Robust Constrained Interpolating Control of Interconnected Systems

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***Abstract*— This paper presents a decentralised interpolating control scheme for the robust constrained control of uncertain linear discrete-time interconnected systems with local state and control constraints. The control law of each distinct subsystem relies on the gentle interpolation between a local high-gain controller with a global low-gain controller. Both controllers benefit from the computation of separable robust invariant sets for local control design, which overcomes the computational burden of large-scale systems. Another advantage is that for each subsystem both low- and high-gain controllers can be efficiently determined off-line, while the inexpensive interpolation between them is performed on-line. For the interpolation, a new low-dimensional linear programming problem is solved at each time instant. Proofs of recursive feasibility and robust asymptotic stability of the proposed control are provided. A numerical example demonstrates the robustness of decentralised interpolating control against model uncertainty and disturbances. The proposed robust control is computationally inexpensive, and thus it is well suited for large-scale applications.**

I. INTRODUCTION

Model Predictive Control (MPC) is one of the most practical approaches for constrained control [1]. An implicit solution can be obtained by solving on-line a static optimisation problem over a finite receding horizon using the current state of the plant as the initial state as well as predicted disturbances. This repetitive optimisation procedure avoids myopic control actions while embedding a dynamic open-loop optimisation problem into a closed-loop structure. An explicit solution in the form of piecewise affine state feedback control law can be obtained off-line using polyhedral manipulations and multiparametric programming for low-order systems [2]. Robust MPC (RMPC) has been introduced to address robustness against model uncertainty and disturbances [3]. RMPC is usually obtained by solving a semidefinite optimisation problem with Linear Matrix Inequalities (LMI) that maximises the trace of an invariant ellipsoid, associated with a state feedback controller. The invariant ellipsoid, which contains the current (observable) state, guarantees the recursive feasibility and robust stability of the overall system. However, the main drawback of LMI-based synthesis methods associated with ellipsoids is that: (a) require substantial on-line computational effort; and, (b) indicate great conservativeness due to the fixed/symmetrical structure of involved ellipsoids and their operations [4], [5], [6], [7], [8], [9], [1]. Set-based MPC can improve its performance and obtain a larger terminal set by incorporating state decomposition within MPC [10].

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Interpolating Control (IC) is an alternative approach for constrained control that significantly reduces the computational effort compared to optimisation-based schemes such as MPC with quadratic performance criterion [11]. The main idea of IC is to blend a local high-gain (inner) controller, which satisfy some user-desired performance specifications, with a global low-gain (outer) vertex controller via interpolation. IC is well suited for the constrained control of polytopic uncertain systems with input and state constraints [12]. Although IC is appealing as an idea, its complexity is in direct relationship with the computational complexity of the low-gain vertex controller, which might be high for large-scale systems. To overcome this difficulty, an improved IC method has been proposed in [13] to reduce computational complexity for time-invariant and uncertain discrete-time linear systems. The global outer controller is determined in an augmented state and control space and thus no vertex representation of the controllable invariant set is needed.

To overcome the computational complexity of the centralised vertex controller [14], [15], this work proposes a Robust decentralised Interpolating Control (RdIC) approach to solve constrained control problems via interpolation in low-dimensional spaces instead of for a large-scale dynamic system; and, to guarantee robustness. The advantages of the proposed RdIC are dimensionality and well-structured decoupled information constraints. Another feature of this approach is the robustness that keeps the system stable to perturbations and uncertainties, both within subsystems and interconnections. Set invariance is important for RdIC to guarantee recursive feasibility and robust asymptotic stability of the closed-loop system. This paper proposes to compute separable robust controlled invariant sets in low-dimensional spaces for local control design, which overcomes the computational burden of large-scale systems. A similar approach is pursued e.g. in [16], [17], [18], where separable invariant sets are also computed for local control design. Moreover, computing IC for the whole system would be difficult because a low-gain high-dimensional controller needs to be computed. Alternatively, it is more convenient to determine local IC for subsystems in a distributed way, where possible interconnections are treated as bounded disturbances [19].

II. PRELIMINARIES

A. Problem Formulation

Consider a discrete-time linear time-varying interconnected dynamical system consisting of N subsystems,

$$\mathcal{S}_i : \begin{cases} x_i(k+1) = A_i(k)x_i(k) + B_i(k)u_i(k) \\ \qquad\qquad + \displaystyle\sum_{j\in\mathcal{M}_i} e_{ij}\bar{A}_{ij}(k)x_j(k), \quad i \in \mathcal{N}, \end{cases} \tag{1}$$

where $x_i(\cdot) \in \mathbb{R}^{n_i}$ and $u_i(\cdot) \in \mathbb{R}^{m_i}$ are, respectively, the (observable) state and control vectors for the subsystem $i \in \mathcal{N} = \{1, 2, \ldots, N\}$; $A_i(k) \in \mathbb{R}^{n_i \times n_i}$ and $B_i(k) \in \mathbb{R}^{n_i \times m_i}$ are the state and control matrices; and, $\bar{A}_{ij}(k) \in \mathbb{R}^{n_i \times n_j}$ is an *interconnection state matrix* between subsystem i and j, where $\mathcal{M}_i$ is the set of neighbour subsystems to i for information exchange; $e_{ij} \in [0, 1]$ are *weighting constants*, which model the strength of adjacent interconnections. If the adjacency matrices are null or $\mathcal{M}_i = \emptyset$, $\forall\, i \in \mathcal{N}$, then system (1) is *decoupled*. The overall system $\mathcal{S} = \bigcup_{i \in \mathcal{N}} \mathcal{S}_i$ involves a global state vector $x^\mathsf{T} = \begin{bmatrix} x_1^\mathsf{T} \; x_2^\mathsf{T} \; \cdots \; x_N^\mathsf{T} \end{bmatrix} \in \mathbb{R}^n$ and a global control vector $u^\mathsf{T} = \begin{bmatrix} u_1^\mathsf{T} \; u_2^\mathsf{T} \; \cdots \; u_N^\mathsf{T} \end{bmatrix} \in \mathbb{R}^m$, where $n = \sum_{i \in \mathcal{N}} n_i$ and $m = \sum_{i \in \mathcal{N}} m_i$.

The family of time-varying matrices in the N subsystems are characterised by polytopic uncertainty

$$
\begin{aligned}
& A_i(k) = \sum_{l=1}^{q_i} \alpha_i^{(l)}(k) A_i^{(l)}, \quad B_i(k) = \sum_{l=1}^{q_i} \alpha_i^{(l)}(k) B_i^{(l)}, \\
& \bar{A}_{ij}(k) = \sum_{l=1}^{\bar{q}_{ij}} \bar{\alpha}_{ij}^{(l)}(k) \bar{A}_{ij}^{(l)}, \quad i \in \mathcal{N},\, j \in \mathcal{M}_i \qquad (2) \\
& \sum_{l=1}^{q_i} \alpha_i^{(l)}(k) = 1, \quad \sum_{l=1}^{\bar{q}_{ij}} \bar{\alpha}_{ij}^{(l)}(k) = 1, \quad i \in \mathcal{N},\, j \in \mathcal{M}_i,
\end{aligned}
$$

where q_i and $\bar{q}_{ij}$ are the number of realisations for the subsystem $i \in \mathcal{N}$ and the number of realisations for the neighbour to $i \in \mathcal{N}$ interconnected subsystems $j \in \mathcal{M}_i$, respectively; $\alpha_i^{(l)}(k)$, $l = 1, \ldots, q_i$, for all $i \in \mathcal{N}$, and $\bar{\alpha}_{ij}^{(l)}(k)$, $l = 1, \ldots, \bar{q}_{ij}$, for all $i \in \mathcal{N}$ and $j \in \mathcal{M}_i$, are unknown and time-varying non-negative constants. The matrices $A_i^{(l)}$ and $B_i^{(l)}$, $l = 1, \ldots, q_i$, for all $i \in \mathcal{N}$, and $\bar{A}_{ij}^{(l)}$, $l = 1, \ldots, \bar{q}_{ij}$, for all $i \in \mathcal{N}$ and $j \in \mathcal{M}_i$, are all given.

The unconstrained decentralised robust control problem of the interconnected system (1) is to design a controller that robustly asymptotically stabilises each subsystem $i \in \mathcal{N} = \{1, 2, \ldots, N\}$ to the origin, where the i-th controller uses the local state vector $x_i(k)$ to generate the local control $u_i(k)$ for the plant. We assume that the state x_i is measurable and available for feedback in each subsystem, and that a robustly asymptotically stabilising state-feedback controller

$$u_i(k) = -K_i x_i(k), \quad i \in \mathcal{N} \qquad (3)$$

exists for each subsystem $i \in \mathcal{N}$.

Consider now the constrained case where the states and controls of the system (1) with polytopic uncertainty (2) are subject to bounded polytopic constraints

$$
\begin{cases}
x_i(k) \in \mathcal{X}_i,\, \mathcal{X}_i = \{x_i \in \mathbb{R}^{n_i} \mid F_{x_i} x_i \leq g_{x_i}\}, \\
u_i(k) \in \mathcal{U}_i,\, \mathcal{U}_i = \{u_i \in \mathbb{R}^{m_i} \mid F_{u_i} u_i \leq g_{u_i}\},
\end{cases} \qquad (4)
$$

$\forall\, k \geq 0$, $i \in \mathcal{N}$, where F_{x_i}, F_{u_i} are constant matrices and g_{x_i}, g_{u_i} are constant vectors of appropriate dimension with positive elements, and the origin is contained in the interior of the sets. The inequalities are component-wise.

To account for couplings between subsystems, we convert the system (1) into a decoupled system following as in [20], and consider an interconnected dynamical system with additive norm-bounded disturbances. Let $w_i(k) = \sum_{j \in \mathcal{M}_i} e_{ij} \bar{A}_{ij}(k) x_j(k)$, $i \in \mathcal{N}$, be the vector of interconnections of (1). Given the $\bar{q}_{ij}$ realisations of $\bar{A}_{ij}(k)$ in (2), perturbations due to couplings are bounded by

$$\|w_i(k)\| \leq \sum_{j \in \mathcal{M}_i} \sum_{l=1}^{\bar{q}_{ij}} \left\| \bar{A}_{ij}^{(l)}\, x_j(k) \right\|, \quad \forall\, i \in \mathcal{N},$$

where each element of the norm is the support function of the compact set $\mathcal{X}_j$ in each of the rows of the matrix $\bar{A}_{ij}^{(l)}$, $j \in \mathcal{M}_i$. The vector of interconnections may now be brought to the general form of polytopic constraints

$$w_i(k) \in \mathcal{W}_i,\, \mathcal{W}_i = \{w_i \in \mathbb{R}^{n_i} \mid F_{w_i} w_i \leq g_{w_i}\}, \qquad (5)$$

$\forall\, k \geq 0$, $i \in \mathcal{N}$, where F_{w_i} and g_{w_i} are suitable. If a state x_j is free, a generous upper bound can be introduced to guarantee connective stability. Finally, the interconnected system (1) can be re-written as:

$$x_i(k+1) = A_i(k) x_i(k) + B_i(k) u_i(k) + w_i(k), \quad i \in \mathcal{N}. \quad (6)$$

The constrained interconnected system (6) with constraints (4) and (5), will be used as a basis for interpolating constrained robust control design in the next sections.

B. Robust Invariant Sets

The following definitions from the invariant set theory will be used in the rest of the paper (see e.g. [2], [21]).

Definition 2.1 (Robust Positively Invariant Set): Given the local controller (3) for each subsystem $i \in \mathcal{N}$ and $A_i^K = (A_i - B_i K_i)$, the set $\Omega_i \subseteq \mathcal{X}_i$ is a robust positively invariant constraint-admissible set with respect to $x_i(k+1) = A_i^K x_i(k) + w_i(k)$ subject to the local constraints (4), (5), if and only if, $\forall\, x_i(k) \in \Omega_i$ and $\forall\, w_i(k) \in \mathcal{W}_i$, the system evolution satisfies $x_i(k+1) \in \Omega_i$ and $K_i x_i(k) \in \mathcal{U}_i$, $\forall\, k \geq 0$.

The largest robust positively invariant set that respects constraints is called *Maximal Admissible Set* (MAS) [22]. The MAS can be defined in polyhedral form as

$$\Omega_i = \{x_i \in \mathbb{R}^{n_i} : F_i^0 x_i \leq g_i^0\}, \quad i \in \mathcal{N}.$$

Definition 2.2 (Robust Controllable Invariant Set): Given the interconnected system (6) and the constraints (4), (5), the set $\Psi_i \subseteq \mathcal{X}_i$ is robust controllable invariant, if and only if, for all $x_i(k) \in \Psi_i$, there exists an admissible control $u_i(k) \in \mathcal{U}_i$ such that $x_i(k+1) \in \Psi_i$, $\forall\, i \in \mathcal{N}$, $\forall\, w_i(k) \in \mathcal{W}_i$, $\forall\, k \geq 0$, i.e. The half-space representation of Ψ_i is given by

$$\Psi_i = \{x_i \in \mathbb{R}^{n_i} : F_i^1 x_i \leq g_i^1\}, \quad i \in \mathcal{N}.$$

Definition 2.3 (M-step Robust Controllable Set): The set $P_i^M \subseteq \mathcal{X}_i$ is the set of all states for which exists an admissible control sequence such that the system (6) reaches the MAS Ω_i in no more than M steps along an admissible trajectory, i.e. one that satisfies (4), (5). The set P_i^M is called *M-step robust controllable set* and can be described by its half-space representation

$$P_i^M = \{x_i \in \mathbb{R}^{n_i} : F_i^M x_i \leq g_i^M\}, \quad i \in \mathcal{N}.$$

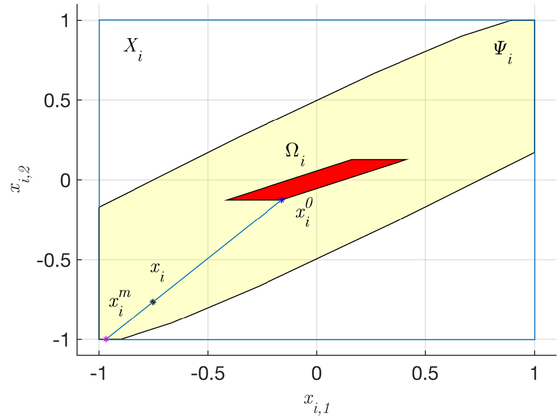

Fig. 1: In each subsystem $i \in \mathcal{N}$, any state $x_i(k)$ can be decomposed as a convex combination of $x_i^0(k) \in \Omega_i$ and $x_i^m(k) \in \Psi_i$.

Iterative algorithms for computing invariant sets are well known [14], [22], [23], [21]. These algorithms have no guarantee of finite-time convergence except if suitable stopping criteria are chosen. A sufficient condition for finite time termination requires the sets $\mathcal{X}_i$, $\mathcal{U}_i$, and $\mathcal{W}_i$ to be bounded and the closed loop system to be asymptotically stable [22]. An algorithm for computing the maximal robust controllable set Ψ_i, $i \in \mathcal{N}$, is as follows [11]:

Algorithm 1: Maximal robust controllable invariant set

input : Number of realisations q_i; matrices $A_i^{(1)}, \ldots, A_i^{(q_i)}$, $B_i^{(1)}, \ldots, B_i^{(q_i)}$; and, sets $\mathcal{X}_i$, $\mathcal{U}_i$, $\mathcal{W}_i$, $i \in \mathcal{N}$.

output: Ψ_i.

Initialise: Set $\ell = 0$, $F_i^0 = F_{x_i}$, $g_i^0 = g_{x_i}$ and $X_i^0 = \mathcal{X}_i$;

1 Let $g_{w_i} = g_i^0 - \max_{w_i \in \mathcal{W}_i} \{F_i^0 w_i\}$, consider the polytope

$$P = \left\{ (x_i, u_i) \in \mathbb{R}^{n_i+m_i} : \begin{bmatrix} F_i^0 \left(A_i^{(1)} x_i + B_i^{(1)} u_i\right) \\ F_i^0 \left(A_i^{(2)} x_i + B_i^{(2)} u_i\right) \\ \vdots \\ F_i^0 \left(A_i^{(q_i)} x_i + B_i^{(q_i)} u_i\right) \end{bmatrix} \leq \begin{bmatrix} g_{w_i} \\ g_{w_i} \\ \vdots \\ g_{w_i} \end{bmatrix} \right\}$$

compute the projection of P onto $\mathbb{R}^{n_i}$-space and check the redundancies with X_i^0,

2 **if** *all the inequalities are redundant with respect to* X_i^0 **then**
  STOP and set $\Psi_i = X_i^0$;
else
  CONTINUE;
end

3 Set $X_i^0 = P \cap X_i^0$;

4 Set $\ell = \ell + 1$ and go to **Step 1**.

Algorithm 1 produces a unique set, whenever it is not empty, see [2] for more details and existence criteria.

III. ROBUST CONSTRAINED CONTROL VIA DISTRIBUTED INTERPOLATION

A. Robust Distributed Interpolation-based Control

Fig. 1 illustrates the interpolation concept in a two-dimensional state space $\mathcal{X}_i$, where the set Ψ_i depicted in yellow and the MAS Ω_i depicted in red. Suppose that any known state $x_i(k) \in \Psi_i$ can be decomposed as follows

$$x_i(k) = s_i(k) x_i^m(k) + (1 - s_i(k)) x_i^0(k), \quad i \in \mathcal{N}, \tag{7}$$

where $x_i^0(k) \in \Omega_i$ and $x_i^m(k)$ is such that there exists a control $u_i^1(k) \in \mathcal{U}_i$ defined in the outer set such that $A_i(k) x_i^m(k) + B_i(k) u_i^1(k) + w_i(k) \in \Psi_i$, $\forall\, w_i \in \mathcal{W}_i$; and $s_i(k) \in [0, 1]$ is an interpolating coefficient. Similarly, the control in each subsystem is decomposed as follows

$$u_i(k) = s_i(k) u_i^1(k) + (1 - s_i(k)) u_i^0(k), \quad i \in \mathcal{N}, \tag{8}$$

where $u_i^0(k) = -K_i^0 x_i^0(k)$ is the inner stabiliser controller (3) of each subsystem $\mathcal{S}_i$, $i \in \mathcal{N}$, and u_i^1 is the outer control. For the interpolation (7), (8), only $x_i(k) \in \Psi_i$ in each subsystem $i \in \mathcal{N}$ is known (the current state of the system). The interpolating vector consisting of coefficients s_i, state vectors $x_i^0 \in \Omega_i$ and $x_i^m \in \Psi_i$, and the outer control vector u_i^1 are all unknown and under-determination. The inner control u_i^0 is known from (3) for given $x_i^0(k)$.

In the proposed decentralised approach, the inner control for each subsystem is defined in the robust maximal admissible set Ω_i for a given feedback control high-gain matrix K_i, $\forall\, i \in \mathcal{N}$. The outer control for each subsystem is defined in the robust controllable invariant set Ψ_i, $\forall\, i \in \mathcal{N}$. The set Ψ_i, $i \in \mathcal{N}$, can be obtained in an extended state and control space as the M-step robust controllable set if M is maximal, i.e., if $P_i^{M+1} = P_i^M$, $\forall\, i \in \mathcal{N}$, similarly to [13] or it can be computed as the maximal robust controllable invariant set. Alternatively, the set Ψ_i for each subsystem $i \in \mathcal{N}$ can be obtained by solving a semi-definite optimisation problem with LMIs that maximises the trace of an invariant ellipsoid, associated with a low-gain controller $u_i^1(k) = -K_i^1 x_i(k)$, $i \in \mathcal{N}$ and local polyhedral constraints (4), (5).

The goal of control is to steer $x_i(k) \in \Psi_i$ as close as possible to the robust positively invariant set Ω_i, i.e. to minimise the local interpolating coefficients s_i, $\forall\, i \in \mathcal{N}$. Clearly, the local controller can steer the system to the origin by definition, if $s_i = 0$, $\forall\, i \in \mathcal{N}$. To solve this interpolation problem, similarly to [11], the following optimisation problem is formulated for each subsystem $i \in \mathcal{N}$ at each discrete time k (index k is omitted for clarity):

$$s_i^*(x_i) = \min_{s_i, x_i^0, x_i^m, u_i^1} s_i$$

subject to:

$$\begin{cases} F_i^0\, x_i^0 \leq g_i^0 \\ F_i^1 \left(A_i^{(l)}\, x_i^m + B_i^{(l)}\, u_i^1\right) \leq g_i^1 - \max\limits_{w_i^{(l)} \in \mathcal{W}_i} F_i^1 w_i^{(l)}, \\ s_i\, x_i^m + (1 - s_i)\, x_i^0 = x_i \\ 0 \leq s_i \leq 1, \quad u_i^1 \in \mathcal{U}_i, \end{cases} \tag{9}$$

where the second inequality holds for $l = 1, \ldots, q_i$, $i \in \mathcal{N}$. This is a bilinear optimisation problem that can be transformed into an LP problem with the change of variables $r_i^0 = (1 - s_i)\, x_i^0$, $r_i^m = s_i\, x_i^m$, and $v_i^1 = s_i\, u_i^1$. It follows that $r_i^0 \in (1 - s_i)\, \Omega_i$, $r_i^m \in s_i\, \Psi_i$ and $v_i^1 \in s_i\, \mathcal{U}_i$. The equality constraints in (9) can be rewritten as $r_i^0 = x_i - r_i^m$. Then, the LP problem for each subsystem $i \in \mathcal{N}$ at each discrete time k reads (index k is omitted for clarity):

$$s_i^*(x_i) = \min_{s_i, r_i^m, v_i^1} s_i$$

$$\text{subject to:} \tag{10}$$

$$\begin{cases} s_i\, g_i^0 - F_i^0\, r_i^m \leq g_i^0 - F_i^0 x_i \\ F_i^1 \left(A_i^{(l)}\, r_i^m + B_i^{(l)}\, v_i^1\right) \leq s_i \Big(g_i^1 - \max\limits_{w_i^{(l)} \in \mathcal{W}_i} F_i^1 w_i^{(l)}\Big), \\ 0 \leq s_i \leq 1, \quad v_i^1 \in s_i \mathcal{U}_i, \end{cases}$$

where the second inequality holds for $l = 1, \ldots, q_i$, $i \in \mathcal{N}$. This LP problem involves n_i, $i \in \mathcal{N}$, less variables and corresponding equality constraints compared to (9). The second inequality in the optimisation problem (10) guarantees that the state $x_i^m(k)$ is robust controllable by u_i^1, i.e., $A_i(k)\, x_i^m(k) + B_i(k)\, u_i^1 + w_i(k) \in \Psi_i$, for all $w_i \in \mathcal{W}_i$. Summarising, for each subsystem $i \in \mathcal{N}$ both Ω_i and Ψ_i are determined off-line while only the interpolation between them is performed on-line. For the interpolation the LP problem (10) is solved on-line at each time step k and its solution is denoted by $s_i^*, r_i^{m*}, v_i^{1*}$, while $r_i^{0*} = x_i - r_i^{m*}$, $i \in \mathcal{N}$. The control in each subsystem can be then recovered from (8), provided the change of variables to convert (9) to (10). The LP problem for distributed interpolating control is less computationally expensive compared to the overall interpolating scheme and appropriate for hardware-embedded or real-time control of large-scale systems.

Remark 1: To apply decentralised interpolating control to system (6), the realisations of the state and control matrices are considered when computing the invariant sets. Intensity of couplings between subsystems is uncertain and depends on the weighting parameters e_{ij} and interconnections. The proposed approach guarantees stability for any convex combination of $\bar{A}_{ij}^{(l)}$, $l = 1, \ldots, \bar{q}_{ij}$, and any value of $e_{ij} \in [0, 1]$, $j \in \mathcal{M}_i, i \in \mathcal{N}$, i.e., any failure in the system.

B. Recursive Feasibility and Robust Asymptotic Stability

This section provides proofs of recursive feasibility and robust asymptotic stability for the overall system $\mathcal{S} = \bigcup_{i \in \mathcal{N}} \mathcal{S}_i$ with decentralised interpolating-based control despite the influence of additive disturbances.

To start with, define the global vectors (for clarity the discrete time index k is omitted)

$$r^0 = \left[{r_1^0}^\mathsf{T}\ {r_2^0}^\mathsf{T}\ \cdots\ {r_N^0}^\mathsf{T}\right]^\mathsf{T}, \ \ r^m = \left[{r_1^m}^\mathsf{T}\ {r_2^m}^\mathsf{T}\ \cdots\ {r_N^m}^\mathsf{T}\right]^\mathsf{T},$$

$$v^0 = \left[{v_1^0}^\mathsf{T}\ {v_2^0}^\mathsf{T}\ \cdots\ {v_N^0}^\mathsf{T}\right]^\mathsf{T}, \ \ v^1 = \left[{v_1^1}^\mathsf{T}\ {v_2^1}^\mathsf{T}\ \cdots\ {v_N^1}^\mathsf{T}\right]^\mathsf{T}.$$

Then the global state and control vectors can be decomposed as follows:

$$x(k) = r^0(k) + r^m(k), \qquad u(k) = v^0(k) + v^1(k),$$

where $v_i^0 = (1 - s_i)\, u_i^0$, $i \in \mathcal{N}$.

The following two theorems summarise the main results.

Theorem 3.1 (Recursive feasibility): The decentralised interpolation problem (7), (8), (10) guarantees recursive feasibility for the overall system (6) with state constraints $\mathcal{X} = \prod_{i \in \mathcal{N}} \mathcal{X}_i$, control constraints $\mathcal{U} = \prod_{i \in \mathcal{N}} \mathcal{U}_i$, and disturbance constraints $\mathcal{W} = \prod_{i \in \mathcal{N}} \mathcal{W}_i$, for all $x \in \Psi = \prod_{i \in \mathcal{N}} \Psi_i \subseteq \mathbb{R}^n$.

Proof: For recursive feasibility, we have to prove that $u(k)$ is in $\mathcal{U}$ and $x(k+1) \in \Psi$, for all $k \geq 0$. Since controls are independent, it is sufficient to prove that $u_i(k) \in \mathcal{U}_i$.

$$\begin{aligned} F_{u_i} u_i(k) &= F_{u_i} \{s_i(k)\, u_i^1(k) + (1 - s_i(k))\, u_i^0(k)\} \\ &= s_i(k)\, F_{u_i}\, u_i^1(k) + (1 - s_i(k))\, F_{u_i}\, u_i^0(k) \\ &\leq\ s_i(k)\, g_{u_i} + (1 - s_i(k))\, g_{u_i} = g_{u_i}. \end{aligned}$$

Since we consider local states and controls, it is sufficient to prove that $x_i(k+1) \in \Psi_i$, for all $i \in \mathcal{N}$

$$\begin{aligned} &x_i(k+1) \\ &= A_i(k)\, x_i(k) + B_i(k)\, u_i(k) + w_i(k) \\ &= A_i(k)\ \left(s_i(k)\, x_i^m(k) + (1 - s_i(k))\ x_i^0(k)\right) \\ &\quad + B_i(k)\ \left(s_i(k)\, u_i^1(k) + (1 - s_i(k))\ u_i^0(k)\right) + w_i(k) \\ &= s_i(k)\ \left(A_i(k)\, x_i^m(k) + B_i(k)\, u_i^1(k) + w_i(k)\right) \\ &\quad + (1 - s_i(k))\ \left(A_i(k)\, x_i^0(k) + B_i(k)\, u_i^0(k) + w_i(k)\right). \end{aligned}$$

Since $A_i^{(l)}\, x_i^m(k) + B_i^{(l)}\, u_i^1(k) + w_i^{(l)} \in \Psi_i$ and $A_i^{(l)}\, x_i^0(k) + B_i^{(l)}\, u_i^0(k) + w_i^{(l)} \in \Omega_i \subseteq \Psi_i$, for any $l = 1, \ldots, q_i$, it follows that $x_i(k+1) \in \Psi_i$, for all $i \in \mathcal{N}$. ■

Theorem 3.2 (Robust stability with additive disturbances): The decentralised interpolation problem (7), (8), (10) guarantees the robust asymptotic stability for the overall system (6) with state constraints $\mathcal{X} = \prod_{i \in \mathcal{N}} \mathcal{X}_i$, control constraints $\mathcal{U} = \prod_{i \in \mathcal{N}} \mathcal{U}_i$, and disturbance constraints $\mathcal{W} = \prod_{i \in \mathcal{N}} \mathcal{W}_i$, for all $x \in \Psi = \prod_{i \in \mathcal{N}} \Psi_i \subseteq \mathbb{R}^n$.

Proof: Let $s(x) = \begin{bmatrix} s_1(x_1) & \cdots & s_N(x_N) \end{bmatrix}^\mathsf{T}$ be the vector Lyapunov function, where $s_i(x_i)$ is the Lyapunov function of the subsystem $\mathcal{S}_i$. Define the following non-negative function $V\colon \mathbb{R}^n \to \mathbb{R}^+$ for all $x \in \Psi$ as $V(x) = d^\mathsf{T} s$, where d is an all-ones vector of dimension N, so $V(x) = \sum_{i \in \mathcal{N}} s_i(x_i)$. Using the asymptotic stability of the subsystems, we know that $s_i^*(k+1) \leq s_i^*(k)$ for all $i \in \mathcal{N}$. It follows that $V(x)$ is a non-increasing function and $\sum_{i \in \mathcal{N}} s_i^*(x_i)(k+1) \leq \sum_{i \in \mathcal{N}} s_i^*(x_i)(k)$. The state vector x reaches (element-wise) any robust positively invariant sets Ω_i, i.e., $s_i = 0$, in finite time and the inner controller u^0 robustly stabilises the system (converges to the origin). ■

Remark 2: A special case of the constrained dynamical system (4), (5), (6) is the system where information structure constraints are absent, i.e. $\mathcal{M}_j = \emptyset, \forall\, j \in \mathcal{N}$. The advantage is that the inner invariant sets Ω_i and the outer invariant sets Ψ_i, $i \in \mathcal{N}$ do not depend on the uncertainty of the interconnections, so the computation of the sets is less expensive. The corresponding proofs of recursive feasibility and robust asymptotic stability are similar to the proofs of Theorem 3.1 and Theorem 3.2, and thus omitted.

IV. NUMERICAL EXAMPLE

This section demonstrates the effectiveness of the proposed robust decentralised interpolating control (RdIC) scheme. We provide a numerical example where RdIC is compared with two other robust control approaches, namely the robust centralised IC (RcIC) [13] and RMPC [3]. RcIC and RdIC were computed by the Interpolating Control Toolbox (ICT), a Matlab toolbox recently developed by [24], which relies on the Invariant Set toolbox [23]. The robust MPC (RMPC) [3] is an implicit MPC approach that is based on convex optimisation and linear matrix inequalities. RMPC was computed by the MUP MATLAB/Simulink toolbox [25].

The example concerns a time-varying uncertain system (six states and three inputs) that can be decomposed into three interconnected subsystems with two states, one input, and four structural constraints each. The state matrix is time-variant and is defined by the following vertices ($q_i = 2$, $\bar{q}_{ij} = 2$, for $i \in \mathcal{N}$, $j \in \mathcal{M}_i$):

$$A_1^{(1)} = A_2^{(1)} = A_3^{(1)} = \begin{bmatrix} 1.1 & 1.0 \\ 0 & 1.0 \end{bmatrix},$$
$$A_1^{(2)} = A_2^{(2)} = A_3^{(2)} = \begin{bmatrix} 0.6 & 1.0 \\ 0 & 1.0 \end{bmatrix}.$$

Let $I_a^{(l)} = I_{ij} \bar{A}_{ij}^{(l)}$ be the interconnection matrices for $i \in \mathcal{N}$, $j \in \mathcal{M}_i$, where $I_{ij} = \mathrm{diag}(e_{ij})$ and $\bar{A}_{ij}^{(l)} = a^{(l)} \times I_2$ with $e_{ij} = 1$, $a^{(1)} = 0.015$, and $a^{(2)} = 0.01$, then the state matrix for the centralised system is

$$A^{(l)} = \begin{bmatrix} A_1^{(l)} & I_a^{(l)} & I_a^{(l)} \\ I_a^{(l)} & A_2^{(l)} & I_a^{(l)} \\ I_a^{(l)} & I_a^{(l)} & A_3^{(l)} \end{bmatrix}, \quad l = 1, 2,$$

and $A(k) = \alpha(k)\, A^{(1)} + (1 - \alpha(k))\, A^{(2)}$. The control matrix is time-invariant: $B_i^{(1)} = B_i^{(2)} = \begin{bmatrix} 0 & 1 \end{bmatrix}^\mathsf{T}$, $i = 1, 2, 3$. The system is paired with state and control constraints:

$$|x_{i,j}| \leq 10, \quad |u_i| \leq 2, \quad i = 1, 2, 3, \ j = 1, 2, \tag{11}$$

where $x_{i,j}$ are elements of $x_i = \begin{bmatrix} x_{i,1} & x_{i,2} \end{bmatrix}^\mathsf{T}$.

For the proposed RdIC, the local high-gain feedback control laws are computed with weighting matrices $Q_\mathrm{d} = I_2$ and $R_\mathrm{d} = 10^{-5}$. The MAS set Ω_i is then computed with respect to (11) and gain matrix $K_i = \begin{bmatrix} 0.7738 & 1.7034 \end{bmatrix}$, $i = 1, 2, 3$. The outer invariant sets Ψ_i, $i = 1, 2, 3$, are computed as the maximal robust control invariant sets. RcIC is designed with respect to constraints (11) and the MAS Ω is computed respect the gain matrix

$$K = \begin{bmatrix} 0.7743 & 1.7035 & 0.0253 & 0.0282 & 0.0253 & 0.0282 \\ 0.0253 & 0.0282 & 0.7743 & 1.7035 & 0.0253 & 0.0282 \\ 0.0253 & 0.0282 & 0.0253 & 0.0282 & 0.7743 & 1.7035 \end{bmatrix}.$$

The Ψ is computed as the maximal robust control invariant set for the overall system, as shown in Fig. 2(c). Both robust interpolating control approaches simulated with the same realisation of $\alpha(k)$, as shown in Fig. 2(a).

Fig. 3 depicts the evolution of states and controls in the three subsystems for the initial state $x_0^\mathsf{T} = \begin{bmatrix} 5.6543 & -3.0 & 0.340 & -3.7635 & -7.0 & 3.2736 \end{bmatrix}$, which belongs to the outer invariant set. As can be seen, both robust IC methods and RMPC have stabilised the system around the origin, albeit with different control actions. These figures also illustrate the faster and smoother convergence of the proposed RdIC scheme to the origin over the previous RcIC approach and the value of decentralised interpolation in local topologies and separable invariant sets. Note that IC is not optimal control in the sense that no objective function is assumed, which offers an explanation to the counterintuitive result of the indicated superiority of RdIC over RcIC. Further, RdIC offers similar performance to RMPC (cf. control trajectories in Fig. 3) with control effort for all subsystems $\|u\|_2 = 7.8$ and $\|u\|_2 = 7.4$, respectively.

Fig. 2(b) shows the interpolating coefficient for RcIC and the three subsystems of RdIC. Clearly, all coefficients are positive and non-increasing Lyapunov functions, and thus the stability of the overall system is guaranteed. Note that $\sum_{i \in \mathcal{N}} s_i(k)$ for RdIC not necessarily equals to $s(k)$ for RcIC. Also the interpolating coefficients $s_i(k)$ of RdIC are vanishing to zero in less steps than $s(k')$ for RcIC, i.e., the states x_i enter Ω_i faster, for $k < k'$. Concluding, RdIC allows for better exploitation of the signal space and offers fast convergence to MAS.

V. CONCLUSIONS

This paper presented a distributed IC scheme for the decentralised robust constrained control of uncertain discrete-time linear time-varying interconnected systems with local state and control constraints. IC schemes rely on the availability of robust controllable invariant sets for the overall system under control, which is computationally expensive. An alternative avenue, which is pursued in this work, is to compute separable robust controlled invariant sets for local control design, which overcomes the computational burden of large-scale systems and centralised IC. Based on this concept, a distributed interpolation scheme is developed for each subsystem to allow for the gentle interpolation between a local high-gain controller with a global low-gain controller. A low-dimensional LP problem is solved on-line for each subsystem at each time step. Proofs of recursive feasibility and robust asymptotic stability of the proposed decentralised interpolating scheme were given. A numerical example, including a comparison with RMPC, demonstrated that the proposed robust control indicates robustness against model uncertainty, fast convergence and smooth control behaviour, and thus it is well suited for large-scale applications.

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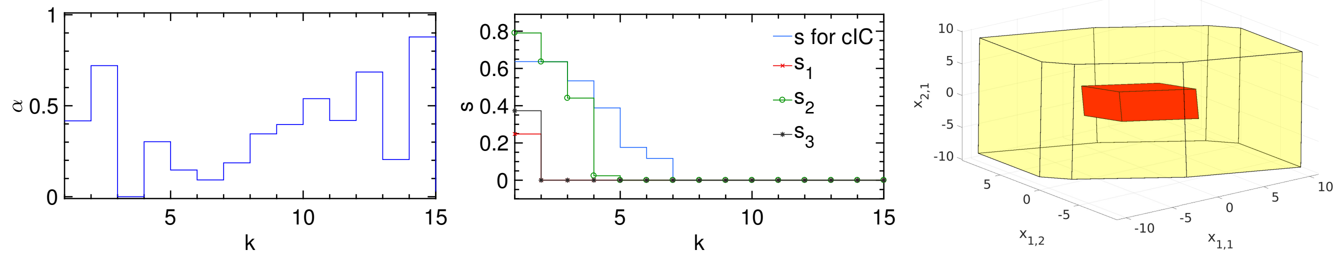

(a) (b) (c)

Fig. 2: Example 2. (a) $\alpha(k)$ realisations; (b) Interpolating coefficients of robust cIC and robust dIC; (c) Invariant set of the overall system $\mathcal{S}$ cut through $x_{2,2} = 0$, $x_{3,1} = 0$, $x_{3,2} = 0$. The yellow set is the maximal robust control invariant set Ψ. The red region is the maximal MAS for the control law $u^0 = -K\,x^0$.

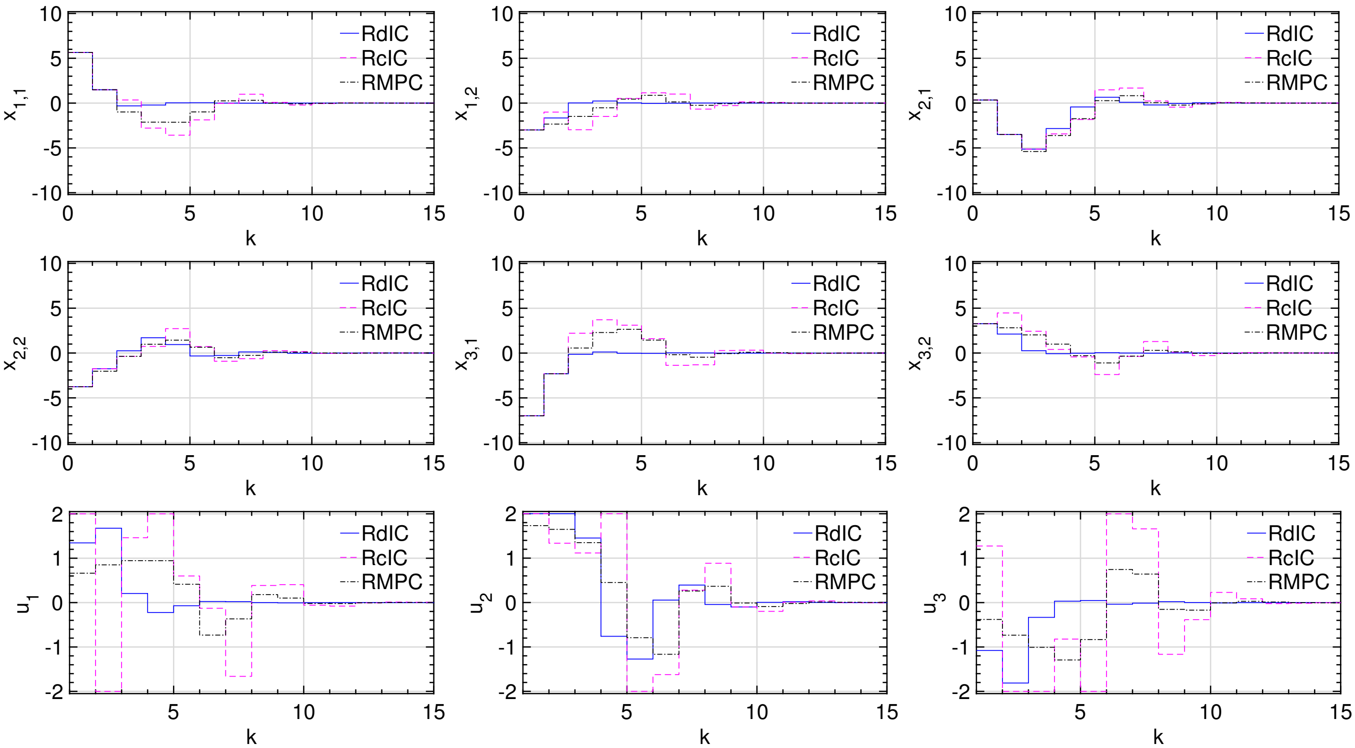

Fig. 3: State and control trajectories for RdIC (solid blue), RcIC (dashed magenta) and RMPC (dot-dashed black).

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