The NC property for Banach algebras, and $C^*$-equivalence

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Abstract

Consider a unital Banach algebra $\mathcal{A}$ having the NC property: that

$$1 - \mathcal{A}_1^+ \subseteq \mathcal{A}_1^+.$$  

Then

$$|h|_\sigma = \|h\| \quad (h \in \mathcal{A}^+).$$

Any unital hermitian Banach $*$-algebra with this property is therefore $C^*$-equivalent.

Introduction

All Banach algebras considered here will be unital, with $\|1\| = 1$.

As usual, given an element $a$ of a unital Banach algebra $\mathcal{A}$ write $\sigma(a)$ for its spectrum and $|a|_\sigma$ for its spectral radius.

Write $\mathcal{A}^r$ for the set $\{h \in \mathcal{A} : \sigma(h) \subseteq \mathbb{R}\}$, the set of elements in $\mathcal{A}$ with real spectrum; write $\mathcal{A}^+$ for the set $\{h \in \mathcal{A} : \sigma(h) \subseteq \mathbb{R}^+\}$ (the positive elements of $\mathcal{A}$); and $\mathcal{A}_1^+$ for the set $\{h \in \mathcal{A}^+ : \|h\| \leq 1\}$. We write $h \gg 0$ to indicate that $\sigma(h) \subseteq \mathbb{R}^+ \setminus \{0\}$ — equivalently, that $h$ is positive and invertible.

The NC property introduced in [4] is that

$$1 - \mathcal{A}_1^+ \subseteq \mathcal{A}_1^+,$$

or, alternatively put,

$$\text{if } \sigma(h) \subseteq \mathbb{R}^+ \& \|h\| \leq 1 \text{ then } \|1 - h\| \leq 1.$$  

As we shall show, given NC one can produce positive square roots of positive elements, with control of their norm, and prove that the norm and spectral radius agree on positive elements. (This was claimed in [4, §11] but the argument there presented was both overclumsy and undercorrect.) The essential tool is the family of real polynomial functions $\pi_n$ defined recursively by:

$$\pi_0(t) = 0,$$

$$2\pi_{n+1}(t) = t + 2\pi_n(t) - \pi_n(t)^2.$$  

Recall that on $[0, 1]$ these polynomials increase monotonically and uniformly to the square root function.

4 April 2017
The NC Square Root Lemma

Let $\mathcal{A}$ be a unital Banach algebra. Given an element $a \in \mathcal{A}$ we define

$$a_n = \pi_n(a) \quad (n = 0, 1, \ldots)$$

and check that then, for $n = 0, 1, \ldots$,

$$2(1 - a_{n+1}) = 1 - a + (1 - a_n)^2$$
$$2(a_{n+2} - a_{n+1}) = (a_{n+1} - a_n)(2 - a_{n+1} - a_n).$$

Lemma 1. Consider a unital Banach algebra $\mathcal{A}$ satisfying NC and an element $h \in \mathcal{A}^+$ for which $h \gg 0$ and $\|h\| < 1$.

Define $h_n = \pi_n(h)$ for $n = 0, 1, \ldots$. Then $(h_n)$ is a Cauchy sequence and its limit $h_\infty$ lies in $\mathcal{A}_1$. Moreover, $h_\infty^2 = h$.

Proof. Choose an $\varepsilon > 0$ such that

$$2\varepsilon \leq h \quad \& \quad \|h\| + 2\varepsilon < 1$$

and put $\lambda = 1 - \varepsilon$. Note that $2\varepsilon < 1 < 2\lambda$.

Now $\|h - \varepsilon\| \leq \|h\| + \varepsilon \leq 1 - \varepsilon = \lambda$, and $1 - h = \lambda \left[\frac{1 - h - \varepsilon}{\lambda}\right]$; so

$$\|1 - h\| \leq \lambda.$$

Also, $\|h - 2\varepsilon\| \leq \|h\| + 2\varepsilon < 1$, and $1 - h_1 = 1 - \frac{h}{2} = \lambda \left[\frac{1 - h - 2\varepsilon}{2\lambda}\right]$; so

$$\|1 - h_1\| \leq \lambda.$$

From this, the induction steps being

$$2\|1 - h_n\| \leq \|1 - h\| + \|1 - h_n\|^2 \leq \lambda + \lambda^2 \leq 2\lambda$$
and

$$2\|h_{n+2} - h_{n+1}\| \leq \|h_{n+1} - h_n\| \|1 - h_n\| + \|1 - h_{n+1}\| \leq 2\lambda \|h_{n+1} - h_n\|,$$
we deduce that

$$\|1 - h_n\| \leq \lambda \quad \text{(whence } \|h_n\| \leq 1) \quad \text{and}$$
$$\|h_{n+1} - h_n\| \leq \lambda^{n+1}$$

for $n = 1, 2, \ldots$. Thus $(h_n)$ is a Cauchy sequence in $\mathcal{A}_1$ and its limit $h_\infty$, also in $\mathcal{A}_1$, must satisfy the relation

$$2h_\infty = h + 2h_\infty - h_\infty^2,$$
that is

$$h_\infty^2 = h.$$
The *NC* Theorem

**Theorem 2.** If $A$ is a unital Banach algebra satisfying *NC* then

$$|h|_\sigma = \|h\| \quad (h \in A^+).$$

Consequently positive square roots of positives exist and are uniquely determined. Further,

$$\|h\| \leq 2|h|_\sigma \quad (h \in A^*).$$

**Proof.** First suppose that $h \gg 0$ and that $\|h^2\| < 1$. Put $k = h^2$ and apply the Lemma to obtain a $k_{\infty} = \lim \pi_n(k)$, such that $k^2_{\infty} = k = h^2$. By construction, $k_{\infty}$ commutes with $h$.

Consider any character $\varphi$ on the unital commutative Banach subalgebra of $A$ generated by $h$. We have $\varphi(k_{\infty})^2 = \varphi(h)^2$; and, since both $k_{\infty}$ and $h$ are positive, we have $\varphi(k_{\infty}) = \varphi(h)$. Thus $\min \sigma(k_{\infty} + h) = 2 \min \sigma(h) > 0$, which shows that $k_{\infty} + h$ is invertible.

Now $(k_{\infty} - h)(k_{\infty} + h) = k^2_{\infty} - h^2 = 0$, whence $k_{\infty} = h$. That is, $h = \lim \pi_n(h^2)$ and therefore

$$\|h\| \leq 1, \text{ so long as } h \gg 0 \text{ and } \|h^2\| < 1.$$ 

Suppose now that $h \geq 0$ and $\|h^2\| < 1$. Then $h + \varepsilon \gg 0$ and $\|(h + \varepsilon)^2\| < 1$ for $\varepsilon$ positive and small enough. Thus $\|h + \varepsilon\| \leq 1$ for $\varepsilon$ small enough: so $\|h\| \leq 1$. The usual scaling argument shows that $\|h\|^2 \leq \|h^2\|$ and therefore

$$|h|_\sigma = \|h\|$$

for all $h \in A^+$.

Since the norm and spectral norm agree on positives we have

$$k = \lim \pi_n(k^2) \quad (k \in A_+^*).$$

Thus, if $k$ and $l$ are positive square roots of the same element (which, without loss of generality, we may assume to be in $A_+^*$) then $k = \lim \pi_n(k^2) = \lim \pi_n(l^2) = l$. That is, positives have unique positive square roots.

Given $h \in A$ with $\sigma(h) \subseteq \mathbb{R}$ we can define its absolute value $|h| = (h^2)^{\frac{1}{2}}$ and then define $h^+$ and $h^-$ by $2h^\pm = |h| \pm h$. This provides the decomposition $h = h^+ - h^-$ where $h^\pm \geq 0$, $h^+h^- = 0$ and $\|h^\pm\| \leq |h|_\sigma \leq \|h\|$. Hence $\|h\| \leq \|h^+\| + \|h^-\| \leq 2|h|_\sigma$. $\square$

Hermitian Banach *-algebras with *NC*

When $A$ is a *-algebra write $A^h = \{ h \in A : h = h^* \}$.

A unital Banach *-algebra $A$ is **hermitian** if $\sigma(h) \subseteq \mathbb{R}$ for all $h \in A^h$; that is, if $A^h \subseteq A^r$. Such an algebra is **symmetric**: $\sigma(a^*a) \subseteq \mathbb{R}^+$ for all $a \in A$ (the Shirali-Ford Theorem).

In his seminal paper [2, §5] Ptak showed that on a hermitian Banach *-algebra the Ptak seminorm, the function $a \mapsto |a|_{\Sigma} = |a^*a|^\frac{1}{2}$, is a $C^*$-seminorm. Moreover, he showed that a hermitian Banach *-algebra $A$ is $C^*$-equivalent if there exists a $\beta > 0$ such that $\|h\| \leq \beta|h|_\sigma$ for all $h \in A^h$. Hence
**Corollary 3.** If $A$ is a unital hermitian Banach $*$-algebra with the NC property then $A$ is $C^*$-equivalent. □

**Remark.** We have $\|a\| \leq 4|a|_\Sigma$ for all $a \in A$. When the involution is isometric, as was hypothesized in [4], we also have $|a|_\Sigma \leq \|a\|$ for all $a$.

**Remark.** One cannot improve this: NC does not imply that the original norm is $C^*$. As remarked by B.A. Barnes, reported by R.B. Burckel in his review of [1], if we take $\mathcal{B} = (\mathbb{R}[0,1], \|\cdot\|_\infty)$ and construct a Banach algebra norm $\|\cdot\|_c$ on $A = \mathbb{C}[0,1]$ as the complexification of $\mathcal{B}$ following [3, Theorem 1.3.2] then, taking $h, k \in \mathcal{B}$, $hk = 0$, $\|h\|_\infty = \|k\|_\infty > 0$, we find that $\|h + ik\|_c \geq \sqrt{2}\|h + ik\|_\infty$. So the norm $\|\cdot\|_c$ is a $C^*$-norm on $A^+$ but is not a $C^*$-norm on $A$.

**References**


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