On endogenous formation of price expectations

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JEL classification:
D53
D81
D82
D84

Keywords:
Perfect foresight
Price expectations
Endogenous borrowing constraints

A B S T R A C T

We study a two-period exchange economy with complete financial markets and endogenous borrowing constraints. Contrary to *perfect foresight* paradigm, we assume that agents are heterogeneous in their ability to forecast future prices. We introduce a new equilibrium concept, called *informationally constrained equilibrium*, where the formation of price expectations is endogenous and reflects the revelation of information from observing bounds on liabilities designed to ensure solvency at any contingency. We prove that, under standard assumptions, an equilibrium always exists and we characterize the degree of information revealed by the endogenous debt limits.

1. Introduction

A remarkable feature of the Arrow and Debreu (1954) model is that it imposes weak informational requirements. Agents are assumed to know only their own characteristics, no assumption is imposed on what they know or believe about others' fundamentals and beliefs. Market prices convey all relevant information and guide agents to take their decisions.

While the informational requirements of the model are minimal, its formal extension to uncertain dynamic environments is burdened with a formidable presence of necessary markets for the delivery of goods at all conceivable states. Arrow (1953) and Radner (1972) showed how the allocation of an Arrow-Debreu equilibrium can be implemented with a more realistic (sequential) structure of commodity and financial markets that permits the delivery of contingent money at future events. However, the transition to a sequential setting is not without cost. It implies that agents make decisions based on expectations of future prices for commodities and contracts, and these decisions and expectations are taken to be collectively compatible. Although agents need not agree on the joint probability distribution of future events, they base their decisions anticipating that a unique price vector will prevail at every contingency. That is, agents have degenerate and common

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[1] Yiannis Vailakis is grateful to Filipe Martins-da-Rocha for several conversations, work and interaction. We thank Yehuda John Levy for helpful comments on technical issues. Yiannis Vailakis acknowledges the financial support of an ERC starting grant (FP7, Ideas specific program, Project 240983 DCFM) and of two ANR research grants (Projects Novo Tempus and FIRE); also the hospitality of Kobe University (RIEB) and Université Paris-Dauphine at early stages of the project.

price expectations. In addition, expectations are fulfilled at equilibrium. A set of degenerate, common and self-fulfilling expectations, together with market clearing, defines a *perfect foresight* equilibrium (Radner (1972)).

Perfect foresight rules out situations where economic variables create informational externalities. Before taking their decisions, agents usually observe private signals that are correlated with states. One expects that the less informed agents will try to infer information from observing variables influenced by the behaviour of all (in particular, the more informed) agents. Radner (1979) (see also Allen (1981a), Grossman (1981)) proposed an equilibrium concept, *rational expectations*, that takes into account the potential informational feedback from market clearing prices. Equilibrium prices are fully revealing: when agents use all available information (private signals and the information revealed by prices) to calculate their demands, market clearing leads precisely to those prices.

Both equilibrium concepts impose strong requirements on the formation of price expectations. Rationality no longer implies that individual behaviour is determined by the maximization of a well ordered utility function, but also that agents know the map from states (perfect foresight) or signals (rational expectations) to equilibrium prices. In either case, this map summarises a huge amount of information. As stressed by Radner (1982) (see also Dutta and Morris (1997)), these knowledge requirements are hard to justify. Decision makers have to fully understand the economic environment and have an implausible capacity for computation and communication.

One way to defend these concepts is to assume that agents draw on past experience to predict the evolution of future prices. Although this may be true in a stable world where similar events occur regularly, it is difficult to accept it in environments where agents are exposed to new and unfamiliar events. In addition, learning does not always provide a satisfactory foundation for either hypothesis. Rational learning requires that agents' prior beliefs assign positive probability on the true model for the entire stochastic process, not just for what would happen after beliefs converged (see Kalai and Lehrer (1993) and Blume and Easley (1998)). When agents base their decisions to statistical models that are misspecified, adaptive learning converges to *self-confirming* equilibria, not to rational expectations equilibria, and the differences between these two concepts might be substantial (see Sargent (2008)). There is also a vast literature (see Wagener (2014) and the references therein) exploring whether perfect foresight and rational expectations provide a good description in a number of expectation formation experiments. The findings suggest that agents' expectations coordinate quickly but not necessarily on the rational expectations value. Moreover, even in situations where the hypotheses perform well, it is more likely that coordination on rational values is brought about by the institutional structure of the market rather than the rationality of agents.

The paper is an attempt to show that institutional restrictions can, to some extent, weaken the coordination requirements underpinning the perfect foresight or rational expectations equilibrium concept. It explores how price expectations are formed in a sequential (two-period) model with endogenous borrowing constraints. Contrary to perfect foresight paradigm, we assume that agents are heterogeneous in their ability to forecast future prices. Specifically, we study an economy with two types of agents. Agents of the first type can perfectly anticipate future prices as in Radner (1972). We refer to those agents as *sophisticated*. Agents of the second type cannot perfectly foresee future prices. We refer to those agents as *minimally rational*.

Minimally rational agents infer information from observing variables that are influenced by the behaviour of all market participants. This might look similar to rational expectations (Radner (1979)), but differentiates from it in a crucial aspect. The transmission of information in our setting is not linked to the observation of prices, but rather relates to endogenous bounds on short-sales. Such a mechanism has the advantage of being simpler and costless (in terms of what agents have to know) as opposed to the map that associates equilibrium prices to signals. In the spirit of Norman (2015), the coordination of price expectations reflects the agents' hypothesis regarding the restrictions they face in financial markets. This is in line with experimental evidence (Wagener (2014)) suggesting that agents are rational in an operational sense, trying to tie their prediction problem to the information revealed by the institutional arrangements that legislate and manage their activities.

Lack of perfect foresight implies that minimally rational agents might be insolvent at some contingencies and have to default on their liabilities. The temporary equilibrium literature (Grandmont (1977)) recognized this issue long ago and argued for the design of explicit default and bankruptcy rules. For instance, in Green (1974), bankrupt agents are allocated the minimum level of consumption. As pointed out by Eichberger (1989), such rules raise additional complications since they lead to budget sets, and consequently to demand correspondences, that are non-convex.

To overcome this issue, we follow the literature on limited commitment (Kehoe and Levine (1993), Zhang (1997) and Alvarez and Jermann (2000)) and impose bounds on short sales designed to preclude default at equilibrium. Rationality in this setting means that all agents know that the bounds ensure solvency according to future prevailing prices. That is, observing the bounds, minimally rational agents are capable of inferring a set of prices that induce non-negative wealth at all contingencies. They basically know the map from prices to bounds compatible with no bankruptcy. If an equilibrium exists, then equilibrium prices will be contained in this set. However, the signal (bounds on short sales) does not allow to discern the equilibrium prices among the other prices in the set. Therefore, expectations may not be common and degenerate. Nevertheless, they are formed endogenously and, as in rational expectations, reflect all available information.

We propose an equilibrium concept, *informationally constrained equilibrium*, that makes precise the way price expectations become compatible with the observation of endogenous debt limits. We show that, under standard assumptions, such an equilibrium always exists and we characterize the degree of the revealed information. In that respect, the analysis relates to a vast literature dealing with the existence and properties of non-revealing or partially revealing equilibria in asymmetric information economies.[2]

A conceptual issue of our equilibrium notion is that there is no unambiguous mechanism that minimally rational agents can employ to assign a probability distribution on the set of possible prevailing prices. Therefore, the choice of a criterion to rank first period actions is not obvious. Inspired by recent advances in decision theory, we assume that minimally rational agents make decisions based on an adapted *variational preferences* or *smooth preferences* criterion.[3] In that respect, the paper relates to a growing literature that incorporates ambiguity aversion in the standard Arrow-Debreu model (see, for instance, Rigotti et al. (2008) and Rigotti and Shannon (2012)) and in asymmetric information economies (see, for instance, Tallon (1998), He and Yannelis (2015), Liu (2016), de Castro et al. (2017)). The crucial difference is that, in our setting, there is ambiguity about endogenous uncertainty (future prevailing prices) as opposed to ambiguity about exogenous uncertainty (prevailing state of nature).

Existence is proven in a constructive way. We first consider an auxiliary economy and show that it attains an equilibrium in the spirit of Radner (1972). We then argue that any such equilibrium is indeed an informationally constrained equilibrium. The main technical hurdle amounts to show that the demand correspondence of minimally rational agents is upper semi-continuous.

The paper is structured as follows. Section 2 presents an example where a sequential equilibrium fails to exist for arbitrary price expectations, and provides insight on the way the same economy can support an informationally constrained equilibrium. Section 3 lays out a general model with endogenous borrowing limits. Section 4 discusses how agents, especially the minimally rational, construct their objectives. Section 5 defines the equilibrium concept and proves its existence. Section 6 characterizes the revelation of information at equilibrium. Section 7 highlights how the model can be extended to more than two periods. Main proofs and technical results, some of which can be of independent interest, are presented in the two appendices.

2. Non-existence of sequential equilibrium with arbitrary price expectations

To motivate the analysis we present below an example (see also Daher et al. (2007)) in which a sequential competitive equilibrium fails to exist even if there is some agreement among agents about the future prevailing prices. Equilibrium is achieved temporarily, since only the actions of the first period are coordinated by spot prices. Any price vector that potentially can clear commodity markets in the second period is not compatible with the given profile of agents' price expectations. That is, for all possible market clearing prices, one agent goes bankrupt and a sequential equilibrium cannot be supported.

Example. We consider a two-period exchange economy with no uncertainty. There is a single commodity, $L_0 = \{\ell_0\}$, the first period $t = 0$ and three commodities, $L_1 = \{\ell_1, \ell_2, \ell_3\}$, the second period $t = 1$. At period $t = 0$, two agents, $J = \{j_1, j_2\}$, trade a single asset that delivers one unit of good ℓ_1 the next period, i.e., the asset's dividend is $\xi = (1, 0, 0)$. Both agents have strictly positive endowments, identical first period utilities, but distinct second period utilities. Specifically, for any $j \in J$,

$$\forall x_0 \in \mathbb{R}_+, \quad u_0^j(x_0) = x_0 \quad \text{and} \quad e_0^j > 0,$$
$$\forall x_1 \in \mathbb{R}_+^3, \quad u_1^{j_1}(x_1) = 2x_1(\ell_1) + x_1(\ell_2) + x_1(\ell_3), \quad u_1^{j_2}(x_1) = x_1(\ell_1) + 2[x_1(\ell_2) + x_1(\ell_3)]$$

and

$$\forall \ell \in L_1, \quad e_1^j(\ell) = e_1^j > 0.$$

We choose the standard normalization for second period prices, i.e.,

$$\Delta_1 := \left\{ p_1 \in \mathbb{R}_+^3 : \sum_{\ell \in L_1} p_1(\ell) = 1 \right\}.$$

Agents have different price expectations defined by the probability[4]

$$\nu^j = P_H \mathbf{1}_{\{p_H^j\}} + P_L \mathbf{1}_{\{p_L^j\}},$$

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[2] When assets are real, there is space for partial revelation of information (see, among others, Allen (1981b, 1985), Jordan (1982), Ausubel (1990) and Pietra and Siconolfi (2008)). With nominal assets, equilibrium prices may be non-revealing (see, among others, Polemarchakis and Siconolfi (1993), Rahi (1995) and Citanna and Villanacci (2000)).

[3] See Maccheroni et al. (2006) and Klibanoff et al. (2005) for an axiomatisation of these criteria.

[4] If K is a finite set and A is a subset of K, then $\mathbf{1}_A$ denotes the vector $(\mathbf{1}_A(k))_{k\in K}$ in $\mathbb{R}^K$ defined by $\mathbf{1}_A(k) = 1$ if $k \in A$ and $\mathbf{1}_A(k) = 0$ elsewhere.

where $P_H > 0$, $P_L > 0$ and $P_H + P_L = 1$. That is, agents expect that only two price vectors may prevail at the second period, i.e., $\operatorname{supp} \nu^j = \{p^j_H, p^j_L\}$.

Both agents believe that goods ℓ_2 and ℓ_3 will be priced equally, i.e., $p^j_H(\ell_2) = p^j_H(\ell_3)$ and $p^j_L(\ell_2) = p^j_L(\ell_3)$. However, agent j_1 believes that the price of good ℓ_1 will be lower than the price of the other two goods. Though agent j_2 assigns a positive probability to this event, she believes that the opposite may also be true.

Formally, let $\rho^j_H = p^j_H(\ell_2)/p^j_H(\ell_1)$ and $\rho^j_L = p^j_L(\ell_2)/p^j_L(\ell_1)$ be the price of good ℓ_2 in units of good ℓ_1, and assume that

$$0 < \rho^{j_2}_L < 1 \quad \text{and} \quad 2 < \rho^{j_1}_L < \rho^{j_2}_H \leq \rho^{j_1}_H. \tag{i}$$

We also impose the following restrictions on fundamentals

$$2 < \frac{e^{j_2}_0}{(1 + 2\rho^{j_1}_L)e^{j_1}_1} < 2\frac{P_L}{\rho^{j_2}_L} + P_H. \tag{ii}$$

Let θ^j denote the asset position of agent $j \in J$. For any $p_1 \in \Delta_1$, the indirect utility function is given by

$$v^j(p_1, \theta^j) = \max\left\{u^j_1(x_1) \,:\, p_1 \cdot x_1 \leq e^j_1 + p_1(\ell_1)\theta^j\right\},$$

while the period-1 demand for consumption is given by

$$d^j_1(p_1, \theta^j) := \operatorname{argmax}\left\{u^j_1(x_1) \,:\; p_1 \cdot x_1 \leq e^j_1 + p_1(\ell_1)\theta^j\right\}.$$

Since $\rho^{j_1}_H > \rho^{j_1}_L > 1$, we have[5]

$$v^{j_1}(p^{j_1}_L, \theta^{j_1}) = 2\left(\theta^{j_1} + \frac{e^{j_1}_1}{p^{j_1}_L(\ell_1)}\right) \quad \text{and} \quad v^{j_1}(p^{j_1}_H, \theta^{j_1}) = 2\left(\theta^{j_1} + \frac{e^{j_1}_1}{p^{j_1}_H(\ell_1)}\right).$$

Since $\rho^{j_2}_L < 1 < \rho^{j_2}_H$, we have[6]

$$v^{j_2}(p^{j_2}_L, \theta^{j_2}) = 2\left(\frac{1}{\rho^{j_2}_L}\theta^{j_2} + \frac{e^{j_2}_1}{p^{j_2}_L(\ell_2)}\right) \quad \text{and} \quad v^{j_2}(p^{j_2}_H, \theta^{j_2}) = \theta^{j_2} + \frac{e^{j_2}_1}{p^{j_2}_H(\ell_1)}.$$

Given a pair $(1, q)$ (with $q > 0$) of first period prices, the budget set $B^j_0(q, \nu^j)$ of agent $j \in J$ contains all pairs (x_0, θ) (with $x_0 \geq 0$ and $\theta \in \mathbb{R}$) such that[7]

$$x_0 + q \cdot \theta \leq e^j_0 \quad \text{and} \quad \theta \geq -\frac{e^j_1}{p^j_L(\ell_1)} = -(1 + 2\rho^j_L)e^j_1.$$

The period-0 demand is given by

$$d^j_0(q, \nu^j) := \operatorname{argmax}\{x_0 + q^j\theta \,:\, (x_0, \theta) \in B^j_0(q, \nu^j)\}$$

where

$$q^{j_1} = 2 \quad \text{and} \quad q^{j_2} = 2\frac{P_L}{\rho^{j_2}_L} + P_H.$$

Because of restriction (ii), we have $q^{j_2} > q^{j_1}$.

The demand for the single asset is given by

$$\begin{aligned} \theta^j(q) &= \{\theta \in \mathbb{R} : \exists x_0 \in \mathbb{R}_+,\ (x_0, \theta) \in d^j_0(q, \nu^j)\} \\ &= \operatorname{argmax}\left\{(q^j - q)\theta \,:\; q\theta \leq e^j_0 \quad \text{and} \quad \theta \geq -\frac{e^j_1}{p^j_L(\ell_1)}\right\}. \end{aligned}$$

Only prices that lie in the open interval (q^{j_1}, q^{j_2}) can clear the asset market, since otherwise both agents will go short or long on the asset simultaneously. For any $q \in (q^{j_1}, q^{j_2})$, the asset positions are as follows

5 Indeed, the optimal choice of agent j_1 is to set $x^{j_1}_1(\ell_2) = x^{j_1}_1(\ell_3) = 0$.

6 If $p^{j_2}_L$ prevails, the optimal choice of agent j_2 is to set $x^{j_2}_1(\ell_1) = 0$. If $p^{j_2}_H$ prevails, the optimal choice is to set $x^{j_2}_1(\ell_2) = x^{j_2}_1(\ell_3) = 0$.

7 Observe that the second inequality implies that $\theta \geq -\frac{e^j_1}{p^j_H(\ell_1)} = -(1 + 2\rho^j_H)e^j_1$.

$$\theta^{j_1}(q) = -\frac{e_1^{j_1}}{p_L^{j_1}(\ell_1)} \quad \text{and} \quad \theta^{j_2}(q) = \frac{e_0^{j_2}}{q}.$$

It follows that

$$q^\star = \frac{p_L^{j_1}(\ell_1)e_0^{j_2}}{e_1^{j_1}} = \frac{e_0^{j_2}}{(1+2\rho_L^{j_1})e_1^{j_1}}$$

is the market clearing price.[8]

Any equilibrium price vector $p_1^\star$ must satisfy $1/5 \leq p_1^\star(\ell_1)$. Given $p_1 \in \Delta_1$, agent j_1's second period wealth is

$$w^{j_1}(p_1) = e_1^{j_1} + p_1(\ell_1)\theta^{j_1}(q).$$

In particular, at any possible market clearing price $p_1^\star$, the wealth is negative. Indeed,

$$w^{j_1}(p_1^\star) = \left(1 - \frac{p_1^\star(\ell_1)}{p_L^{j_1}(\ell_1)}\right) e_1^{j_1} \leq \left[1 - \frac{1}{5}(1+2\rho_L^{j_1})\right] e_1^{j_1} < 0.$$

Therefore, agent j_1's demand for consumption is not defined and no sequential equilibrium can exist for this specification of price expectations.

In this example, one agent undertakes an extreme position in the asset market. This position is accommodated by the other agent and markets clear the first period. The problem arises in the second period, since any potential equilibrium price vector does not belong in the support of agent j_1's expectations. If such a price realises, then the agent goes bankrupt and no sequential equilibrium can be supported.

There is a trivial way to make arbitrary price expectations compatible with the existence of a sequential equilibrium. This amounts to impose real solvency on investment strategies.[9] We take a different route and explore whether there are minimal restrictions on asset trades that preclude bankruptcy, but allow agents to form endogenous, though not necessarily common and degenerate, price expectations.

To provide intuition, we consider the previous example but without specifying arbitrarily agents' price expectations. Assume that asset positions are subject to debt limits given by

$$\forall j \in J, \quad \Theta^j = \{\theta \in \mathbb{R} : \theta \geq b^j\} \quad \text{where} \quad b^j = -5e_1^j.$$

Suppose also that all agents understand that the bound is imposed to ensure solvency at a prevailing, yet unknown, price vector $p_1^\star$. Then, every agent can infer the following set of possible period-1 prices[10]

$$\Pi^j(b^j) = \left\{\pi_1 \in \Delta_1^\circ : b^j = -\frac{\pi_1 \cdot e_1^j}{\pi_1(\ell_1)}\right\}.$$

Equivalently, observing the bound b^j, every agent understands that the equilibrium price vector should lie on the set

$$\Pi^j(b^j) = \left\{\pi_1 \in \Delta_1^\circ : \pi_1(\ell_1) = 1/5 \quad \text{and} \quad \pi_1(\ell_2) + \pi_1(\ell_3) = 4/5\right\}.$$

Notice that the price vector $(1/5, 2/5, 2/5)$ belongs to this set.

To rank actions the first period both agents need to define their indirect utility. Since there is no unambiguous mechanism to assign a probability distribution on the set $\Pi^j(b^j)$, agents have to appeal to a decision criterion that is subject to no probabilistic information.

A possible choice is to assume that both agents are pessimist, expecting the worst outcome with the highest probability. That is, given $\theta^j \in \Theta^j$ and π_1 in $\Pi^j(b^j)$, agent j computes

$$v^j(\pi_1, \theta^j) = \max\left\{u^j(x_1) : \pi_1 \cdot x_1 \leq e_1^j(\ell_1) + \pi_1(\ell_1)\theta^j\right\},$$

and then she takes the infimum value among all prices in $\Pi^j(b^j)$.

8 Condition (ii) implies that $q^\star \in (q^{j_1}, q^{j_2})$.

9 In the example this is equivalent to restricting asset positions as follows

$$\forall j \in J, \quad \Theta^j \subseteq \{\theta \in \mathbb{R} : e_1^j + \theta \geq 0\}.$$

If agents are *prudent* (Svensson (1981)), in the sense that ν^j has full support, then investment strategies satisfy real solvency.

10 $\Delta_1^\circ := \Delta_1 \cap \mathbb{R}_{++}^{L_1}$ is the relative interior of the convex set Δ_1.

We can easily see that the infimum is attained at the price vector $p_1^\star = (1/5, 2/5, 2/5)$. A sequential equilibrium can now be supported. Indeed, for $q^\star = 1$, the optimal asset positions are $\theta^{j_1}(q^\star) = -5e_1^{j_1}$ and $\theta^{j_2}(q^\star) = 5e_1^{j_1}$. Period-0 consumption is $e_0^{j_1} + 5e_1^{j_1}$ for agent j_1 and $e_0^{j_2} - 5e_1^{j_1} > 0$ for agent j_2.[11] At period $t = 1$, agent j_1 consumes nothing while agent j_2 consumes the total endowment of all goods. These choices are optimal under the price vector $p_1^\star$.

In what follows we extend the analysis to a general two-period model with complete markets and propose an equilibrium concept that makes precise the way non-degenerate price expectations relate to the observation of endogenous debt limits.

3. The economy

We consider a pure exchange economy that extends over two periods $t \in \{0, 1\}$. There is exogenous uncertainty about the economic environment (i.e., the consumers' characteristics) at date $t = 1$, captured by a finite set S of states of nature. The economy consists of a finite set H of agents, indexed by $h \in H$. There are two types of agents who differ on the way they form expectations about endogenous variables. More precisely, a subset I of agents is assumed to have perfect knowledge of the economy's fundamentals. These agents, called *sophisticated*, can perfectly forecast future equilibrium prices. The remaining set $J = H \setminus I$ consists of agents who cannot observe all aspects of the environment. These agents, called *minimally rational*, may not anticipate correctly future equilibrium prices. We allow for economies that are solely populated by minimally rational agents, i.e., the set I may be empty.

3.1. Fundamentals and commodity markets

Commodity markets open sequentially. At every period $t \in \{0, 1\}$, there is a finite set L_t of commodities available for trade. Let $X_t := \mathbb{R}_+^{L_t}$ denote the set of commodity bundles, and $P_t := \mathbb{R}_+^{L_t}$ be the set of commodity prices at period t.

Every agent $h \in H$ has a utility function $u_0^h : X_0 \to \mathbb{R}$ for first period consumption and an endowment of goods $e_0^h \in X_0$. For any possible realization of exogenous uncertainty $s \in S$, every agent has a utility function $u_s^h : X_1 \to \mathbb{R}$ for consumption and is endowed with a bundle of goods $e_s^h \in X_1$. Expectations about the states of nature are captured by a probability measure μ^h.

Assumptions. For each agent $h \in H$,

(a) endowments are strictly positive, i.e., $e_0^h \in \mathbb{R}_{++}^{L_0}$ and $e_s^h \in \mathbb{R}_{++}^{L_1}$ for every $s \in S$;
(b) the functions $x \mapsto u_0^h(x)$ and $x \mapsto u_s^h(x)$ are continuously differentiable, concave, strictly increasing, unbounded from above but bounded from below;
(c) The probability μ^h has full support, i.e., $\operatorname{supp} \mu^h = S$.

3.2. Financial markets

Agents trade at the initial period a finite set S of elementary Arrow securities. The index of an Arrow security is identical to the state $s \in S$ it pays off. Securities are assumed to be *numéraire*, that is, there exists a non-zero bundle of goods $\xi \in \mathbb{R}_+^{L_1}$ such that each security $s \in S$ delivers (the market value of) the bundle ξ in state s and nothing in other states.

We call a portfolio $\theta = (\theta_s)_{s \in S}$ an asset bundle in $\Theta := \mathbb{R}^S$, and denote by $Q := \mathbb{R}_+^S$ the space of asset prices $q = (q_s)_{s \in S}$. A position θ_s is a claim if $\theta_s \geq 0$ and a liability if $\theta_s \leq 0$. We impose bounds on short sales, denoted by $b^h = (b_s^h)_{s \in S}$ where $b_s^h \in [-\infty, 0]$. The set of investment strategies is a subset of Θ defined by

$$\Theta^h(b^h) := \{\theta \in \Theta : \theta \geq b^h\}.$$

Agents may not honour their debt obligations at period $t = 1$. Following Livshits et al. (2007) (see also Eichberger (1989)) we assume that default induces the full seizure of endowments and investment returns. That is, upon default, agents receive the minimum consumption level (equal to zero), their endowment is sold, and the remaining wealth is handed over to the creditors to at least partially redeem their debt. This may prevent agents to default strategically, i.e., if they have the resources, they may choose to repay their debt.[12] Nevertheless, default can be non-strategic. Since minimally rational agents are unable to perfectly foresee equilibrium prices, they can be insolvent at some contingencies.

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[11] The fact that $e_0^{j_2} > 5e_1^{j_1}$ follows from restrictions (i)-(ii). If $e_0^{j_2} \leq 5e_1^{j_1}$ a sequential equilibrium still exists free of these restrictions. Indeed, let $q^\star = e_0^{j_2}/5e_1^{j_1}$ and $\theta^{j_1}(q^\star) = -5e_1^{j_1}$ and $\theta^{j_2}(q^\star) = 5e_1^{j_1}$. Agent j_1 consumes the total endowment of the good at period $t = 0$ and nothing at period $t = 1$.

[12] A complete description of the bankryptcy rule should also specify the utiltiy penalties upon default. We do not provide details in this direction since we will introduce bounds on liabilities that preclude default at equilibrium.

It is well known that such a bankruptcy rule gives rise to non-convex budget sets, and consequently, to non-convex demand correspondences.[13] To overcome this complication we argue for borrowing constraints that preclude default at equilibrium.[14] Specifically, we assume that

$$\forall h \in h, \quad \forall \theta \in \Theta^h(b^h), \quad \forall s \in S, \quad p_s \cdot e_s^h + (p_s \cdot \xi)\theta_s \geq 0,$$

where p_s is the prevailing price vector at state $s \in S$. If, for instance, short-selling is forbidden, i.e., $b^h = 0$ for each $h \in H$, then default is trivially precluded. Obviously, we argue for borrowing constraints that prevent default but allow for maximal risk sharing.[15]

Given a prevailing price vector p_s, the maximum level of debt an agent can issue (subject to no default) at state $s \in S$ is given by

$$\beta_s^h(p_s) = \min\{\theta_s \in \mathbb{R} : \ p_s \cdot e_s^h + (p_s \cdot \xi)\theta_s \geq 0\}.$$

Since only relative prices matter, without any loss of generality, we can restrict contingent prices to the simplex Δ_1 of $\mathbb{R}^{L_1}$, i.e.,

$$\Delta_1 := \left\{ p_s \in \mathbb{R}_+^{L_1} : \ p_s \cdot \mathbf{1}_{L_1} = 1 \right\}.$$

We subsequently consider the function β_s^h from Δ_1 to $[-\infty, 0]$ given by

$$\forall \pi_s \in \Delta_1, \quad \beta_s^h(\pi_s) = \begin{cases} -(\pi_s \cdot e_s^h)/(\pi_s \cdot \xi) & \text{if} \quad \pi_s \cdot \xi > 0 \\ -\infty & \text{if} \quad \pi_s \cdot \xi = 0. \end{cases}$$

We assume that all agents understand that the bounds are imposed according to the family of functions $(\beta_s^h)_{s \in S}$, i.e.,

$$\forall h \in H, \quad \forall s \in S, \quad b_s^h = \beta_s^h(p_s),$$

where $p_s \in \Delta_1$ is an equilibrium price vector.

Standard general equilibrium theory assumes a centralised market in which all trades are anonymous and an auctioneer orchestrates the equilibration of supply and demand. We may employ a similar reasoning to justify the determination of the proposed borrowing constraints. Imagine, for instance, there is a hypothetical clearinghouse that is passive in the sense that it does not take an active position itself. Its purpose is to equilibrate supply and demand in financial markets in a way that diminishes search and information costs, while reducing default risk by acting as a third party warrantor to every contract. From that perspective, the borrowing constraints can be interpreted as an endogenous rule that restricts the trades of agents to a set that the clearinghouse foresees no possibility of bankruptcy.

4. Individual optimization

Actions at the initial period $t = 0$ involve commitments for the future period $t = 1$. Agents take into account the consequences of their choices, so decision making has to be described retrospectively, that is, by backward induction. Agents act rationally in the sense that they use all available information to make decisions. They understand that debt limits convey information about second period prevailing prices. For the sophisticated agents this information is irrelevant. For the minimally rational agents this information is useful, but, the dimensionality of the relevant information is, in general, larger than the dimensionality of the revealed information.

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[13] Indeed, if the realized state of nature is $s \in S$, the investment strategy is θ_s, and the prevailing commodity price is p_s, then the budget set contains all consumption bundles $x_s \in X_1$ such that

$$p_s \cdot x_s \leq \max\{0, p_s \cdot e_s^h + (p_s \cdot \xi)\theta_s\}.$$

[14] The temporary equilibrium literature recognized long ago the need for institutional restrictions to deal with insolvency and bankruptcy problems. Milne (1980) proposes exogenous borrowing constraints that reflect lenders' perception for default risk. In Stahl (1985) a clearinghouse has subjective beliefs about the realization of uncertainty and restricts each individual to a set of portfolios that ensures solvency for all realizations of uncertainty in the support of its expectations. A similar idea is behind the credit rationing schemes proposed by Eichberger (1989). A bank that has more pessimistic beliefs (relative to agents' beliefs) imposes a credit rationing scheme that is consistent with its own expectations. Bankruptcy is still possible though its likelihood is clearly reduced.

[15] In the spirit of Alvarez and Jermann (2000) (see also Kehoe and Levine (1993) and Zhang (1997)), we look for debt limits that are self-enforcing.

4.1. Actions at period $t = 1$

At the initial period $t = 0$, every agent $h \in H$ chooses an action $a = (x_0, \theta)$, where $x_0 \in X_0$ is a consumption bundle and $\theta \in \Theta^h(b^h)$ is a portfolio. At period $t = 1$, given θ, a realized state of nature $s \in S$ and a commodity price vector $p_s \in \Delta_1$, the budget set $B^h_s(p_s, \theta)$ contains all consumption bundles $x_s \in X_1$ such that

$$p_s \cdot x_s \leq p_s \cdot e^h_s + (p_s \cdot \xi)\theta_s.$$

An agent $h \in H$ chooses among consumption bundles $x_s \in B^h_s(p_s, \theta)$ to maximize her payoff. The demand set is given by

$$d^h_s(p_s, \theta) := \operatorname{argmax} \left\{ u^h_s(x_s) \; : \; x_s \in B^h_s(p_s, \theta) \right\}.$$

What differentiates the *minimally rational* from the *sophisticated* agents is how they construct the preference relation at the initial period.

4.2. Actions at period $t = 0$

Given spot prices (p_0, q) and debt limits b^h, let $B^h_0(p_0, q, b^h)$ denote agent h's budget set at period $t = 0$, containing all actions $a = (x_0, \theta) \in X_0 \times \Theta^h(b^h)$ such that

$$p_0 \cdot x_0 + q \cdot \theta \leq p_0 \cdot e^h_0.$$

To choose among actions in this set, agents need to form expectations about future commodity prices.

We assume that agents know that utility functions are strictly increasing, so they infer the trivial non-arbitrage condition that any equilibrium price belongs to the set Δ°_1.[16] Given $\theta \in \Theta^h(b^h)$, we denote by $v^h_s(\cdot, \theta)$ the map defined by

$$\forall \pi_s \in \Delta^\circ_1, \quad v^h_s(\pi_s, \theta) = \sup\{u^h_s(x_s) \; : \; x_s \in B^h_s(\pi_s, \theta)\}.$$

4.2.1. Sophisticated agents

These agents are assumed to have perfect foresight, therefore, they are able to foresee, at period $t = 0$, which price vector p_s will prevail at state $s \in S$. Specifically, for every agent $i \in I$, the preference relation over actions in the set $X_0 \times \Theta^i(b^i)$ is captured by the function $U^i_0(\cdot; p_1)$ defined by

$$\forall a = (x_0, \theta), \quad U^i_0(a; p_1) = u^i_0(x_0) + \sum_{s \in S} \mu^i(s) v^i_s(p_s, \theta),$$

where $p_1 = (p_s)_{s \in S}$ is the (correctly) anticipated family of contingent prices. Given first-period prices (p_0, q), the period-0 demand set is given by

$$d^i_0(p_0, q; p_1) := \operatorname{argmax} \left\{ U^i_0(a; p_1) \; : \; a \in B^i_0(p_0, q, b^i) \right\}.$$

Observe that the demand correspondence does not explicitly depend on the bounds $b^i = (b^i_s)_{s \in S}$. Since the bounds are defined by the relation $b^i_s = \beta^i_s(p_s)$ for any $s \in S$, and all sophisticated agents anticipate perfectly the prevailing prices $(p_s)_{s \in S}$, observing the bounds does not improve their information set.

4.2.2. Minimally rational agents

These agents cannot perfectly foresee the prevailing commodity prices at period $t = 1$. Nevertheless, they understand that debt limits ensure solvency given period-1 market clearing prices. That is, every agent $j \in J$ understands that the bounds $b^j = (b^j_s)_{s \in S}$ on elementary Arrow securities are imposed according to the family of functions $(\beta^j_s)_{s \in S}$. Therefore, observing the bound b^j_s at period $t = 0$, agent j infers a set of possible prices given by

$$\Pi^j_s(b^j_s) = \{\pi_s \in \Delta^\circ_1 \; : \; b^j_s = \beta^j_s(\pi_s)\}.$$

A family of contingent prices $\pi_1 = (\pi_s)_{s \in S}$ belongs to

$$\Pi^j(b^j) = \prod_{s \in S} \Pi^j_s(b^j_s).$$

We notice that minimally rational agents use all available information to forecast future prices. However, as opposed to sophisticated agents, they do not perfectly foresee the function that relates the states of nature to exact equilibrium prices. They instead know the map from prices to bounds that preclude bankruptcy.

[16] Strict increasingness of u^h_s implies that the set $d^h_s(p_s, \theta)$ is well defined if and only if the price p_s is strictly positive, i.e., $p_s \in \Delta^\circ_1$.

This ambiguity raises a conceptual issue regarding the decision rule minimally rational agents can employ to make their first-period decisions. A priori, there is no unambiguous mechanism that can be used to assign a probability distribution on the set $\Pi^j(b^j)$. Equivalently, decision making is subject to no probabilistic information about possible prevailing prices.

Recent advances in decision theory offer a plethora of alternatives to model ambiguity about exogenous uncertainty.[17] For our purposes, we adapt well-established decision criteria to capture ambiguity aversion related to endogenous uncertainty. To appeal to standard general equilibrium tools, we restrict attention to convex preferences represented by functionals that give rise to concave indirect utility functions.[18]

The first criterion has a variational representation in the spirit of the preferences axiomatised by Maccheroni et al. (2006).[19] More precisely, given first-period prices (p_0, q) and bounds b^j, agent j ranks actions a in $B_0^j(p_0, q, b^j)$ according to $V_0^j(\cdot; b^j)$ defined by

$$\forall a = (x_0, \theta), \quad V_0^j(a; b^j) = u_0^j(x_0) + \inf\left\{ v^j(\pi_1, \theta) + c^j(\pi_1) : \pi_1 \in \Pi^j(b^j) \right\}, \tag{†}$$

where $v^j(\pi_1, \theta) = \sum_{s\in S} \mu^j(s) v_s^j(\pi_s, \theta)$, and $c^j : \Delta_1^S \to [0, \infty]$ is an upper semi-continuous and convex function interpreted as the intensity of price ambiguity aversion. Equivalently, $c^j(\pi_1)$ represents the cost of selecting the vector of contingent prices $\pi_1 \in \Pi^j(b^j)$.

If $c^j(\pi_1) = 0$ for all $\pi_1 \in \Pi^j(b^j)$ and $c^j(\pi_1) = \infty$ otherwise, then the criterion is simply maxmin expected utility (extreme pessimism). If on the other hand, $c^j(\hat{\pi}_1) = 0$ for some $\hat{\pi}_1 \in \Pi^j(b^j)$ and $c^j(\pi_1) = \infty$ otherwise, then the criterion is simply expected utility with respect to the price vector $\hat{\pi}_1$.

The analogue to multiplier preferences (Hansen and Sargent (2008)) obtains when $c^j(\pi_1) = \sum_{s\in S} c_s^j(\pi_s)$ and

$$\forall s \in S, \quad c_s^j(\pi_s) = \begin{cases} \kappa_s^j R(\pi_s || \hat{\pi}_s), & \text{if } \pi_s \in \Pi_s^j(b_s^j) \\ \infty, & \text{otherwise,} \end{cases}$$

where $R(\pi_s || \hat{\pi}_s) = \sum_{\ell\in L_1} \pi_s(\ell) \log \frac{\pi_s(\ell)}{\hat{\pi}_s(\ell)}$. That is, minimally rational agents have in their mind a benchmark family of contingent prices $\hat{\pi}_1 \in \Pi^j(b^j)$, and the cost function is a measure of the distance of other prices in $\Pi^j(b^j)$ with respect to this benchmark. The constant $\kappa_s^j > 0$ reflects the weight agent j is giving to the possibility that $\hat{\pi}_s$ is not the correct price.

Another specification relates to ε-contamination preferences. In this case, $c^j(\pi_1) = \sum_{s\in S} c_s^j(\pi_s)$ and

$$\forall s \in S, \quad c_s^j(\pi_s) = \begin{cases} 0, & \text{if } \pi_s \in (1-\varepsilon)\hat{\pi}_s + \varepsilon \Pi_s^j(b_s^j) \\ \infty, & \text{otherwise.} \end{cases}$$

That is, minimally rational agents have in their mind a combination between a price benchmark $\hat{\pi}_1 \in \Pi^j(b^j)$ and anything else in the set $\Pi^j(b^j)$.

The second criterion has a two-layer expected utility representation in the spirit of the smooth preferences axiomatised by Klibanoff et al. (2005). More precisely, given first-period prices (p_0, q) and bounds b^j, agent j ranks actions a in $B_0^j(p_0, q, b^j)$ according to $V_0^j(\cdot; b^j)$ defined by

$$\forall a = (x_0, \theta), \quad V_0^j(a; b^j) = u_0^j(x_0) + \int_{\Pi^j(b^j)} \Phi^j\left(v^j(\pi_1, \theta)\right) \nu^j(\pi_1 | b^j), \tag{‡}$$

where as before $v^j(\pi_1, \theta) = \sum_{s\in S} \mu^j(s) v_s^j(\pi_s, \theta)$.

Minimally rational agents first evaluate the expected utility of their decision with respect to a family of contingent prices $\pi_1 \in \Pi^j(b^j)$. Then, instead of taking the infimum over expected utilities, they take an expectation of distorted expected utilities according to a probability measure $\nu^j(\cdot | b^j)$. For our purposes the function $\Phi^j : \mathbb{R} \to \mathbb{R}$ is assumed to be increasing, bounded from above and concave. For instance, an agent $j \in J$ may choose a uniform measure over $\Pi^j(b^j)$ and exhibit constant ambiguity aversion in the sense that $\Phi^j(x) = -\frac{1}{\alpha^j} e^{-\alpha^j x}$ for some $\alpha^j > 0$.

For both criteria, the associated period-0 demand correspondence is given by

$$d_0^j(p_0, q, b^j) := \operatorname{argmax}\{V_0^j(a; b^j) : a \in B_0^j(p_0, q, b^j)\}.$$

[17] We refer to Etner et al. (2012) and Gilboa and Marinacci (2013) for comprehensive surveys of the literature.

[18] This requirement rules out some interesting criteria, for instance the α-maxmin criterion, that are not in general concave. α-maxmin preferences is a special class of biseparable preferences studied in Ghirardato and Marinacci (2002) and Ghirardato et al. (2004). The only invariant biseparable preferences that are convex are actually maxmin preferences and these are included in our analysis.

[19] This class is general enough to nest many models of ambiguity aversion including the Gilboa and Schmeidler (1989) maxmin preferences, Schmeidler (1989) Choquet expected utility, Hansen and Sargent (2008) multiplier preferences, and ε-contamination preferences (see, for instance, Etner et al. (2012)). For applications of variational preferences in general equilibrium we refer to Rigotti et al. (2008) and Rigotti and Shannon (2012) who study the market implications of ambiguity in the standard Arrow-Debreu model.

5. Equilibrium concept

5.1. Informationally constrained equilibrium

We next introduce an equilibrium concept that makes precise the way agents' decisions become compatible with the formation of endogenous, but not necessarily common and degenerate, price expectations.

Definition 5.1. A family $\{(p_0, q, \boldsymbol{b}, \boldsymbol{a}), (p_s, \boldsymbol{x}_s)_{s \in S}\}$ such that

- p_0 is the period-0 consumption price vector
- q is the period-0 asset price vector
- $\boldsymbol{b} = (b^h)_{h \in H}$ is the family of period-0 debt limits
- $\boldsymbol{a} = (a^h)_{h \in H}$ is the family of period-0 actions

and for each possible state $s \in S$

- p_s is the consumption price vector
- $\boldsymbol{x}_s = (x_s^h)_{h \in H}$ is the allocation of consumption bundles

is called an *informationally constrained equilibrium* (ICE), if the following conditions are satisfied:

(a) period-0 actions are optimal, i.e., for every $j \in J$,

$$a^j \in d_0^j(p_0, q, b^j) \quad \text{where} \quad b^j = (b_s^j)_{s \in S} \quad \text{and} \quad b_s^j = \beta_s^j(p_s),$$

and for every $i \in I$,[20]

$$a^i \in d_0^i(p_0, q; p_1) \quad \text{where} \quad p_1 = (p_s)_{s \in S};$$

(b) period-0 markets for consumption and assets clear, i.e.,

$$\sum_{h \in H} x_0^h = \sum_{h \in H} e_0^h \quad \text{and} \quad \sum_{h \in H} \theta^h = 0;$$

(c) period-1 actions are optimal, i.e., for every $h \in H$, for every $s \in S$,

$$x_s^h \in d_s^h(p_s, \theta^h);$$

(d) period-1 consumption markets clear, i.e., for every $s \in S$,

$$\sum_{h \in H} x_s^h = \sum_{h \in H} e_s^h.$$

Theorem 5.1. *Under the stated assumptions on primitives (preferences and endowments), an Informationally Constrained Equilibrium (ICE) always exists.*

The following section presents a sketch of the existence proof and discusses in detail the challenges associated to the presence of minimally rational agents. The key insight derives from considering first an auxiliary economy and show that it attains a competitive equilibrium similar to Radner (1972) perfect foresight equilibrium. By construction, any selection from the set of Radner-like equilibria is indeed an ICE of the original economy. The proofs of the main results are postponed to Appendix A. Technical results are presented in Appendix B.

We encompass both representations of ambiguity aversion to price uncertainty discussed in Section 4.2.2. That is, we provide a proof assuming that minimally rational agents use either (†) or (‡) to rank first period actions. Establishing upper semi-continuity of the optimal demand correspondence is the main technical hurdle under both representations.

Though the proof strategy requires initially the presence of at least one sophisticated agent, this is not essential for our existence result. An ICE exists even in settings where all agents are assumed to be minimally rational. We discuss below where the presence of a sophisticated agent bits initially and propose an argument to dispense with it.

Finally, we assume that the payoff vector is strictly positive, i.e., $\xi \in \mathbb{R}^{L_1}_{++}$. This is without any loss of generality since we can easily approximate any equilibrium of an economy with positive payoffs by the limit of equilibria of economies with strictly positive payoffs.

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[20] Recall that sophisticated agents face also bounds on short sales, i.e., for all $i \in I$, $b_s^i = \beta_s^i(p_s)$ for any $s \in S$. However, observing the bounds is irrelevant for their optimal decisions.

5.2. Existence of ICE

We denote by **P** the set of prices $\mathrm{P} = (p_0, q, p_1)$, where $p_1 = (p_s)_{s\in S}$ is a family of contingent prices with $p_s \in \Delta_1$ for all $s \in S$, and $(p_0, q) \in \Delta_0$ where

$$\Delta_0 := \left\{(p_0, q) \in \mathbb{R}_+^{L_0} \times \mathbb{R}_+^S \ : \ p_0 \cdot \mathbf{1}_{L_0} + q \cdot \mathbf{1}_S = 1\right\}.$$

Observe that the set **P** is non-empty, compact and convex.

Recall that $X_0 = \mathbb{R}_+^{L_0}$, $X_1 = \mathbb{R}_+^{L_1}$ and $\Theta = \mathbb{R}^S$. Let $(e_0, e_1) \in \mathbb{R}_+^{L_0} \times \mathbb{R}_+^{L_1}$ be the aggregate endowments, i.e., $e_0 := \sum_{h\in H} e_0^h$ and $e_1 := \sum_{s\in S} e_s$ with $e_s := \sum_{h\in H} e_s^h$. We fix initially a family of non-empty, compact and convex sets $(X_0^h, X_1^h, \Theta^h)_{h\in H}$ such that, for each $t \in \{0, 1\}$, the set X_t^h is a subset of X_t containing the aggregate endowment e_t in its interior, and Θ^h is a subset of Θ containing 0 in its interior.

We next consider the economy $\mathcal{E} = (X_0^h, X_1^h, \Theta^h)_{h\in H}$, where the intertemporal budget sets are given by[21]

$$B^h(\mathrm{P}) = \left\{(x_0, \theta, (x_s)_{s\in S}) \in X_0^h \times \Theta^h \times [X_1^h]^S \ : \ (1) \quad \text{and} \quad (2)\right\},$$

where

$$p_0 \cdot x_0 + q \cdot \theta \le p_0 \cdot e_0^h \tag{1}$$

and

$$\forall s \in S, \quad p_s \cdot x_s \le p_s \cdot [e_s^h + \theta_s \xi]. \tag{2}$$

The intertemporal demand of a sophisticated agent $i \in I$ is given by

$$\delta^i(\mathrm{P}) := \operatorname{argmax}\left\{u_0^i(x_0) + \sum_{s\in S} \mu^i(s) u_s^i(x_s) \ : \ (x_0, \theta, (x_s)_{s\in S}) \in B^i(\mathrm{P})\right\}.$$

The intertemporal demand of a minimally rational agent $j \in J$ is given by

$$\delta^j(\mathrm{P}) := \operatorname{argmax}\left\{W_0^j(x_0, \theta; p_1) + \sum_{s\in S} \mu^j(s) u_s^j(x_s) \ : \ (x_0, \theta, (x_s)_{s\in S}) \in B^j(\mathrm{P})\right\},$$

where

$$W_0^j(x_0, \theta; p_1) = \begin{cases} u_0^j(x_0) + \inf\left\{v^j(\pi_1, \theta) + c^j(\pi_1) \ : \ \pi_1 \in P^j(p_1)\right\} \\ \qquad\qquad \text{or} \\ u_0^j(x_0) + \int_{P^j(p_1)} \Phi^j\left(v^j(\pi_1, \theta)\right) \nu^j(\pi_1 | p_1). \end{cases}$$

The first specification of W_0^j applies when actions are ranked according to criterion (†), while the second one applies when actions are ranked according to criterion (‡). In both specifications, $v^j(\pi_1, \theta) = \sum_{s\in S} \mu^j(s) v_s^j(\pi_s, \theta)$ and $P^j(p_1) := \prod_{s\in S} P_s^j(p_s)$, where for any $s \in S$,

$$v_s^j(\pi_s, \theta) = \sup\left\{u_s^j(x_s) \ : \ x_s \in X_1 \quad \text{and} \quad \pi_s \cdot x_s \le \max\{0, \pi_s \cdot [e_s^j + \theta_s \xi]\}\right\}$$

and

$$P_s^j(p_s) := \Pi_s^j(\beta_s^j(p_s)) = \left\{\pi_s \in \Delta_1^\circ \ : \ \pi_s \cdot [e_s^j(p_s \cdot \xi) - \xi(p_s \cdot e_s^j)] = 0\right\}.$$

The following remark clarifies two important aspects of the function v_s^j.

Remark 5.1. First, the consumption bundles x_s are not restricted to lie on the compact set X_1^j. Since minimally rational agents discern only a set of possible prevailing prices, they anticipate to consume ex-ante ($t = 0$) bundles that they will not effectively choose to consume ex-post (after the state s is realized and the equilibrium price p_s is revealed). So the reason to take the supremum over the unbounded set X_1.

Second, the budget constraint is written taking the "max" on the right hand side, i.e.,

$$\pi_s \cdot x_s \le \max\{0, \pi_s \cdot [e_s^j + \theta_s \xi]\}.$$

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[21] To be rigorous we should write $B^h(X_0^h, X_1^h, \Theta^h, \mathrm{P})$. Our choice to omit the sets X_0^h, X_1^h, Θ^h is driven by a desire for notational simplicity.

This accounts for the fact that a portfolio choice $\theta \in \Theta^j$ may lead to bankruptcy, i.e., there may be a state $s \in S$ and a price vector $\pi_s \in P_s^j(p_s)$ such that $\pi_s \cdot [e_s^j + \theta_s \xi] < 0$. This means that v_s^j may not be concave in θ_s. Nevertheless, when $(x_0, \theta, (x_s)_{s\in S}) \in B^j(\mathrm{P})$, the budget restriction (2) implies that

$$\forall s \in S, \quad \theta_s \geq \beta_s^j(p_s) = -\frac{p_s \cdot e_s^j}{p_s \cdot \xi}.$$

Since by definition

$$P_s^j(p_s) = \left\{ \pi_s \in \Delta_1^\circ : \beta_s^j(p_s) = -\frac{\pi_s \cdot e_s^j}{\pi_s \cdot \xi} \right\},$$

it follows that $\pi_s \cdot [e_s^j + \theta_s \xi] \geq 0$ for all $\pi_s \in P_s^j(p_s)$. Therefore, v_s^j is indeed concave on budget feasible investment strategies.[22]

The next remark discusses the restrictions imposed to the family of probability measures in the second specification of W_0^j.

Remark 5.2. We assume that $\nu^j(\cdot|p_1)$ is an absolutely continuous probability measure on Δ_1^S with full support on the set $P^j(p_1)$. That is, ν^j assigns zero probability to the difference $\Delta_1^S \setminus P^j(p_1)$ and $\nu^j(\pi_1|p_1) = h^j(\pi_1|p_1)m(\pi_1)$, where h^j is the density of ν^j and m is the Lebesgue measure on the affine space G defined by

$$G := \left\{ (\pi_s)_{s\in S} \in \mathbb{R}^{L_1 \times S} : \forall s \in S, \quad \pi_s \cdot \mathbf{1}_{L_1} = 1 \right\}.$$

In addition, the map $(\pi_1, p_1) \mapsto h^j(\pi_1|p_1)$ is jointly continuous on $\tilde{P}^j(p_1) \times [\Delta_1^\circ]^S$.[23]

A *competitive equilibrium* of the economy $\mathcal{E} = (X_0^h, X_1^h, \Theta^h)_{h\in H}$ is a family $\{(p_0, q, \boldsymbol{a}), (p_s, \boldsymbol{x}_s)_{s\in S}\}$ such that $\mathrm{P} = (p_0, q, (p_s)_{s\in S}) \in \mathbf{P}$ and

$$\forall h \in H, \quad (a^h, (x_s^h)_{s\in S}) \in \delta^h(\mathrm{P})$$

together with

$$\sum_{h\in H} ((x_0^h, \theta^h), (x_s^h)_{s\in S}) = \sum_{h\in H} ((e_0^h, 0), (e_s^h)_{s\in S}).$$

Proposition 5.1. *Under the assumptions imposed on primitives (preferences and endowments), the economy $\mathcal{E} = (X_0^h, X_1^h, \Theta^h)_{h\in H}$ attains a competitive equilibrium.*

The detailed proof of Proposition 5.1 is presented in Appendix A. The following remark discusses the main challenges.

Remark 5.3. The key step amounts to show that every demand correspondence δ^h is upper semi-continuous. Indeed, standard arguments imply that δ^i (the demand of sophisticated agents) is upper semi-continuous on $\mathbf{P}$. This follows from Berge's Maximum Theorem as the map

$$(x_0, (x_s)_{s\in S}) \mapsto u_0^i(x_0) + \sum_{s\in S} \mu^i(s) u_s^i(x_s)$$

is continuous on $X_0^i \times [X_1^i]^S$, and the correspondence B^i is non-empty, continuous and compact-valued on $\mathbf{P}$. However, for the minimally rational agents, we only succeeded to prove a weaker result.

Lemma 5.1. *For any agent $j \in J$, the demand correspondence δ^j is upper semi-continuous on $\Delta_0 \times [\Delta_1^\circ]^S$.*

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[22] The concavity of v_s^j with respect to θ_s implies that W_0^j is concave in θ, and that δ^j is a convex-valued correspondence. The later property is essential for an application of Kakutani's fixed point theorem to prove existence of an equilibrium in economy $\mathcal{E}$.

[23] $\tilde{P}^j(p_1)$ stands for the closure of $P^j(p_1)$. See Appendix A.

To prove the lemma it suffices to show that the function W_0^j is continuous on $X_0^j \times \Theta^j \times [\Delta_1^\circ]^S$. For the first specification of W_0^j, it is sufficient that the map

$$(p_1, \theta) \mapsto \inf\left\{v^j(\pi_1, \theta) + c^j(\pi_1) \,:\, \pi_1 \in P^j(p_1)\right\}$$

is continuous on $[\Delta_1^\circ]^S \times \Theta^j$. Proving continuity with respect to p_1 is challenging since the set $P^j(p_1)$ is not compact, preventing the application of the standard Maximum Theorem. Claim A.1 in Appendix A shows how to overcome this issue by exploiting recent advances of the Maximum Theorem to problems with non-compact image sets. For the second specification of W_0^j, it suffices to show that the map

$$(p_1, \theta) \mapsto \int_{P^j(p_1)} \Phi^j\left(v^j(\pi_1, \theta)\right) v^j(\pi_1|p_1)$$

is continuous on $[\Delta_1^\circ]^S \times \Theta^j$. Claim A.2 in Appendix A provides a constructive argument that can be of independent interest.

The proof of Proposition 5.1 assumes initially the presence of at least one sophisticated agent, i.e., the set I is non-empty. This permits to show that fixed-point prices are strictly positive, a property that is subsequently used to argue that these prices clear markets. However, as we argue in the second part of the proof, this assumption is not essential. An ICE exists even when the economy is solely composed of minimally rational agents. The key insight rests on considering a sequence of artificial economies, each one having a sophisticated agent whose endowments are chosen to converge to zero along the sequence. Each artificial economy has an ICE with strictly positive equilibrium prices. Most importantly, the sequence of second period commodity prices is uniformly bounded away from zero. Extracting a subsequence if necessary, we show that the limit is indeed an equilibrium of the original economy where all agents are minimally rational.

Equipped with Proposition 5.1, we next look the *frictionless* economy $\mathcal{E}^f$ which is nothing other than the economy $\mathcal{E}$ when, for every agent $h \in H$, the family of consumption and investment sets (X_0^h, X_1^h, Θ^h) is replaced by the initial one (X_0, X_1, Θ). The relevant observation for our purposes is that we can always find a family $(X_0^h, X_1^h, \Theta^h)_{h\in H}$ of non-empty, compact and convex sets such that an equilibrium of the economy $\mathcal{E}$ is an equilibrium of the frictionless economy $\mathcal{E}^f$.

Indeed, for every agent $h \in H$, let

$$X_0^h := \{x_0 \in X_0 \,:\, x_0 \le e_0 + \mathbf{1}_{L_0}\} \quad \text{and} \quad X_1^h := \{x_1 \in X_1 \,:\, x_1 \le e_1 + \mathbf{1}_{L_1}\}.$$

The choice of Θ^h is more subtle. Let $\widetilde{\Theta}^h$ be the subset of Θ given by

$$\widetilde{\Theta}^h := \{\theta \in \Theta : \forall s \in S, \quad \exists p_s \in \Delta_1 \quad \text{such that} \quad p_s \cdot [e_s^h + \xi\theta_s] \ge 0\}.$$

This set is bounded from below. Indeed, let $\gamma := \min\{p_s \cdot \xi : p_s \in \Delta_1\}$. Since $p_s \cdot \xi > 0$, it follows that $\gamma > 0$.[24] It is also true that $p_s \cdot e_s^h \le \overline{e}_s^h := \sum_{\ell\in L_1} e_s^h(\ell)$. It follows that

$$\theta \in \widetilde{\Theta}^h \Rightarrow \forall s \in S, \quad \theta_s \ge -\max_{s\in S} \overline{\theta}_s^h,$$

where $\overline{\theta}_s^h := \overline{e}_s^h/\gamma$.

Observe that the set $\widetilde{\Theta}^h$ is not necessarily convex. Nevertheless, we can work with its convex hull $\mathrm{co}(\widetilde{\Theta}^h)$ and define the set

$$\widehat{\Theta} := \left\{(\theta^h)_{h\in H} \in \prod_{h\in H} \mathrm{co}(\widetilde{\Theta}^h) : \sum_{h\in H} \theta^h = 0\right\}.$$

This set is closed and convex. This is also true for the projection of $\widehat{\Theta}$ on $\mathrm{co}(\widetilde{\Theta}^h)$, denoted $\widehat{\Theta}^h$. We can choose a real number r such that $\widehat{\Theta}^h \subset B(0, r)$.[25] It follows that the set $\Theta^h := \mathrm{co}(\widetilde{\Theta}^h) \cap \overline{B(0, r)}$ is non-empty, compact, convex and contains 0 in its interior.

A standard convexity argument (see Appendix A) allows us to prove the following claim.

Proposition 5.2. *Given a family of sets $(X_0^h, X_1^h, \Theta^h)_{h\in H}$ specified as above, any equilibrium of the bounded economy $\mathcal{E}$ is an equilibrium of the frictionless economy $\mathcal{E}^f$.*

Theorem 5.1 is then a direct corollary of the last proposition since, by construction, any equilibrium of the frictionless economy $\mathcal{E}^f$ is in fact an informationally constrained equilibrium (ICE).

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[24] Here we exploit the fact that $\xi \in \mathbb{R}_{++}$.

[25] $B(0, r)$ is the open ball in $\mathbb{R}^S$ with centre 0 and radius r.

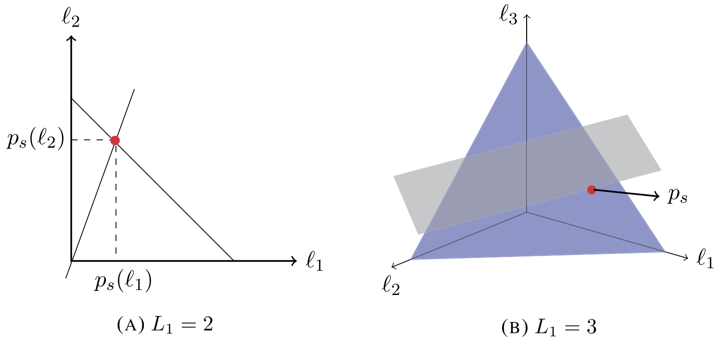

Fig. 1. The set $\Pi_s^j(\beta_s^j(p_s))$. (For interpretation of the colours in the figure(s), the reader is referred to the web version of this article.)

6. Revelation of information

It is natural to investigate the *degree* of information revealed at equilibrium. We propose below a complete characterisation.

Definition 6.1. An informationally constrained equilibrium is said to be:

(a) *fully revealing* at state $s \in S$, if every agent $j \in J$, after observing the bound b_s^j, can infer the associated equilibrium price, i.e.,

$$\Pi_s^j(\beta_s^j(p_s)) = \{p_s\};$$

(b) *fully non-revealing* at state $s \in S$, if every agent $j \in J$, after observing the bound b_s^j, obtains no information about the equilibrium price, i.e.,

$$\Pi_s^j(\beta_s^j(p_s)) = \Delta_1^\circ;$$

(c) *partially revealing* at state $s \in S$, if it is neither fully revealing nor fully non-revealing.

We introduce the concept of the dimension of a convex set. If A is a convex subset of $\mathbb{R}^{L_1}$, the dimension of A is the dimension of the smallest affine space containing A. For instance, the dimension of Δ_1 is $|L_1| - 1$. If the set A is a singleton, then its dimension is 0.

Proposition 6.1. *Let $(p_s)_{s\in S}$ be the family of second period prices of an informationally constrained equilibrium. For every agent $j \in J$, the information $\Pi_s^j(b_s^j)$ revealed by the bound $b_s^j = \beta_s^j(p_s)$ is a convex subset of Δ_1°. More precisely, we have*

$$\Pi_s^j(\beta_s^j(p_s)) = \left\{\pi_s \in \Delta_1^\circ \,:\, \pi_s \cdot \left[e_s^j(p_s \cdot \xi) - \xi(p_s \cdot e_s^j)\right] = 0\right\}.$$

Remark 6.1. If agent j's endowment is marketable at state $s \in S$, that is, $e_s^j \in \{\lambda\xi : \lambda > 0\}$, then no information is revealed.

If agent j's endowment is not marketable at state $s \in S$, that is, $e_s^j \notin \{\lambda\xi : \lambda > 0\}$, then the information set $\Pi_s^j(\beta_s^j(p_s))$ is the intersection of the simplex Δ_1° with a hyperplane passing through p_s. In particular, it is of dimension $|L_1| - 2$. Fig. 1 illustrates the cases for $L_1 = 2$ and $L_1 = 3$. If there are at most two goods at period $t = 1$, then an informationally constrained equilibrium is fully revealing, otherwise, it is only partially revealing.

7. Formation of expectations over time

We can easily extend the model to more than two periods and investigate the formation of price expectations over time. For notational simplicity, we undertake this exercise in a simple setting with four periods, i.e., $t \in \mathcal{T} = \{0, 1, 2, 3\}$, no exogenous uncertainty, one consumption good per per period and one asset delivering a unit of the consumption good as return. Any agent $h \in H$ is characterized by a family of instantaneous utility functions $u_t^h : [0, \infty) \to \mathbb{R}$ and endowments $e_t^h > 0$. Each period $t \in \mathcal{T}$, the price of the consumption good is normalized to 1, so the endogenous uncertainty is about the future path of asset prices $(q_{t+1}, \ldots, q_3)$.

At any period $t \in \mathcal{T}$, given $q_t > 0$ and $b_t^h \leq 0$, agent $h \in H$ chooses an action

$$(x_t^h, \theta_t^h) \in [0, \infty) \times [b_t^h, \infty).$$

Subject to no default at any period, the sequence of budget restrictions is given by

$$\forall t \in \{0,1,2\}, \quad x_t^h + q_t\theta_t^h \le e_t^h + \theta_{t-1}^h$$

and

$$x_3^h \le e_3^h + \theta_2^h,$$

where by convention $q_3 = 0$ and $\theta_3^h = 0$. Since we look for bounds that ensure solvency, we consider the family of functions $(\beta_t^h)_{t\in\{0,1,2\}}$ such that

$$\beta_2^h = -e_3^h$$
$$\forall \rho_2 \in (0,\infty), \quad \beta_1^h(\rho_2) = -e_2^h + \rho_2\beta_2^h = -e_2^h - \rho_2 e_3^h$$

and

$$\forall (\rho_1,\rho_2) \in (0,\infty)^2, \quad \beta_0^h(\rho_1,\rho_2) = -e_1^h + \rho_1\beta_1^h(\rho_2) = -e_1^h - \rho_1 e_2^h - \rho_1\rho_2 e_3^h.$$

We assume that minimally rational agents understand that bounds are determined by the family of functions $(\beta_t^j)_{t\in\{0,1,2\}}$. At $t=0$, agent j observes the equilibrium price $q_0 > 0$ and the bound $b_0^j = \beta_0^j(q_1,q_2)$ where (q_1,q_2) are the future equilibrium prices. The agent needs to form expectations about (q_1,q_2). Actually, he can infer that the pair (q_1,q_2) belongs to the set

$$\Pi_0^j(b_0^j) = \{(\rho_1,\rho_2) \in (0,\infty)^2 : \ \beta_0^j(\rho_1,\rho_2) = b_0^j\}.$$

That is, $\Pi_0^j(b_0^j)$ is the following hyperbola

$$\Pi_0^j(b_0^j) = \{(\rho_1,\rho_2) \in (0,\infty)^2 : \ \rho_2 e_3^j = -e_2^j - (e_1^j + b_0^j)/\rho_1\}.$$

Since $b_0^j = \beta^j(q_1,q_2)$, we must have

$$b_0^j + e_1^j = -q_1 e_2^j - q_1 q_2 e_3^j < 0.$$

This implies that agent j infers that the equilibrium price q_1 belongs to the interval

$$\left(0, -\frac{b_0^j + e_1^j}{e_2^j}\right),$$

but has no information about the equilibrium price q_2.

At period $t=1$, agent j observes the equilibrium price q_1 together with the bound $b_1^j = \beta_1^j(q_2)$. He infers that the price q_2 lies in the set

$$\Pi_1^j(b_1^j) = \{\rho_2 > 0 : \ \beta_1^j(\rho_2) = b_1^j\} = \left\{-\frac{b_1^j + e_2^j}{e_3^j}\right\} = \{q_2\}.$$

That is, the bound at period $t=1$ reveals the exact equilibrium price at period $t=2$.

We notice that

$$\underbrace{\Pi_1^j(b_1^j)}_{\substack{\text{information}\\ \text{revealed at } t=1}} = \underbrace{\Pi_0^j(b_0^j)}_{\substack{\text{information}\\ \text{revealed at } t=0}} \bigcap \underbrace{[\{q_1\}\times(0,\infty)]}_{\substack{\text{signal observed at}\\ t=1}}.$$

Equivalently, the information revealed to minimally rational agents is time consistent.

8. Conclusion

The paper studies an economy in which agents differ in their ability to anticipate future prices. Contrary to temporary equilibrium paradigm that takes expectations as part of the primitives of the economy, we explore a channel that allows for the endogenous formation of price expectations. Agents make inferences about future prevailing prices by observing bounds on short-sales that ensure solvency at any contingency. They know the map from prices to bounds that are compatible with no bankruptcy, so, they are capable of inferring a set of prices assuring non-negative wealth the second period. Provided that an equilibrium exists, the vector of equilibrium prices will be contained in this set. Though expectations are endogenous, they are, in general, neither common nor degenerate.

Our aim is to show that institutional arrangements can, to some extent, diminish the coordination requirements on the realization of future prices. A shortcoming of our approach is that we do not explain how the debt limits are enforced at equilibrium.[26] We implicitly assume that there is an entity that is able to compute the aggregate demand and solve the fixed-point problem to perfectly anticipate market clearing prices. This is clearly not satisfactory. Nevertheless, it is generally argued that financial institutions acquire superior information about the state of the economy than individual traders. If their objectives assimilate somehow those of the hypothetical entity, the analysis may provide a first approximation of the rationale underlying the enforcement of borrowing constraints.

Appendix A. Proofs

This appendix contains the formal proofs of the results presented in Section 5.2. It exploits some technical lemmas proven in Appendix B.

For our purposes, given a vector of contingent prices $p_1 = (p_s)_{s\in S} \in [\Delta_1^\circ]^S$, we define the correspondence $\tilde{P}^j(p_1) := \prod_{s\in S} \tilde{P}^j_s(p_s)$ where

$$\forall s \in S, \quad \tilde{P}^j_s(p_s) := \left\{\pi_s \in \Delta_1 : \pi_s \cdot \left[e^j_s(p_s \cdot \xi) - \xi(p_s \cdot e^j_s)\right] = 0\right\}.$$

Notice that $\tilde{P}^j$ is non-empty (since $p_1 \in P^j(p_1) \subset \tilde{P}^j(p_1)$), closed and compact-valued. Lemma B.3 shows that it is also continuous on $[\Delta_1^\circ]^S$.

A.1. Proof of Lemma 5.1

The result follows if we show that W^j_0 is continuous on $X^j_0 \times \Theta^j \times [\Delta_1^\circ]^S$. The following claims show that this is true for both specifications of W^j_0.

Claim A.1. *The map*

$$(p_1, \theta) \mapsto \inf\left\{v^j(\pi_1, \theta) + c^j(\pi_1) \;:\; \pi_1 \in P^j(p_1)\right\}$$

is continuous on $[\Delta_1^\circ]^S \times \Theta^j$.

Proof. Since $v^j(\pi_1, \theta) = +\infty$ at any $\pi_1 = (\pi_s)_{s\in S}$ with $\pi_s \in \partial\Delta_1$ for some $s \in S$, taking the infimum over $P^j(p_1)$ or its closure $\tilde{P}^j(p_1)$ makes no difference. Upper semi-continuity is then a direct consequence of Lemma B.2 and Lemma B.3.

To show that the map is lower semi-continuous is more subtle. The argument exploits an extension of Berge's Theorem to problems with non-compact image sets (see Theorem 1.2 in Feinberg et al. (2013)). More precisely, lower semi-continuity follows if we show that $v^j + c^j$ is $\mathbb{K}$-inf-compact on $\mathrm{gph}(\tilde{P}^j) \times \Theta^j$.[27]

Let $\lambda \in \mathbb{R}$ and K be a compact set of $[\Delta_1^\circ]^S$. By Lemma B.2, $v^j + c^j$ is continuous on $[\Delta_1^\circ]^S \times \Theta^j$, so the set

$$\Lambda(\lambda) := \left\{(\pi_1, \theta) \in [\Delta_1^\circ]^S \times \Theta^j : v^j(\pi_1, \theta) + c^j(\pi_1) \le \lambda\right\}$$

is a closed subset of $[\Delta_1^\circ]^S \times \Theta^j$. The set

$$\left\{(p_1, \pi_1, \theta) \in K \times \tilde{P}^j(p_1) \times \Theta^j : v^j(\pi_1, \theta) + c^j(\pi_1) \le \lambda\right\}$$

is a subset of $\mathrm{gph}_K(\tilde{P}^j) \times \Theta^j$. It is also compact since $\Lambda(\lambda)$ is closed and $\tilde{P}^j$ is compact-valued. This suffices to prove the claim. □

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[26] This is always an issue in sequentially trading economies with borrowing constraints.

[27] For a topological space $\mathbb{X}$, we denote by $\mathbb{K}(\mathbb{X})$ the family of all nonempty compact subsets of $\mathbb{X}$. For an extended real-valued function f, defined on a nonempty subset K of a topological space $\mathbb{X}$, consider the level sets

$$\mathcal{D}_f(\lambda; K) = \{y \in K : f(y) \le \lambda\}, \quad \lambda \in \mathbb{R}.$$

We say that a function f is *inf-compact* on K if all these sets are compact.

Let $\Phi : \mathbb{X} \to 2^{\mathbb{Y}} \setminus \{\emptyset\}$ be a correspondence with its graph on a subset K of $\mathbb{X}$ be defined as follows

$$\mathrm{gph}_K(\Phi) = \left\{(x, y) \in K \times \mathbb{Y} : y \in \Phi(x)\right\}.$$

A function $u : \mathbb{X} \times \mathbb{Y} \to \overline{\mathbb{R}}$ is called $\mathbb{K}$-*inf-compact* on $\mathrm{gph}(\Phi)$, if for every $K \in \mathbb{K}(\mathbb{X})$ this function is inf-compact on $\mathrm{gph}_K(\Phi)$.

Claim A.2. *The map*

$$(p_1,\theta)\mapsto \int\limits_{P^j(p_1)} \Phi^j\left(\nu^j(\pi_1,\theta)\right)\nu^j(\pi_1|p_1)$$

is continuous on $[\Delta_1^\circ]^S\times\Theta^j$.

Proof. The function Φ^j is assumed to be continuous and bounded from above. It follows that $\Phi^j\circ\nu^j$ is continuous and bounded on $[\Delta_1^\circ]^S\times\Theta^j$ (since ν^j is continuous and bounded from below), so, without any loss of generality, we can assume that $\Phi^j\circ\nu^j$ is positive. Since the measure of the boundary of Δ_1^S is zero, taking the integral over $P^j(p_1)$ or its closure $\tilde{P}^j(p_1)$ makes no difference.

Let $\overline{B(0,\epsilon)}$ be the closed ball in $\mathbb{R}^{L_1\times S}$ given by

$$\overline{B(0,\epsilon)}:=\left\{(\pi_s)_{s\in S}\in\mathbb{R}^{L_1\times S}:\forall s\in S,\quad \|\pi_s\|\le\epsilon\quad\text{and}\quad \pi_s\cdot\mathbf{1}_{L_1}=0\right\}.$$

Given $p_1\in[\Delta_1^\circ]^S$ and $\tilde{P}^j(p_1)$, we consider the set $B_\epsilon(\tilde{P}^j(p_1))\subset G$ defined by[28]

$$B_\epsilon(\tilde{P}^j(p_1)):=\tilde{P}^j(p_1)+\overline{B(0,\epsilon)}.$$

Let $\epsilon>0$ and $(p_1^n,\theta^n)_{n\in\mathbb{N}}$ be a sequence on $[\Delta_1^\circ]^S\times\Theta^j$ that converges to $(p_1,\theta)\in[\Delta_1^\circ]^S\times\Theta^j$.

By Lemma B.4, there exists N such that $\tilde{P}^j(p_1^n)\subset B_\epsilon(\tilde{P}^j(p_1))$ for all $n>N$. Let $A_n:=\{\pi_1\in\tilde{P}^j(p_1^n):\pi_1\notin\tilde{P}^j(p_1^n)\cap\tilde{P}^j(p_1)\}$. For n large enough, A_n is a subset of the difference $B_\epsilon(\tilde{P}^j(p_1))\setminus\tilde{P}^j(p_1)$.

Given an integer N, let Z stand for the closure of $\cup_{n>N}\tilde{P}^j(p_1^n)$. By Lemma B.5, there exists N such that $\tilde{P}^j(p_1)\subset B_\epsilon(\tilde{P}^j(p_1^n))\subset B_\epsilon(Z)\subset G$ for all $n>N$.[29] Let $A'_n:=\{\pi_1\in\tilde{P}^j(p_1):\pi_1\notin\tilde{P}^j(p_1^n)\cap\tilde{P}^j(p_1)\}$. For n large enough, A'_n is a subset of the difference $B_\epsilon(Z)\setminus Z$.

Observe that

$$\begin{aligned}
&\int\limits_{\tilde{P}^j(p_1^n)} \Phi^j\left(\nu^j(\pi_1,\theta^n)\right)\nu^j(\pi_1|p_1^n)-\int\limits_{\tilde{P}^j(p_1)} \Phi^j\left(\nu^j(\pi_1,\theta)\right)\nu^j(\pi_1|p_1)\\
&=\left[\int\limits_{A_n}\Phi^j\left(\nu^j(\pi_1,\theta^n)\right)\nu^j(\pi_1|p_1^n)+\int\limits_{\tilde{P}^j(p_1^n)\cap\tilde{P}^j(p_1)}\Phi^j\left(\nu^j(\pi_1,\theta^n)\right)\nu^j(\pi_1|p_1^n)\right]\\
&\quad-\left[\int\limits_{A'_n}\Phi^j\left(\nu^j(\pi_1,\theta)\right)\nu^j(\pi_1|p_1)+\int\limits_{\tilde{P}^j(p_1^n)\cap\tilde{P}^j(p_1)}\Phi^j\left(\nu^j(\pi_1,\theta)\right)\nu^j(\pi_1|p_1)\right].
\end{aligned}$$

Since $\Phi^j\circ\nu^j$ is bounded and continuous, the compactness of $\tilde{P}^j(p_1)$ and the continuity of the density h^j implies that, for n large enough,[30]

$$\begin{aligned}
&\int\limits_{\tilde{P}^j(p_1^n)\cap\tilde{P}^j(p_1)}\Phi^j\left(\nu^j(\pi_1,\theta^n)\right)\nu^j(\pi_1|p_1^n)-\int\limits_{\tilde{P}^j(p_1^n)\cap\tilde{P}^j(p_1)}\Phi^j\left(\nu^j(\pi_1,\theta)\right)\nu^j(\pi_1|p_1)\\
&\le\int\limits_{\tilde{P}^j(p_1)}|\Phi^j(\nu^j(\pi_1,\theta^n))-\Phi^j(\nu^j(\pi_1,\theta))|h^j(\pi_1|p_1^n)m(\pi_1)\\
&\quad+\int\limits_{\tilde{P}^j(p_1)}\Phi^j(\nu^j(\pi_1,\theta))|h^j(\pi_1|p_1^n)-h^j(\pi_1|p_1)|m(\pi_1)\\
&\le B\int\limits_{\tilde{P}^j(p_1)}|\Phi^j(\nu^j(\pi_1,\theta^n))-\Phi^j(\nu^j(\pi_1,\theta))|m(\pi_1)
\end{aligned}$$

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28 Recall that G is the affine space spanned by Δ_1^S defined in Remark 5.2.

29 $B_\epsilon(Z):=Z+\overline{B(0,\epsilon)}$.

30 The last inequality follows from an application of the Lebesgue Dominated Convergence Theorem.

$$+B' \int\limits_{\tilde{P}^j(p_1)} |h^j(\pi_1|p_1^n) - h^j(\pi_1|p_1)| m(\pi_1)$$
$$\leq \epsilon.$$

Therefore, for n large enough,

$$\int\limits_{\tilde{P}^j(p_1^n)} \Phi^j\left(\nu^j(\pi_1,\theta^n)\right)\nu^j(\pi_1|p_1^n) - \int\limits_{\tilde{P}^j(p_1)} \Phi^j\left(\nu^j(\pi_1,\theta)\right)\nu^j(\pi_1|p_1)$$
$$\leq \int\limits_{A_n} \Phi^j\left(\nu^j(\pi_1,\theta^n)\right)\nu^j(\pi_1|p_1^n) + \epsilon$$
$$\leq \int\limits_{B_\epsilon(\tilde{P}^j(p_1))\setminus\tilde{P}^j(p_1)} \Phi^j\left(\nu^j(\pi_1,\theta^n)\right)h^j(\pi_1|p_1^n)m(\pi_1) + \epsilon$$
$$\leq C\left[m(B_\epsilon(\tilde{P}^j(p_1))) - m(\tilde{P}^j(p_1))\right] + \epsilon,$$

where the last inequality is due to the boundedness of $\Phi^j \circ \nu^j$, the compactness of $B_\epsilon(\tilde{P}^j(p_1))$ and the continuity of h^j.

We can now claim that the map is upper semi-continuous. Indeed,

$$\limsup_n \int\limits_{\tilde{P}^j(p_1^n)} \Phi^j\left(\nu^j(\pi_1,\theta^n)\right)\nu^j(\pi_1|p_1^n) \leq \int\limits_{\tilde{P}^j(p_1)} \Phi^j\left(\nu^j(\pi_1,\theta)\right)\nu^j(\pi_1|p_1)$$
$$+C\left[m(B_\epsilon(\tilde{P}^j(p_1))) - m(\tilde{P}^j(p_1))\right] + \epsilon,$$

and the term inside the brackets disappears when ϵ goes to zero.

A specular argument shows that, for n large enough,

$$\int\limits_{\tilde{P}^j(p_1)} \Phi^j\left(\nu^j(\pi_1,\theta)\right)\nu^j(\pi_1|p_1) - \int\limits_{\tilde{P}^j(p_1^n)} \Phi^j\left(\nu^j(\pi_1,\theta^n)\right)\nu^j(\pi_1|p_1^n)$$
$$\leq \int\limits_{A_n'} \Phi^j\left(\nu^j(\pi_1,\theta)\right)\nu^j(\pi_1|p_1) + \epsilon$$
$$\leq \int\limits_{B_\epsilon(Z)\setminus Z} \Phi^j\left(\nu^j(\pi_1,\theta)\right)\nu^j(\pi_1|p_1) + \epsilon$$
$$\leq C'\left[m(B_\epsilon(Z)) - m(Z)\right] + \epsilon.$$

It follows that

$$\liminf_n \int\limits_{\tilde{P}^j(p_1^n)} \Phi^j\left(\nu^j(\pi_1,\theta^n)\right)\nu^j(\pi_1|p_1^n) \geq \int\limits_{\tilde{P}^j(p_1)} \Phi^j\left(\nu^j(\pi_1,\theta)\right)\nu^j(\pi_1|p_1)$$
$$-C'\left[m(B_\epsilon(Z)) - m(Z)\right] - \epsilon.$$

This proves that the map is lower semi-continuous as the term inside the brackets disappears when ϵ goes to zero. □

A.2. Proof of Proposition 5.1

For each integer $n \in \mathbb{N}$, let Δ_1^n be the non-empty, compact and convex subset of Δ_1° given by

$$\Delta_1^n := \left\{ p_s \in \Delta_1 \ : \ p_s \geq \frac{1}{|L_1|(n+1)}\mathbf{1}_{L_1} \right\}.$$

We consider the correspondence $\chi_0 : \prod_{h\in H} X_0^h \times \Theta^h \longrightarrow 2^{\Delta_0}$ defined by

$$\chi_0((x_0^h,\theta^h)_{h\in H}) := \operatorname{argmax}\left\{ \sum_{h\in H} p_0 \cdot (x_0^h - e_0^h) + q\cdot\theta^h \ : \ (p_0,q)\in\Delta_0 \right\}.$$

For $n \in \mathbb{N}$ and $s \in S$, we also consider the correspondence $\chi_s^n : \prod_{h\in H} X_1^h \longrightarrow 2^{\Delta_1^n}$ defined by

$$\chi_s^n((x_s^h)_{h\in H}) := \operatorname{argmax} \left\{ \sum_{h\in H} p_s \cdot (x_s^h - e_s^h) : \ p_s \in \Delta_1^n \right\}.$$

Denote by K^n the non-empty, compact and convex set

$$K^n = \Delta_0 \times [\Delta_1^n]^S \times \prod_{h\in H} X_0^h \times \Theta^h \times [X_1^h]^S,$$

and let φ^n be the correspondence from K^n to K^n defined by

$$\varphi^n(\mathrm{P}, (x_0^h, \theta^h, (x_s^h)_{s\in S})_{h\in H}) = \chi_0((x_0^h, \theta^h)_{h\in H}) \times \prod_{s\in S} \chi_s^n((x_s^h)_{h\in H}) \times \prod_{h\in H} \delta^h(\mathrm{P}).$$

The correspondence φ^n is upper semi-continuous with non-empty, compact and convex values. Kakutani's Theorem implies that φ^n has a fixed-point $\{(p_0^n, q^n, \boldsymbol{a}^n), (p_s^n, \boldsymbol{x}_s^n)_{s\in S}\}$. Since every set K^n is a subset of the compact set

$$K = \Delta_0 \times \Delta_1^S \times \prod_{h\in H} X_0^h \times \Theta^h \times [X_1^h]^S,$$

passing to a subsequence if necessary, $\{(p_0^n, q^n, \boldsymbol{a}^n), (p_s^n, \boldsymbol{x}_s^n)_{s\in S}\}_{n\in\mathbb{N}}$ converges to an element $\{(p_0, q, \boldsymbol{a}), (p_s, \boldsymbol{x}_s)_{s\in S}\}$ in K.

Claim A.3. *The family* $\{(p_0, q, \boldsymbol{a}), (p_s, \boldsymbol{x}_s)_{s\in S}\}$ *is an equilibrium of the economy* $\mathcal{E} = (X_0^h, X_1^h, \Theta^h)_{h\in H}$.

Proof. We consider two cases.

Case 1. Assume that the set of sophisticated agents is non-empty, i.e., $I \neq \emptyset$.

Since the demand correspondence of sophisticated agents is upper semi-continuous on $\Delta_0 \times \Delta_1^S$, we have that

$$\forall i \in I, \quad (x_0^i, \theta^i, (x_s^i)_{s\in S}) \in \delta^i(p_0, q, (p_s)_{s\in S}).$$

Summing up the budget constraints at period $t = 0$ across all agents, and using the fact that $(p_0, q) \in \chi_0((x_0^h, \theta^h)_{h\in H})$ we obtain that

$$\forall (p_0', q') \in \Delta_0, \quad p_0' \cdot \sum_{h\in H}(x_0^h - e_0^h) + q' \cdot \sum_{h\in H} \theta^h \leq \ p_0 \cdot \sum_{h\in H}(x_0^h - e_0^h) + q \cdot \sum_{h\in H} \theta^h \leq 0.$$

The above implies that

$$\sum_{h\in H}(x_0^h - e_0^h) \leq 0 \quad \text{and} \quad \sum_{h\in H} \theta^h \leq 0.$$

In particular, for any sophisticated agent $i \in I$, we have that $x_0^i \leq e_0$, where e_0 is the aggregate endowment that lies in the interior of X_0^i. Since the utility functions are strictly increasing, we infer that $p_0 \in \mathbb{R}_{++}^{L_0}$.

Fix a state $s \in S$. Summing up the budget constraints across all agents, and using the fact that $p_s^n \in \chi_s^n((x_s^h)_{h\in H})$ we obtain that

$$\forall p_s \in \Delta_1^n, \quad \sum_{h\in H} p_s \cdot (x_s^{h,n} - e_s^h) \leq \sum_{h\in H} p_s^n \cdot (x_s^{h,n} - e_s^h) \leq 0.$$

Let $p_s' \in \Delta_1^\circ$. There exists n_0 such that, for any $n \geq n_0$, we have $p_s' \in \Delta_1^n$. Hence,

$$\forall n \geq n_0, \quad \sum_{h\in H} p_s' \cdot (x_s^{h,n} - e_s^h) \leq \sum_{h\in H} p_s^n \cdot (x_s^{h,n} - e_s^h) \leq 0.$$

Passing to the limit, we get that

$$\sum_{h\in H} p_s' \cdot (x_s^h - e_s^h) \leq \sum_{h\in H} p_s \cdot (x_s^h - e_s^h) \leq 0.$$

Since p_s' was arbitrary, the inequality holds for any $p_s' \in \Delta_1^\circ$ and, by continuity, for any $p_s' \in \Delta_1$. Therefore, we get that

$$\sum_{h\in H}(x_s^h - e_s^h) \leq 0.$$

In particular, for any sophisticated agent $i \in I$, we have that $x^i_s \leq e_s$, where e_s is the aggregate endowment at state $s \in S$ that lies in the interior of X^i_1. Since the utility functions are strictly increasing, we infer that $p_s \in \mathbb{R}^{L_1}_{++}$.

Observe that any security delivers, contingent to a state $s \in S$, the bundle ξ. Since p_s is strictly positive, the asset delivers the strictly positive value $p_s \cdot \xi$. By non-arbitrage, the price q_s at period $t = 0$ must be strictly positive.

Lemma 5.1 shows that the demand correspondence δ^j is upper semi-continuous on $\Delta_0 \times [\Delta^n_1]^S$. Therefore, as $(p_0, q, (p_s)_{s\in S})$ is a family of strictly positive prices, we obtain

$$\forall j \in J, \quad (x^j_0, \theta^j, (x^j_s)_{s\in S}) \in \delta^j(p_0, q, (p_s)_{s\in S}).$$

To conclude the proof, we show that all markets clear.

As utility functions are strictly increasing, the budget constraints must be binding at the optimal solution. This implies that

$$p_0 \cdot \sum_{h\in H}(x^h_0 - e^h_0) = 0 \quad \text{and} \quad q \cdot \sum_{h\in H}\theta^h = 0,$$

together with

$$\forall s \in S, \quad p_s \cdot \sum_{h\in H}(x^h_s - e^h_s) = 0.$$

Market clearing follows as all prices are strictly positive.

Case 2. Assume that the set of sophisticated agents is empty, i.e., $I = \emptyset$.

We consider a sequence of artificial economies, each one populated by the minimally rational agents and a single sophisticated agent, indexed by $i \in \mathbb{N}$. The fundamentals of the artificial agent are as follows: at every $s \in \{0\} \cup S$ endowments equal $e^i_s(\ell) = \alpha/i$, where $\ell \in L_0 \cup L_1$ and $\alpha \in (0,1)$; the utilities $(u^i_s)_{s\in\{0\}\cup S}$ are assumed to be linear with all good coefficients equal to 1.

Any artificial economy has a competitive equilibrium $\{(p^i_0, q^i, \boldsymbol{a}^i), (p^i_s, \boldsymbol{x}^i_s)_{s\in S}\}$ with all prices $(p^i_0, q^i, (p^i_s)_{s\in S})$ be strictly positive. We claim that the second period commodity prices are, in fact, uniformly bounded away from zero, with the bound be independent of the index i. Indeed, fix $s \in S$ and $\ell \in L_1$. From the optimality conditions of the artificial agent i we infer that

$$\frac{p^i_s(\ell)}{p^i_s(k)} \geq 1, \text{ for all } k \in L_1 \setminus \{\ell\}.$$

Summing over $k \in L_1 \setminus \{\ell\}$ and exploiting the fact that $p^i_s \in \Delta_1$ we obtain that

$$p^i_s(\ell) \geq \frac{1}{|L_1|}.$$

This proves that p^i_s is uniformly bounded away from zero.

Passing to a subsequence if necessary, $\{(p^i_0, q^i, \boldsymbol{a}^i), (p^i_s, \boldsymbol{x}^i_s)_{s\in S}\}_{i\in\mathbb{N}}$ converges to an element $\{(p_0, q, \boldsymbol{a}), (p_s, \boldsymbol{x}_s)_{s\in S}\}$. Since $p_s \in \Delta^\circ_1$, by Lemma 5.1 (the demand correspondence δ^j is upper semi-continuous on $\Delta_0 \times [\Delta^\circ_1]^S$), we get that

$$\forall j \in J, \quad (x^j_0, \theta^j, (x^j_s)_{s\in S}) \in \delta^j(p_0, q, (p_s)_{s\in S}).$$

Since the utility functions are strictly increasing we can infer that $p_0 \in \mathbb{R}^{L_0}_{++}$ and $q \in \mathbb{R}^S_{++}$.

We claim that the family $\{(p_0, q, \boldsymbol{a}), (p_s, \boldsymbol{x}_s)_{s\in S}\}$ is an equilibrium of the original economy that is solely populated by minimally rational agents. It suffices to show that the optimal consumption and investment decisions of the artificial agent converge to zero as i approaches to infinite.

Indeed, for any $i \in \mathbb{N}$,

$$p^i_0 \cdot (x^i_0 - e^i_0) + q^i \cdot \theta^i = 0.$$

In addition, for all $s \in S$,

$$\theta^i_s \geq \frac{-\sum_{\ell\in L_1} e^i(\ell)}{p^i_s \cdot \xi} = \frac{-|L_1|(\alpha/i)}{p^i_s \cdot \xi}$$

and

$$p^i_s \cdot (x^i_s - e^i_s) = (p^i_s \cdot \xi)\theta^i_s.$$

Since $e^i_s(\ell) = \alpha/i$ for any $s \in \{0\} \cup S$ and $\ell \in L_0 \cup L_1$, taking the limit as i goes to infinite proves the claim. □

A.3. Proof of Proposition 5.2

Assume that $\{(p_0, q, \boldsymbol{a}), (p_s, \boldsymbol{x}_s)_{s\in S}\}$ is an equilibrium of the bounded economy $\mathcal{E}$. Fix and agent $h \in H$. To prove the result it suffices to show that $(a^h, (x^h_s)_{s\in S})$ is still the optimal choice in the frictionless economy $\mathcal{E}^f$.

Assume on the contrary that there exists another allocation $(a^{h\prime}, (x^{h\prime}_s)_{s\in S}) \in X_0 \times \Theta \times [X_1]^S$ that is budget feasible and gives agent h higher utility. It is then easy to find $\lambda \in (0, 1]$ such that the allocation

$$(\bar{a}^h, (\bar{x}^h_s)_{s\in S}) = \lambda(a^h, (x^h_s)_{s\in S}) + (1-\lambda)(a^{h\prime}, (x^{h\prime}_s)_{s\in S})$$

is still budget feasible and lies on the set $X^h_0 \times \Theta^h \times [X^h_1]^S$. As utility functions are assumed to be concave, we conclude that $(\bar{a}^h, (\bar{x}^h_s)_{s\in S})$ is strictly preferable to $(a^h, (x^h_s)_{s\in S})$. This contradicts the optimality of $(a^h, (x^h_s)_{s\in S})$ in the bounded economy $\mathcal{E}$.

Appendix B. Preliminary results

We here state some technical results that are used in the main proofs. To this purpose, for any integer $n \in \mathbb{N}$, let Δ^n_1 be a non-empty, compact and convex subset of Δ°_1 given by

$$\Delta^n_1 := \left\{\pi \in \Delta_1 \ : \ \pi \geq \frac{1}{|L_1|(n+1)}\mathbf{1}_{L_1}\right\}.$$

Lemma B.1. *Fix $s \in S$. For any $n \in \mathbb{N}$, the correspondence*

$$B^j_s : \Delta_1 \times \Theta^j \to 2^{X_1}$$

defined by

$$B^j_s(\pi_s, \theta) = \left\{x_s \in X_1 \ : \pi_s \cdot x_s \leq \max\{0, \pi_s \cdot [e^j_s + \theta_s \xi]\}\right\}$$

is continuous on $\Delta^n_1 \times \Theta^j$ with non-empty, compact and convex values.

Proof. Notice that there exists a compact set $X^{j,n}_1$ of X_1 such that

$$\forall (\pi_s, \theta) \in \Delta^n_1 \times \Theta^j, \quad B^j_s(\pi_s, \theta) \subset X^{j,n}_1.$$

It is also true that the set

$$\operatorname{gph}(B^j_s) \cap \left[\Delta^n_1 \times \Theta^j \times X^{j,n}_1\right]$$

is closed. Let $B^{j,n}_s : \Delta^n_1 \times \Theta^j \to 2^{X^{j,n}_1}$ be the correspondence defined by

$$B^{j,n}_s(\pi_s, \theta) = B^j_s(\pi_s, \theta).$$

The Closed Graph Theorem (Aliprantis and Border, 2006, Theorem 17.11) applies to $B^{j,n}_s$, so the correspondence B^j_s is upper semi-continuous on $\Delta^n_1 \times \Theta^j$.

The proof of lower semi-continuity deserves more attention. If $\pi_s \cdot [e^j_s + \theta_s \xi] > 0$, it is easy to show that B^j_s is lower semi-continuous at (π_s, θ). Assume next that $\pi_s \cdot [e^j_s + \theta_s \xi] \leq 0$, and choose a sequence $(\pi^k_s, \theta^k)_{k\in\mathbb{N}}$ converging to (π_s, θ). Fix x_s in $B^j_s(\pi_s, \theta)$. Since π_s belongs to Δ^n_1, we must have $x_s = 0$. Choose now the sequence $(x^k_s)_{k\in\mathbb{N}}$ defined by $x^k_s = 0$ for each k. It is obvious that $(x^k_s)_{k\in\mathbb{N}}$ converges to x_s and, for each k, we have $x^k_s \in B^j_s(\pi^k_s, \theta^k)$. This proves the claim. □

Lemma B.2. *The map $(\pi_1, \theta) \mapsto v^j(\pi_1, \theta) + c^j(\pi_1)$ is upper semi-continuous on $[\Delta_1]^S \times \Theta^j$ and continuous on $[\Delta^\circ_1]^S \times \Theta^j$.*

Proof. Observe that $v^j_s(\pi_s, \theta) = +\infty$ when $\pi_s \in \partial\Delta_1$.[31] Let $(\pi^n_s, \theta^n)_{n\in\mathbb{N}}$ be a sequence on $\Delta_1 \times \Theta^j$ that converges to (π_s, θ), where $\pi_s \in \partial\Delta_1$. It follows that

$$\limsup_n v^j_s(\pi^n_s, \theta^n) \leq v^j_s(\pi_s, \theta) = +\infty.$$

Fix an integer $n \in \mathbb{N}$. Given Claim B.1, continuity of v^j_s on $\Delta^n_1 \times \Theta^j$ follows from the Maximum Theorem. Since this is true for any $n \in \mathbb{N}$, we get that v^j_s is continuous on $\Delta^\circ_1 \times \Theta^j$.

The claim is true as v^j is the weighted sum of $(v^j_s)_{s\in S}$, and by assumption c^j is upper semi-continuous and convex. □

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[31] Indeed, if $\pi_s(\ell) = 0$ for some good $\ell \in L_1$, then the demand for that good is infinite, i.e., $x_s(\ell) = +\infty$, and $u^j_s(x_s) = +\infty$.

Lemma B.3. *The correspondence $\tilde{P}^j$ is continuous on $[\Delta_1^\circ]^S$.*

Proof. Fix $s \in S$. The correspondence $\tilde{P}_s^j$ is non-empty, compact-valued and closed. Therefore, it is upper semi-continuous by an application of the Closed Graph Theorem (Aliprantis and Border, 2006, Theorem 17.11). Upper semi-continuity of $\tilde{P}^j$ follows from the fact that it is the product of compact-valued upper semi-continuous correspondences.

It is also true that the product of lower semi-continuous correspondences is lower semi-continuous. However, proving that $\tilde{P}_s^j$ is lower semi-continuous is slightly more involved.

Let $(p_s^n)_{n\in\mathbb{N}}$ be a sequence in Δ_1° that converges to some $p_s \in \Delta_1^\circ$. Fix $\pi_s \in \tilde{P}_s^j(p_s)$. If $e_s^j = \lambda\xi$ for some $\lambda > 0$, then $\tilde{P}_s^j(p_s) = \Delta_1$ and the claim is trivially true. We impose $e_s^j \neq \lambda\xi$ for any $\lambda > 0$ and consider three cases:

(i) Suppose, passing to a subsequence if necessary, that $\pi_s \cdot \left[e_s^j(p_s^n \cdot \xi) - \xi(p_s^n \cdot e_s^j)\right] = 0$ for all $n \in \mathbb{N}$. In this case, simply set $\pi_s^n = \pi_s$ for all $n \in \mathbb{N}$.

(ii) Suppose, passing to a subsequence if necessary, that $\pi_s \cdot \left[e_s^j(p_s^n \cdot \xi) - \xi(p_s^n \cdot e_s^j)\right] > 0$ for all $n \in \mathbb{N}$. We claim that there exists $\pi_s' \in \Delta_1^\circ$ such that

$$\pi_s' \cdot \left[e_s^j(p_s \cdot \xi) - \xi(p_s \cdot e_s^j)\right] < 0. \qquad (*)$$

Indeed, if this is not true, then, for all $\pi_s' \in \Delta_1^\circ$,

$$\pi_s' \cdot \left[e_s^j(p_s \cdot \xi) - \xi(p_s \cdot e_s^j)\right] \geq 0.$$

By continuity, the inequality is true for all $\pi_s' \in \Delta_1$. Define next

$$\forall \ell \in L_1, \quad \gamma_\ell(p_s) := e_s^j(\ell)(p_s \cdot \xi) - \xi(\ell)(p_s \cdot e_s^j).$$

Observe that $\gamma_\ell(p_s) \geq 0$. Since $p_s \in \Delta_1^\circ \cap \tilde{P}^j(p_s)$, we have that $\sum_{\ell\in L_1} p_s(\ell)\gamma_\ell(p_s) = 0$. It follows that $\gamma_\ell(p_s) = 0$ for any $\ell \in L_1$. This in turn implies that $e_s^j(\ell) = \frac{p_s \cdot e_s^j}{p_s \cdot \xi}\xi(\ell)$, contradicting the fact that $e_s^j \neq \lambda\xi$ for any $\lambda > 0$.

Due to inequality (*), for n large enough, we have that $\pi_s' \cdot \left[e_s^j(p_s^n \cdot \xi) - \xi(p_s^n \cdot e_s^j)\right] < 0$. Let $\lambda_n \in (0, 1)$ be as follows

$$\lambda_n := \frac{\sum_{\ell\in L_1} \pi_s'(\ell)\gamma_\ell(p_s^n)}{\sum_{\ell\in L_1} \pi_s'(\ell)\gamma_\ell(p_s^n) - \sum_{\ell\in L_1} \pi_s(\ell)\gamma_\ell(p_s^n)},$$

and observe that

$$\lambda_n \sum_{\ell\in L_1} \pi_s(\ell)\gamma_\ell(p_s^n) + (1-\lambda_n)\sum_{\ell\in L_1} \pi_s'(\ell)\gamma_\ell(p_s^n) = 0.$$

Define next $\pi_s^n = \lambda_n\pi_s + (1-\lambda_n)\pi_s'$. By construction, for n large enough, $\pi_s^n \in \Delta_1$ and $\pi_s^n \cdot \left[e_s^j(p_s^n \cdot \xi) - \xi(p_s^n \cdot e_s^j)\right] = 0$. That is, $\pi_s^n \in \tilde{P}_s^j(p_s^n)$. Letting $n \to \infty$, we get that $\lambda_n \to 1$ and $\pi_s^n \to \pi_s$.

(iii) The case where $\pi_s \cdot \left[e_s^j(p_s^n \cdot \xi) - \xi(p_s^n \cdot e_s^j)\right] < 0$, for all $n \in \mathbb{N}$, unravels in a similar manner, therefore, the argument is omitted. □

We next exhibit two additional properties of the correspondence $\tilde{P}^j$. The first one follows from the fact that $\tilde{P}^j$ is upper semi-continuous with convex and compact values.

Lemma B.4. *Let $(p_1^n)_{n\in\mathbb{N}}$ be a sequence in $[\Delta_1^\circ]^S$ that converges to $p_1 \in [\Delta_1^\circ]^S$. Then, for any $\epsilon > 0$, there exists an integer N, such that, for any $n > N$, $\tilde{P}^j(p_1^n) \subset \tilde{P}^j(p_1) + \overline{B(0,\epsilon)}$, where $\overline{B(0,\epsilon)}$ is a closed ball in $\mathbb{R}^{L_1\times S}$ centered at 0 with radius ϵ.*

The second property is less obvious so it requires a proof.[32]

Lemma B.5. *Let $(p_1^n)_{n\in\mathbb{N}}$ be a sequence in $[\Delta_1^\circ]^S$ that converges to $p_1 \in [\Delta_1^\circ]^S$. Then, for any $\epsilon > 0$, there exists an integer N, such that, for any $n > N$, $\tilde{P}^j(p_1) \subset \tilde{P}^j(p_1^n) + \overline{B(0,\epsilon)}$, where $\overline{B(0,\epsilon)}$ is a closed ball in $\mathbb{R}^{L_1\times S}$ centered at 0 with radius ϵ. Equivalently, for any $\epsilon > 0$, there exists N, such that,*

$$\forall n > N, \quad \forall \pi_1 \in \tilde{P}^j(p_1), \quad \exists \pi_1^n \in \tilde{P}^j(p_1^n) \cap \overline{B(\pi_1,\epsilon)}.$$

[32] We are not aware of any general result that implies this property.

Proof. Assume that the statement is not true. Then, there exists $\epsilon > 0$, such that, given N, there exists $n(N) > N$ and $\pi_1(N) \in \tilde{P}^j(p_1)$ such that

$$\forall \pi_1 \in \tilde{P}^j(p_1^{n(N)}), \quad \|\pi_1 - \pi_1(N)\| > \epsilon.$$

Since the set $\tilde{P}^j(p_1)$ is compact, we can assume that $\pi_1(N) \to \bar{\pi}_1 \in \tilde{P}^j(p_1)$ as $N \to \infty$. That is, there exists M such that $\|\bar{\pi}_1 - \pi_1(N)\| < \epsilon/2$ for all $N > M$.

Hence,

$$\forall N > M, \quad \forall \pi_1 \in \tilde{P}^j(p_1^{n(N)}), \quad \|\pi_1 - \bar{\pi}_1\| \geq \|\pi_1 - \pi_1(N)\| - \|\bar{\pi}_1 - \pi_1(N)\| \geq \epsilon/2.$$

This delivers a contradiction. Indeed, since $\tilde{P}^j$ is lower semi-continuous, there exists a sequence $\{\pi_1^{n(N)}\}_{N\in\mathbb{N}}$ such that $\pi_1^{n(N)} \in \tilde{P}^j(p_1^{n(N)})$ for all N, and $\pi_1^{n(N)} \to \bar{\pi}_1$ as $N \to \infty$. □

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