The vacuum friction paradox and related puzzles

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ABSTRACT
The frequency of light emitted by a moving source is shifted by a factor proportional to its velocity. We find that this Doppler shift requires the existence of a paradoxical effect: that a moving atom radiating in otherwise empty space feels a net or average force acing against its direction motion and proportional in magnitude to is speed. Yet there is no preferred rest frame, either in relativity or in Newtonian mechanics, so how can there be a vacuum friction force?

KEYWORDS
Friction; photons; spontaneous emission; atomic recoil; relativity

1. Introduction

Light emitted by a moving source has a frequency that is blue-shifted if the source is moving towards us but red-shifted if it is moving away. This optical Doppler shift [1] is the close relative of the acoustic Doppler shift that can be heard as a drop in frequency when an emergency vehicle, with its siren blaring, passes you and changes from approaching you to moving away. Photons of a higher frequency carry more energy and also more momentum than those of a lower frequency and this suggests that a radiating source emits more energy and, in particular, more momentum in its direction of motion than it does in the backwards direction. The difference per photon will typically be a tiny amount, but for an object as light as an atom the difference is significant.

Consider then a moving atom in an excited electronic state undergoing a transition to its ground state by the spontaneous emission of a photon. The photon carries away momentum and to balance this the atom experiences a recoil in the opposite direction which changes its velocity, Fig. 1. If the photon is emitted in the direction of motion then the photon will carry away more momentum than if it is emitted in the opposite direction. Momentum conservation means, necessarily, that the atom will receive a bigger recoil kick if the photon is emitted forward than if it is sent backwards and this suggests the existence of a force: the vacuum friction force.

This friction force seems to be a simple consequence of the Doppler shift together with momentum conservation but its existence is somewhat paradoxical. Why is this?

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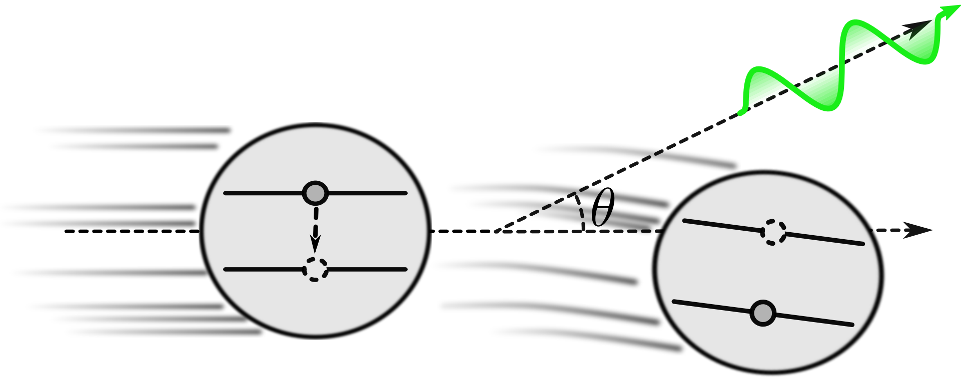

Figure 1. The emission of a photon causes a corresponding change in the velocity of the atom so as to conserve momentum.

First recall that both special relativity and, indeed, Newtonian mechanics tells us that there is no preferred state of rest, or no preferred inertial frame. If we consider the spontaneous emission in the frame of the atom then the effect is spatially isotropic and there is no forward or backward direction and no Doppler shift. This means in turn that the average change in the momentum of the atom is zero. Yet how can there be a force (the vacuum friction force) that exists in a frame in which the atom is moving but not in another: its rest frame? This is the paradoxical situation we address here.

We work throughout in the non-relativistic regime in which the speed of all matter is many orders of magnitude less than c, the speed of light. For this reason we need employ only non-relativistic mechanics and can safely neglect any term in our calculations of order $(v/c)^2$ or higher. Despite this, we find that relativistic ideas still have a role to play.

The form of the vacuum friction force can be derived by careful application of the techniques of quantum optics [2], but we can also arrive at the correct form of the force using only simple ideas from mechanics together with the expressions of Planck and Einstein for the energy and momentum of a photon [3]. This is the approach we take here. We start with a physical justification for the vacuum friction force before providing an explicit derivation of its form. Remarkably, getting to the correct answer requires, in addition to the other elements mentioned above, an eighteenth century expression for the observed aberration of light from distant stars [4]. The resolution of the paradox requires the introduction of a key idea, perhaps the most famous one, from relativity.

We conclude by turning our attention from emission to the absorption and then to the reflection of light [5]. The Doppler shift is important in the former because the resonant absorption of the light will depend on the motion of the atom: if the atom is moving towards the source then it will see the light as blue shifted, but if it is moving away from the source then it will see the light as red-shifted. This difference is the key idea in the field of laser cooling and has been used to great effect to cool atoms [6–10], ions [11–13] and even small mechanical devices [14–16]. In absorbing the light the atom will experience a recoil and hence change its velocity. The question we address is which velocity, the initial one or that after the absorption is the relevant one for determining the correct Doppler shift. Again only simple ideas from mechanics together with the forms of the energy and momentum for a photon provide the answer [5]. Finally, we can take the complexities of the radiative process out of the analysis and ask a simpler question: what happens to the frequency of a photon reflected from a small mirror, one that this light enough that its velocity is changed by momentum exchanged with the photon?

2. Vacuum friction: a paradox

A stationary excited atom in otherwise empty space can decay to its ground state by the emission of a single photon. The radiated photon carries away energy matching the difference in the internal energy between the excited and ground states. If the angular frequency of the photon is ω then this energy is $\hbar\omega$[1]. The photon carries also a linear momentum

$$\mathbf{p} = \hbar\mathbf{k}, \tag{1}$$

where $\mathbf{k}$ is the wavevector, the magnitude of which is simply related to the angular frequency: $|\mathbf{k}| = \omega/c$. This is in accord with the dictates of special relativity, which require the ratio of the energy and momentum of a massless particle to be the speed of light: $E/p = c$.

Momentum conservation requires that an initially stationary atom recoils in the direction opposite that of the motion of the photon. Indeed this idea was appreciated by Einstein very early in the development of quantum theory [18] and used to establish the equilibrium between blackbody radiation and the dynamics of a gas of radiating and absorbing molecules [19]. It is also a central feature in the phenomena than make up the field of laser cooling.

The isotropy of space requires that there is no preferred direction of emission and hence the probability that a photon will be emitted in the positive z direction, for example, must be the same as the probability for it to be emitted in the opposite, negative z, direction. More precisely, the emission by an atom (moving at the non-relativsitic speeds under consideration) will have reflection symmetry, with the probability of emission in any given direction being the same as for the opposite direction. It follows that the average momentum change in the emission process will be zero for our initially stationary atom, as depicted in the first panel of Fig. 2. This does not mean that the atom remains stationary: it will recoil in the random direction opposite to that of the motion of the photon. The zero *average* is simply a consequence of the random direction of emission.

A typical example from cold atom physics illustrates how small the recoil effect is: a rubidium atom emitting a red photon will experience a change in velocity of about $7\ \mathrm{mm\,s^{-1}}$ or, in more familiar units, about $1/40\ \mathrm{km\,h^{-1}}$. For comparison, the characteristic (or root mean square) velocity for a rubidium atom in the gas phase at room temperature is about $40,000$ times bigger than this.

If the atom is in motion then the situation is different because the Doppler shift requires a stronger recoil if the emission is in the direction of motion of the atom and hence the existence of a friction force. The analysis presented here is based on that given in [3] where a more complete account can be found.

[1] This is not strictly true as there is a very small recoil shift [6,7,17], of which more later, but including this here would complicate the analysis without altering the conclusion.

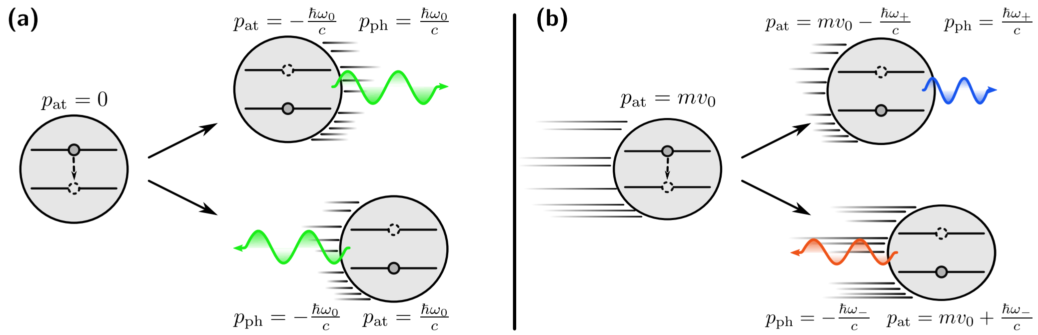

Figure 2. The momentum change for an atom due to its recoil on spontaneously emitting a photon. This recoil is in the opposite direction to the random direction of emission. (a) For an initially stationary atom the average change in the momentum of the atom is zero. (b) For an initially moving atom the Doppler shift requires that the average momentum change of the atom will not be zero. Here ω_+ and ω_- are the Doppler-shifted frequencies defined in Eq. (2).

2.1. *A physical justification*

To get a feel for the origin of vacuum friction it suffices to consider just a single spatial dimension, which we choose to be the z direction. Let the excited atom be moving in the positive z direction with speed v. This means that light emitted in the direction of motion will be shifted up in frequency due to the linear or first-order Doppler shift. If it is emitted in the opposite direction then it will be shifted down in frequency, as depicted in the second panel of Fig. 2. If ω_0 is the transition frequency then it will also be the frequency of the light emitted in the frame of the atom. For emission in the positive z direction the Doppler shifted frequency will be ω_+ and for emission in the opposite direction it will be ω_- where

$$\omega_\pm = \omega_0 \left(1 \pm \frac{v}{c}\right). \tag{2}$$

If the photon is emitted in the positive z direction then the atom's momentum will change by $-\hbar\omega_+/c$ and if in the negative z direction then it will change by $\hbar\omega_-/c$. If the emission is equally likely to take place in either direction then the average change in the momentum of the atom is

$$\begin{aligned} \langle \Delta p \rangle &= \frac{1}{2}\left(-\frac{\hbar\omega_+}{c} + \frac{\hbar\omega_-}{c}\right) \\ &= -\frac{\hbar\omega_0}{c^2}v. \end{aligned} \tag{3}$$

This momentum change is an impulse proportional to the speed of the atom but acting in the opposite direction to the velocity. The associated force will also be proportional to the velocity and acting in the opposite direction and is, therefore, a frictional force.

It seems that there is a net average change in the momentum of the atom but that this is proportional to the relative velocity between the atom and the observer and this should raise concerns, for how can there be a force acting in one intertial frame, that of the observer, but not in another, that of the atom itself? It does not help to suggest that perhaps the atom is more likely to emit in the backward rather than the forward direction so as to balance out the average momentum, for were this the case then it would have to be true also when viewed from the frame of the atom, and this would be at odds with the isotropy of space. This, in essence, is the paradox. Before attempting to resolve it, however, we should carry out a more careful derivation so as to ensure that we have not been led astray by the simplicity of this line of reasoning.

2.2. *A derivation*

In order to keep our treatment simple we avoid, as far as is possible, the complications of special relativity. To this end we restrict the relative speed of the atom and observer to be very much less than that of light. In practice this means that v/c is a very small number and that we may neglect terms or order v^2/c^2 and higher.

A stationary atom in an excited electronic state will decay to its ground state with a characteristic decay rate Γ. The fact that the atom is moving will reduce the observed decay rate or, equivalently, increase the lifetime of the excited state, due to time-dilation [20,21], but this effect depends on the Lorentz factor $\gamma = \left(1 - v^2/c^2\right)^{-1/2}$ which differs from unity by a term of order v^2/c^2. Hence we can safely neglect this so that Γ is also the decay rate for our moving atom.

For simplicity we take the emission to be fully isotropic so that the photon is equally likely to be emitted in any direction. It is possible to take account of the particular emission pattern associated with the transition but this complication makes no essential difference to the result or the analysis [3]. As the emission is taken to be isotropic, we can define the decay rate into an infinitesimal solid angle, $d\Omega = \sin\theta d\theta d\phi$, to be

$$d\Gamma = \Gamma \frac{d\Omega}{4\pi} \tag{4}$$

so that $\Gamma = \int d\Gamma$.

To proceed let us assume that the atom is moving in the positive z direction or, equivalently, let the position and direction of the motion serve to define the z axis. If the photon is emitted in the (θ, ϕ) direction then the Doppler shift means that the frequency of the emitted photon will be $\omega(1 + v\cos\theta/c)$, where θ is the angle between the direction of the atom's motion and that of the emitted photon. The atom will experience a corresponding recoil kick in the opposite direction. We can exploit the rotational symmetry about the z axis to deduce that the net force, averaged over all possible emission directions must be in the z direction. To evaluate this we need to integrate the components of the recoil momentum in the z direction multiplied by the rate of emission at time t, which is simply the product of Γ and the probability that a decay has not yet happened: $\Gamma e^{-\Gamma t}$. Bringing all this together we find a first expression for the force

$$\mathbf{F}_1 = -e^{-\Gamma t} \int \frac{\hbar\omega_0}{c} \left(1 + \frac{v}{c}\cos\theta\right) \cos\theta d\Gamma \, \hat{\mathbf{z}}. \tag{5}$$

Here the final factor of $\cos\theta$ accounts for the projection of the recoil onto the z direction. Note that in any given realisation the atom will receive a single kick or impulse at the time of emission. If we average this impulse with the rate of emission, then we find an average force and it is this average force that is of interest here. It may be helpful to think of an ensemble of moving excited atoms: each will decay and emit a photon, receiving a recoil kick as it does so. The average of these events will produce a net force on the centre of mass of the atoms and this is the average force that we seek.

There is one further effect that needs to be accounted for before we complete our derivation of the net force and this has its origins in the aberration of light due to the finite speed of light. This is familiar from special relativity [20] but is, in fact, a far older effect, which was first derived by an eighteenth century astronomer, James Bradley [4]. This requires us to replace the angle θ by the aberration-corrected angle θ' for which, to first order in v/c the cosine becomes

$$\cos\theta' = \cos\theta + \frac{v}{c}\sin^2\theta. \tag{6}$$

If we take account of this aberration due to motion then we find the total vacuum friction force to be

$$\begin{aligned}
\mathbf{F} &= -e^{-\Gamma t}\int \frac{\hbar\omega_0}{c}\left(1+\frac{v}{c}\cos\theta'\right)\cos\theta' d\Gamma\,\hat{\mathbf{z}} \\
&= -e^{-\Gamma t}\int \frac{\hbar\omega_0}{c}\left(\cos\theta+\frac{v}{c}\right)d\Gamma\,\hat{\mathbf{z}} \\
&= -e^{-\Gamma t}\Gamma\frac{\hbar\omega_0}{c^2}\mathbf{v}.
\end{aligned} \tag{7}$$

This is clearly a force, proportional to the atomic velocity and opposing the motion: a friction force.

2.2.1. Bradley's theory of stellar aberration

It is interesting to see how Bradley arrived at his aberration formula 176 years before the advent of special relativity. Bradley noticed consistent discrepancies in measurements of the distances to stars based on parallax [4]. His ingenious explanation of this was based on the finite, but very large, value of the speed of light and the very great distances involved. This meant that the actual position of a star at the observation time was not that at which it was seen to appear in the night sky. Let us suppose that the star is observed to be at an angle θ' to the direction of the relative velocity. During the time taken for the light to reach the observer, the star has moved to a position at an angle θ to the relative velocity, as depicted in Fig. 3. A simple exercise in trigonometry, using the fact that $v/c \ll 1$, gives[2]

$$\begin{aligned}
\sin(\theta-\theta') &= \frac{v}{c}\sin\theta' \\
\Rightarrow \cos\theta' &= \cos\theta + \frac{v}{c}\sin^2\theta.
\end{aligned} \tag{8}$$

In our expression for the force it is θ' that should be used. To see this we need only replace the moving, radiating star Bradley's idea with our moving and spontaneously emitting atom.

It is remarkable to note that Bradley's observations led him to determine the time taken for light to travel from the sun to the earth. Converting Bradley's value to a

[2]The manipulation in this equation uses the fact that $v \ll c$ so that $\sin(\theta-\theta') \approx \theta-\theta'$. It follows that

$$\cos\theta' \approx \cos\left(\theta - \frac{v}{c}\sin\theta'\right) \approx \cos\theta + \frac{v}{c}\sin^2\theta',$$

where we have used the Taylor series to expand the cosine and again used the fact that $v \ll c$.

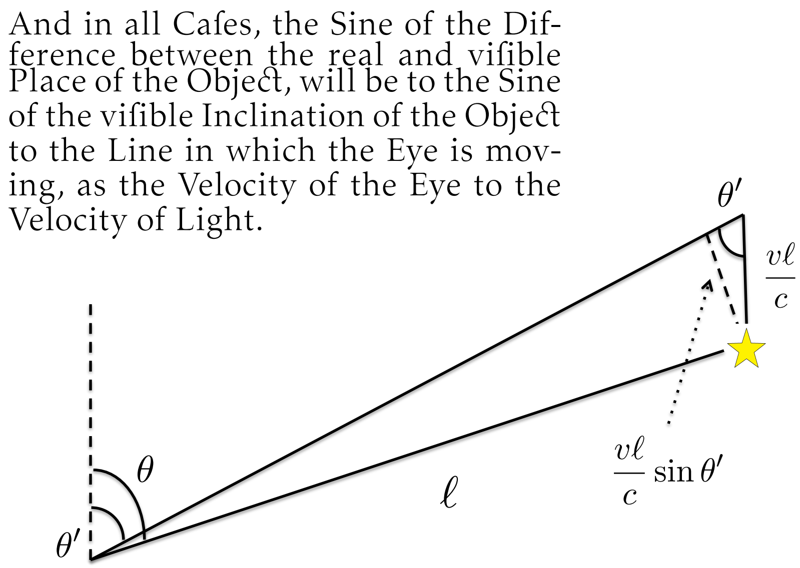

Figure 3. Bradley's original statement of the effect of aberration due to motion and the finite speed of light [4]. The star is moving vertically down the page relative to the observer and thus appears to be at an angle θ' to the direction of relative motion. During time of flight to the observer, at distance ℓ, the star moves a distance $v\ell/c$ to the point at which we have placed a yellow star. A simple exercise in trigonometry gives for the sine of the angle between the true and observed positions: $\sin(\theta - \theta') = (v/c)\sin\theta'$.

speed for light we arrive at $3.03 \times 10^8 \,\mathrm{m\,s^{-1}}$, a mere 1 % above the value obtained in subsequent more accurate measurements.

2.3. *A paradox resolved*

Newton's second law of motion is usually stated as "force equals mass times acceleration" and if we apply this with our force then there is, necessarily, an acceleration that opposes the motion of the atom. This is paradoxical in that there is then a net or average change in the motion of the atom but this change is due to the relative motion between the atom and the observer: an observer would see a change in the motion of the atom if they were moving relative to the atom but see no change if they were in a frame in which the atom was initially stationary. These statements cannot both be true, neither in Newtonian mechanics nor in special relativity.

The resolution of this paradox starts with a more precise statement of Newton's second law than the one we usually use. When in doubt it is often best to return to original sources, and it is interesting to consider what Newton actually wrote for his famous second law [22]:

"Mutationem motus proportionalem eſſe vi motrici impreſſae & fieri ſecundum lineam rectam qua vis illa imprimitur."

This was translated by Ball [23] as "The change of momentum [per unit time] is always proportional to the moving force impressed, and takes place in the direction in which the force is impressed" and by Chandrasekhar [24] as "The change of motion is proportional to the motive force impressed; and is made in the direction of right line in which the force is impressed". Chandrasekhar goes on to express this in current terminology as:

$$\begin{aligned} \textbf{force} &= \text{change in } \textbf{motion} \\ &= \text{change in [mass } \times \textbf{ velocity}] \\ &= \text{mass } \times \text{ change in } \textbf{velocity} \\ &= \text{mass } \times \textbf{ acceleration}. \end{aligned} \tag{9}$$

Yet there are two ways in which the momentum, $\mathbf{p} = m\mathbf{v}$, can change: the velocity can change or the mass can change:

$$\frac{d\mathbf{p}}{dt} = \frac{dm}{dt}\mathbf{v} + m\frac{d\mathbf{v}}{dt}. \tag{10}$$

Herein lies the resolution of the paradox of vacuum friction.

We have seen that the existence of the optical Doppler shift together with the kinematical properties of photons leads directly to the existence of a vacuum friction force and, moreover, that this is proportional to the velocity of the atom. The paradox is the existence of a corresponding acceleration is at odds with the principle that there is no preferred frame of absolute rest, either in Newton's mechanics or in Einstein's. The only way to reconcile these is that the rate of change of momentum associated with the vacuum friction force corresponds to a change in the mass of the atom rather than its velocity:

$$\begin{aligned}\mathbf{F} &= \frac{dm}{dt}\mathbf{v} \\ &= -e^{-\Gamma t}\Gamma\frac{\hbar\omega_0}{c^2}\mathbf{v} \\ \Rightarrow \frac{dm}{dt} &= -e^{-\Gamma t}\Gamma\frac{\hbar\omega_0}{c^2}. \end{aligned} \tag{11}$$

To see what this suggestion leads to, let us integrate the equation to find the change in the mass of the atom at long times. We find

$$m(\infty) = m(0) - \frac{\hbar\omega_0}{c^2}, \tag{12}$$

which suggests that the atom is lighter at long times than it is initially and, moreover, gives an explicit form for this change in mass: it is the energy of the emitted photon divided by the square of the speed of light. After a sufficiently long time the atom will have decayed and the energy carried away by the emitted photon corresponds to a reduction in the mass of the atom. The resolution of our paradox is a manifestation of the relativistic principle that energy has inertia, encapsulated in surely the most famous of all equations (certainly in physics): $E = mc^2$. It is interesting to note that this equivalence was the first idea to attract Einstein's attention following the publication of his original paper on special relativity [25].

The remarkable feature of our analysis, however, is that we have arrived at the necessity for the quantitative equivalence of energy and inertia by working with a non-relativistic theory. The Doppler shift and Bradley's aberration formula assume a finite speed of light, but not that has the same constant value in all intertial frames nor other concepts associated with special relativity, and yet they suffice to find Einstein's famous result.

So the resolution of the paradox is that there is indeed a vacuum friction force but that this force imparts no net acceleration to the atom. Rather it induces a change in the mass of the atom, reducing it by the energy of the emitted photon divided by c^2.

2.4. *An analogy from rocket science*

On closer inspection, the resolution to our paradox has two elements, one is the relativistic relationship between energy and inertia but the second in the fact that a precise statement of Newton's second law of motion, as opposed to the usual $\mathbf{F} = m\mathbf{a}$, includes the fact that the force can be associated with a change in the mass of the object. This second feature is not primarily a relativistic one and to make this point let us revisit a classic problem from Newtonian mechanics, that of the dynamics of a space rocket. In doing so, we contrast the predictions for the speed of the rocket using the correct form of Newton's second law and the, in this case, incorrect form: $\mathbf{F} = m\mathbf{a}$. An analysis in one spatial dimension suffices and the one presented here is based on that given by Kibble [26].

Let us suppose, as depicted in Fig. 4, that we have a rocket of mass M moving with velocity v in a force-free region, far from any gravitating bodies. From this state the rocket can accelerate by ejecting material from its engines. Consider the ejection of a small amount of mass dm in the form of exhaust will be at velocity u relative to that

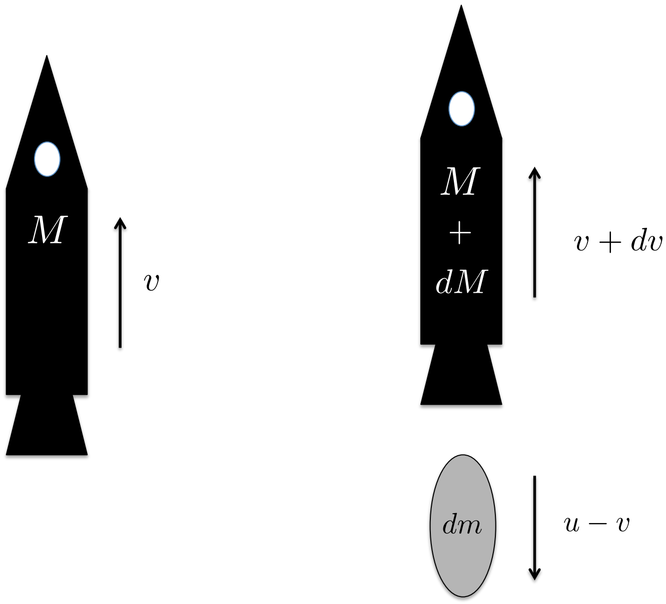

Figure 4. The motion of a rocket accelerated by the ejection of exhaust (adapted from Kibble's text [26]).

of the rocket. The ejected mass reduces the overall mass of the rocket,

$$dM = -dm \tag{13}$$

and the effect of ejecting the mass will be to increase the velocity of the rocket to $v + dv$. Momentum has to be conserved and so we find

$$\begin{aligned} Mv &= (M - dm)(v + dv) + dm(v - u) \\ &= Mv + Mdv - udm, \end{aligned} \tag{14}$$

where we have dropped terms of second order in the infinitesimal quantities, dm and dv. If we cancel the large term, Mv, divide by dt and rearrange the equation then we end up with

$$\begin{aligned} M\frac{dv}{dt} - u\frac{dm}{dt} &= 0 \\ \Rightarrow M\frac{dv}{dt} + v\frac{dM}{dt} &= \frac{dp_{\text{rocket}}}{dt} \\ &= (u - v)\frac{dm}{dt}. \end{aligned} \tag{15}$$

This has the form of Newton's second law of motion with the ejection of the exhaust corresponding to a force $(u - v)dm/dt$ applied to the rocket. The first term on the left-hand side is the familiar Ma and the second is the part of the force law associated with the change in mass of the rocket. Let us see what effect this second term has in this case.

We begin by replacing the ejected mass, dm, by the mass lost from the rocket, $-dM$, to give

$$\begin{aligned} M\frac{dv}{dt} + v\frac{dM}{dt} &= (v-u)\frac{dM}{dt} \\ \Rightarrow M\frac{dv}{dt} &= -u\frac{dM}{dt}. \end{aligned} \tag{16}$$

We can rewrite this in integral form as

$$\int \frac{dM}{M} = -\int \frac{dv}{u}. \tag{17}$$

We can evaluate these integrals this simply if we assume that the velocity of the ejected fuel relative to the rocket, u, is constant. This gives an expression for the velocity of the rocket in terms of its initial value, the velocity at which the exhaust is ejected and the fraction of the initial combined mass of the rocket and its fuel that remains:

$$v = v_0 + u \ln\left(\frac{M_0}{M}\right), \tag{18}$$

where v_0 and M_0 are, respectively, the initial velocity and mass of the rocket. This result is remarkable in that it is independent of the time taken to change the velocity. It tells us, moreover, that expelling sufficient mass the rocket can reach (at least within this Newtonian theory) an unbounded velocity. If the rocket starts from rest it will reach the relative velocity of the exhaust, u, by reducing its mass to M_0/e (about 37% of the initial mass).

It is instructive to see what would be the result if we, erroneously, neglected the mass change required by Newton's second law, and analysed this problem using $\mathbf{F} = m\mathbf{a}$. This would mean dropping the term vdM/dt so that our equation of motion would be

$$M\frac{dv}{dt} = (v-u)\frac{dM}{dt}. \tag{19}$$

We can integrate this equation simply if we assume, as before, that u is a constant corresponding to a uniform operation of the engine:

$$\begin{aligned} \int \frac{dv}{v-u} &= \int \frac{dM}{M} \\ \Rightarrow \ln\left(\frac{v-u}{v_0-u}\right) &= \ln\left(\frac{M}{M_0}\right) \\ \Rightarrow v &= u + \frac{M}{M_0}(v_0-u), \end{aligned} \tag{20}$$

Two things are clear from this (incorrect) treatment: the first is that the limiting speed, as eventually all of the mass is eaten up, is u and the second, more worrying conclusion, is that this is the limiting velocity even if the initial velocity, v_0 was greater than u and, in this case, the rocket engine ejecting material backwards would slow the rocket rather than accelerate it!

The mass change associated with an external force is important in relativity but there exist examples, such as that of the rocket, where it is also important in Newtonian mechanics.

3. Doppler shift in absorption: a puzzle

The Doppler shift plays an important role in the absorption of light by an atom as well as in emission. As we have seen, if the atom is moving towards the source of light then it will experience a frequency that is up-shifted and if it is moving away then the light in the atom's frame will be down-shifted. This means that resonant absorption of the light requires it to be of a lower frequency than the natural transition frequency if the atom is moving towards the source, but at a higher frequency if the atom is moving away form the source. The absorption of the photon imparts a recoil and changes the velocity of the atom. This raises a simple question: which velocity do we put into the Doppler-shift formula, the initial velocity prior to the absorption, the final velocity after absorption or, perhaps, some other velocity? We can answer this using the simple principles of the conservation of energy and momentum. Our presentation is based on that given previously by one of us [5] and uses an approach proposed by Fermi [27] in the early days of the modern quantum theory.

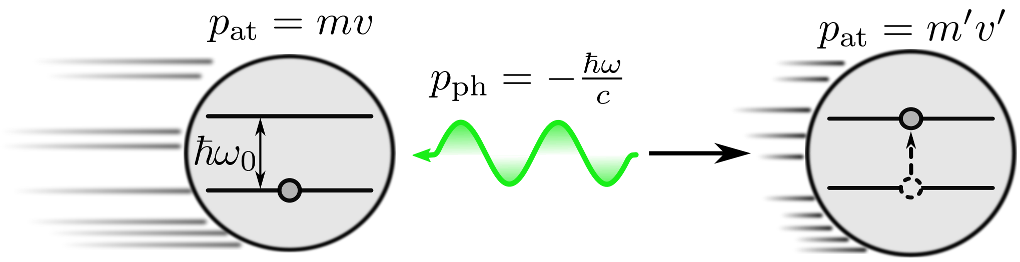

Figure 5. Left: an atom moving towards a light source can absorb a photon. Right: following the absorption, the atom's energy and also its momentum are changed.

Once again it suffices to consider a single spatial dimension and let the atom, of mass m, be moving initially with a velocity v towards a light source, as in Fig. 5. We denote the atomic transition angular frequency by ω_0 and let the frequency of the light be ω. The recoil of the atom in the absorption event will reduce its velocity to v'. If we impose the conservation of energy and momentum by comparing the pre- and post-absorption values then we find

$$\hbar\omega_0 + \frac{1}{2}m'v'^2 = \hbar\omega + \frac{1}{2}mv^2 \quad (21)$$

$$m'v' = mv - \frac{\hbar\omega}{c}, \quad (22)$$

where, as we have seen, $m' = m + \hbar\omega_0/c^2$ accounts for the increase in mass of the atom due to absorbing the energy of the photon. For our low velocity, non-relativistic analysis, however, we may safely drop terms of order v^2/c^2 and so set $m' = m$.

We are seeking the resonance condition, that is the value we should choose for the frequency of the light, ω, to produce resonant absorption. To find this we can rearrange our energy and momentum conservation laws

$$\begin{aligned}\frac{1}{2}mv'^2 - \frac{1}{2}mv^2 &= \hbar\omega - \hbar\omega_0 &(23)\\ mv' - mv &= -\frac{\hbar\omega}{c} &(24)\end{aligned}$$

and divide the first equation by the second to remove the mass:

$$\begin{aligned}\frac{v'+v}{2c} &= \frac{\omega-\omega_0}{-\omega}\\ \Rightarrow \omega_0 &= \omega\left(1+\frac{v'+v}{2c}\right). &(25)\end{aligned}$$

This the familiar Doppler shift but the relevant velocity in this is not the initial velocity of the atom, v, nor its final velocity, v', but rather the average of the two. This is a pleasingly democratic result, with neither velocity having a privileged role.

There is another way to write our Doppler shift in terms of the initial velocity of the atom and the ratio of the energy of the photon and the rest-mass energy of the atom. To see this we can eliminate v' using our energy and momentum conservation laws and arrive at

$$\omega_0 = \omega\left(1+\frac{v}{c}-\frac{\hbar\omega}{2mc^2}\right). \qquad (26)$$

The final term is a very small correction due to the recoil of the atom in absorption and, appropriately, is called the recoil shift [6,7]. Despite its small magnitude, it has been measured in experiment [17].

4. Doppler shift in reflection: the same puzzle?

Do the same considerations arise for light reflected from a moving object? Certainly the light will be Doppler shifted and it is this mechanism that is used in traffic enforcement to detect the speed of motor vehicles. Here the Doppler shift obliges us by being twice as large as in absorption or emission. If the light is reflected from an object moving towards us with velocity v then the frequency of the light returned to us, ω', is related to that of the light sent towards the reflecting object by

$$\omega' = \omega\left(1+\frac{2v}{c}\right). \qquad (27)$$

This means that to measure the speed in this way with a precision of $1\,\mathrm{km\,h^{-1}}$ requires a measurement of the fractional frequency shift, $(\omega'-\omega)/\omega$, with an accuracy of roughly 2 parts in 10^9, or approximately a 50 Hz shift for a 25 GHz radar gun.

To address the phenomenon under discussion, let us suppose that the mirror is of a low mass, so that the momentum imparted to the mirror by the reflection of a photon suffices to change its velocity, as depicted in Fig. 6. As before let v be the

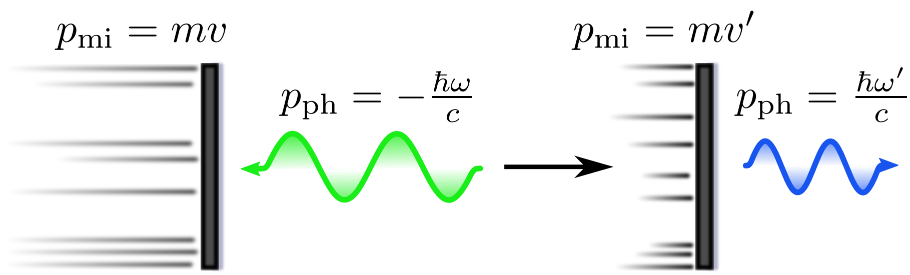

Figure 6. Left: a low-mass mirror moving towards a light source. Right: following the reflection, the mirror's velocity is reduced and the frequency of the light is upshifted.

initial velocity of the mirror and this be towards the source of the light. The velocity after the light has been reflected is v' and the initial and post-reflection frequencies of the photon are, respectively, ω and ω'. If we apply the conservation of energy and momentum to this problem then we find

$$\frac{1}{2}mv'^2 + \hbar\omega' = \frac{1}{2}mv^2 + \hbar\omega \tag{28}$$

$$mv' + \frac{\hbar\omega'}{c} = mv - \frac{\hbar\omega}{c}. \tag{29}$$

Rearranging these to remove the mass, gives

$$\omega\left(1 + \frac{v' + v}{2c}\right) = \omega'\left(1 - \frac{v' + v}{2c}\right)$$

$$\Rightarrow \omega' \approx \omega\left(1 + \frac{v' + v}{c}\right), \tag{30}$$

where we have made use of the fact that the mirror velocity is very much less than that of light. The effect of the reflection is a Doppler shift at twice a characteristic velocity of the mirror but, as in the case of absorption, this characteristic velocity is the average of the mirror's velocity before and after the reflection.

So the Doppler shift on reflection appears to be simply twice what it would be for absorption or emission, and this seems sensible as, from the point of view of the kinematics of the light, we can think of the reflection as an absorption followed by a re-emission in the direction back towards the source. Strictly speaking, however, this is not quite true but the difference between the two effects arises when the atom or mirror moves at a relativistic velocity. The problem can be addressed using relativistic rather than Newtonian kinematics but we leave this, preferably, as an exercise for the readers or, alternatively, refer them to [5].

5. Conclusion

We have seen how the first order Doppler shift, an effect that is not usually thought of as being relativistic in nature, leads directly to a paradox. This paradox, which is inherent in the interaction between light and atoms, is the existence of a vacuum friction force; a force that opposes the motion of an atom. For such a force to induce an acceleration, however, would be to require the existence of a preferred frame of absolute rest, an idea that is in conflict both with relativity and with Newtonian mechanics. Resolving this paradox relies on two points. The first is that Newton's second law of motion is not $\mathbf{F} = m\mathbf{a}$ but rather $\mathbf{F} = \dot{\mathbf{p}}$ and hence a force can act to change the mass rather than induce an acceleration; this is the case for the vacuum friction force. The second is that the change in the mass of the atom is associated with the loss of energy in the form of the emitted photon; a consequence of the quintessential relativistic idea of the inertial nature of energy as embodied in the famous law $E = mc^2$. It is remarkable that we arrived at this result without explicitly introducing special relativity; our analysis was based on non-relativistic mechanics together with the first order Doppler shift which, like its acoustical counterpart, is usually thought of as a non-relativistic effect.

There are further subtleties in the relationship between the Doppler shift and the motion of matter in interaction with light. In the absorption of light or its reflection by an object of small mass, the interaction with the light changes the object's velocity. It is natural to ask, therefore, whether it is the velocity of the object before the interaction or afterwards that is the one relevant to the Doppler formula. We have seen that the correct velocity for inclusion, both in absorption and in reflection, is simply the average of these two velocities. If we picture these processes as taking a finite time then this is the interval during which the material body changes its velocity. What could be more natural, therefore, than for the average velocity during this period to be that which appears in the required Doppler shift?

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