Curved spacetime from interacting gauge theories

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Curved spacetime from interacting gauge theories

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Abstract. Phonons in a Bose-Einstein condensate can be made to behave as if they propagate in curved spacetime by controlling the condensate flow speed. Seemingly disconnected to this, artificial gauge potentials can be induced in charge neutral atomic condensates by for instance coupling two atomic levels to a laser field. In this work, we connect these two worlds and show that synthetic interacting gauge fields, i.e., density-dependent gauge potentials, induce a non-trivial spacetime structure for the phonons. Whilst the creation of effective horizons for phonons solely depends on the flow speed of the condensate, this allows for the creation of new spacetime geometries which can be easily designed by tuning the transverse laser phase. By exploiting this new degree of freedom we show that effectively charged phonons in 2+1 dimensions can be simulated, which behave as if they move under the influence of both a gravitational and an electromagnetic field.

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1. Introduction

Analogue gravity has in the recent decades provided us with a powerful simulator of quantum fields in curved spacetime. In this discipline, we seek physical systems whose underlying dynamics might be different from gravity, but where an apparent spacetime picture emerges for some field in the system [1, 2, 3]. Consequently, a quantum field would then behave as though it is in curved spacetime. In this way, it is possible to study physics that relies on the spacetime structure, but not the gravitational dynamics. Thus, by analogy, effects such as Hawking radiation, the process in which black holes evaporate by emitting thermal radiation, proposed by S Hawking [4, 5], the related Unruh effect [6], and cosmological particle creation [7, 8] are no longer unattainable experimentally. For instance, this has enabled the experimental study of the Hawking effect in flowing Bose-Einstein Condensates (BEC) [9, 10, 11], flowing water [12] and nonlinear optics [13], although in the latter the observed radiation might have had a more complicated origin. In-depth theoretical studies in BECs [14, 15, 16, 17, 18, 19, 20, 21], ultracold fermions [22], moving/nonlinear optical media [23, 24, 25, 26, 27], rings of trapped ions [28] and superconducting circuits [29] to name a few, has exposed a multitude of properties of analog Hawking radiation. For example, this unveiled the existence of density-density correlations in a BEC with a sub/supersonic transition [30] that are vital for experimental investigation [9]. Another example is the deep connection between seemingly unrelated quantum field phenomena such as the dynamical Casimir effect, Hawking radiation and time-refraction [31, 32, 33]. Although Hawking radiation has held most of the attention of the community, analogue gravity is not restricted to this phenomenon alone. For example cosmological particle creation was studied in [34, 35, 3, 36, 37].

However, so far analogue gravity has been restricted to the study of spacetimes created using only *one* degree of freedom [1, 2, 3]. For instance, the current of a BEC has as of yet been the sole determiner of the analogue spacetime felt by the phonons propagating in it. In this article, we will introduce a *second* degree of freedom in the form of a gauge potential, by which richer physics can in principle be simulated. From an experimental perspective, this allows the experimenter to overcome some limitations in spacetime geometry design related to the physical velocity of the system. Despite recent progress in creating sonic horizons in a BEC [9, 10, 11], the fine-tuning of an effective spacetime is still exceptionally challenging with current techniques. The reason, apart from the technical aspect, is that velocity profiles achievable are always restricted to satisfy the physical constraint provided by the continuity equation. By using the degree of freedom introduced by the gauge potential, the effective spacetime geometry experienced by phonons can be designed with much more flexibility, as it only relies on the phase profile of the laser beam coupling the internal states of atoms, which can be chosen almost at will by the experimenter. Synthetic gauge potentials emerge in cold-atom systems by for instance coupling the centre-of-mass motion of the particles to their internal degrees of freedom with a laser [38, 39]. A Berry connection [40] then emerges in the form of a vector potential for the centre-of-mass dynamics. The coupled atomic states must be long-lived compared to the characteristic time scales of the investigated analogue phenomena. Atomic losses due to spontaneous decay, happening on competing time scales, would overwhelm the sought physics and eventually destroy the condensate. Working with atomic optical transitions, promising candidates could for instance be Ytterbium and Strontium which have extremely long lived states of the order of seconds [38]. An alternative route in order to avoid spontaneous emission and heating is to use dark states and three-level atoms [41, 38].

In this paper we start from a density-dependent gauge potential, first described in [42], and demonstrate how this adds a new degree of freedom to analogue gravity models. We show that the nonlinearity of the vector potential introduces a new mechanism for designing effective spacetime geometries in a condensate. This new degree of freedom is potentially very powerful, owing to the design flexibility of effective spacetimes. We then discuss how the additional degree of freedom introduced by the nonlinear gauge potential allows us to extend analogue gravity beyond what is currently possible. By choosing a particular configuration for the system we show for example that effectively charged particles in $2+1$ dimensions can be simulated with phonons, which behaves as particles subjected to both a gravitational and an electromagnetic field.

This article is structured as follows. In Section 2, we introduce artificial gauge fields for Bose-Einstein condensates, and in particular discuss density-dependent gauge fields. We derive the effective acoustic metric for phonons in a BEC under the influence of density-dependent gauge fields in Section 3. Furthermore, in Section 4 we show that the acoustic metric can be cast into a Kaluza-Klein framework, and that the phonons acquire an effective charge. Finally we discuss the experimental consideration in Section 5 and conclude in Section 6.

In this work, we will be using a $(-,+,+,+)$ metric signature, and spatial vector quantities are denoted with boldface typesetting, for instance $\mathbf{u} = (u_1, u_2, u_3)$. The magnitude of a spatial vector is denoted as $u = \sqrt{\mathbf{u}\cdot\mathbf{u}}$ according to the underlying physical metric in the laboratory frame (i.e. not the effective metric), unless otherwise stated. Furthermore, a component of a vector will be denoted as u_ζ for some set of indices ζ. Throughout this work, we will use capital Roman indices M, N to indicate range $\{0,1,2,3\}$, whereas Greek indices μ,ν run over $\{0,1,2\}$ and finally use lower case Roman indices i,j to refer to the respective spatial components. Furthermore, we will make use of Cartesian coordinates, unless otherwise stated.

2. The physical system

In this Section we describe how a density-dependent gauge potential [42, 43, 44, 45, 46] can be created in a Bose-Einstein condensate. We consider a BEC of two-level atoms, where the collisional interactions are modelled by a zero-range pseudo potential. The two internal atomic levels are coupled by an external laser. The microscopic N-body

Hamiltonian which describes the dynamics of the system is then given by [42]

$$H = \sum_{n=1}^{N} \left[\left(\frac{\mathbf{p}_n^2}{2m} + V_{\text{ext}}(\mathbf{r}_n) \right) \otimes \mathbb{I}_n + H_{\text{int}}(\mathbf{r}_n) \right] \otimes \mathbb{I}_{\mathcal{H}\setminus n} + \sum_{n<\ell}^{N} G_{n,\ell}\left(\mathbf{r}_n, \mathbf{r}_\ell\right) \otimes \mathbb{I}_{\mathcal{H}\setminus\{n,\ell\}}. \quad (1)$$

The first term in Eq. (1) is the sum of the non-interacting Hamiltonians, with $\mathbf{p}_n = -(i/\hbar)\,\boldsymbol{\nabla}_n$ and $\mathbf{r}_n$ the quantum mechanical momentum operator, where $\boldsymbol{\nabla}$ is the standard *nabla* operator (e.g. $\boldsymbol{\nabla} = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$ in Cartesian space with x_i being the spatial coordinates), and the position of the nth particle. The identity operators $\mathbb{I}_{\mathcal{H}\setminus\{n,\ell,\ldots\}}$ act on the subspace which excludes the particles $n, \ell, \ldots$. The coupling between the two internal levels $|1\rangle$ and $|2\rangle$ is given by

$$H_{\text{int}}(\mathbf{r}) = \frac{\hbar\Omega(\mathbf{r})}{2} \begin{pmatrix} 0 & e^{-i\phi(\mathbf{r})} \\ e^{i\phi(\mathbf{r})} & 0 \end{pmatrix} \quad (2)$$

where $\Omega(\mathbf{r})$ is the Rabi frequency which characterizes the strength of the light-matter coupling and $\phi(\mathbf{r})$ is the phase of the laser phase, both depending in general on position $\mathbf{r}$. We further assumed that the laser detuning from the atomic resonance is zero. The second term in Eq. (1) represents the two-body interaction between the particles which takes the diagonal form $G_{n,\ell}(\mathbf{r}_n, \mathbf{r}_\ell) = \text{diag}\,[U_{11}, U_{12}, U_{12}, U_{22}]\;\delta(\mathbf{r}_n - \mathbf{r}_\ell)$, with the coupling constants given by $U_{ij} = 4\pi\hbar^2 a_{ij}/m$ and where a_{ij} are the scattering lengths relative to the three different collision channels. The Lagrangian is written in terms of the Hamiltonian as

$$L = \prod_{i=1}^{N} \left(\int d^3 r_i \right) \left[\boldsymbol{\Psi}^\dagger \left(i\hbar\partial_t - H \right) \boldsymbol{\Psi} \right]. \quad (3)$$

Here $d^3 r_i$ is the volume element occupied by the i-th particle (Riemann measure of the integral). In the limit of weakly interacting atoms, $\rho_i a_{ij}^3 \ll 1$ (with $i, j = 1, 2$), the many-body wavefunction $\boldsymbol{\Psi}(\mathbf{r}_1, \mathbf{r}_2, \ldots \mathbf{r}_N)$ is written as the symmetrized product of the single particle (pseudo-)spinor wave function $\boldsymbol{\phi}(\mathbf{r})$, which satisfies the normalization condition $\int d^3 r\, \boldsymbol{\phi}^\dagger \boldsymbol{\phi} = 1$, so that $\boldsymbol{\Psi}(\mathbf{r}_1, \mathbf{r}_2, \ldots \mathbf{r}_N) = \prod_{i=1}^{N} \phi(\mathbf{r}_i)$. By using this meanfield ansatz for the many-body wave function, the Lagrangian takes the form

$$L_{MF} = \int d^3 r \left[\boldsymbol{\psi}^\dagger \left(i\hbar\partial_t - H_{MF} \right) \boldsymbol{\psi} \right], \quad (4)$$

where we defined the condensate wave function $\boldsymbol{\psi}(\mathbf{r}) = \sqrt{N}\boldsymbol{\phi}(\mathbf{r})$ and the single particle mean field Hamiltonian H_{MF}

$$H_{MF} = \left(\frac{\mathbf{p}^2}{2m} + V_{\text{ext}}(\mathbf{r}) \right) \otimes \mathbb{I} + H_{\text{int}}(\mathbf{r}) + U_{\text{MF}}. \quad (5)$$

Hereafter we will drop the explicit space-dependence from the equations, unless needed for clarity's sake. Here $\mathbb{I}$ the 2×2 identity operator which acts in the space of the atomic internal degrees of freedom. In Eq. (5) the operator $U_{\rm MF}$ describes the mean field collisional effects, and is given by

$$U_{\rm MF} = \frac{1}{2}\begin{pmatrix}\Delta_1 & 0\\ 0 & \Delta_2\end{pmatrix} \tag{6}$$

with

$$\Delta_1 = U_{11}\rho_1 + U_{12}\rho_2 \tag{7}$$
$$\Delta_2 = U_{12}\rho_1 + U_{22}\rho_2 \tag{8}$$

where $\rho_i = |\psi_i|^2$ is the density of atoms in level $|i\rangle$ $(i = 1, 2)$, and ψ_i the corresponding component of the order parameter.

In the weakly interacting limit, the coupling energy $\hbar\Omega$ between the internal states can be much larger than the collisional mean field shifts. This allows us to treat the meanfield interaction as a small perturbation to the atom-light coupling. To first order in $\mathcal{O}(\rho_{ij}U_{ij}/\hbar\Omega)$ (with $i, j = 1, 2$), the eigenstates can be written as

$$|\chi_\pm\rangle = |\chi_\pm^{(0)}\rangle \pm \frac{\Delta_1 - \Delta_2}{\hbar\Omega}|\chi_\mp^{(0)}\rangle, \tag{9}$$

where $|\chi_\pm^{(0)}\rangle = \left(|1\rangle \pm e^{i\phi}|2\rangle\right)/\sqrt{2}$ are the so called dressed states. The interacting dressed states in Eq. (9), represent a basis for the internal Hilbert space of the atoms. The condensate wave function $|\boldsymbol{\psi}(\mathbf{r},t)\rangle$ can therefore be written as $|\boldsymbol{\psi}(\mathbf{r},t)\rangle = \sum_{i=\{+,-\}}\psi_i(\mathbf{r},t)|\chi_i\rangle$.

Our goal is to describe the center-of-mass dynamics of the atoms whose internal state is given by $|\chi_\pm\rangle$ in Eq. (9). In order to do that we use the adiabatic assumption where $\psi_\mp(\mathbf{r},t) \equiv 0$. This assumption is valid as long as the collisionally induced detuning Δ_i $(i = 1, 2)$ is small compared to $\hbar\Omega$. The projection of the mean field Lagrangian in Eq. (4) onto the dressed states ($|\chi_\pm\rangle$) then results in the Lagrangian

$$L_\pm = \int d^3r \left[\psi_\pm^\dagger\left(i\hbar\partial_t - H_\pm\right)\psi_\pm\right], \tag{10}$$

where

$$H_\pm = \frac{(\mathbf{p} - \mathbf{A}_\pm)^2}{2m} + W \pm \frac{\hbar\Omega}{2} + \frac{U}{2} \tag{11}$$

is the Hamiltonian describing the dynamics of the $\pm$ component of the condensate with $U = (U_{11} + U_{22} + 2U_{12})/4$. The vector potential $\mathbf{A}_\pm = -\langle\chi_\pm|\,\mathbf{p}\,|\chi_\pm\rangle$ and the scalar potential $W = |\langle\chi_+|\,\mathbf{p}\,|\chi_-\rangle|^2/2m$ stem from the adiabatic projection of the full system onto one of the subspaces spanned by the dressed states.

If we now substitute the dressed states from Eq. (9) into the expressions for the vector and scalar potential, together with $\Delta_1 = \rho_\pm(u_{11} + U_{12})/2$, $\Delta_2 = \rho_\pm(U_{22} + U_{12})/2$ from Eqs.(7) and (8), we obtain, to leading order,

$$\mathbf{A}_\pm = \mathbf{A}^{(0)} \pm \mathbf{a}_1\rho_\pm(\mathbf{r}), \tag{12}$$

$$W = \frac{\left|\mathbf{A}^{(0)}\right|^2}{2m}. \tag{13}$$

Here we define by m the mass of the atoms, $\mathbf{A}^{(0)} = -\frac{\hbar}{2}\boldsymbol{\nabla}\phi$ the single particle component of the vector potential, with $\boldsymbol{\nabla}\phi$ the gradient of the phase ϕ of the laser beam and $\rho_\pm(\mathbf{r}) = |\psi_\pm|^2$ the density of the dressed state, with $\psi_\pm$ the condensate order parameter. The vector $\mathbf{a}_1 = [(U_{11} - U_{22})/(8\Omega)]\ \boldsymbol{\nabla}\phi$ controls the direction and strength of the first order nonlinear, density-dependent contribution.

By minimizing the action $S_\pm = \int dt L_\pm$ with respect to $\psi^*_\pm$, we obtain a Gross-Pitaevskii equation (GPE) for the condensate wave function, of the form

$$i\hbar\frac{\partial\psi_\pm}{\partial t} = \left[\frac{(\mathbf{p} - \mathbf{A}_\pm)^2}{2m} - \mathbf{a}_1\cdot\mathbf{j} + W + U\rho_\pm\right]\psi_\pm \tag{14}$$

where a current nonlinearity appears,

$$\mathbf{j} = \frac{\hbar}{2mi}\left[\psi^*_\pm\left(\nabla - \frac{i}{\hbar}\mathbf{A}_\pm\right)\psi_\pm - \psi_\pm\left(\nabla + \frac{i}{\hbar}\mathbf{A}_\pm\right)\psi^*_\pm\right]. \tag{15}$$

For the sake of readability we drop in what follows the subscript in the quantities defined above, that is we write $\psi_\pm \to \psi$ and similarly for $\mathbf{A}$ and ρ. The following equations are the same weather we consider the $(+)$ or $(-)$ component of the condensate, except for the sign of some terms. To distinguish between the two cases we use the convention that upper signs in the equations refer to the $(+)$ component, while lower signs refer to the $(-)$ one. The choice of working with one or the other component does not affect the generality of the following arguments. Working with the $(+)$ or $(-)$ component would be the same provided we exchange $U_{11} \leftrightarrow U_{22}$.

3. Effective metric

The GPE in Eq. (14) provides the framework from which the effective spacetime felt by phonons can be derived. To this aim, we work in the hydrodynamic formalism, and write the order parameter in terms of the particle density ρ and its phase S, as $\psi = \sqrt{\rho}e^{iS}$. In terms of these quantities, the Eq. (14) is equivalent to the continuity and the (quantum) Euler equations

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{v}) = 0 \tag{16}$$

and

$$\hbar\frac{\partial S}{\partial t} = \frac{\hbar^2}{2m}\frac{\boldsymbol{\nabla}^2\sqrt{\rho}}{\sqrt{\rho}} - \frac{\hbar^2}{2m}(\boldsymbol{\nabla}S)^2 \pm 2\rho\,\mathbf{a}_1\cdot\mathbf{v} + \rho\left(\frac{\rho\,|\mathbf{a}_1|^2}{2m} - U\right). \tag{17}$$

Here $\mathbf{v} = (\hbar/m)\boldsymbol{\nabla}S - \mathbf{A}/m$ is the physical velocity in the condensate. Phonons represent long wavelength excitations of the system above its mean-field component. At the classical level, the dynamics of these excitations can be derived by linearizing Eqs. (16) and (17) on top of the condensate background. Given the density of particles in the condensate (ρ_0) and the phase of the order parameter (S_0), we include small perturbations in the theory by writing the density and the phase as $\rho = \rho_0 + \rho_1$ and $S = S_0 + S_1$, respectively. Here ρ_1 and S_1 account for small deviations from the condensate component. Retaining only the first order terms in S_1, and ρ_1, and working in the hydrodynamic regime in which the quantum pressure can be neglected, the dynamics of the excitations is described by the equation

$$\Box S_1 \equiv \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^M} \left(\sqrt{-g} g^{MN} \frac{\partial S_1}{\partial x^N} \right) = 0, \tag{18}$$

where g is the metric determinant, and x^M is a set of Cartesian coordinate in the reference Minkowski spacetime of the laboratory (remind that capital Latin indices span the full 3+1 spacetime). This is the equation for a scalar field propagating in a curved spacetime [47] and describes the evolution of acoustic fluctuations living in the effective background geometry described by the metric g_{MN} tensor

$$g_{MN} = \frac{\rho_0}{c_s} \begin{pmatrix} -\left(c^2 - v_0^2\right) & -\left(v_0 \mp v_a\right)_j \\ -\left(v_0 \mp v_a\right)_i & \delta_{ij} \end{pmatrix}. \tag{19}$$

Here $\mathbf{v}_0 = \hbar \boldsymbol{\nabla} S_0/m - \mathbf{v}_a$ is the physical velocity field in the condensate, and $\mathbf{v}_a = \rho_0 \mathbf{a}_1/m$ is the effective velocity induced by the nonlinear vector potential. Also, $c_s = \left(c^2 - 2\mathbf{v}_a \cdot \mathbf{v}_0 + v_a^2\right)^{1/2}$ is the local speed of sound in the condensate, while $c = \left(U\rho_0/m\right)^{1/2}$ is the value it would take in absence of the potential. It is worth pointing out here that, by working in the hydrodynamic limit, we neglected dispersive effects in the model. The latter can be taken into account by retaining the quantum pressure term in Eq. (17). In terms of gravity analogue, this would lead to a momentum dependence in the metric in Eq. (19), that is encoded in the value of the speed of sound c. We limit the discussion in what follows to considering the low momentum limit of the model, and refer to the literature for what concerns the problems related to the "rainbow" character of the metric at high momenta [1].

Equation (19) shows that cross terms in the metric, mixing the space and time coordinates in the laboratory reference frame, are induced by the nonlinear vector potential. Interestingly, the time-time component (as well as the pure spatial components) of the metric is not affected by the nonlinear potential, and solely the physical velocity of the condensate is responsible for the establishment of an ergo-region and eventually the appearence of acoustic horizons in the system [1]. The nonlinear potential thus provides an extra degree of freedom that can be exploited in order to design effective spacetimes for phonons. In particular, Eq. (19) reveals that a nontrivial curved spacetime can be induced even for a static condensate, for which the physical velocity $\mathbf{v}_0$ of particles is zero. Moreover, effects such as cosmological particle creation [47], or dynamical Casimir effect [48], where the time-dependent mode density of the vacuum manifests itself as particle creation, can be relatively easily implemented by simply modulating in time the light-matter interaction parameters, such as the Rabi frequency or the detuning. Ultimately, a time-modulation of the light-matter interaction leads to a time-dependent $\mathbf{a}_1$, and thus a time-dependent metric.

We show in the next section how the new degree of freedom provided by the nonlinear vector potential can be used to simulate the dynamics of charged particles in two-dimensions, moving under the combined influence of both a gravitational and an electromagnetic field.

4. Simulating effectively charged phonons

We start with the metric in Eq. (19) and make the following time-coordinate transformation

$$dt' = dt + \frac{(\mathbf{v}_0 - \mathbf{v}_a)}{c^2 - v_0^2} \cdot d\mathbf{r}. \tag{20}$$

This is a type of co-moving coordinates, and in this new set of coordinates the $dtdx_i$ components of the metric are absorbed into the $dx_i dx_j$ components. Not all metrics can be put into this form, but includes interesting cases such as both neutral and charged black holes, see for instance the well-known transformation connecting the Schwarzschild and the Gullstrand-Painlevé metrics [1]. Disregarding the conformal factor, the effective metric seen by phonons now reads

$$ds^2 = -\left(c^2 - v_0^2\right) dt'^2 + \left[\delta_{ij} + \frac{(v_0 - v_a)_i}{\sqrt{c^2 - v_0^2}} \frac{(v_0 - v_a)_j}{\sqrt{c^2 - v_0^2}}\right] dx_i dx_j. \tag{21}$$

Let us now define the scalar $\varphi^2 \equiv (v_0 - v_a)_3^2/(c^2 - v_0^2)$, the two-dimensional vector $\mathcal{A}_i \equiv (v_0 - v_a)_i / (v_0 - v_a)_3$ and the $2+1$-dimensional metric $h_{\mu\nu} = \mathrm{diag}\left[-\left(c^2 - v_0^2\right), 1, 1\right]$ (note that Greek indexes take the values $\mu, \nu, ... = 0, 1, 2$, while lowercase Latin indexes run through the corresponding two-dimensional space sector ($i = 1, 2$)). In terms of these quantities, the full $3+1$ metric can be written as

$$g_{MN} = \begin{pmatrix} h_{\mu\nu} + \varphi^2 \mathcal{A}_\mu \mathcal{A}_\nu & \varphi^2 \mathcal{A}_\mu \\ \varphi^2 \mathcal{A}_\mu & 1 + \varphi^2 \end{pmatrix}. \tag{22}$$

Interestingly, if we assume that $\varphi^2 \gg 1$ (such that $1+\varphi^2 \simeq \varphi^2$), this metric has the same form as the ansatz introduced by Klein [49] in the context of dimensional reduction in the Kaluza-Klein (KK) theory (up to the conformal factor in Eq. (19)). Such a theory represents the first attempt of pursuing the unification of electromagnetism and gravity, in the framework of a more general theory of gravity in higher dimensions [50, 51, 49].

One of the simpler implementation of this idea postulates the existence of a hidden fifth dimension, whose characteristic scale is of the order of the Planck length, and generalizes the Einstein theory of gravity to this higher dimensional spacetime. Electromagnetism then emerges upon dimensional reduction of the theory, i.e. "compactifying", or in other words integrating out, the extra dimension on a circle [50]. It is worth clarifying that the effective metric Eq. (22) seen by the phonons has of course nothing to do with the KK theory itself and its fundamental motivation. We are not looking here at any fundamental theory unifying electromagnetism and gravity. Moreover, the effective metric g_{MN}, or in other words, the $2+1$ metric $h_{\mu\nu}$ and the vector potential A_μ are not dynamical objects, as is usual in analogue models. What is of interest here is that the effective metric seen by phonons in the condensate has the same form as the KK ansatz, and we expect to see analogous physics at the *kinematic* level.

By assuming that the *dilaton* field (in the KK language) φ is constant, and that the metric $h_{\mu\nu}$ and the vector potential $\mathcal{A}_\mu$ are both independent of (for instance) the coordinate x_3, the effective metric Eq. (22) can be seen as a KK metric in a $2+1$ physical spacetime, in which the role of the extra dimension is played by the third spatial component. Following this reasoning, $\mathcal{A}_\mu$ takes the role of an effective electromagnetic potential for phonons in the $2+1$ spacetime. Phonons in the $x_1 - x_2$ plane of the condensate are thus expected to behave as the fundamental excitations of the KK theory. In other words, the phonons in the $x_1 - x_2$ plane take on the character of charged particles subjected to both a gravitational and an electromagnetic field.

To prove this assertion we start from the action

$$S = -\frac{1}{2}\int dt d^3 r\sqrt{-g}g^{MN}\left(\partial_M S_1\right)\left(\partial_N S_1\right), \tag{23}$$

from which the wave equation in Eq. (18) can be deduced. The factor g here is the determinant of the metric tensor in Eq. (22), which can also be expressed in terms of the determinant of the $2+1$ metric tensor $h = \det\left[h_{\mu\nu}\right]$, as

$$\begin{aligned} g &= \det\left[g_{MN}\right] \\ &= \det\left[h^{\mu\nu}\right]^{-1}\left[\left(\frac{1}{\varphi^2} + \mathcal{A}^2\right) - \mathcal{A}^\mu h_{\mu\nu}\mathcal{A}^\nu\right]^{-1} \\ &= h\varphi^2, \end{aligned} \tag{24}$$

where in this case the inner product in $\mathcal{A}^2$ uses the effective metric $h_{\mu\nu}$. Since, by assumption, the metric in Eq. (22) does not depend on the coordinate x_3, we can expand S_1 in terms of a suitable basis $\{\xi_n(x_3)\}$ in the x_3 direction

$$S_1 = \sum_n s_n\left(x^\nu\right)\xi_n(x_3). \tag{25}$$

We consider for simplicity periodic boundary conditions and write the eigenmodes as $\xi_n(x_3) = e^{ik_3^n x_3}/\sqrt{L}$, where L is the length of the system in the x_3 direction, and $k_3^n = 2\pi n/L$ for $n \in \mathbb{Z}$. By inserting this expansion into Eq. (23) and integrating over x_3, we find the reduced action in the form

$$\mathcal{S} = \varphi\sum_n\left\{-\frac{1}{2}\int dt d^2 x\sqrt{-h}\left[\left(h^{\mu\nu}\left[\left(\partial_\mu - ik_3^n\mathcal{A}_\mu\right)s_n\right]^2\right) - \left(\frac{k_3^n}{\varphi}\right)^2 s_n^2\right]\right\}. \tag{26}$$

It is given by the sum of an infinite number of actions, describing the dynamics of massive charged particles in the 2+1 dimensional transverse $x_1 - x_2$ plane, and is what is called the *Kaluza-Klein tower*, in the KK numenclature. The values of the effective charges q_n and masses m_n for each mode are set by the value of the momentum of the mode in the x_3 direction, that is $q_n = \hbar k_3^n$, $m_n = \hbar k_3^n / c\varphi$. The ratio $q/m = c\varphi$, which is related to the cyclotron frequency ω_c by $\omega_c = Bq/m$, is however the same for all the modes. Whilst other (more complicated) configuration in the x_3-direction could have been chosen, this simplest scenario already offers some of the hallmarks of KK theory, that is, the previously mentioned Kaluza-Klein tower of mass and charge modes.

5. Experimental considerations

For an experimental realisation of the analog gravity effects discussed above, a number of criteria must be fulfilled. Firstly, the density-dependent gauge potential relies on the Rabi frequency and the corresponding energy scale to dominate over any collisional interaction energies. In practice this means $\hbar\Omega$ must be larger than the chemical potential μ of the Bose-Einstein condensate. Secondly, we need to ensure a suitable choice of atomic states and scattering lengths, such that a non-zero current nonlinearity can be obtained. For this one needs $U_{11} \neq U_{22}$ which, if not readily available, can be achieved using Feshbach resonances. In other words, one needs to be in the adiabatic regime, where the dressed states arising from the light-matter interaction are not coupled. This not only requires that $\hbar\Omega \gg \mu$, but the atomic states must also be long lived. Finally, the nonlinear gauge potential should take some specific spatial form in order to emulate a specific spacetime structure in Eq. (19) or effective vector potential in Eq. (21). Similarly to the proposal in Ref. [52], the relevant phase profiles required to this purpose can be easily obtained using standard beam shaping technologies.

Let us consider the example of a constant effective magnetic field for the implementation of effectively charged phonons on curved spacetime. More complex configurations can be obtained by opportunely choosing the space and time dependence of the experimental parameters. In what follows we use the cylindrical coordinates (r, θ, z), defined as

$$r = \sqrt{x_1^2 + x_2^2}, \tag{27}$$

$$\theta = \arctan(x_2/x_1), \tag{28}$$

$$z = x_3. \tag{29}$$

In order to clearly separate the mean-field dynamics from the dynamics of phonons, we tailor the laser beam in such a way that $\boldsymbol{\nabla}\phi \sim (1/r)\hat{e}_\theta$, with $\hat{e}_\theta$ the unit vector in the tangent direction. In this case, the zero-order synthetic vector potential gives rise to a zero effective magnetic fields for the atoms. We choose the remaining experimental parameters, that is, the Rabi frequency and the mean-field coupling constants $U_{11} - U_{22}$, in such a way that $\boldsymbol{\mathcal{A}} = (B/2)r\,\hat{e}_\theta$. In this way, the corresponding effective magnetic

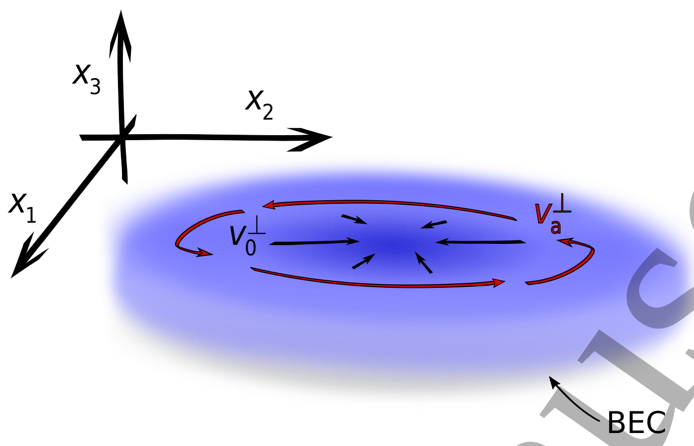

Figure 1. Scheme of the experimental set-up discussed in the text, needed to simulate effectively charged phonons in a BEC under the action of a homogeneous effective magnetic field. The velocity profile of the physical velocity in the plane of the condensate $\mathbf{v}_0^\perp$ is used to induce a curvature in the effective metric $h_{\mu\nu}$ experienced by phonons in the plane of the condensate. The internal states of the two-level atoms are coupled by a laser beam, whose phase gradient ϕ is circularly directed. The Rabi frequency Ω and/or the mean-field coupling constants U_{ij} are finally tuned in order to implement a constant effective magnetic field on phonons.

field felt by phonons is constant and equal to B. Note here that, if we let $U_{11} - U_{22}$ vary in space, we need to suitably shape U_{12} as well, in order to keep the mean-field interaction parameter U constant and ensure that the speed of sound c_s is homogeneous and the conformal factor in the metric (19) constant.

The curvature of the reduced (2+1) metric $h_{\mu\nu}$ is encoded in its time-time component $h_{00} = -(c^2 - v_0^2)$. Let us denote $f(t, r)$ the spacetime profile we want to implement for this term. This condition, together with the constraint provided by the definition of the dilaton field φ (here just a constant), completely determine the spacetime profile of the physical velocity $\mathbf{v}_0$ in the system. Accordingly, the component of the velocity in the longitudinal direction is determined by the condition $K\,(v_0 - v_a)_3^2 = c^2\varphi f(t, r)$, where K is some large non-dimensional constant in order to ensure that $\varphi^2 \gg 1$ also. The need of tailoring both the longitudinal and transverse components of the physical velocity might be experimentally challenging.

To overcome this difficulty we could carefully design the longitudinal component of $\mathbf{v}_a$ such that the above constraints are satisfied, keeping the longitudinal component of the physical velocity constant. This is most easily done by tailoring the profile of the phase of the laser and/or of the Rabi frequency. We will here assume it being zero for simplicity in the following. The transverse (the $x_1 - x_2$) component $\mathbf{v}_0^\perp$ of the velocity takes thus the form $\left(v_0^\perp\right)^2 = c^2\,[1 - f(t, r)]$. It is this component of the velocity that provides the curvature of the $h_{\mu\nu}$ metric in the proposed configuration. We require c_s^2 (or simply c^2 if $v_a \ll c$) to be fixed in order to maintain a homogeneous conformal factor of the metric throughout space. This constraint can be relaxed in the geometrical acoustic limit (or in other terms in the eikonal approximation) [1], since the physics is insensitive to the conformal factor of the metric in this case.

With this choice of parameters, the coordinate transformation in Eq. (20) takes the differential form

$$dt' = dt + \frac{1}{c}\left(\frac{\varphi}{f(t,r)}\right)^{1/2}\left(dz + \frac{B}{2}r^2 d\theta\right). \tag{30}$$

A possible set-up for this is the two-dimensional pancake shaped condensate illustrated in Fig. 1. Here, the phonons living on the plane of the condensate gain an effective mass and charge once we populate a mode with a non-null longitudinal momentum. By imprinting an asymmetric perturbation in the transverse plane, we excite effectively charged phonons, which in the presence of a constant magnetic field will precess with the cyclotron frequency ω_c.

The set-up discussed here does not satisfy periodic boundary conditions in the x_3 direction, but we are still able to excite a plane wave in this direction, providing phonons in the transverse plane with charge and mass. Nonetheless, periodic boundary conditions in (at least) one direction required for simple mass/charge modes follows naturally from most experimentally available atom traps. For instance, a harmonic trap is indeed cylindrically symmetric. Distinct mass and charge modes in the Kaluza-Klein tower for the phonons are therefore well-defined.

More interesting physics can in principle be simulated, and further study needs to be done in order to address specific cosmological scenarios. For example, the evaporation of a black hole via the emission of massive (and charged) particles in the different KK tower of modes can be simulated by creating a (sonic) horizon in the system. Moreover, high intensity effective electromagnetic fields can be implemented by opportunely tuning the condensate velocity and the light-matter interaction parameters. This opens the door to investigating the high intensity physics of quantum fields, such as the Schwinger pair production [53]. In the analogue case, phonons pair with positive and negative norm are created because of the vacuum fluctuations in the system, which are pulled apart by the intense effective field acting on them.

6. Summary and Conclusion

We have shown that density-dependent synthetic gauge fields acting on neutral atoms can be exploited in order to enrich the physics that can be simulated in a BEC implementation of gravity analogues. Such nonlinear fields opens up the possibility of introducing a new, independent, and versatile degree of freedom, useful for designing effective spacetime for phonons. The structure of this effective spacetime depends on the details of the atom-field interaction parameters and can be easily adjusted by designing the experimental set-up. As an application of the model, we showed that effectively charged particles subjected to both a gravitational and an electromagnetic field can be simulated with phonons. This provides an example of the novelty of the model and the enriched physics that the framework of analogue gravity is capable to simulate when considering synthetic electromagnetic field. We pointed out that non-trivial analogue physics can be simulated even with a static condensate, using exclusively the degree of freedom provided by the nonlinear synthetic vector potentials acting on the atoms.

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References

[1] Carlos Barceló, Stefano Liberati, and Matt Visser. Analogue gravity. *Living Reviews in Relativity*, 14(1):3, May 2011.

[2] Mário Novello, Matt Visser, and Grigori Volovik. *Artificial black holes*. World Scientific, 2002.

[3] Daniele Faccio, Francesco Belgiorno, Sergio Cacciatori, Vittorio Gorini, Stefano Liberati, and Ugo Moschella. *Analogue Gravity Phenomenology: Analogue Spacetimes and Horizons, from Theory to Experiment*, volume 870. Springer, 2013.

[4] Stephen W Hawking. Black hole explosions. *Nature*, 248(5443):30–31, 1974.

[5] Stephen W Hawking. Particle creation by black holes. *Communications in mathematical physics*, 43(3):199–220, 1975.

[6] William George Unruh. Experimental black-hole evaporation? *Phys. Rev. Lett.*, 46(21):1351, 1981.

[7] Erwin Schrödinger et al. The proper vibrations of the expanding universe. *Physica*, 6(7):899–912, 1939.

[8] Leonard Parker. Particle creation in expanding universes. *Phys. Rev. Lett.*, 21(8):562, 1968.

[9] Jeff Steinhauer. Observation of thermal hawking radiation and its entanglement in an analogue black hole. *arXiv preprint arXiv:1510.00621*, 2015.

[10] Jeff Steinhauer. Observation of self-amplifying hawking radiation in an analogue black-hole laser. *Nature Physics*, 10(11):864–869, 2014.

[11] Oren Lahav, Amir Itah, Alex Blumkin, Carmit Gordon, Shahar Rinott, Alona Zayats, and Jeff Steinhauer. Realization of a sonic black hole analog in a bose-einstein condensate. *Phys. Rev. Lett.*, 105(24):240401, 2010.

[12] Germain Rousseaux, Christian Mathis, Philippe Maïssa, Thomas G Philbin, and Ulf Leonhardt. Observation of negative-frequency waves in a water tank: a classical analogue to the hawking effect? *New Journal of Physics*, 10(5):053015, 2008.

[13] E Rubino, F Belgiorno, SL Cacciatori, M Clerici, V Gorini, G Ortenzi, L Rizzi, VG Sala, M Kolesik, and D Faccio. Experimental evidence of analogue hawking radiation from ultrashort laser pulse filaments. *New Journal of Physics*, 13(8):085005, 2011.

[14] Matt Visser. Acoustic black holes: horizons, ergospheres and hawking radiation. *Classical and Quantum Gravity*, 15(6):1767, 1998.

[15] B Reznik. Origin of the thermal radiation in a solid-state analogue of a black hole. *Phys. Rev. D*, 62(4):044044, 2000.

[16] Luis Javier Garay, JR Anglin, JI Cirac, and P Zoller. Sonic analog of gravitational black holes in bose-einstein condensates. *Phys. Rev. Lett.*, 85(22):4643, 2000.

[17] U Leonhardt, T Kiss, and P Öhberg. Bogoliubov theory of the hawking effect in bose–einstein condensates. *Journal of Optics B: Quantum and Semiclassical Optics*, 5(2):S42, 2003.

[18] U Leonhardt, T Kiss, and P Öhberg. Theory of elementary excitations in unstable bose-einstein condensates and the instability of sonic horizons. *Phys. Rev. A*, 67(3):033602, 2003.

[19] Stefano Liberati, Matt Visser, and Silke Weinfurtner. Analogue quantum gravity phenomenology from a two-component bose–einstein condensate. *Classical and Quantum Gravity*, 23(9):3129, 2006.

[20] Carlos Barceló, Stefano Liberati, Sebastiano Sonego, and Matt Visser. Quasi-particle creation by analogue black holes. *Classical and Quantum Gravity*, 23(17):5341, 2006.

[21] Stefano Finazzi and Renaud Parentani. Black hole lasers in bose–einstein condensates. *New Journal of Physics*, 12(9):095015, 2010.

[22] S Giovanazzi. Hawking radiation in sonic black holes. *Phys. Rev. Lett.*, 94(6):061302, 2005.

[23] U Leonhardt and P Piwnicki. Optics of nonuniformly moving media. *Phys. Rev. A*, 60(6):4301, 1999.

[24] Ulf Leonhardt and Paul Piwnicki. Relativistic effects of light in moving media with extremely low group velocity. *Phys. Rev. Lett.*, 84(5):822, 2000.

[25] WG Unruh and R Schützhold. On slow light as a black hole analogue. *Phys. Rev. D*, 68(2):024008, 2003.

[26] Mike Petev, Niclas Westerberg, Daniel Moss, Elenora Rubino, C Rimoldi, SL Cacciatori, F Belgiorno, and D Faccio. Blackbody emission from light interacting with an effective moving dispersive medium. *Phys. Rev. Lett.*, 111(4):043902, 2013.

[27] David Vocke, Calum Maitland, Angus Prain, Kali E. Wilson, Fabio Biancalana, Ewan M. Wright, Francesco Marino, and Daniele Faccio. Rotating black hole geometries in a two-dimensional photon superfluid. *Optica*, 5(9):1099–1103, Sep 2018.

[28] Birger Horstmann, Benni Reznik, Serena Fagnocchi, and J Ignacio Cirac. Hawking radiation from an acoustic black hole on an ion ring. *Phys. Rev. Lett.*, 104(25):250403, 2010.

[29] PD Nation, MP Blencowe, AJ Rimberg, and E Buks. Analogue hawking radiation in a dc-squid array transmission line. *Phys. Rev. Lett.*, 103(8):087004, 2009.

[30] Roberto Balbinot, Alessandro Fabbri, Serena Fagnocchi, Alessio Recati, and Iacopo Carusotto. Nonlocal density correlations as a signature of hawking radiation from acoustic black holes. *Phys. Rev. A*, 78(2):021603, 2008.

[31] Jose Tito Mendonca, Gert Brodin, and Mattias Marklund. Vacuum effects in a vibrating cavity: time refraction, dynamical casimir effect, and effective unruh acceleration. *Physics Letters A*, 372(35):5621–5624, 2008.

[32] PD Nation, JR Johansson, MP Blencowe, and Franco Nori. Colloquium: Stimulating uncertainty: Amplifying the quantum vacuum with superconducting circuits. *Reviews of Modern Physics*, 84(1):1, 2012.

[33] Ralf Schützhold. Refractive index perturbations–unruh effect, hawking radiation or dynamical casimir effect? *arXiv preprint arXiv:1110.6064*, 2011.

[34] Petr O. Fedichev and Uwe R. Fischer. Gibbons-hawking effect in the sonic de sitter space-time of an expanding bose-einstein-condensed gas. *Phys. Rev. Lett.*, 91:240407, Dec 2003.

[35] Piyush Jain, Silke Weinfurtner, Matt Visser, and C. W. Gardiner. Analog model of a friedmann-robertson-walker universe in bose-einstein condensates: Application of the classical field method. *Phys. Rev. A*, 76:033616, Sep 2007.

[36] N Westerberg, S Cacciatori, F Belgiorno, F Dalla Piazza, and D Faccio. Experimental quantum cosmology in time-dependent optical media. *New Journal of Physics*, 16(7):075003, 2014.

[37] A. Prain, S. Vezzoli, N. Westerberg, T. Roger, and D. Faccio. Spontaneous photon production in time-dependent epsilon-near-zero materials. *Phys. Rev. Lett.*, 118:133904, Mar 2017.

[38] J. Dalibard, F. Gerbier, G. Juzeliūnas, and P. Öhberg. *Rev. Mod. Phys.*, 83:1523–1543, 2011.

[39] N. Goldman, G. Juzeliūnas, P. Öhberg, and I. B. Spielman. *Reports on Progress in Physics*, 77:126401, 2014.

[40] Michael V Berry. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 392:45–57, 1984.

[41] H. Theuer, R.G. Unanyan, C. Habscheid, K. Klein, and K. Bergmann. *Opt. Express*, 4:77, 1999.

[42] M. J. Edmonds, M. Valiente, G. Juzeliūnas, L. Santos, and P. Öhberg. *Phys. Rev. Lett.*, 110:085301, 2013.

[43] M. J. Edmonds, M. Valiente, and P. Öhberg. *EPL (Europhysics Letters)*, 110:36004, 2015.

[44] Jun-hui Zheng, Bo Xiong, Gediminas Juzeliūnas, and Daw-Wei Wang. *Phys. Rev. A*, 92:013604, 2015.

[45] Salvatore Butera, Manuel Valiente, and Patrik Öhberg. Quantized vortices in interacting gauge theories. *Journal of Physics B: Atomic, Molecular and Optical Physics*, 49(1):015304, 2015.

[46] Yvan Buggy and Patrik Öhberg. On the hydrodynamics of bose-condensed fluids subject to density-dependent gauge potentials. *arXiv preprint arXiv:1708.07712*, 2017.

[47] Nicholas David Birrell and Paul Charles William Davies. *Quantum fields in curved space*. Number 7. Cambridge university press, 1984.

[48] Gerald T. Moore. Quantum theory of the electromagnetic field in a variable?length one?dimensional cavity. *Journal of Mathematical Physics*, 11(9):2679–2691, 1970.

[49] Oskar Klein. Quantentheorie und fünfdimensionale relativitätstheorie. *Zeitschrift für Physik*, 37(12):895–906, Dec 1926.

[50] D Bailin and A Love. Kaluza-klein theories. *Reports on Progress in Physics*, 50(9):1087, 1987.

[51] Theodor Kaluza. Zum unitätsproblem der physik. *Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.)*, 1921(arXiv: 1803.08616):966–972, 1921.

[52] Francesco Marino. Acoustic black holes in a two-dimensional photon fluid. *Phys. Rev. A*, 78(6):063804, 2008.

[53] Julian Schwinger. On gauge invariance and vacuum polarization. *Phys. Rev.*, 82:664–679, Jun 1951.