$  is a stably self-dual  $\mathcal{A}(1)$ -module. Here  $\mathcal{A}(1)$  is not a free right  $\mathcal{B}$ -module since for example

$$\text{Sq}^1 P_1^1 = 1(\text{Sq}^2 \text{Sq}^2).$$

The self-dual cyclic module  $\mathcal{A}(1)//\mathbb{F}_2(P_1^1) = \mathcal{A}(1)/\mathcal{A}(1)\{P_1^1\}$  is a quotient Hopf algebra since  $P_1^1$  is central in  $\mathcal{A}(1)$ . This module can be realised, see Example 5.6 for details.

**Constructions of some more  $k\mathbf{O}$ -modules.** Collapsing  $H\mathbb{Z}$  onto its top cell gives a map  $H\mathbb{Z} \rightarrow S_{k\mathbf{O}}^5$  and composing with a map of degree 2 to the bottom cell of  $\Sigma^5 H\mathbb{Z}$  gives a  $k\mathbf{O}$ -module map  $H\mathbb{Z} \rightarrow \Sigma^5 H\mathbb{Z}$  whose mapping cone has cohomology as shown.



We can repeat this using a suitable map from  $\Sigma^{-4}HZ$  to this by using a degree 2 map to the bottom cell and so on. Repeatedly using such constructions and their Spanier-Whitehead duals leads to a ‘daisy chain’ of copies of suspensions of  $\mathcal{A}(1)//\mathcal{A}(0)$  glued together by actions of  $Sq^1$ . Such an  $\mathcal{A}(1)$ -module is realised as the cohomology of a  $kO$ -module. This process can either be stopped after finitely many iterations or continued indefinitely in either positive or negative directions. Of course these  $\mathcal{A}(1)$ -modules are well known.

In fact  $H\mathbb{Z}$  can itself be realised in a similar way. The  $kO$ -module  $kU$  has cohomology

$$(5.12) \quad H_{kO}^* kU = \mathcal{A}(1)//\mathcal{E}(1) = \mathcal{A}(1)/\mathcal{A}(1)\{Sq^1, Sq^1 Sq^2\} \quad Sq^2 \begin{array}{c} \bullet \\ \curvearrowright \\ \bullet \end{array}$$

so  $D_{kO} kU \sim \Sigma^{-2}kU$ . A similar construction to the above yields a complex whose cohomology has the following form.



By iterating we can obtain familiar  $\mathcal{A}(1)$ -modules such as



which can also be extended both upwards and downwards.

### Some sample calculations and examples.

**Example 5.2.** Consider the Adams spectral sequence

$$E_2^{s,t} = \text{Ext}_{\mathcal{A}(1)}^{s,t}(H_{kO}^* H\mathbb{Z}, H_{kO}^* H) \implies \mathcal{D}_{kO}(H, H\mathbb{Z})^{s-t}.$$

Since

$$H_{kO}^* H \cong \mathcal{A}(1) \cong D(\mathcal{A}(1))[6]$$

we have

$$\begin{aligned} E_2^{*,*} &\cong \text{Ext}_{\mathcal{A}(1)}^{*,*}(H_{kO}^* H\mathbb{Z}, D(\mathcal{A}(1))[6]) \\ &\cong \text{Ext}_{\mathcal{A}(1)}^{*,*}(\mathcal{A}(1) \otimes H_{kO}^* H\mathbb{Z}, \mathbb{F}_2[6]) \\ &\cong \text{Hom}^*(H_{kO}^* H\mathbb{Z}, \mathbb{F}_2[6]) \\ &\cong \text{Hom}^6(H_{kO}^* H\mathbb{Z}, \mathbb{F}_2[6]) \cong \text{Hom}(H_{kO}^* H\mathbb{Z}, \mathbb{F}_2) \cong \mathbb{F}_2 \end{aligned}$$

where the generator is the dual map to the non-zero element in  $H_{kO}^0 H\mathbb{Z}$ . So  $D_{kO}(H, H\mathbb{Z})^* \cong \mathbb{F}_2[0]$  and the generator corresponds to the map  $H \rightarrow H\mathbb{Z}$  which collapses  $H$  onto its top cell composed with inclusion of the bottom cell of  $\Sigma^6 H\mathbb{Z}$ .

By Spanier-Whitehead duality,

$$\mathcal{D}_{kO}(H\mathbb{Z}, H)^* \cong \mathcal{D}_{kO}(\Sigma^{-6}H, \Sigma^{-5}H\mathbb{Z})^* \cong \mathcal{D}_{kO}(H, \Sigma H\mathbb{Z})^* \cong \mathbb{F}_2[5].$$

This time the generator involves collapse of  $H\mathbb{Z}$  onto its top cell composed with inclusion of the bottom cell of  $\Sigma^5 H$ , i.e., a composition

$$H\mathbb{Z} \rightarrow S_{kO}^5 \rightarrow \Sigma^5 H.$$

The reader may like to compare and contrast this calculation with that using the classical Adams spectral sequence

$$E_2^{s,t} = \text{Ext}_{\mathcal{A}}^{s,t}(H^* H\mathbb{Z}, H^* H) \implies \mathcal{D}_S(H, H\mathbb{Z}).$$

**Example 5.3.** Consider  $\mathcal{D}_{kO}(H, kO)^*$ . By Spanier-Whitehead duality this is isomorphic to

$$\mathcal{D}_{kO}(kO, \Sigma^{-6}H)^* \cong \mathcal{D}_{kO}(kO, H)^{* -6}.$$

The Adams spectral sequence

$$E_2^{s,t} = \text{Ext}_{\mathcal{A}(1)}^{s,t}(H_{kO}^* H, \mathbb{F}_2) \implies \mathcal{D}_{kO}(H, kO)^{s-t}$$

has

$$\text{Ext}_{\mathcal{A}(1)}^{*,*}(H_{kO}^* H, \mathbb{F}_2) \cong \text{Ext}_{\mathcal{A}(1)}^{*,*}(\mathcal{A}(1), \mathbb{F}_2) = \text{Hom}(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2$$

whose generator detects the inclusion of the bottom cell of  $H$  whose Spanier-Whitehead dual is collapse onto the top cell of  $H$ , i.e., a map  $H \rightarrow S_{kO}^6 \sim \Sigma^6 kO$ . The fibre of this realises the cyclic module

$$DI(1)[5] = \mathcal{A}(1)/\mathcal{A}(1)\{\text{Sq}^1 \text{Sq}^2 \text{Sq}^1 \text{Sq}^2\},$$

a suspension of the dual of the counit, see (5.1).

**Example 5.4.** Recall the Joker module  $\mathcal{J}$  of (5.2). In [Bak18] we showed that there is a spectrum  $J$  for which the  $kO$ -module  $kO \wedge J$  has cohomology is

$$H_{kO}^*(kO \wedge J) \cong H^*(J) \cong \mathcal{J}$$

as an  $\mathcal{A}(1)$ -module. Actually there are two inequivalent such spectra which are Spanier-Whitehead dual and their cohomology realises the  $\mathcal{A}$ -modules with the two possible  $\text{Sq}^4$  actions. For our present purposes we may choose  $J'$  and  $J''$  to be either of them.

First we will determine  $\mathcal{D}_{kO}(kO \wedge J', kO \wedge J'')$ . Using duality we have

$$\begin{aligned} \mathcal{D}_{kO}(kO \wedge J', kO \wedge J'')^* &\cong \mathcal{D}_{kO}(kO, (kO \wedge DJ') \wedge_{kO} (kO \wedge J''))^* \\ &\cong \mathcal{D}_{kO}(kO, kO \wedge DJ' \wedge J'')^*. \end{aligned}$$

Now  $\mathcal{J}$  is stably self-dual and in fact as  $\mathcal{A}(1)$ -modules,

$$(5.13) \quad \mathcal{J} \otimes \mathcal{J} \cong \mathbb{F}_2[0] \oplus \mathcal{A}(1)[-4] \oplus \mathcal{A}(1)[-3] \oplus \mathcal{A}(1)[-2].$$

There is an Adams spectral sequence

$$E_2^{s,t} = \text{Ext}_{\mathcal{A}(1)}(H_{kO}^*(kO \wedge DJ' \wedge J''), \mathbb{F}_2) \implies \mathcal{D}_{kO}(kO, kO \wedge DJ' \wedge J'')^*$$

and its  $E_2$ -term is given by

$$\begin{aligned} E_2^{*,*} &\cong \text{Ext}_{\mathcal{A}(1)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) \oplus \text{Hom}^*(\mathbb{F}_2[-4], \mathbb{F}_2) \oplus \text{Hom}^*(\mathbb{F}_2[-3], \mathbb{F}_2) \oplus \text{Hom}^*(\mathbb{F}_2[-2], \mathbb{F}_2) \\ &\cong \text{Ext}_{\mathcal{A}(1)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) \oplus \mathbb{F}_2[4] \oplus \mathbb{F}_2[3] \oplus \mathbb{F}_2[2]. \end{aligned}$$

The reader is invited to describe the stable maps  $kO \wedge J' \rightarrow kO \wedge J''$  corresponding to  $\mathbb{F}_2$ -summands in cellular terms. The generator of  $\text{Ext}_{\mathcal{A}(1)}^{0,0}(\mathbb{F}_2, \mathbb{F}_2)$  is an infinite cycle in the spectral sequence showing there is indeed a weak equivalence of  $kO$ -modules  $kO \wedge J' \rightarrow kO \wedge J''$ , hence the choices of spectra  $J'$  and  $J''$  do not affect the  $kO$ -module up to homotopy equivalence and from now on we just write  $kO \wedge J$  for any such  $kO$ -module.

Notice that  $\text{Ext}_{\mathcal{A}(1)}^{1,2}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2$ , and this corresponds to the map  $k\mathbb{O} \wedge \Sigma J \rightarrow k\mathbb{O} \wedge J$  induced by multiplication by  $\eta$ . This shows that the cofibre sequence

$$k\mathbb{O} \wedge J \rightarrow k\mathbb{O} \wedge C_\eta \wedge J \rightarrow k\mathbb{O} \wedge \Sigma^2 J$$

cannot split since the short exact sequence

$$\begin{array}{ccccccc} 0 & \longleftarrow & H_{k\mathbb{O}}^*(k\mathbb{O} \wedge J) & \longleftarrow & H_{k\mathbb{O}}^*(k\mathbb{O} \wedge C_\eta \wedge J) & \longleftarrow & H_{k\mathbb{O}}^*(k\mathbb{O} \wedge \Sigma^2 J) \longleftarrow 0 \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ & & H^*(J) & & H^*(C_\eta \wedge J) & & H^*(\Sigma^2 J) \end{array}$$

represents the corresponding element of  $\text{Ext}_{\mathcal{A}(1)}^{1,2}(\mathcal{J}, \mathcal{J}) \cong \text{Ext}_{\mathcal{A}(1)}^{1,2}(\mathcal{J} \otimes \mathcal{J}, \mathbb{F}_2)$ . Of course this also implies the well-known fact that the cofibre sequence of spectra

$$J \rightarrow C_\eta \wedge J \rightarrow \Sigma^2 J$$

does not split.

**Example 5.5.** Consider the  $k\mathbb{O}$ -module  $H\mathbb{Z} \wedge_{k\mathbb{O}} H\mathbb{Z}$ . The cohomology of this is

$$H_{k\mathbb{O}}^*(H\mathbb{Z} \wedge_{k\mathbb{O}} H\mathbb{Z}) \cong H_{k\mathbb{O}}^* H\mathbb{Z} \odot H_{k\mathbb{O}}^* H\mathbb{Z}$$

where the factors are given in (5.6). Since  $H\mathbb{Z}$  is a unital  $k\mathbb{O}$ -algebra,  $H\mathbb{Z}$  is a retract.

Notice that  $H\mathbb{Z}$  and hence  $H\mathbb{Z} \wedge_{k\mathbb{O}} H\mathbb{Z}$  are Spanier-Whitehead stably self dual with

$$D_{k\mathbb{O}} H\mathbb{Z} \sim \Sigma^{-5} H\mathbb{Z}, \quad D_{k\mathbb{O}}(H\mathbb{Z} \wedge_{k\mathbb{O}} H\mathbb{Z}) \sim \Sigma^{-10} H\mathbb{Z} \wedge_{k\mathbb{O}} H\mathbb{Z}.$$

Moreover,  $H_{k\mathbb{O}}^* H\mathbb{Z} \odot H_{k\mathbb{O}}^* H\mathbb{Z}$  must have  $\mathcal{A}(1)$ -module summands  $H_{k\mathbb{O}}^* H\mathbb{Z}$  and  $H_{k\mathbb{O}}^* H\mathbb{Z}[5]$ ; in fact a routine calculation shows that

$$H_{k\mathbb{O}}^* H\mathbb{Z} \odot H_{k\mathbb{O}}^* H\mathbb{Z} \cong H_{k\mathbb{O}}^* H\mathbb{Z} \oplus \mathcal{A}(1)[2] \oplus H_{k\mathbb{O}}^* H\mathbb{Z}[5].$$

Now it is straightforward to show that

$$H\mathbb{Z} \wedge_{k\mathbb{O}} H\mathbb{Z} \sim H\mathbb{Z} \vee \Sigma^2 H \vee \Sigma^5 H\mathbb{Z}.$$

In particular it follows that  $H\mathbb{Z}_*^{k\mathbb{O}} H\mathbb{Z}$  is not a free  $\mathbb{Z}_{(2)}$ -module; it is known that  $H\mathbb{Z}_* H\mathbb{Z}$  also has simple 2-torsion, see [Koc82]. Of course the  $\mathbb{Z}_{(2)}$ -algebra structures of  $H\mathbb{Z}_*^{k\mathbb{O}} H\mathbb{Z}$  and  $H\mathbb{Z}_{k\mathbb{O}}^* H\mathbb{Z}$  are both trivial for degree reasons.

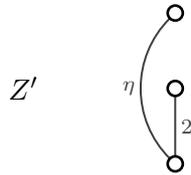
In the planned sequel we will consider the tmf-algebras  $H\mathbb{Z} \wedge_{\text{tmf}} H\mathbb{Z}$  and  $k\mathbb{O} \wedge_{\text{tmf}} k\mathbb{O}$ .

**Example 5.6.** There is a CW complex  $Z$  such that

$$H_{k\mathbb{O}}^*(k\mathbb{O} \wedge Z) \cong H^* Z \cong \mathcal{A}(1)/\mathcal{A}(1)\{\mathbb{P}_1^1\},$$

here is a construction.

Consider the complex  $Z'$  for which  $\pi_2(Z') \cong \mathbb{Z} \oplus \mathbb{Z}/2$ .



Then  $Z$  is obtained by attaching a 3-cell to  $Z'$  using the sum of the generators of  $\pi_2(Z') \cong \mathbb{Z} \oplus \mathbb{Z}/2$  (see Figure 2). There is a map  $Z \rightarrow H$  which extends to a  $k\mathbb{O}$ -module morphism  $k\mathbb{O} \wedge Z \rightarrow H$  inducing an epimorphism of  $\mathcal{A}(1)$ -modules making the following diagram commute.

$$\begin{array}{ccccc} H_{k\mathbb{O}}^* H & \longrightarrow & H_{k\mathbb{O}}^*(k\mathbb{O} \wedge Z) & \xleftarrow{\cong} & H^* Z \\ \cong \updownarrow & & \updownarrow \cong & & \\ \mathcal{A}(1) & \xrightarrow{\text{quo}} & \mathcal{A}(1)/\mathcal{A}(1)\{\mathbb{P}_1^1\} & & \end{array}$$

The kernel of the quotient homomorphism is isomorphic to  $\mathcal{A}(1)/\mathcal{A}(1)\{P_1^1\}[3]$  and there is a cofibre sequence of  $kO$ -modules

$$kO \wedge Z \rightarrow H \rightarrow kO \wedge \Sigma^3 Z.$$

We can also splice together infinitely many copies of the short exact sequence

$$0 \longleftarrow H^*Z \longleftarrow \mathcal{A}(1) \longleftarrow \mathcal{A}(1)/\mathcal{A}(1)\{P_1^1\}[3] \longleftarrow 0$$

to obtain

$$0 \longleftarrow H^*Z \longleftarrow \mathcal{A}(1) \longleftarrow \mathcal{A}(1)/\mathcal{A}(1)[3] \longleftarrow \mathcal{A}(1)/\mathcal{A}(1)[6] \longleftarrow \dots$$

which is a periodic resolution of  $H^*Z$  by cyclic free  $\mathcal{A}(1)$ -modules. This can be used to determine the Adams  $E_2$ -term for computing  $\pi_*(kO \wedge Z)$ , see Figure 3. The homotopy groups of  $kO \wedge Z$  are given by

$$\pi_*(kO \wedge Z) = \mathbb{F}_2[u_2]$$

where  $u_2 \in \pi_2(kO \wedge Z)$  is represented in the Adams spectral sequence by the element of  $\text{Ext}_{\mathcal{A}(1)}^{1,3}(H^*Z, \mathbb{F}_2)$  corresponding to the algebraic extension

$$0 \rightarrow \mathbb{F}_2[3] \rightarrow \mathcal{A}(1) \rightarrow H^*Z \rightarrow 0$$

with non-trivial  $P_1^1$  action.

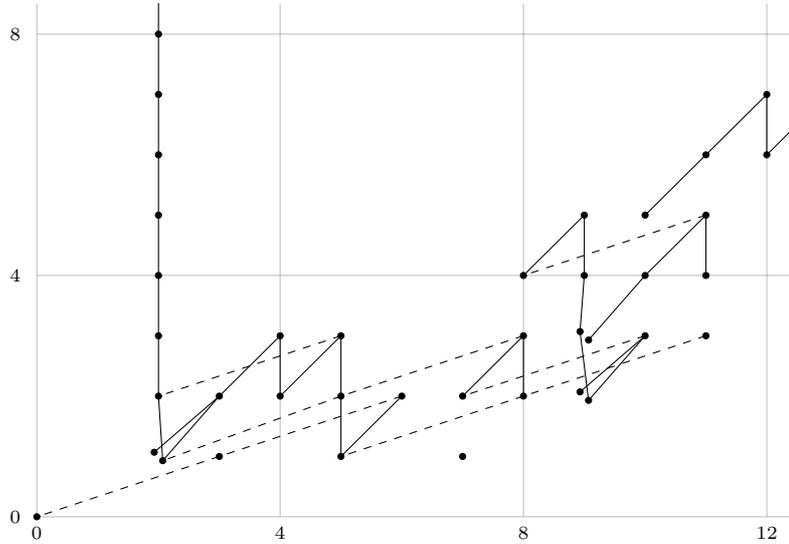


FIGURE 2.  $\text{Ext}_{\mathcal{A}(1)}^{s,t}(H^*Z', \mathbb{F}_2)$  for  $0 \leq s \leq 8$ ,  $0 \leq t - s \leq 12$

Given two realisations of  $\mathcal{A}(1)//\mathbb{F}_2(P_1^1)$  by  $kO$ -modules  $W_1, W_2$ , consider the Adams spectral sequence

$$E_2^{s,t} = \text{Ext}_{\mathcal{A}(1)}^{s,t}(\mathcal{A}(1)//\mathbb{F}_2(P_1^1), \mathcal{A}(1)//\mathbb{F}_2(P_1^1)) \implies \mathcal{D}_{kO}(W_1, W_2).$$

Then by Proposition 3.7,

$$E_2^{s,t} \cong \text{Ext}_{\mathbb{F}_2(P_1^1)}^{s,t}(\mathcal{A}(1)//\mathbb{F}_2(P_1^1), \mathbb{F}_2[3]).$$

The  $\mathbb{F}_2(P_1^1)$ -module structure of  $\mathcal{A}(1)//\mathbb{F}_2(P_1^1)$  has a non-trivial  $P_1^1$ -action linking the generators in degrees 0 and 3.

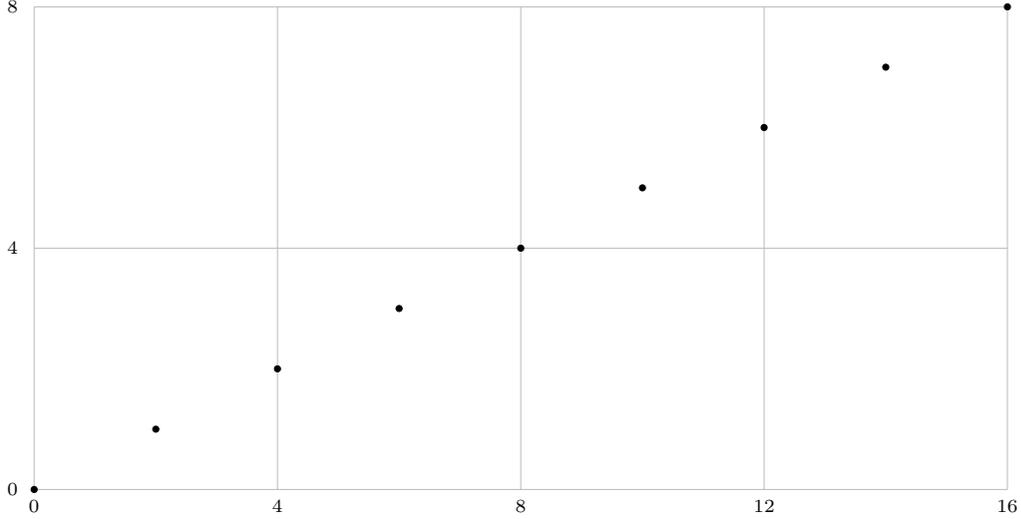
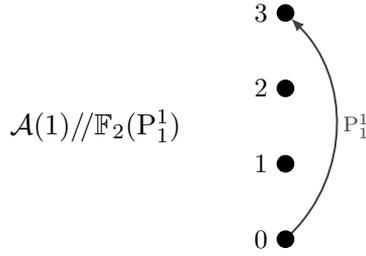


FIGURE 3.  $\text{Ext}_{\mathcal{A}(1)}^{s,t}(H^*Z, \mathbb{F}_2)$  for  $0 \leq s \leq 12$ ,  $0 \leq t - s \leq 20$



Therefore

$$\mathcal{A}(1)//\mathbb{F}_2(P_1^1) \cong \mathbb{F}_2(P_1^1) \oplus \mathbb{F}_2[1] \oplus \mathbb{F}_2[2]$$

and since

$$\text{Ext}_{\mathbb{F}_2(P_1^1)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2[w]$$

with  $w$  in bidegree  $(1, 3)$ , we have

$$E_2^{*,*} \cong \mathbb{F}_2[-3] \oplus \mathbb{F}_2[w][-2] \oplus \mathbb{F}_2[w][-1].$$

There can be no non-trivial Adams differentials, in particular, the generator of  $\mathbb{F}_2[-3]$  which corresponds to the identity homomorphism can be realised by a weak equivalence  $W_1 \rightarrow W_2$  of  $kO$ -modules. This shows that this  $kO$ -module is well defined up to weak equivalence and also stably self dual.

The  $\mathcal{A}(1)$ -module obtained by inducing up the  $\mathbb{F}_2(P_1^1)$ -module above has the form

$$\mathcal{A}(1) \otimes_{\mathbb{F}_2(P_1^1)} \mathcal{A}(1)//\mathbb{F}_2(P_1^1) \cong \mathcal{A}(1) \oplus \mathcal{A}(1)//\mathbb{F}_2(P_1^1)[1] \oplus \mathcal{A}(1)//\mathbb{F}_2(P_1^1)[2],$$

and this is isomorphic to  $H^*(Z \wedge Z)$ .

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