Universal Relations in Coupled Electro-magneto-elasticity

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Abstract

In the present work, we develop a class of the coupled universal relations with the possible forms of electro-magneto-elastic (EME) deformation families in smart materials. In line with that, we adopt a classical continuum mechanics-based approach following the second law of thermodynamics. More precisely, we first formulate the deformation of an EME continua through the fundamental laws of physics with an amended form of energy function. This amended energy function successfully resolves the physical interpretation of the Maxwell stress tensor under large deformations.Next, we develop the EME coupling type of universal relations through a new inequality $\mathbf{Tb} - \mathbf{bT} \neq 0$ for a class of an EME material parallel to an equation $\mathbf{Tb} - \mathbf{bT} = 0$ for an isotropic elastic material existing in the literature. Wherein, $\mathbf{T}$ and $\mathbf{b}$ denote the total Cauchy stress tensor and left Cauchy-Green deformation tensor, respectively. Further, we propose the possible forms of EME deformation families in smart materials for some standard experimental arrangements. At last, we also apply the above findings to a magnetostriction phenomenon in order to check the practical feasibility of the same and a good agreement is achieved successfully.

Keywords: Smart Materials, Electro-magneto-elasticity, Coupled Universal Relations, Deformation Families.

1. Introduction

In general, an incompressible isotropic electro-magneto-elastic (EME) material is an important class of the man-made material. In line with that, the material is referred to as EME or smart material whose mechanical response is coupled with an electromagnetic effect [1, 2, 3]. Electrostriction and magnetostriction are the fundamental examples of electro-magneto-mechanical phenomenon shown by smart materials with an application of electromagnetic field [4, 5, 6, 7]. In current scenario, EME coupling effect got the current industrial attention due to inherent applications like electronic packaging, medical ultrasonic imaging, sensors, and actuators [8, 9, 10] etc. These applications recently demand the development of the constitutive relationships to study their EME deformations, effectively. The constitutive relationships of smart materials may be directly used to obtain the general solutions of EME coupled problems.

In particular, the coupling behavior of smart materials in presence of an electromagnetic field is a challenging task to describe accurately, because their elastic properties quickly get affected with external fields [11, 12]. But, this can be studied through the development of the coupling type of universal relations. In general, constitutive modeling provides the fundamental steps to model the exact EME coupled behaviors of smart materials up to a certain level. These fundamentals steps to model the EME coupled behaviors of smart materials may be seen in Figure 1. These steps are based on the fundamental laws of physics alongside with thermodynamics. Further, the constitutive modeling provides important relationships known as constitutive relations through which we may formulate the connections between the stress tensor to the deformation as well as an applied electromagnetic field. The constitutive modeling specifically tries to answer the question that in what manner the material will behave with an applied stimulus. In addition, some mathematical equations that hold for every material in a specified class are known as the universal relations. And, these relations also correlate the components of total Cauchy stress tensor with the deformation and applied electromagnetic field variables

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[13, 14, 15] similar to the constitutive relations. A key objective of the universal relations is to help the material experimentalist to specify the material in a particular constitutive class.

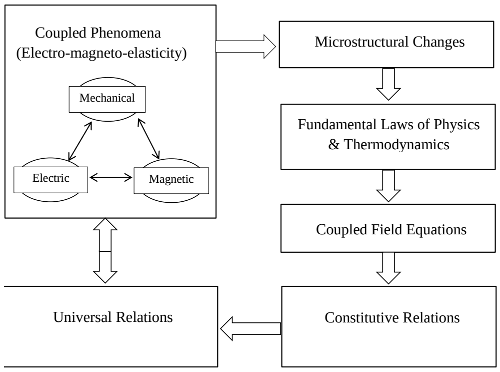

Figure 1: Fundamental constitutive modeling approach for the smart materials

In the literature, Van Suchtelen [9] proposed a new class of material behavior known as electro-magneto-elastic coupling effect that may be observed with the combination of piezo-electric and piezo-magnetic composites. Then, the micro-structural properties and the relationships between them for these class of smart materials, are studied by different authors [10, 16, 17]. Next, the additional coupled properties with the introduction of discontinuous reinforcement in these materials, are addressed by Bracke and Van Vliet [18], Nan [19], Benveniste [20] and Tong et al. [21]. Further, the first standardised approach to develop the constitutive model for smart materials were presented by Hayes and Knops [22], and most recently by Dorfmann et al. [23] and Bustamante et al. [24]. Specifically, very few number of short-notes [13, 25, 26, 27, 28] on the EME coupled universal relations are available in the literature for incompressible isotropic EME materials starting from the original incompressible isotropic elastic deformation work by Rivlin [25], Beatty [13, 26] and isotropic electro-elastic and magneto-elastic deformation works by Bustamante et al. [27] and Dorfmann et al. [28].

From the literature, we may conclude that limited research have been performed to model the coupled behavior of EME materials. However, most of them lack of simplicity in their formulation and a combined analysis that considers the general electro-magneto-elastic coupling effect lacks in the literature. Therefore, the research area of EME coupled constitutive modeling is broad and still open to researchers for the accurate prediction of the electro-magneto-mechanical coupling behavior of smart materials.

The primary aim of the current study is to develop the universal relations in coupled electro-magneto-elasticity. In line with that, we first introduce a new inequality $\mathbf{Tb} - \mathbf{bT} \neq 0$ on non-coaxiality of tensors $\mathbf{T}$ and $\mathbf{b}$ parallel to an equation $\mathbf{Tb} - \mathbf{bT} = 0$ on coaxiality one existing in the literature. Wherein, $\mathbf{T}$ and $\mathbf{b}$ denote the total Cauchy stress tensor and left Cauchy green deformation tensor for the considered electro-magneto-elastic deformation of continua, respectively. As a first step for the same, we adopt a unification of electro-magnetic theory and continuum mechanics in line with Kumar and Sarangi [29], Jordan [30], Bustamante et al. [27] and Dorfmann et al. [28]. In addition, a new class of the possible deformation families are also proposed through the formulated coupled universal relations. These deformation families provide the direct connections between the physical coupling components of electro-elastic and magneto-elastic coupling parameters of smart materials. And, these deformation families may also be counted as an additional class of controllable deformation families proposed by Beatty [26, 13] under external electromagnetic field in every homogeneous incompressible isotropic EME material.

2. Field equations

In current section, we summarize the electro-magneto-elastic deformation-based physical equations alongside with the principles of thermodynamics for smart materials [31].

2.1. Kinematics

Consider an electro-magneto-elastic deformable solid that occupy the material space β_0 in a stress-free configuration. Wherein, any material point is taken with respect to an arbitrarily chosen origin that may be represented by the position $\mathbf{R}$. During deformation in current configuration β, the same material point $\mathbf{R}$ occupies another position $\mathbf{r} = \kappa(\mathbf{R})$. Wherein, κ denotes the one-to-one type of deformation mapping. And, the corresponding deformation tensor $\mathbf{F}$ for an incompressible isotropic EME material is defined as follows

$$\mathbf{F} = \text{Grad}\kappa = \frac{\partial \mathbf{r}}{\partial \mathbf{R}}, \tag{1}$$

wherein Grad operator represents the gradient with respect to the reference position $\mathbf{R}$. In addition, we herein focus an incompressible isotropic electro-magneto-elastic deformation of continua through which the condition $J = \det\mathbf{F} = 1$ has been accounted throughout the paper.

2.2. *Electromagnetic field equations*

2.2.1. *Eularian form*

In the current configuration β, the body is assumed as free of currents, free of electric charge and time independent. The electro-magneto-elastic deformation field variables are the electric field $\mathbf{E}$, polarization density $\mathbf{P}$, electrical displacement $\mathbf{D}$ and magnetic field $\mathbf{H}$, magnetic induction $\mathbf{B}$, magnetization density $\mathbf{M}$. From the literature [32, 33], we have the expressions of the electrical displacement $\mathbf{D}$ and the magnetic displacement $\mathbf{B}$ in material media as follows

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}, \quad \mathbf{B} = \mu_0 [\mathbf{H} + \mathbf{M}], \tag{2}$$

wherein, ϵ_0 and μ_0 denote the electric permittivity and magnetic permeability of free space. Additionally, we herein specifically consider the constant electric field as well as constant magnetic field effects on the elastic behavior of an electro-magneto-elastic material. Therefore, the direct coupling effect between electric and magnetic fields that induced through each-other may be ignored. For quasi-static case, the electric field $\mathbf{E}$ and the electric displacement $\mathbf{D}$ successfully satisfy the Maxwell's equations [34] in the corresponding form

$$\text{curl}\mathbf{E} = 0, \quad \text{div}\mathbf{D} = 0, \quad \text{curl}\mathbf{H} = 0, \quad \text{div}\mathbf{B} = 0, \tag{3}$$

wherein curl and div operators represent the curl and divergence with respect to the position vector $\mathbf{r}$ in the current configuration β.

2.2.2. *Lagrangian form*

In the preceding sub-section, the expressions are presented in Eulerian form. However, we may also formulate the similar expressions in the reference configuration β_0. In line with that, the new parallel operators Div, Curl with the same physics are defined with respect to the reference variable $\mathbf{R}$. And, the electromagnetic field variables in Lagrangian form may be defined as $\mathbf{E}^l$, $\mathbf{D}^l$, $\mathbf{P}^l$ and $\mathbf{H}^l$, $\mathbf{B}^l$, $\mathbf{M}^l$. Next, the relations between these field variables for an incompressible isotropic electro-magneto-elastic deformation of continua in Eulerian as well as in Lagrangian form may be represented as follows

$$\begin{aligned} \mathbf{E}^l &= \mathbf{F}^T\mathbf{E}, \quad \mathbf{D}^l = \mathbf{F}^{-1}\mathbf{D}, \\ \mathbf{H}^l &= \mathbf{F}^T\mathbf{H}, \quad \mathbf{B}^l = \mathbf{F}^{-1}\mathbf{B}. \end{aligned} \tag{4}$$

The above relations (4) successfully assure that the equations (3) are equivalent to the given equations as follows

$$\begin{aligned} \text{Curl}\mathbf{E}^l &= 0, \quad \text{Div}\mathbf{D}^l = 0, \\ \text{Curl}\mathbf{H}^l &= 0, \quad \text{Div}\mathbf{B}^l = 0. \end{aligned} \tag{5}$$

Finally, in this section, we summarized the field equations of electromagnetism following the fundamental laws of physics related to an isotropic EME materials.

3. Constitutive theories

In current section, an EME deformation of a continua is formulated followed by the fundamental laws of thermodynamics. Additionally, a new concept of an amended form of energy density function [29] in electro-magneto-elasticity parallel to [23] in electro-elasticity is adopted, which profitably undertakes the physical insight of the Maxwell stress tensor under large deformations.

In general, the constitutive relationships for an isotropic EME material may be formulated following the theory of electro-magneto-elasticity with the help of EME independent field variables. In line with that, we start with the free energy density function represented as the function of the deformation tensor $\mathbf{F}$, electric field $\mathbf{E}$ and magnetic field $\mathbf{H}$, which depends on a number of factors. Now, if we take $\mathbf{F}$, $\mathbf{E}$ and $\mathbf{H}$ as the independent field variables in Eularian form, then the energy density function in an isothermal condition may be written as follows

$$\varphi = \varphi(\mathbf{F}, \mathbf{E}, \mathbf{H}). \tag{6}$$

Now, the rate of the free energy density function $\dot{\varphi}$ or, equivalently, the power density may also be obtained as follows

$$\dot{\varphi} = \frac{\partial \varphi}{\partial \mathbf{F}} : \dot{\mathbf{F}} + \frac{\partial \varphi}{\partial \mathbf{E}}.\dot{\mathbf{E}} + \frac{\partial \varphi}{\partial \mathbf{H}}.\dot{\mathbf{H}}, \tag{7}$$

wherein the superposed dot $\dot{()}$ in the corresponding variables represents the material time derivative. Next, we may consider $\mathbf{F}$, $\mathbf{E}^l$ and $\mathbf{H}^l$ as independent field variables for an EME material in Lagrangian form. Then, the free energy density function $\phi(\mathbf{F}, \mathbf{E}^l, \mathbf{H}^l)$ may be defined using relations (4) as follows

$$\phi(\mathbf{F}, \mathbf{E}^l, \mathbf{H}^l) = \varphi(\mathbf{F}, \mathbf{F}^{-T}\mathbf{E}^l, \mathbf{F}^{-T}\mathbf{H}^l). \tag{8}$$

Following [23, 35, 36], the total Cauchy stress tensor $\mathbf{T}$ from the Maxwell's idea for an EME material with the Cauchy stress tensor $\mathbf{S}$ may be directly recovered the expression given as follows

$$\begin{aligned} \mathbf{T} = \mathbf{S} + \mathbf{P} \otimes \mathbf{E} + \epsilon_0 [\mathbf{E} \otimes \mathbf{E} - \frac{1}{2}(\mathbf{E}.\mathbf{E})\mathbf{I}] \\ + \mu_0^{-1} [\mathbf{B} \otimes \mathbf{B} - \frac{1}{2}(\mathbf{B}.\mathbf{B})\mathbf{I}]. \end{aligned} \tag{9}$$

Herein, we may note that the above total Cauchy stress tensor (9) through the Maxwell's idea followed the stress superposition principle in an EME deformation of the continua. This explicit approach to recovering the total Cauchy stress tensor through the superposition of stresses may lead to a conceptual inaccuracy, especially in large deformations of an electro-magneto-elastic continua [37, 38]. Additionally, the stress superposition is physically irrelevant in terms of the definition of stress. This should be avoided in line with the use of Maxwell stress tensor due to its failure in providing insights to the problem. For the detailed discussions about the physical objectivity of the Maxwell stress tensor, one may refer to [37, 38] and references therein. In line with that, the above issue may be overcome successfully followed by a new concept of an amended energy density function Ω [29] in electro-magneto-elasticity parallel to [23] in electro-elasticity. This amended energy density function $\Omega = \Omega(\mathbf{F}, \mathbf{E}^l, \mathbf{H}^l)$ does not require use of the notion of Maxwell stress tensor within the material. And, the same $\Omega = \Omega(\mathbf{F}, \mathbf{E}^l, \mathbf{H}^l)$ successfully resolved the stress superposition issue. Additionally, this also systematically incorporated the Maxwell stress contribution for EME materials as follows

$$\Omega(\mathbf{F}, \mathbf{E}^l, \mathbf{H}^l) = \rho\phi(\mathbf{F}, \mathbf{E}^l, \mathbf{H}^l) - \frac{1}{2}\epsilon_0 \mathbf{E}^l \cdot (\mathbf{b}^{-1}\mathbf{E}^l) + \frac{1}{2}\mu_0^{-1}\mathbf{B}^l \cdot (\mathbf{b}\mathbf{B}^l). \tag{10}$$

Now, we may define the set of constitutive laws in generalized form through the amended energy density function directly for EME materials as follows

$$\mathbf{T} = -p\mathbf{I} + \mathbf{F}\frac{\partial\Omega}{\partial\mathbf{F}}, \quad \mathbf{D}^l = -\frac{\partial\Omega}{\partial\mathbf{E}^l}, \quad \mathbf{B}^l = -\frac{\partial\Omega}{\partial\mathbf{H}^l}, \tag{11}$$

wherein p denotes an indeterminate hydrostatic pressure arising from the incompressibility constraint associated with the Cauchy stress tensor.

Next, an isotropic EME material may also be defined as one for which the total energy density function Ω is an isotropic function of three tensors, namely, the left Cauchy Green deformation tensor $\mathbf{b}$, $\mathbf{E}^l \otimes \mathbf{E}^l$, and $\mathbf{H}^l \otimes \mathbf{H}^l$. Additionally, the energy function $\Omega(I_1, I_2 \ldots I_9)$ for an incompressible EME material depends on the following invariants given as follows

$$\begin{aligned} I_1 &= \mathrm{tr}\mathbf{b}, \quad I_2 = \frac{1}{2}[(\mathrm{tr}\mathbf{b})^2 - \mathrm{tr}(\mathbf{b}^2)], \quad I_3 = \det\mathbf{b}, \\ I_4 &= [\mathbf{E}^l \otimes \mathbf{E}^l] : \mathbf{I}, \quad I_5 = [\mathbf{E}^l \otimes \mathbf{E}^l] : \mathbf{b}^{-1}, \\ I_6 &= [\mathbf{E}^l \otimes \mathbf{E}^l] : \mathbf{b}^{-2}, \quad I_7 = [\mathbf{H}^l \otimes \mathbf{H}^l] : \mathbf{I}, \\ I_8 &= [\mathbf{H}^l \otimes \mathbf{H}^l] : \mathbf{b}^{-1}, \quad I_9 = [\mathbf{H}^l \otimes \mathbf{H}^l] : \mathbf{b}^{-2}. \end{aligned} \tag{12}$$

Herein, we may also include an additional I_{10}^{th} invariant representing the direct coupling effect between electric and magnetic field. But, we consider the constant electric field as well as constant magnetic field for the sake of current simplification. From the relations (10), (11) and the definitions of the invariants in (12), the explicit expressions of the total Cauchy stress tensor $\mathbf{T}$, electric displacement vector $\mathbf{D}$ and magnetic displacement vector $\mathbf{B}$ may be expressed as follows

$$\begin{aligned} \mathbf{T} &= -p\mathbf{I} + 2\Omega_1\mathbf{b} + 2\Omega_2[I_1\mathbf{b} - \mathbf{b}^2] - 2\Omega_5\mathbf{E}\otimes\mathbf{E} \\ &\quad -2\Omega_6[\mathbf{b}^{-1}\mathbf{E}\otimes\mathbf{E} + \mathbf{E}\otimes\mathbf{b}^{-1}\mathbf{E}] + 2\Omega_8\mathbf{bH}\otimes\mathbf{bH} \\ &\quad +2\Omega_9[\mathbf{bH}\otimes\mathbf{b}^2\mathbf{H} + \mathbf{b}^2\mathbf{H}\otimes\mathbf{bH}], \\ \mathbf{D} &= -2[\Omega_4\mathbf{b} + \Omega_5\mathbf{I} + \Omega_6\mathbf{b}^{-1}]\mathbf{E}, \\ \mathbf{B} &= -2[\Omega_7\mathbf{b} + \Omega_8\mathbf{b}^2 + \Omega_9\mathbf{b}^3]\mathbf{H}, \end{aligned} \tag{13}$$

wherein the notation Ω_i is defined by $\Omega_i = \dfrac{\partial\Omega}{\partial I_i}$, $i = 1, 2, 3.., 9$.

Finally, in this section, we successfully formulated the constitutive theories for an incompressible isotropic EME material deformation under external electromagnetic field.

4. Universal relations

In current section, we develop the EME coupling type of universal relations through a new inequality $\mathbf{Tb} - \mathbf{bT} \neq 0$ for a class of an EME material parallel to an equation $\mathbf{Tb} - \mathbf{bT} = 0$ for a class of an elastic material existing in the literature.

In general, a class of universal relations for isotropic electro-magneto-elastic coupling are fully depend on the total Cauchy stress tensor $\mathbf{T}$ and the left Cauchy Green tensor $\mathbf{b}$. The symmetry and skew-symmetry of the total Cauchy stress tensor decide the different class of universal relations for the smart materials. Now, introducing the suitable notations for simplicity in the constitutive relations (13)

$$\begin{aligned} &\gamma_1 = 2(\Omega_1 + \Omega_2 I_1), \quad \gamma_2 = -2\Omega_2, \quad \gamma_4 = -2\Omega_4, \\ &\gamma_5 = -2\Omega_5, \quad \gamma_6 = -2\Omega_6, \\ &\gamma_7 = 2\Omega_7, \quad \gamma_8 = 2\Omega_8, \quad \gamma_9 = 2\Omega_9, \end{aligned} \tag{14}$$

so that the relations (13) may be rewritten as follows

$$\begin{aligned} \mathbf{T} &= -p\mathbf{I} + \gamma_1\mathbf{b} + \gamma_2\mathbf{b}^2 + \gamma_5\mathbf{E}\otimes\mathbf{E} + \gamma_6[\mathbf{b}^{-1}\mathbf{E}\otimes\mathbf{E} \\ &\quad +\mathbf{E}\otimes\mathbf{b}^{-1}\mathbf{E}] + \gamma_8\mathbf{bH}\otimes\mathbf{bH} + \gamma_9[\mathbf{bH}\otimes\mathbf{b}^2\mathbf{H} \\ &\quad +\mathbf{b}^2\mathbf{H}\otimes\mathbf{bH}], \\ \mathbf{D} &= \gamma_4\mathbf{bE} + \gamma_5\mathbf{E} + \gamma_6\mathbf{b}^{-1}\mathbf{E}, \\ \mathbf{B} &= -\gamma_7\mathbf{bH} - \gamma_8\mathbf{b}^2\mathbf{H} - \gamma_9\mathbf{b}^3\mathbf{H}. \end{aligned} \tag{15}$$

Beatty [13] proposed a special class of universal relations for pure isotropic elastic materials. In this case, both of the tensors $\mathbf{T}$ and $\mathbf{b}$ denote the total Cauchy stress tensor and left Cauchy green deformation tensor, respectively are symmetric in nature. And, there exist an important tensorial relation in the form of $\mathbf{Tb} - \mathbf{bT} = 0$ that helps to develop the universal relations in pure elasticity, is also symmetric in nature. However in isotropic electro-magneto-elasticity case, this tensorial relation $(\mathbf{Tb} - \mathbf{bT})$ term is no longer be symmetric as well as $\mathbf{Tb} - \mathbf{bT} = 0$. Now, this $(\mathbf{Tb} - \mathbf{bT})$ term exists in inequality form as $\mathbf{Tb} - \mathbf{bT} \neq 0$. Therefore, a new class of universal relations must exist for an isotropic electro-magneto-elastic material that will help to model the coupled behavior in electro-magneto-elasticity.

A possible structure to generate the coupled universal relations for an isotropic EME material is to work systematically with this tensorial relation $\mathbf{Tb} - \mathbf{bT} \neq 0$. We then calculate an $\mathbf{Tb} - \mathbf{bT}$ expression for an EME deformation of continua given as follows

$$\begin{aligned}\mathbf{Tb} - \mathbf{bT} = \gamma_5[\mathbf{E} \otimes \mathbf{bE} - \mathbf{bE} \otimes \mathbf{E}] + \gamma_6[\mathbf{b}^{-1}\mathbf{E} \otimes \mathbf{bE} \\ -\mathbf{bE} \otimes \mathbf{b}^{-1}\mathbf{E}] + \gamma_8[\mathbf{bH} \otimes \mathbf{b}^2\mathbf{H} - \mathbf{b}^2\mathbf{H} \otimes \mathbf{bH}] \\ +\gamma_9[\mathbf{bH} \otimes \mathbf{b}^3\mathbf{H} - \mathbf{b}^3\mathbf{H} \otimes \mathbf{bH}].\end{aligned} \tag{16}$$

In addition, due to skew-symmetry of the tensorial term $\mathbf{Tb} - \mathbf{bT}$ there exist an axial-vector $(\mathbf{Tb} - \mathbf{bT})_\times$ from which we may obtain the associated coupling type of universal relations for an isotropic EME material. Therefore, the associated axial-vector $(\mathbf{Tb} - \mathbf{bT})_\times$ from equation (16) in electro-magneto-elasticity is given as follows

$$\begin{aligned}(\mathbf{Tb} - \mathbf{bT})_\times = (\mathbf{bE}) \times (\gamma_5\mathbf{E} + \gamma_6\mathbf{b}^{-1}\mathbf{E}) \\ +(\gamma_8\mathbf{b}^2\mathbf{H} + \gamma_9\mathbf{b}^3\mathbf{H}) \times (\mathbf{bH}).\end{aligned} \tag{17}$$

Now, we obtain the coupled universal relations through which the coupling phenomena of the smart materials may be defined as follows

$$\begin{aligned}(\mathbf{Tb} - \mathbf{bT})_\times \cdot [\mathbf{bE}] = [(\gamma_8\mathbf{b}^2\mathbf{H} + \gamma_9\mathbf{b}^3\mathbf{H}) \times (\mathbf{bH})] \\ \cdot[\mathbf{bE}], \\ (\mathbf{Tb} - \mathbf{bT})_\times \cdot [\mathbf{bH}] = [(\mathbf{bE}) \times (\gamma_5\mathbf{E} + \gamma_6\mathbf{b}^{-1}\mathbf{E})] \\ \cdot[\mathbf{bH}].\end{aligned} \tag{18}$$

The above coupled universal relations (20) in isotropic electro-magneto-elasticity is quite different in terms of suitable coupling from the existing universal relations in isotropic electro-elasticity [27], magneto-elasticity [28] and pure-elasticity [13] cases, respectively. Beatty [13] explored the rule of tensorial expression $\mathbf{Tb} - \mathbf{bT}$ on the coaxiality of the tensors $\mathbf{T}$ and $\mathbf{b}$ for isotropic elastic materials. Further, the same is also used by Bustamante et al. [27] and Dorfmann et al. [28] in order to deduce the universal relations for isotropic electro-elastic and magneto-elastic materials, respectively. Now, we also follow the same for isotropic coupled electro-magneto-elastic materials. But, the coaxiality of the tensors $\mathbf{T}$ and $\mathbf{b}$ for the current case vanishes.

The coupled universal relations (18) are able to provide the guidelines to the material scientist, at the moment of proposing a constitutive equation for an EME material. Additionally, the obtained coupled universal relations (18) may also be theoretically validated by considering the corresponding field variable as zero with the universal relations proposed by Bustamante et al. [27] and Dorfmann et al. [28] for isotropic electro-elastic and magneto-elastic materials, respectively. Similarly in isotropic elasticity, the identical universal relations with Beatty [13] may also be recovered by considering both as zero electromagnetic field.

For simplification, if we represent the axial-vector as $(\mathbf{Tb} - \mathbf{bT})_\times = \mathbf{A} = A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + A_3\mathbf{e}_3$ from equation (17) and considering two physical coupling components defined as follows

$$\begin{aligned}\mathbf{Z} = [(\mathbf{bE}) \times (\gamma_5\mathbf{E} + \gamma_6\mathbf{b}^{-1}\mathbf{E})], \\ \mathbf{Q} = [(\gamma_8\mathbf{b}^2\mathbf{H} + \gamma_9\mathbf{b}^3\mathbf{H}) \times (\mathbf{bH})],\end{aligned} \tag{19}$$

wherein $\mathbf{Z}$ and $\mathbf{Q}$ are considered as two electro-elastic and magneto-elastic coupling parameters for an incompressible isotropic EME material, respectively. And, these coupling parameters physically define the corresponding field forces contribution for an electro-magneto-elastic deformation of continua. Then, the developed coupled universal relation (18) may be rewritten as follows

$$\mathbf{A} \cdot [\mathbf{bE}] = \mathbf{Q} \cdot [\mathbf{bE}], \quad \mathbf{A} \cdot [\mathbf{bH}] = \mathbf{Z} \cdot [\mathbf{bH}]. \tag{20}$$

Finally, in this section, we successfully developed the coupled universal relations for an incompressible isotropic EME deformable solid to model the coupling behavior of EME materials.

5. Electro-magneto-elastic deformation families

In current section, we propose the possible electro-magneto-elastic deformation families based on the developed coupled universal relations (18).

In order to define the deformation families for an electro-elastic deformation of continua, we consider a

Beatty [13] type deformation given as follows

$$\mathbf{b} = \begin{bmatrix} b_{11} & b_{12} & 0 \\ b_{21} & b_{22} & 0 \\ 0 & 0 & b_{33} \end{bmatrix}, b_{12} = b_{21}, \quad \mathbf{T} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix},$$

and

$$\mathbf{E} = \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} H_1 \\ H_2 \\ H_3 \end{bmatrix}. \tag{21}$$

Now, the coupled universal relation (18) for an above Beatty [13] type deformation may be rewritten in a simplified form given as follows

$$\begin{aligned} (Q_1 - A_1)(b_{11}E_1 + b_{12}E_2) + (Q_2 - A_2)(b_{21}E_1 \\ + b_{22}E_2) + (Q_3 - A_3)(b_{33}E_3) = 0, \\ (Q_1 - A_1)(b_{11}H_1 + b_{12}H_2) + (Q_2 - A_2)(b_{21}H_1 \\ + b_{22}H_2) + (Q_3 - A_3)(b_{33}H_3) = 0, \end{aligned} \tag{22}$$

wherein (Z_1, Z_2, Z_3) and (Q_1, Q_2, Q_3) are the associated components of two electro-elastic and magneto-elastic coupling parameters, respectively. Herein, the relation (22) represents the second form of the obtained coupled universal relation (18) for the Beatty [13] type deformation. These relations (18) may provide a new class of the controllable deformation families. These deformation families may also be counted as an additional class of controllable deformation families proposed by Beatty [13] under external electromagnetic field for an incompressible isotropic EME material. Now, considering a special case given as

$$\mathbf{E} = \begin{bmatrix} 0 \\ E_0 \\ 0 \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} 0 \\ 0 \\ H_0 \end{bmatrix},$$

for which the above relation (22) may be expressed as follows

$$\frac{b_{22}}{b_{12}} = -\frac{Q_1 - A_1}{Q_2 - A_2}, \quad Z_3 = A_3. \tag{23}$$

Next, for the faithful classification of the deformation families, we consider all the possible special cases based on the standard experimental arrangements. Therefore, a new class of incompressible isotropic EME deformation families may be obtained with an application electromagnetic field given as follows

5.1. *Class I: Electric field E and Magnetic field H are in Parallel combinations*

Family 1.

$$\left(\mathbf{E} = \begin{bmatrix} E_1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{H} = \begin{bmatrix} H_1 \\ 0 \\ 0 \end{bmatrix}\right), \left(\mathbf{E} = \begin{bmatrix} 0 \\ E_2 \\ 0 \end{bmatrix}, \mathbf{H} = \begin{bmatrix} 0 \\ H_2 \\ 0 \end{bmatrix}\right)$$

In this family, both of the cases follow similar type of coupled universal relation obtained from the relation (22) given as follows

$$\frac{Q_1 - A_1}{Q_2 - A_2} = \frac{Z_1 - A_1}{Z_2 - A_2}. \tag{24}$$

Family 2.

$$\left(\mathbf{E} = \begin{bmatrix} 0 \\ 0 \\ E_3 \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} 0 \\ 0 \\ H_3 \end{bmatrix}\right)$$

This case follow the coupled universal relation obtained from the relation (22 given as follows

$$A_3 = Z_3 = Q_3. \tag{25}$$

5.2. *Class II: Electric field E and Magnetic field H are in orthogonal combinations*

Family 3.

$$\left(\mathbf{E} = \begin{bmatrix} E_1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{H} = \begin{bmatrix} 0 \\ H_2 \\ 0 \end{bmatrix}\right), \left(\mathbf{E} = \begin{bmatrix} 0 \\ E_2 \\ 0 \end{bmatrix} \mathbf{H} = \begin{bmatrix} H_1 \\ 0 \\ 0 \end{bmatrix}\right)$$

In this family, both of the cases follow similar type of coupled universal relation obtained from the relation (22) with $b_{12} = b_{21}$ given as follows

$$\frac{b_{22}}{b_{11}} = \frac{(Z_1 - A_1)(Q_1 - A_1)}{(Z_2 - A_2)(Q_2 - A_2)}. \tag{26}$$

Family 4.

$$\left(\mathbf{E} = \begin{bmatrix} E_1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} 0 \\ 0 \\ H_3 \end{bmatrix}\right), \left(\mathbf{E} = \begin{bmatrix} 0 \\ E_2 \\ 0 \end{bmatrix}, \mathbf{H} = \begin{bmatrix} 0 \\ 0 \\ H_3 \end{bmatrix}\right)$$

Herein, both of the cases follow similar type of coupled universal relation obtained from the relation (22) with $b_{12}b_{21} = b_{11}b_{22}$ given as follows

$$A_3 = Z_3. \tag{27}$$

Family 5.

$$\left(\mathbf{E} = \begin{bmatrix} 0 \\ 0 \\ E_3 \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} H_1 \\ 0 \\ 0 \end{bmatrix}\right), \left(\mathbf{E} = \begin{bmatrix} 0 \\ 0 \\ E_3 \end{bmatrix}, \mathbf{H} = \begin{bmatrix} 0 \\ H_2 \\ 0 \end{bmatrix}\right)$$

In this family, both of the cases follow similar type of coupled universal relation obtained from the relation (22) with $b_{12}b_{21} = b_{11}b_{22}$ given as follows

$$A_3 = Q_3. \tag{28}$$

Finally, we note that the above deformation families provide the direct connections between the physical coupling components of the electro-magneto-elastic materials through which the materialist may easily specify the exact class of the smart materials. In general, if any particular class of experimental data with similar assumptions follow these formulated coupled universal relations upto a certain level, then that type of material falls under this sub-class of the smart materials. Therefore, the developed deformation families and their corresponding simplified coupled universal (22) relations may provide an important step to model the coupling behavior of smart materials.

6. Application to a magnetostriction phenomenon

In current section, we apply the developed coupled universal relations (18) to determine the possible electro-elastic and magneto-elastic coupled forces generated on the system of an incompressible isotropic EME material. Additionally, we also checked the possible deformation family for the same.

For that, we may consider a magnetostriction phenomenon, which was experimentally analysed by Lopez-Lopez et al. [39]. In that experiment, a circular plate-plate geometry of the rheometer shown in Figure 2 was used to characterize the MR (Magneto Rheological) properties of the suspensions. For that, a longitudinal magnetic field was applied in the rheometer. And, an application of magnetic field generated the axial force developed by a MR fluid.

We begin our theoretical analysis of the same circular plate geometry rheometer as shown in Figure 2 with an applied magnetic field. And, an additional electric field application is also considered in the same system in order to generalize the same problem. However, the similar problem was also theoretically analysed by us in our previous publication [29]. Therein [29], we followed the fundamental equilibrium force balancing in order to obtain the axial force calculation. For the deformation analysis of system geometry, the coordinate any point of the system in the reference configuration β_0 may be adopted as (Θ, Y, R). Herein, we prefer (Θ, Y, R) than (R, Θ, Y) in order to make consistency in the form of deformation gradient tensor **b** similar to what we have considered in equation (21). And, with an application of electric field E_0 as well as magnetic field H_0 in same direction, that point gets a new position (θ, y, r) in the current configuration β. The cross-section of the rheometer in the reference configuration is defined as $0 \leq \Theta \leq 2\pi$ and $0 \leq R \leq A$, wherein A is the radius of the circular cross-section. The deformation mapping for the constrained axial-length may be expressed as follows

$$\theta = \Theta + \psi Y, \quad y = Y, \quad r = R, \tag{29}$$

wherein ψ is the finite torsional twist per unit length

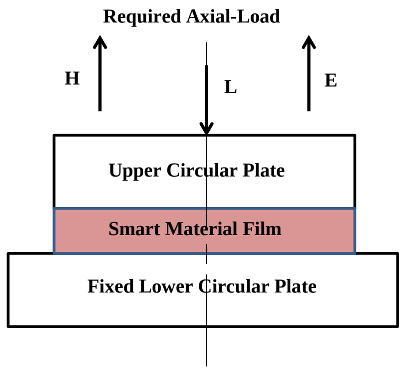

Figure 2: Schematic experimental set-up of the circular plate-plate geometry of the rheometer for magnetostriction [29].

of a cylindrical tube. The associated left Cauchy-green deformation gradient tensor **b** and the applied electromagnetic field vectors for the rheometer are expressed as follows

$$\mathbf{b} = \begin{bmatrix} 1 + r^2\psi^2 & r\psi & 0 \\ r\psi & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} :\Leftrightarrow \begin{bmatrix} b_{11} & b_{12} & 0 \\ b_{21} & b_{22} & 0 \\ 0 & 0 & b_{33} \end{bmatrix},$$

$$\mathbf{E} = \begin{bmatrix} 0 \\ E_0 \\ 0 \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} 0 \\ H_0 \\ 0 \end{bmatrix}. \tag{30}$$

Now, in order to obtain the simplified form of universal relation parallel to (22) and Family 2 for the considered shear deformation case (30), we first try to obtain the universal relation in general sense. For that, the coupled universal relations (18) or (20) is represented in term of the general components of the axial-vector **A**, coupling parameters vectors **Z**, **Q** and left Cauchy green deformation tensor **b** as follows

$$\begin{aligned} A_r b_{12} + A_\theta b_{22} &= Q_r b_{12} + Q_\theta b_{22} \\ &= Z_r b_{12} + Z_\theta b_{22}. \end{aligned} \tag{31}$$

The above equation (31) represents the universal mathematical statement for all type of shear deformations similar to (30) irrespective of any material model consideration for an incompressible isotropic EME material class.

6.1. Validation of the developed universal relation

Herein, we may successively check the validity of the developed coupled universal relation (31) by obtaining the corresponding expressions for the considered deformation (30) of a magnetostriction phenomenon experimentally analysed by Lopez-Lopez et al. [39]. And, the corresponding expressions of the axial-vector $\mathbf{A}$, coupling parameters vectors $\mathbf{Z}$, $\mathbf{Q}$ for the considered shear deformation case (30) irrespective of any material model consideration are given as follows

$$\begin{aligned}\mathbf{A} = [(-2\Omega_5 b_{12} - 2\Omega_6 b_{12} b'_{22} + 2\Omega_6 b_{22} b'_{12}) {E_0}^2 \\ +(2\Omega_8 b_{22} b''_{12} - 2\Omega_8 b_{12} b''_{22} + 2\Omega_9 b_{22} b'''_{12} \\ -2\Omega_9 b_{12} b'''_{22}) {H_0}^2]\mathbf{e}_y, \\ \mathbf{Z} = [(-2\Omega_5 b_{12} - 2\Omega_6 b_{12} b'_{22} + 2\Omega_6 b_{22} b'_{12}) {E_0}^2]\mathbf{e}_y, \\ \mathbf{Q} = [(2\Omega_8 b_{22} b''_{12} - 2\Omega_8 b_{12} b''_{22} + 2\Omega_9 b_{22} b'''_{12} \\ -2\Omega_9 b_{12} b'''_{22}) {H_0}^2]\mathbf{e}_y,\end{aligned} \tag{32}$$

wherein $()'$, $()''$ and $()'''$ terms represent the corresponding components of $\mathbf{b}^{-1}$, $\mathbf{b}^2$ and $\mathbf{b}^3$ tensors, respectively. Therefore, the above relations (32) successfully validate the developed coupled universal relation (31) for the considered deformation (30) of a magnetostriction phenomenon experimentally analysed by Lopez-Lopez et al. [39]. This experimental magnetostriction phenomenon successfully allows the feasibility of the existence of the developed coupled universal relations (18) or (20).

6.2. Coupled force deduction

From the axial-vector expression $(32)_1$, we note that axial-vector is existing in normal direction of the rheometer plates in Figure 2. Wherein, the normal force will act in the plates due to the MR fluid deformation with an application of magnetic field. This physically shows that the stresses due to field applications must exist in the normal direction, which may also be verified experimentally analyzed magnetostriction phenomenon is shown in Figure 2. This is the main use of axial-vector, which can provide direct information of the tractions due to the applied field easily. And, we may determine that generated coupled forces due to electric field as well as magnetic field application for a finite arc deformation of smart materiel film between the plates in axial-direction from the expressions of $\mathbf{Z}$ and $\mathbf{Q}$, respectively. This shows an alternative approach to deduce the coupled forces with an application of fields as compared to our previous forces determination [29].

To deduce the coupled force expression, we first need to obtain the expressions of Ω_i in the axial-vector as well as in the coupling vectors. For that, we require any invariant form of amended energy density function (10). In line with that, we use an amended energy density function proposed by us in our article [29] given as follows

$$\begin{aligned}\Omega = C_1(I_1 - 3) + C_2(I_2 - 3) + \frac{\epsilon_0}{2}(C_3 I_4 + C_4 I_5) \\ + \frac{\mu_0}{2}(C_5 I_7 + C_6 I_8),\end{aligned} \tag{33}$$

wherein $C_1, C_2, ..., C_6$ are the material constant parameters. By substituting this invariant form of amended energy density function (33), we may rewrite the previous relations (32) as follows

$$\begin{aligned}\mathbf{A} = [(-C_4\epsilon_0 b_{12}){E_0}^2 + C_6\mu_0(b_{22}b''_{12} - b_{12}b''_{22}){H_0}^2]\mathbf{e}_y, \\ \mathbf{Z} = [(-C_4\epsilon_0 b_{12}){E_0}^2]\mathbf{e}_y, \\ \mathbf{Q} = [C_6\mu_0(b_{22}b''_{12} - b_{12}b''_{22}){H_0}^2]\mathbf{e}_y,\end{aligned} \tag{34}$$

wherein again $()'$, $()''$ and $()'''$ terms represent the corresponding components of $\mathbf{b}^{-1}$, $\mathbf{b}^2$ and $\mathbf{b}^3$ tensors, respectively. For the numerical validation with the Lopez-Lopez et al. [39] experimental data, we assume $E_0 = 0$. And, the theoretical generated force using the above relations (34) and (30) through the magnetic field application only is given as follows

$$Q_y = C_6\mu_0(r\psi){H_0}^2. \tag{35}$$

For a particular arc length $(r\psi)$, the above theoretical generated force expression (35) is equivalent to the load $L = C{H_0}^2$ required to maintain the reference configuration. To get the best fit value of the constant C, we have $C = 0.000041$ that is obtained by using the least square method. For this obtained value of $C = 0.000041$, the theoretical load expression Q_y shows a good agreement with the Lopez-Lopez et al. [39] experimental data shown in Figure 3.

Finally, in this section, we successfully applied the developed coupled universal relations to a magnetostriction phenomenon. Following that, we successfully obtained the generated coupled forces with the corresponding field application. Additionally, we also checked the practical feasibility of the above findings to a magnetostriction phenomenon and a good agreement is achieved successfully.

7. Concluding remarks

In the present paper, the modeling of coupled electro-magneto-elastic (EME) behavior of smart materials is

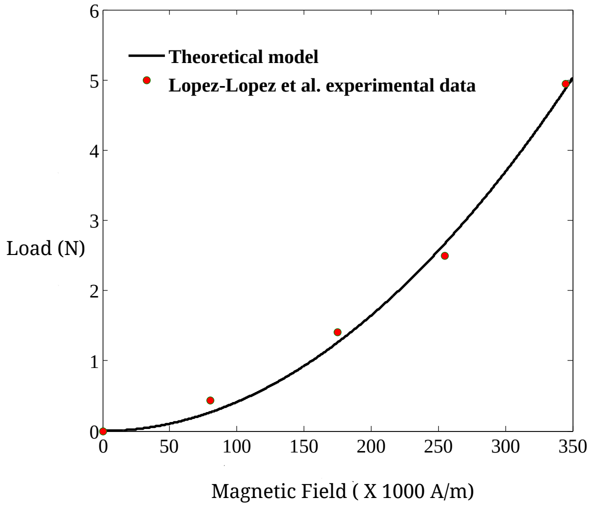

Figure 3: Comparison of the generated axial-load Q_1 with Lopez-Lopez et al. [39] experimental data.

formulated through the finite deformation of EME continua. A new concept of amended energy density function [29, 23] that depends on nine invariants (12) is adopted to obtain the total Cauchy stress tensor, electric displacement vector and the magnetic induction vector for an EME material. The developed coupled universal relations (18), (22) and their corresponding electro-magneto-elastic deformation families establish a direct connection among the stress, deformation and applied fields in smart materials. Specifically, the present study generalizes the continuum concept to coupled electro-magneto-elasticity and the primary works by Rivlin [25], Beatty [26, 13] on elastic deformation and Bustamante et al. [27], Dorfmann et al. [28] on electro-elastic and magneto-elastic deformations. Their works are obtainable from our developed equations as a special case.

The coupled universal relations are developed through a new inequality $\mathbf{Tb} - \mathbf{bT} \neq 0$ rather than an equation $\mathbf{Tb} - \mathbf{bT} = 0$ existing in the literature. In addition, a special class of EME deformation families is also obtained through the formulated coupled universal relations. Herein, the proposed deformation families may be counted as an additional class of controllable deformation families proposed by Beatty [13] under an electromagnetic field. In general, the mathematical relationship among the axial-vector, stress field, deformation field, and externally applied field may directly help the material experimentalist to classify the coupling behavior of smart materials.

At last, we, unfortunately, do not have the parallel electro-magneto-elastic deformation-based experimental results that could lend support to the proposed coupled universal relation-based conjecture. Therefore, no comparison of the results of the proposed EME deformation-based study with the experimental data is currently possible. This is plainly an issue that invites additional study. However, the validity of coupled universal relations (18) is successfully confirmed by obtaining the existing theoretical results in the literature as a special case of the developed theory. Additionally, by applying the above findings to an existing magnetostriction phenomenon, we checked the practical feasibility of the same and a good agreement is achieved successfully.

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