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Highlights

- The effect of the alternation bias on decisions over repeated lotteries is studied
- A new explanation for Samuelson's "fallacy of large numbers" is provided
- The alternation bias effect interacts with intrinsic risk preferences
- If subjects are risk averse, we find an increase in willingness to risk
- If subjects are risk prone, we find a decrease in willingness to risk

Alternation Bias and Sums of Identically Distributed Monetary Lotteries

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Abstract

The alternation bias is the tendency of people to believe that random events alternate more often than statistical laws imply. This paper examines the theoretical effect of this psychological bias on preferences over repeated investments by using a model of the belief in the law of small numbers. An alternation bias agent (ABA) has a different perception to a rational agent (RA) about the outcome distribution of the sum of *n* realizations of a lottery. The results show that an ABA, that maximises expected utility, could reject a single realization of a lottery while accepting several repetitions in accordance with Paul Samuelson's fallacy of large numbers. Furthermore, the explanation of this type of preference, based on the alternation bias, is compatible with previous behavioural accounts. A more general result shows that the alternation bias increases (decreases) the expected utility of the perceived sum of identically distributed lotteries if individuals are risk averse (risk seekers).

Keywords: alternation bias; repeated lotteries, expected utility, risk aversion, behavioural economics.

JEL: D03, D81, G02, G11.

1. Introduction

The alternation bias is the tendency of people to believe that random events alternate more often than statistical laws imply. This psychological bias is also called the "gambler's fallacy" and is a manifestation of the "belief in the law of small numbers" or "local representativeness" (Kahneman and Tversky, 1972; Tversky and Kahneman, 1971). There is wide empirical evidence of the alternation bias in experiments where subjects produce or judge random sequences (Zhao et al., 2014; Gilovich et al., 1985; Kahneman and Tversky, 1972; see Bar-Hillel and Wagenaar, 1991, and Oskarsson et al., 2009, for a survey).

An interesting research question is to what extent this psychological bias has an impact on economic behaviour and decision making. Rabin (2002) pointed out that an immediate consequence of the alternation bias in lottery play is people having incentives to bet on numbers that have not recently won. Rabin also finds that people believing in the law of small numbers make wrong inferences about the underlying probability of an event, concluding that people could believe in non-existent variation in the quality of managers' performance in the context of mutual-fund management. Kaivanto (2008) shows that assuming decision makers display alternation bias we can explain the St. Petersburg Paradox when considering conventional parameterisations of Cumulative Prospect Theory (Tversky and Kahneman, 1992). In another paper, Kaivanto and Kroll (2012) argue that the alternation bias could be an explanation for violations of the reduction of compound lotteries axiom.

In this paper, we study the theoretical effect of the alternation bias on preferences over sums of identically distributed (not necessarily independent) monetary lotteries. For that

task, we use the model of the "belief in the law of small numbers" (Rabin, 2002) to describe the perception of sums of monetary lotteries with two or more outcomes. We focus on the classic example in which Paul Samuelson's colleague (SC) was willing to accept 100 repetitions of a lottery while rejecting one single realization (Samuelson, 1963). Samuelson showed that a switch in acceptance between single and repeated lotteries is a mistake from a normative point of view, which he called a "fallacy of large numbers" (SC's fallacy hereafter). Our analysis shows that the alternation bias could account for SC decisions. A key point is that an alternation bias believer (ABA) will overestimate the perceived certainty around the average value of repeated lotteries. More specifically, an ABA will overestimate the probability of having five heads out of ten tosses. Our explanation of SC decisions is innovative because it focuses on how people judge probabilities rather than on properties of the utility function (see previous work in: Lippman and Mamer, 1988; Nielsen, 1985; Ross, 1999). Furthermore, this explanation is compatible with previous behavioural accounts based on loss aversion (Tversky and Bar-Hillel, 1983; Benartzi and Thaler, 1995; Gneezy and Potters, 1997). Specifically, the alternation bias increases the possibility that a loss averse individual behaves according to SC decisions.

In addition, we show a general result about how the alternation bias affects preferences on sums of lotteries. We find that a risk averse (risk seeking) individual will have a higher (lower) expected utility for a sum of lotteries if she is an ABA rather than a rational agent (RA). The result holds for the sum of $n \geq 2$ realizations of any lottery with $C \geq 2$ monetary outcomes. More formally, we show that the alternation bias reduces the perceived risk of sum of lotteries according to the definition in Rothschild and Stiglitz's (1970 and 1971). The psychological intuition for this result is that believers in the law of small numbers perceive a reduced variability in their (repeated)

investments with respect to the rational case. The ultimate effect of the alternation bias relies on whether people like or dislike risk. To the author's knowledge, this result has never been addressed in the literature.

In the next section, we present the model of the belief in the law of small numbers. Then we apply the model to the perceived sum of monetary lotteries. In section 4, results about SC's fallacy are presented. Section 5 contains a general result for the effect of the alternation bias. Finally, section 6 adds a discussion and conclusion.

2. Belief in the law of small numbers

Rabin (2002) describes an intuitive model to account for the well-documented gambler's fallacy in the case of random variables with two outcomes.[1] [2] For this study, we extend his model to the case of multinomial random variables. The true process is given by a sequence of *iid* variables X_i, $i = 1, \ldots, n$, that take values $S = \{1,2, \ldots, C\}$ with probabilities $p = \{p_1, p_2, \ldots, p_C\}$. An RA has correct beliefs about the independence and identical distribution of each random variable. However, an ABA misunderstands the independence property and believes X_i and X_{i-1} $(i = 2, \ldots, n)$ are negatively correlated. An ABA's beliefs can be described as if each X_i is the ith draw *without replacement* from an urn with an integer number, $N < \infty$, of signals.

The number of signals representing each value $k \in S$ is $p_k N$ such that the probability for first draw is $P(X_1 = k) = \frac{p_k N}{N} = p_k$. We will assume that the urn is renewed every two draws so that each odd realization is a draw from an N-signal urn independent of

[1] Although Rabin uses a specific intuitive description of the alternation bias, his model is mathematically equivalent to previous strategies (Budescu, 1987; Oskarsson et al., 2009).

[2] In this study, we will focus on the case that the underlying distribution function of the *iid* random variables is known and the alternation bias occurs. Nonetheless, Rabin's model (2002) has interesting implications for situations with uncertainty about the true probability of the signals. Indeed, it has been experimentally shown to be consistent with prediction tasks and investment decisions (Altmann and Burns, 2005; Huber et al., 2010) and to be a better fit than competing theories (Asparouhova et al., 2009).

previous ones. Let i be an odd number, then we can generalize $P(X_i = q|X_{i-1} = k) = p_q \ \forall q, k \in S$. On the other hand, each even realization depends on the preceding draw. Let i be an even number then the probability of repetition is $P(X_i = q|X_{i-1} = k) = \frac{p_q N-1}{N-1} < p_q$, for $q = k \in S$, and the probability of alternation is $P(X_i = q|X_{i-1} = k) = \frac{p_q N}{N-1} > p_q$, $\forall q, k \in S$ and $q \neq k$. The renewal assumption has been acknowledged by Rabin (2002) to be artificial but, on the other hand, coherent with N being a finite number (i.e. the urn will be empty after N draws if we do not impose renovation at some point) and convenient to improve the model tractability. For the analysis presented in this paper, we are aware that renewal simplifies the analysis while it captures the essential belief in higher alternation for an ABA than for an RA. The practical implication is that each pair (X_i, X_{i-1}), i being even, are *iid* so that we can focus on the correlation within each pair of draws.

Notice that the unconditional probability distribution of X_i is the same for all i and identical for an ABA and an RA (see appendix). So, the two agents believe each X_i, $\forall i$, is identically distributed but not necessarily independent. The size of the urn determines how much an ABA is biased. The smaller N is the higher the alternation bias is and the more negative the correlation between realizations is expected. Indeed, the correlation between an even realization and the previous odd draw can be computed as $\pi = -\frac{1}{N-1}$ (see appendix). An RA is an individual with N being infinite so that she correctly expects $\pi = 0$. In what follows, we describe conditional probabilities in terms of the

correlation parameter for the ease of exposition. We can model the perceived random process with the next transition probabilities for odd and even periods respectively:[3]

$$P(X_i = q|X_{i-1} = k) = p_q, \text{ for } i > 1 \text{ being odd,} \quad (1)$$

$$P(X_i = q|X_{i-1} = k) = \pi^B \delta_{kq} + (1 - \pi^B)p_q, \text{ for } i > 1 \text{ being even,} \quad (2)$$

where δ_{kq} is the Kronecker delta, i.e. $\delta_{kq} = 1$ if $k = q$, and $\delta_{kq} = 0$ if $k \neq q$. Also, π^B is the correlation coefficient between even and odd realizations for an individual with beliefs B, such that $\pi^{ABA} < \pi^{RA} = 0$. This model can account for different levels of alternation bias; for example the most extended finding in coin tossing that $P(H|T) = P(T|H) = \frac{3}{5}$ (Oskarsson et al., 2009; Rabin, 2002; Bar-Hillel and Wagenaar, 1991; Budescu, 1987; Falk, 1981) by choosing the next parameters: $C = 2$; $p_1 = p_2 = \frac{1}{2}$; $N = 6$; $\pi^{RA} = 0$; $\pi^{ABA} = -\frac{1}{5}$.[4]

3. Monetary lotteries

In this section, we apply the model of the belief in the law of small numbers to realizations of monetary lotteries. Consider a sequence of monetary lotteries l_i, $i = 1, \ldots, n$, that give outcome x_k whenever $X_i = k$. We are interested in studying preferences on the sum of these lotteries, $S_n = \sum_{i=1}^{n} l_i$.

3.1. Perceived sum of monetary lotteries

[3]This model has the same transition probabilities, for even realizations, as the Multinomial Model (Wang and Yang, 1995) or its binomial version in Edwards (1960). Indeed Budescu (1987) used Edward's model for describing people's misperception of random binary sequences.

[4] Notice that even when coin tossing is a sequence of independent random variables ($\pi = 0$), the bias generally found is a probability of alternation higher than the probability of repetition of outcomes, i.e. $P(H|T) = P(T|H) = 3/5 > P(H|H) = P(T|T) = 2/5$.

We will define the *perceived sum of lotteries* as ${S_n}^{\mathrm{B}}$, $\mathrm{B} = \mathrm{ABA}, \mathrm{RA}$. An agent's beliefs will affect the perceived outcome distribution of sum of lotteries as can be illustrated by an example. Assume a random process like coin tossing with $C = 2$; $p_1 = p_2 = \frac{1}{2}$ and $\pi^{RA} = 0$. Assume also that an ABA has beliefs represented by an urn with $N = 6$, such that $\pi^{ABA} = -\frac{1}{5}$. Imagine two lotteries l_1 and l_2 that give monetary outcomes $x_1 = \$200$ and $x_2 = -\$100$. Then the perceived sum of lotteries are ${S_2}^{\mathrm{RA}} = (\$400, \frac{1}{4};\ \$100; \frac{1}{2}, -\$200, \frac{1}{4})$ and ${S_2}^{\mathrm{ABA}} = (\$400, \frac{1}{5};\ \$100, \frac{3}{5};\ -\$200, \frac{1}{5})$. In Figure 1 the aggregation of lotteries is presented. The distribution of l_1 is the same for both types of agent, i.e. there is a 50/50 chance of obtaining \$200/-\$100. However, the conditional probabilities of l_2 vary for an ABA; the probability of wining \$200 is lower after a win in l_1. Notice that the expected monetary outcome is the same for ${S_2}^{\mathrm{ABA}}$ and ${S_2}^{\mathrm{RA}}$ (\$100), however the riskiness is lower for the former with lower probability mass in the extreme results (\$400 and -\$200).

3.2. Expected utility

We will consider that decision makers are expected utility maximizers such that the utility of ${S_n}^{\mathrm{B}}$ is given by:

$$EU\left({S_n}^{B}\right) = \sum_{x \in X} {P_x}^{B} u(x)\,, B = ABA, RA, \tag{3}$$

where X is the set of all possible monetary outcomes of ${S_n}^{\mathrm{B}}$. Individuals derive a specific utility from each monetary outcome, $u(x)$, that is weighted with perceived probabilities, ${P_x}^{B}$. Therefore, preferences depend on two distinct aspects of the decision process: 1) the value attached to each monetary outcome, i.e. the characteristics of the utility function, and; 2) the perceived distribution of outcomes given by the specific

beliefs (RA or ABA). This paper focuses on the role of the latter; although it turns out that the effect of beliefs interacts with the properties of the utility function.

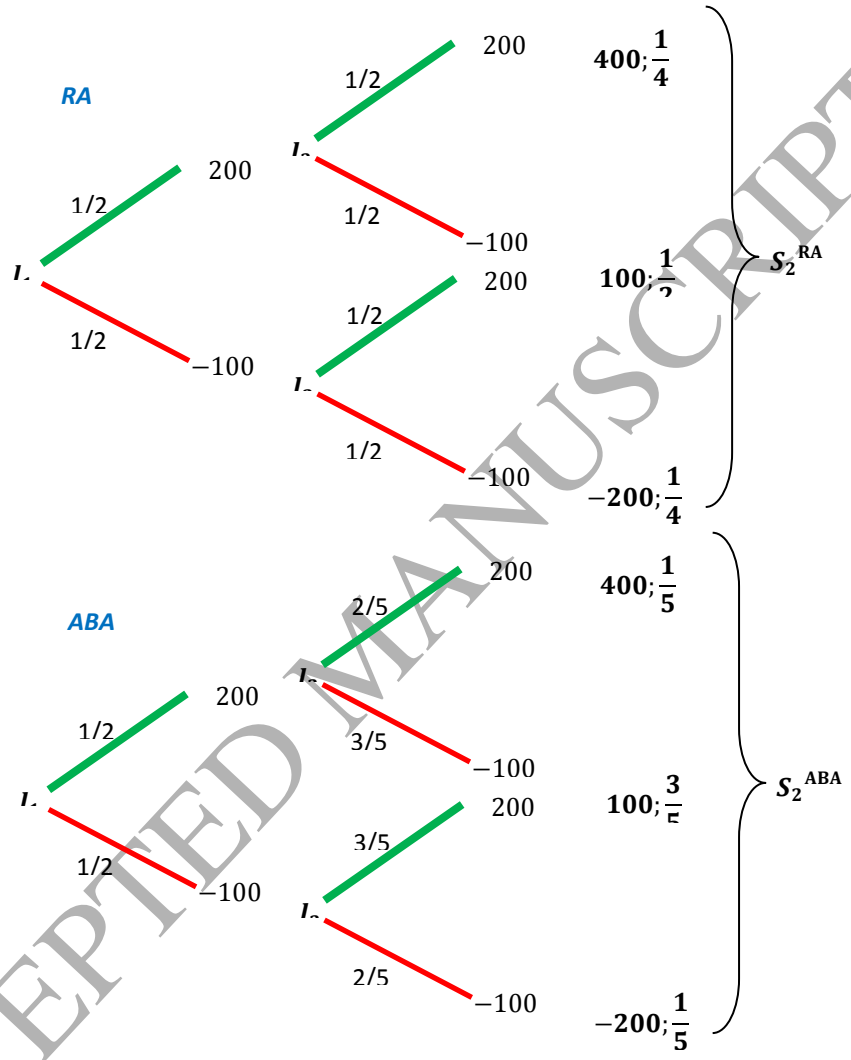

Figure 1. Perceived sum of lotteries for an RA and an ABA

4. SC's fallacy

Samuelson (1963) discussed the (lack of) rationality of his colleague by showing that a sequence of independent ventures of a lottery will not have a favourable expected utility if one single instance of the lottery is worse than abstention at each possible level of wealth given by that sequence. Formally:[5]

Samuelson theorem. *An expected utility maximiser should reject to play n independent repetitions of a lottery l if the next assumption holds:*

A1. Given the minimum and maximum wealth obtainable after playing n repetitions of lottery l, w_{min} and w_{max} respectively, we have that

$$EU(w) > EU(w + l)\ \forall w \in [w_{min}, w_{max}]. \tag{4}$$

This theoretical result is striking given the evidence that people are willing to accept repeated lotteries and, at the same time, reject a single realization of those (Liu and Colman, 2009; DeKay and Kim, 2005; Redelmeier and Tversky, 1992; Montgomery and Adelbratt, 1982). Even more, SC decisions could be considered sensible given the riskiness of the lottery offered by Samuelson and the sound investment opportunity of 100 realizations (Lopes, 1981 and 1996; Benartzi and Thaler, 1999).

Previous accounts of this type of behaviour have been based on relaxing assumption A1, either within normative theory (Lippman and Mamer, 1988; Nielsen, 1985; Ross, 1999) or from behavioural economics (Tversky and Bar-Hillel, 1983; Benartzi and Thaler, 1995; Gneezy and Potters, 1997). Both approaches point to the way individuals assess monetary outcomes. For example, Ross (1999) established sufficient conditions for the utility function that lead to the acceptance of lotteries when these are repeated a

[5] A formal proof of the theorem can be found in Samuelson (1963) or Ross (1999).

sufficient number of times. On the other hand, the behavioural economics literature has focused on loss aversion as the psychological factor behind SC decisions. People are reluctant to take lotteries with a large probability of losses; 50% in the case of Samuelson lottery. However, when this lottery is played several times the probability of losses is reduced – e.g. the probability of a loss drops to 25% when it is played twice - and people are more willing to take the risk.

In this section, we show that even if A1 holds we can explain SC behaviour if she believes in the law of small numbers. Therefore, we depart from previous account in that we focus on the way individuals perceive probabilities rather than on how they value monetary outcomes. Furthermore, we show that alternation bias and loss aversion are two mutually compatible explanations of SC decisions.

4.1. Alternation bias as an explanation

Samuelson called a fallacy the belief that it is almost a sure thing that there will be a million heads when two million symmetric coins are tossed. This fallacy makes people feel that adding risks will improve the certainty of a result. On the contrary, the sum of independent lotteries will increase the actual range of possible outcomes. A similar fallacy will appear if a person is a believer in the law of small numbers. An ABA will overestimate the probability of having five heads out of ten tosses, therefore increasing the perceived certainty around the average value. An ABA will be biased even for short sequences of a bet and will not necessarily need one hundred tosses to have misperception of risk.

The alternation bias affects one key assumption of Samuelson theorem: lotteries are *independent*. As shown next, Samuelson theorem does not hold if we consider that people are ABAs with a misunderstanding of the independence property. An ABA will

perceive one single realization of Samuelson lottery with no bias ${S_1}^{\mathrm{ABA}} = (\$200, \frac{1}{2};\ -\$100, \frac{1}{2})$. Consider now that the individual has a constant absolute risk aversion (CARA) utility function $u(x) = 1 - e^{-\alpha x}$ satisfying A1. Notice that we can assume wealth level to be zero without loss of generality. Then by numerical solution we have that ${S_1}^{\mathrm{ABA}}$ will be rejected when the parameter of absolute risk aversion is higher than 0.00481:

$$EU\left({S_1}^{ABA}\right) = \frac{1}{2}u(200) + \frac{1}{2}u(-100) < 0, \qquad \forall \alpha > 0.00481 \tag{5}$$

Assume now that the same individual with $\alpha > 0.00481$ is offered two realizations. Following equations (1) and (2), the perceived outcome distribution is ${S_2}^{ABA} = (£400, \frac{1+\pi^{\mathrm{ABA}}}{4};\ £100, \frac{1-\pi^{\mathrm{ABA}}}{2};\ -£200, \frac{1+\pi^{\mathrm{ABA}}}{4})$. Then, we can show that acceptance is guaranteed for a sufficiently large alternation bias. Acceptance occurs whenever the next expression holds:

$$EU\left({S_2}^{ABA}\right) = \frac{1+\pi^{\mathrm{ABA}}}{4}u(400) + \frac{1-\pi^{\mathrm{ABA}}}{2}u(100) + \frac{1+\pi^{\mathrm{ABA}}}{4}u(-200) > 0. \tag{6}$$

Rearranging we obtain:

$$EU\left({S_2}^{ABA}\right) =$$

$$\pi^{\mathrm{ABA}}\left[\underbrace{\frac{1}{4}u(400) - \frac{1}{2}u(100) + \frac{1}{4}u(-200)}_{<0\ by\ concavity\ of\ u(x)}\right] + \underbrace{\frac{1}{4}u(400) + \frac{1}{2}u(100) + \frac{1}{4}u(-200)}_{=EU\left({S_2}^{RA}\right)<0\ given\ Samuelson\ theorem} > 0 \tag{7}$$

Given that $\pi^{\mathrm{ABA}} \in [-1,0)$, the expected utility of the two repetitions is a linear function of π^{ABA} with its maximum at $EU\left({S_2}^{ABA}\right) = u(100) > 0$ whenever $\pi^{\mathrm{ABA}} = -1$ and lower limit at $EU\left({S_2}^{ABA}\right) = EU\left({S_2}^{RA}\right) < 0$ when π^{ABA} approaches rationality of zero correlation, i.e. when the independence property is correctly perceived. Notice that

$EU(S_2{}^{RA}) < 0$ because Samuelson theorem holds for rational individuals. The implication is that the alternation bias, if sufficiently large, could result in decisions consistent with SC's fallacy.[6]

4.2. Alternation bias and loss aversion working together

The analysis above shows that loss aversion is not the only behavioural explanation of SC decisions. Furthermore, the alternation bias works together with loss aversion by increasing the range of loss averse individuals that will behave as SC. Loss aversion explains SC behaviour because the likelihood of losses drops as the lottery is repeated. However, if loss aversion is too high a patient could reject both single and multiple realizations. The alternation bias makes the probability of losses to decrease faster when lotteries are repeated. Therefore, alternation bias makes SC behaviour to be plausible even if loss aversion is too high.

Consider an RA with the next utility function

$$u(x) = \begin{cases} x & if\ x \geq 0 \\ \lambda x & if\ x < 0 \end{cases}, \quad \lambda > 1, \tag{8}$$

where x is defined as variation of wealth and λ is the loss aversion parameter. It is easy to show that for $\lambda \in (2,3)$ this subject will reject one single realization of Samuelson lottery and accept two repetitions of it in accordance with SC's fallacy.

[6] Notice that given the renewal assumption of the model any pair of adjacent lotteries are perceived as independent. Hence, applying the same logic from Samuelson theorem, acceptance of two realizations of Samuelson lottery, $EU(S_2{}^{ABA}) > 0$, is sufficient for the acceptance of any number r of repetitions of the two realizations, i.e. $EU(S_{2r}{}^{ABA}) > 0$.

Consider now an ABA with the same utility function in (8). She will reject the single lottery for $\lambda > 2$ as in the RA case. However, she will accept two repetitions of the lottery when the expected utility is positive:

$$EU\left(S_2{}^{ABA}\right) = \frac{1+\pi^{\mathrm{ABA}}}{4}(400) + \frac{1-\pi^{\mathrm{ABA}}}{2}(100) + \frac{1+\pi^{\mathrm{ABA}}}{4}(-200\lambda) > 0, \tag{9}$$

which leads to the next equivalent condition

$$\lambda < \frac{1-\pi^{ABA}}{1+\pi^{ABA}} + 2. \tag{10}$$

Condition (10) implies that for $\lambda \in (2, \frac{1-\pi^{ABA}}{1+\pi^{ABA}} + 2)$ an ABA will behave according to SC's fallacy. Given $\pi^{ABA} < 0$ this interval is wider than the one for an RA. The interpretation is that the alternation bias makes SC behaviour plausible for a wider range of loss averse individuals. Moreover, when the alternation bias approaches perfect negative correlation, $\pi^{\mathrm{ABA}} = -1$, the upper bound of the interval tends to infinity meaning that the individual will accept two realizations of the single lottery no matter how large the degree of loss aversion is.

5. A general result

Given $S_n{}^{\mathrm{ABA}}$ and $S_n{}^{\mathrm{RA}}$, we want to show the next proposition based on Rothschild and Stiglitz's (1970 and 1971) definition of increasing risk:

Proposition. *There exists a lottery W with $E\left[W \middle| S_n{}^{ABA} = x\right] = 0, \forall x \in X$, such that*

$$S_n{}^{\mathrm{RA}} \underset{d}{=} S_n{}^{\mathrm{ABA}} + W, \tag{11}$$

where "$\underset{d}{=}$" has to be interpreted as "has the same distribution as".

Expression (11) implies that the perceived sum for an ABA is less risky than the one for an RA. Therefore, we will show that ${S_n}^{\mathrm{ABA}}$ second-order stochastically dominates ${S_n}^{\mathrm{RA}}$. Notice that for this result to apply we need $\pi^{ABA} < \pi^{RA}$ but we do not need $\pi^{RA} = 0$, i.e. the actual lotteries are not necessarily independent. We first show the case of $n = 2$ and then generalize the result to $n \geq 2$.

5.1. The case of $n = 2$

Consider the case of two successive lotteries $\{l_1, l_2\}$ with $C \geq 2$ possible outcomes $\{x_1, \dots, x_C\}$ with unconditional probabilities $\{p_1, \dots, p_C\}$. Let us call ${S_2}^{\mathrm{B}} = {l_1}^{B} + {l_2}^{\mathrm{B}}$ the lottery formed by the sum of these lotteries when beliefs are π^{B}, $\mathrm{B} = ABA, RA$.

Proposition ($n = 2$):

$$ {S_2}^{RA} \underset{d}{=} {S_2}^{ABA} + Y, \tag{12} $$

where $E\left[Y \middle| {S_2}^{ABA} = x\right] = 0, \forall x \in X$.

Proof. See appendix for full details.

The proof of the proposition is based on showing that the probability distribution of Y after the realization of ${l_1}^{ABA} = x_k$ and ${l_2}^{ABA} = x_q$ is:

$$ If\ x_k \neq x_q, \begin{cases} P(Y = 0) = \frac{(1-\pi^{RA})}{(1-\pi^{ABA})} \\ P\left(Y = x_k - x_q\right) = \frac{(\pi^{RA}-\pi^{ABA})}{2(1-\pi^{ABA})} \\ P\left(Y = x_q - x_k\right) = \frac{(\pi^{RA}-\pi^{ABA})}{2(1-\pi^{ABA})} \end{cases} \qquad If\ x_k = x_q, P(Y = 0) = 1. \tag{13} $$

The intuitive reasoning for finding Y relies on the fact that ${S_2}^{ABA}$ has more probability mass than ${S_2}^{RA}$ for monetary quantities derived from alternation of outcomes.

Therefore, summing Y to ${S_2}^{ABA}$ must reduce the probability of outcomes given by alternation (i.e. when $x_k \neq x_q$) while increases the probability of monetary quantities derived from repetition (i.e. when $x_k = x_q$). Notice, that this is what we do by defining Y in (13). Even more, it is easy to see that $E\left[Y \middle| {S_2}^{ABA} = x\right] = 0\ \forall x \in X$ because its conditional distribution, after alternation or repetition, is symmetrical around zero.

5.2. The case of $n \geq 2$

Consider the case of n successive lotteries $\{l_1, \dots, l_n\}$ with $C \geq 2$ possible outcomes $\{x_1, \dots, x_C\}$ with unconditional probabilities $\{p_1, \dots, p_C\}$. Let us call ${S_n}^{\mathrm{B}} = {l_1}^{B} + \cdots + {l_n}^{\mathrm{B}}$ the aggregate lottery formed by the sum of these lotteries when belief is π^{B}, $\mathrm{B} = ABA, RA$.

Proposition (n $\geq$ 2):

$$ {S_n}^{RA} \underset{d}{=} {S_n}^{ABA} + W. \tag{14} $$

where $E\left[W \middle| {S_n}^{ABA} = x\right] = 0\ \forall x \in X$.

Proof. Consider the sum of lotteries for an ABA:

$$ {S_n}^{ABA} = {l_1}^{\mathrm{ABA}} + {l_2}^{\mathrm{ABA}} + {l_3}^{\mathrm{ABA}} + {l_4}^{\mathrm{ABA}} + \cdots + {l_{n-1}}^{\mathrm{ABA}} + {l_n}^{\mathrm{ABA}}. \tag{15} $$

Notice that given transition probabilities in (1) and (2) the sum of each pair of lotteries ${l_{i-1}}^{\mathrm{ABA}} + {l_i}^{\mathrm{ABA}}$, i being an even number, is independent of the remaining lotteries and identically distributed to ${S_2}^{\mathrm{ABA}} = {l_1}^{\mathrm{ABA}} + {l_2}^{\mathrm{ABA}}$. Assume n is an even number, then using Proposition (n=2) in (12) we will have:[7]

[7] The extension of the result when n is an odd number is straightforward.

$$S_n{}^{ABA} + W = \underbrace{\left(S_2{}^{\mathrm{ABA}} + Y\right) + \left(S_2{}^{\mathrm{ABA}} + Y\right) + \cdots + \left(S_2{}^{\mathrm{ABA}} + Y\right)}_{n/2\ times} \underset{d}{=}$$

$$\underset{d}{=} \underbrace{S_2{}^{\mathrm{RA}} + S_2{}^{\mathrm{RA}} + \cdots + S_2{}^{\mathrm{RA}}}_{n/2\ times} \underbrace{=}_{given\ (1)\ and\ (2)} S_n{}^{RA}, \tag{16}$$

where $W = \underbrace{Y + Y + \cdots + Y}_{n/2\ times}$ and Y has the same probability distribution as in (13).[8] ■

6. Discussion and conclusion

Incorporation of a well-studied cognitive bias to the standard economic model allowed us to make specific predictions about preferences for repeated investments. We have formally stated the intuitive result that a believer in the law of small numbers perceives repeated monetary lotteries as less risky. Given that the alternation bias only affects perception of lotteries if multiple plays are considered we can explain inconsistencies between single and multiple-play lotteries implied by SC's fallacy (1963). Moreover, our behavioural account is innovative because it relies on the psychological mechanism of perception of probabilities rather than on the evaluation of monetary outcomes (Tversky and Bar-Hillel, 1983; Benartzi and Thaler, 1995; Gneezy and Potters, 1997). Alternation bias is sufficient for explaining SC decisions with no need for rejecting other behavioural conditions (Lippman and Mamer, 1988; Nielsen, 1985; Ross, 1999) that could be considered as reasonable (Lopes, 1996; Benartzi and Thaler, 1999).

Our general result that the effect of the alternation bias interacts with the shape of the utility function offers new opportunities for empirical research. Any empirical evidence could be interpreted in terms of the theoretical findings. For example, a positive (negative) relationship between alternation bias and preferences for repeated lotteries could be interpreted as evidence for concavity (convexity) of the utility function.

[8] Notice that each lottery Y is different and conditional on each adjacent $S_2{}^{\mathrm{ABA}}$. Also notice that given $E\left[Y \middle| S_2{}^{ABA} = x\right] = 0$, $x \in X$, we will also have $E\left[W \middle| S_n{}^{ABA} = x\right] = 0$, $x \in X$.

Nonetheless, an empirical test of the theoretical results should consider measurement of preferences, alternation bias and the utility function. Future work could contribute to the existing literature testing theoretical predictions and economic consequences of the belief in the law of small numbers (Altmann and Burns, 2005; Huber et al., 2010; Croson and Sundali, 2005; Asparouhova et al., 2009).

Previous work on randomness perception showed that individuals could have varying levels of alternation bias (Budescu, 1987; Rapoport and Budescu, 1992; Ayton and Fischer, 2004). For example, in Budescu's *experiment 1* subjects were classified according to whether they showed overall negative or positive recency, according to whether they produced more or less alternation than expected for an independent random process. Most of them showed beliefs consistent with the alternation bias (13 out of 18 subjects), however some individuals presented a positive recency or no systematic bias at all.[9] We should expect that the higher the alternation bias is the higher the effect on choices for repeated lotteries will be. Even more, in the case of the so called "positive recency" bias, which implies that $\pi^{ABA} > \pi^{RA}$, the theoretical results are straightforward and opposite to the case of $\pi^{ABA} < \pi^{RA}$ assumed in this paper. Those with "positive recency" should be less (more) willing to play sum of lotteries than RAs if they are risk averse (seekers).

A relevant question is whether the analysis here exposed can be applied to lotteries that are played in parallel rather than sequentially. Several factors can be considered here on both theoretical and empirical grounds. The presented model can be valid if the perception of the aggregated parallel lotteries is correctly predicted. In this sense, we

[9] Comparable results are found in Rapoport and Budescu (1992). Ayton and Fischer (2004) found evidence that the same subjects could believe in positive or negative recency for distinct types of random process. In their experiments individuals showed negative recency in their expectations for the outcome of a roulette game and, simultaneously, positive recency in expectations for their success and failures of their predictions for the outcomes of the same game.

should wonder whether parallel lotteries exist in practice. If two subjects agreed to toss two coins at the same time point some elapsed time between the two tosses is expected (at least few milliseconds). Therefore, an individual could interpret parallel lotteries as a special sequence of lotteries played quickly one after another. In any case, the applicability of the model to parallel lotteries relies on an empirical question: How do people perceive simultaneously played lotteries? Imagine that people perceived two parallel lotteries as if they were two simultaneous draws from an urn in such a way that the two drawn signals must be different. In this case the two draws are negatively correlated as if they were sequential draws without replacement; i.e. as our model predicts.

Furthermore, the belief in the "law of small numbers" asserts that the law of large numbers applies to small samples as well, so that people expect small samples to be highly representative of the population from which they are drawn. For example, researchers are overconfident about findings derived from small samples (Tversky and Kahneman, 1971) and their estimations of sampling distribution or posterior probabilities are independent of sample size (Kahneman and Tversky, 1972). In principle, this belief is applied to any sample formed by either sequential or simultaneous draws. However, evidence on perception of randomness has been mainly focused on sequences. Some empirical work, close to the parallel case, is found in tasks of judgment of the distribution of two colours among cells of two-dimensional grids. Notice that whether the distribution is the result of a sequence or parallel realizations is not a relevant factor. Interestingly, a grid is judged most random when alternation rate is about 0.6, i.e. when the colour changes in 60% of the horizontal and vertical transitions (Zhao et al., 2014; Falk and Konold, 1997).

Some extensions of the present analysis could be considered for its applicability to actual data. For example, the model of "beliefs in the law of small numbers" (Rabin, 2002) limits the effect of the alternation bias to adjacent realizations of lotteries given the renewal assumption. Applications of this model to actual decisions should consider relaxation of this assumption to extend the alternation bias effect to subsequent realizations; for example, people believe that chances for head are higher after three consecutive tosses resulting in tails than after only one toss resulting in tails (see Rabin, 2002, or Rapoport and Budescu, 1997). Another possible path to continue this work is by considering other theories of decision under risk like Rank Dependent Utility (RDU, Quiggin, 1981) or Cumulative Prospect Theory (CPT, Tversky and Kahneman, 1992). Different results could be found under these theories because the effect of the alternation bias on preferences for repeated lotteries would not only depend on the shape of the utility function (concave or convex) but also on the properties of the probability weighting function like overweighting of small probabilities, underweighting of large probabilities, or upper and lower subadditivity (see Tversky and Wakker, 1995, or Gonzalez and Wu, 1999).

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Appendix

Unconditional probabilities

For an RA it is obvious that $P(X_i = q) = p_q, \forall i, \ \forall q \in S$. The same applies to odd realizations for ABAs.

For ABAs and i being even, we have:

$$P(X_i = q) = E[P(X_i = q|X_{i-1} = k)] = \sum_{k=1}^{C} p_k \times P(X_i = q|X_{i-1} = k) =$$

$$\underbrace{p_q \times \frac{p_q N-1}{N-1}}_{Case\ of\ q=k} + \underbrace{\sum_{k\neq q} p_k \times \frac{p_q N}{N-1}}_{Cases\ of\ q\neq k} = p_q \times \frac{p_q N}{N-1} - \frac{p_q}{N-1} + (1-p_q) \times \frac{p_q N}{N-1} = -\frac{p_q}{N-1} +$$

$$+\frac{p_q N}{N-1} = \frac{p_q(N-1)}{N-1} = p_q, \forall q \in S. \qquad \text{(A1)}$$

Correlation between realizations

The correlation coefficient between even realization i and the previous one $i-1$ can be computed as:

$$\pi = Corr(X_i, X_{i-1}) = \frac{Cov(X_i, X_{i-1})}{\sqrt{Var(X_i)Var(X_{i-1})}} = \frac{Cov(X_i, X_{i-1})}{Var(X_i)} = \frac{E[X_i X_{i-1}] - E[X_i]^2}{E[{X_i}^2] - E[X_i]^2}. \qquad \text{(A2)}$$

Before continuing, the following expressions make the algebra easier:

$$E[X_i]^2 = \sum_{k=1}^{C} \sum_{q=1}^{C} p_k p_q kq. \qquad \text{(A3)}$$

$$E[{X_i}^2] = \sum_{k=1}^{C} p_k k^2. \qquad \text{(A4)}$$

$$E[X_i X_{i-1}] = \sum_{k=1}^{C} \sum_{q=1}^{C} p_k P(X_i = q|X_{i-1} = k) kq =$$

$$\sum_{k=1}^{C} p_k \frac{p_k N-1}{N-1} k^2 + \sum_{k=1}^{C} \sum_{q\neq k} p_k \frac{p_q N}{N-1} kq = \frac{N}{N-1} \sum_{k=1}^{C} {p_k}^2 k^2 - \frac{1}{N-1} \sum_{k=1}^{C} p_k k^2 +$$

$$\frac{N}{N-1}\sum_{k=1}^{C}\sum_{q\neq k} p_k p_q kq =$$

$$\frac{N}{N-1}\sum_{k=1}^{C}\sum_{q=1}^{C} p_k p_q kq - \frac{1}{N-1}\sum_{k=1}^{C} p_k k^2 \underbrace{=}_{(A3)\ and\ (A4)} \frac{N}{N-1}E[X_i]^2 - \frac{1}{N-1}E\left[X_i{}^2\right]. \quad \text{(A5)}$$

Now combining (A2) and (A5)

$$\pi = \frac{\frac{1}{N-1}E[X_i]^2 - \frac{1}{N-1}E\left[X_i{}^2\right]}{E\left[X_i{}^2\right] - E[X_i]^2} = \frac{-\frac{1}{N-1}\left(E\left[X_i{}^2\right] - E[X_i]^2\right)}{E\left[X_i{}^2\right] - E[X_i]^2} = -\frac{1}{N-1}. \quad \text{(A6)}$$

Proof of Proposition ($\boldsymbol{n = 2}$)

$$S_2{}^{RA} \underset{d}{=} S_2{}^{ABA} + Y, \quad \text{(A7)}$$

where $E\left[Y \middle| S_2{}^{ABA} = x\right] = 0, \forall x \in X.$

Proof. Specifically, we will prove that Y is obtained with the next procedure:

Generation of $\boldsymbol{Y}$. *Given the realization of* $l_1{}^{ABA} = x_k$ *and* $l_2{}^{ABA} = x_q$ *we will add lottery* Y *with the next distribution:*

$$If\ x_k \neq x_q, \begin{cases} P(Y=0) = \frac{(1-\pi^{RA})}{(1-\pi^{ABA})} \\ P\left(Y = x_k - x_q\right) = \frac{(\pi^{RA}-\pi^{ABA})}{2(1-\pi^{ABA})} \\ P\left(Y = x_q - x_k\right) = \frac{(\pi^{RA}-\pi^{ABA})}{2(1-\pi^{ABA})} \end{cases} \qquad If\ x_k = x_q, P(Y=0) = 1. \quad \text{(A8)}$$

It can be easily seen that $E\left[Y \middle| S_2{}^{ABA} = x\right] = 0, \forall x \in X$. Hence, it will be shown that $S_2{}^{ABA} + Y$ has the same distribution as $S_2{}^{RA}$.

Let us first define the set of all combinations of non-equal monetary outcomes (x_k, x_q) such that their sum is s, i.e. $A = \left\{(k,q): x_k + x_q = s, k \neq q\right\}$. Also we will define the

outcome whose repetition sums up s, $R = \{k: 2x_k = s\}$. Notice that R is a singleton. This way we can represent the probability of s, $P\left({S_2}^{\mathrm{B}} = s\right)$, by adding the probability of all elements in A and R. Given transition probabilities in expression (1) and (2) the outcome distribution of ${S_2}^{\mathrm{B}}$ is determined by:

$$P\left({S_2}^{\mathrm{B}} = s\right) = \sum_{(k,q)\in A} P\left({l_1}^{B} = x_k, {l_2}^{\mathrm{B}} = x_q\right) + P\left({l_1}^{B} = x_R, {l_2}^{\mathrm{B}} = x_R\right) = \sum_{(k,q)\in A}\left[p_k \times \left\{(1-\pi^{\mathrm{B}})p_q\right\}\right] + p_R \times [\pi^{\mathrm{B}} + (1-\pi^{\mathrm{B}})p_R]. \quad \text{(A9)}$$

We can follow a similar procedure to obtain $P\left({S_2}^{\mathrm{ABA}} + Y = s\right)$. In this case s could be the outcome of ${S_2}^{\mathrm{ABA}} + Y$ due to four mutually exclusive events:

a) ${l_1}^{B} = x_k$, ${l_2}^{\mathrm{B}} = x_q$ and $Y = 0$, for $(k, q) \in A$;

b) ${l_1}^{B} = x_R$, ${l_2}^{\mathrm{B}} = x_R$ and $Y = 0$;

c) ${l_1}^{B} = x_R$, ${l_2}^{\mathrm{B}} = x_q$ and $Y = x_R - x_q$, for $q \neq R$;

d) ${l_1}^{B} = x_q$, ${l_2}^{\mathrm{B}} = x_R$ and $Y = x_R - x_q$, for $q \neq R$.

So that we can compute the probability of s as the summation of the probabilities for events a) to d):

$$P\left(S_2{}^{\mathrm{ABA}} + Y = s\right) =$$

$$\underbrace{\sum_{(k,q)\in A}\left[P\left(l_1{}^{ABA} = x_k, l_2{}^{\mathrm{ABA}} = x_q\right) \times P\left(Y = 0 | l_1{}^{ABA} = x_k, l_2{}^{\mathrm{ABA}} = x_q\right)\right]}_{Prob(a)} +$$

$$\underbrace{P\left(l_1{}^{ABA} = x_R, l_2{}^{\mathrm{ABA}} = x_R\right)}_{Prob(b)} +$$

$$\underbrace{\sum_{q\neq R} P\left(l_1{}^{ABA} = x_R, l_2{}^{\mathrm{ABA}} = x_q\right) \times P\left(Y = x_R - x_q | l_1{}^{ABA} = x_R, l_2{}^{\mathrm{ABA}} = x_q\right)}_{Prob(c)} +$$

$$\underbrace{\sum_{q\neq R} P\left(l_1{}^{ABA} = x_q, l_2{}^{\mathrm{ABA}} = x_R\right) \times P\left(Y = x_R - x_q | l_1{}^{ABA} = x_q, l_2{}^{\mathrm{ABA}} = x_R\right)}_{Prob(d)} = \quad \text{(A10)}$$

$$= \sum_{(k,q)\in A}\left[p_k \times \left\{(1-\pi^{\mathrm{ABA}})p_q\right\} \times \frac{(1-\pi^{RA})}{(1-\pi^{ABA})}\right] +$$

$$p_R \times [\pi^{\mathrm{ABA}} + (1-\pi^{\mathrm{ABA}})p_R] +$$

$$\sum_{q\neq R}\left[p_R \times \left\{(1-\pi^{\mathrm{ABA}})p_q\right\} \times \frac{(\pi^{RA}-\pi^{ABA})}{2(1-\pi^{ABA})}\right] +$$

$$\sum_{q\neq R}\left[p_q \times \left\{(1-\pi^{\mathrm{ABA}})p_R\right\} \times \frac{(\pi^{RA}-\pi^{ABA})}{2(1-\pi^{ABA})}\right].$$

We can divide expression (A10) in two parts:

1) $\sum_{(k,q)\in A}\left[p_k \times \left\{(1-\pi^{\mathrm{ABA}})p_q\right\} \times \frac{(1-\pi^{RA})}{(1-\pi^{ABA})}\right] = \sum_{(k,q)\in A}\left[p_k \times \left\{(1-\pi^{\mathrm{RA}})p_q\right\}\right].$

2) $p_R \times [\pi^{\mathrm{ABA}} + (1-\pi^{\mathrm{ABA}})p_R] + \sum_{q\neq R}\left[p_R \times \left\{(1-\pi^{\mathrm{ABA}})p_q\right\} \times \frac{(\pi^{RA}-\pi^{ABA})}{2(1-\pi^{ABA})}\right] +$

$\sum_{q\neq R}\left[p_q \times \left\{(1-\pi^{\mathrm{ABA}})p_R\right\} \times \frac{(\pi^{RA}-\pi^{ABA})}{2(1-\pi^{ABA})}\right] = p_R \times [\pi^{\mathrm{ABA}} + (1-\pi^{\mathrm{ABA}})p_R] +$

$\sum_{q\neq R}\left[p_R \times p_q \times \frac{(\pi^{RA}-\pi^{ABA})}{2}\right] + \sum_{q\neq R}\left[p_R \times p_q \times \frac{(\pi^{RA}-\pi^{ABA})}{2}\right] = p_R \times$

$[\pi^{\mathrm{ABA}} + (1-\pi^{\mathrm{ABA}})p_R] + \sum_{q\neq R}\left[p_R \times p_q \times (\pi^{RA}-\pi^{ABA})\right] = p_R \times$

$[\pi^{\mathrm{ABA}} + (1-\pi^{\mathrm{ABA}})p_R] + p_R \times (1-p_R) \times (\pi^{RA}-\pi^{ABA}) = p_R \times$

$[\pi^{\mathrm{RA}} + (1-\pi^{\mathrm{RA}})p_R].$

Now we combine the two parts,

$$P\left({S_2}^{\mathrm{ABA}} + Y = s\right) = \sum_{(k,q)\in A}\left[p_k \times \left[(1-\pi^{\mathrm{RA}})p_q\right]\right] + p_R \times [\pi^{\mathrm{RA}} + (1-\pi^{\mathrm{RA}})p_R] = P\left({S_2}^{\mathrm{RA}} = s\right), \quad \text{(A11)}$$

therefore, according to expression (A11) the probability of outcome s is the same for ${S_2}^{\mathrm{ABA}} + Y$ and ${S_2}^{\mathrm{RA}}$, so they have the same distribution.■

References

Altmann, E. M., & Burns, B. D. (2005). Streak biases in decision making: Data and a memory model. *Cognitive Systems Research*, *6*(1), 5-16.

Asparouhova, E., Hertzel, M., & Lemmon, M. (2009). Inference from streaks in random outcomes: Experimental evidence on beliefs in regime shifting and the law of small numbers. *Management Science*, *55*(11), 1766-1782.

Ayton, P., & Fischer, I. (2004). The hot hand fallacy and the gambler's fallacy: Two faces of subjective randomness?. *Memory & cognition*, *32*(8), 1369-1378.

Bar-Hillel, M., &Wagenaar, W. A. (1991). The perception of randomness. *Advances in applied mathematics*, *12*(4), 428-454.

Benartzi, S., & Thaler, R. H. (1999). Risk aversion or myopia? Choices in repeated gambles and retirement investments. Management science, 45(3), 364-381.

Benartzi, S., & Thaler, R. H. (1995). Myopic Loss Aversion And The Equity Premium Puzzle. The Quarterly Journal of Economics, 110(1), 73-92.

Budescu, D. V. (1987). A Markov model for generation of random binary sequences. *Journal of Experimental Psychology: Human perception and performance*, *13*(1), 25.

Croson, R., & Sundali, J. (2005). The gambler's fallacy and the hot hand: Empirical data from casinos. Journal of risk and uncertainty, 30(3), 195-209.

DeKay, M. L., & Kim, T. G. (2005). When Things Don't Add Up The Role of Perceived Fungibility in Repeated-Play Decisions. *Psychological Science*,*16*(9), 667-672.

Edwards, A.W.F. (1960) The meaning of binomial distribution. *Nature*, 186, 1074.

Falk, R. (1981) The perception of randomness. Proceedings, Fifth International Conference for the Psychology of Mathematical Education, Grenoble, France, pp. 222–229.

Falk, R., & Konold, C. (1997). Making sense of randomness: Implicit encoding as a basis for judgment. Psychological Review, 104(2), 301.

Gilovich, T., Vallone, R., & Tversky, A. (1985). The hot hand in basketball: On the misperception of random sequences. *Cognitive psychology*, *17*(3), 295-314.

Gonzalez, R., & Wu, G. (1999).On the shape of the probability weighting function. *Cognitive psychology*, *38*(1), 129-166.

Gneezy, U., & Potters, J. (1997). An experiment on risk taking and evaluation periods. The Quarterly Journal of Economics, 631-645.

Huber, J., Kirchler, M., & Stöckl, T. (2010).The hot hand belief and the gambler's fallacy in investment decisions under risk. *Theory and Decision*, *68*(4), 445-462.

Kahneman, D., & Tversky, A. (1972). Subjective probability: A judgment of representativeness. *Cognitive Psychology*, *3*, 430–454.

Kaivanto, K. (2008). Alternation bias and the parameterization of cumulative prospect theory. In: Abdellaoui, M., Hey, J.D. (Eds.), *Advances in Decision Making Under Risk and Uncertainty*. Springer-Verlag, Heidelberg, pp. 91–107.

Kaivanto, K., & Kroll, E. B. (2012). Negative recency, randomization device choice, and reduction of compound lotteries. *Economics Letters*, *115*(2), 263-267.

Markowitz, H. (2014). Mean–variance approximations to expected utility. *European Journal of Operational Research*, *234*(2), 346-355.

Lippman, S. A., &Mamer, J. W. (1988).When many wrongs make a right. *Probability in the Engineering and Informational Sciences*, *2*(01), 115-127.

Liu, H. H., & Colman, A. M. (2009). Ambiguity aversion in the long run: Repeated decisions under risk and uncertainty. *Journal of Economic Psychology*, *30*(3), 277-284.

Lopes, L. L. (1996). When time is of the essence: Averaging, aspiration, and the short run. *Organizational Behavior and Human Decision Processes*, *65*(3), 179-189.

Lopes, L. L. (1981). Decision making in the short run. Journal of Experimental Psychology: Human Learning and Memory, 7(5), 377.

Markowitz H. Portfolio selection, efficient diversification of investments. New Haven, USA: Yale University Press; 1959.

Montgomery, H., &Adelbratt, T. (1982).Gambling decisions and information about expected value. *Organizational Behavior and Human Performance*, *29*(1), 39-57.

Nielsen, L. T. (1985). Attractive compounds of unattractive investments and gambles. *The Scandinavian Journal of Economics*, 463-473.

Oskarsson, A. T., Van Boven, L., McClelland, G. H., & Hastie, R. (2009). What's next? Judging sequences of binary events. *Psychological bulletin*, *135*(2), 262.

Quiggin, J. (1981). Risk perception and risk aversion among Australian farmers. *Australian Journal of Agricultural Economics*, *25*(2), 160-69.

Rabin, M. (2002), Inference by believers in the law of small numbers, *Quarterly Journal of Economics*, 117:775−816

Rapoport, A., &Budescu, D. V. (1997). Randomization in individual choice behavior. *Psychological Review*, *104*(3), 603.

Rapoport, A., &Budescu, D. V. (1992). Generation of random series in two-person strictly competitive games. *Journal of Experimental Psychology: General*, *121*(3), 352.

Redelmeier, D. A., &Tversky, A. (1992). On the framing of multiple prospects. *Psychological Science*, *3*(3), 191-193.

Ross, S. A. (1999). Adding risks: Samuelson's fallacy of large numbers revisited. *Journal of Financial and Quantitative Analysis*, *34*(03), 323-339.

Rothschild, M., & Stiglitz, J. E. (1970). Increasing risk: I. A definition. *Journal of Economic theory*, *2*(3), 225-243.

Rothschild, M., & Stiglitz, J. E. (1971). Increasing risk II: Its economic consequences. *Journal of Economic theory*, *3*(1), 66-84.

Samuelson, P. A. (1963). Risk and uncertainty-a fallacy of large numbers. *Scientia*, *98*(612), 108.

Tversky, A., & Bar-Hillel, M. (1983). Risk: the long and the short. Journal of experimental psychology. Learning, memory, and cognition, 9(4), 713-717.

Tversky, A., & Kahneman, D. (1992). Advances in prospect theory: Cumulative representation of uncertainty. *Journal of Risk and uncertainty*, *5*(4), 297-323.

Tversky, A., & Kahneman, D. (1971). Belief in the law of small numbers. *Psychological bulletin*, *76*(2), 105.

Tversky, A., & Wakker, P. (1995). Risk attitudes and decision weights. *Econometrica: Journal of the Econometric Society*, 1255-1280.

Wang, Y. H., & Yang, Z. (1995). On a Markov multinomial distribution. *Mathematical Scientist*, *20*(1), 40-49.

Zhao, J., Hahn, U., & Osherson, D. (2014). Perception and identification of random events. *Journal of experimental psychology: human perception and performance*, *40*(4), 1358.