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The Numerical Range of a Simple Compression

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Abstract

The numerical range of the contraction $K : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ acting on $L(\mathbb{C}^2)$ is identified, so allowing one to exhibit a hermitian projection that is not ultrahermitian.

An explicit formula for the norm of the operator $\kappa_m := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} ma & b \\ c & d \end{bmatrix}$ ($m \in \mathbb{C}$), translates into a family of inequalities in four complex variables.

Introduction

Although the product of hermitian operators on a Hilbert space is also hermitian if (and only if) they commute, this does not extend to hermitian operators on a Banach space. Indeed, the square of a hermitian need not be hermitian: and even the product of two commuting hermitian projections need not be hermitian.

Here I identify the numerical range of the simplest nontrivial compression operator $K : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$, and so can exhibit hermitian projections that are not ultrahermitian.

The norms of the related operators $\kappa_m := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} ma & b \\ c & d \end{bmatrix}$ are calculated explicitly (as $m$ varies in the complex plane).

Perhaps surprisingly, the quantity $a^2 + b^2 + c^2 + d^2 + \sqrt{(a^2 + b^2 + c^2 + d^2)^2 - 4(ad \pm bc)^2}$ does not necessarily decrease when one replaces $a$ by 0 ($a$, $b$, $c$ and $d$ being arbitrary real numbers), but may increase by up to the factor $\|\kappa_0\|$. 

1 Numerical range

I follow the standard notation and rehearse only a few salient details, referring the reader to [BD], for example, for a full exposition and other references.

Given a Banach space $X$ we say that

\[ f \in X' \text{ supports } x \in X \text{ if } \langle x, f \rangle = \|x\| = \|f\| = 1. \]

The **supporting set** for $X$ is

\[ \Pi_X := \{(x, f) \in X \times X' : \langle x, f \rangle = \|x\| = \|f\| = 1 \}. \]

The **(spatial) numerical range** of the operator $T(\in L(X))$ is

\[ V(T) := \{\langle Tx, f \rangle : (x, f) \in \Pi_X \}. \]

**Definition 1.1** $H$ in $L(X)$ is hermitian if its numerical range is real: equivalently, if $\|e^{iH}\| = 1$ ($\forall r \in \mathbb{R}$): equivalently, if $\|I_X + irH\| \leq 1 + o(r)$ ($\mathbb{R} \ni r \to 0$).
2 The Banach space $L(\mathbb{C}^2)$ and some linear algebra

My example lives on $L(\mathbb{C}^2)$ with the operator norm. Facts to notice about this Banach space:

- Given $f \in L(\mathbb{C}^2)$ we can define a functional $\omega_f : y \mapsto \text{tr}(yf)$ in $L(\mathbb{C}^2)'$: here $\text{tr}$ is the unnormalised trace: and

  $$\|\omega_f\| = \text{tr}|f| = \text{tr}(f^*f)^{\frac{1}{2}}.$$ 

  Since any functional must be of this form we see that the [pre]dual of $L(\mathbb{C}^2)$ is, as a set, the same space as $L(\mathbb{C}^2)$: but with the trace norm.

- $\Pi_{L(\mathbb{C}^2)}$ is biunitarily invariant in the sense that

  $$(uxv, v^*fu^*) \in \Pi_{L(\mathbb{C}^2)} \iff (x, f) \in \Pi_{L(\mathbb{C}^2)}$$

  for any unitaries $u$ and $v$.

- $\Pi_{L(\mathbb{C}^2)}$ is invariant under complex conjugation too — so $V(T)$ is symmetric in the real axis when $T$ has real entries.

Given an element $x = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ of $L(\mathbb{C}^2)$ define

$$\sigma^2 = |a|^2 + |b|^2 + |c|^2 + |d|^2, \quad \nu^2 = |ad - bc|, \quad \text{and} \quad \rho^4 = \sigma^4 - 4\nu^4.$$ 

Then (routine computation!) the eigenvalues of $x^*x$ are $(\sigma^2 \pm \rho^2)/2$ from which we have

$$\|x\|_{L(\mathbb{C}^2)}^2 = \frac{\sigma^2 + \rho^2}{2} \quad \text{and} \quad \text{tr}|x| = \left[\sigma^2 + 2\nu^2\right]^{\frac{1}{2}}.$$ 

**Singular value decomposition**

Given $x \in L(\mathbb{C}^2)$ there are unitaries $u$ and $v$ such that

$$uxv = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

where $\lambda_1$ and $\lambda_2$ ($\lambda_1 \geq \lambda_2$) are the eigenvalues of $|x|$. In particular, if $\|x\| = 1$, there are $u$, $v$ such that

$$uxv = \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix} =: x_\lambda$$

with $0 \leq \lambda \leq 1$: and $\lambda = 1$ precisely when $x$ itself is unitary.
The supporting set $\Pi_{L(\mathbb{C}^2)}$

Define

$$f(\alpha) = \begin{bmatrix} \alpha & 0 \\ 0 & 1 - \alpha \end{bmatrix}.$$ 

**Lemma 2.1** The functionals $f(\alpha)$ ($0 \leq \alpha \leq 1$) support $x_1$: and only these. The functional $f(1)$ is the only support of $x_\lambda$ when $0 \leq \lambda < 1$. □

Hence

**Lemma 2.2**

$$\Pi_{L(\mathbb{C}^2)} = \{(u^* x \lambda^*, v f(\alpha) u)\}$$

where $u, v$ are unitary, $0 \leq \lambda \leq 1$.

\section{The compression $K$
}

Consider the selfadjoint projection $P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ in $L(\mathbb{C}^2)$. Then the left and right multiplication operators

$$L = L_P \quad \& \quad R = R_P$$

are hermitian projections in $L(L(\mathbb{C}^2))$, for $\|e^{irL_P}\| = \|e^{irR_P}\| = \|e^{irP}\| = 1$ ($r \in \mathbb{R}$). They commute, and their product

$$K = LR = RL$$

is a norm 1 projection on $L(\mathbb{C}^2)$, the compression $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$.

**Theorem 3.1** $K$ is not hermitian.

\textbf{Proof.} Note that $\|I - 2Q\| = \|e^{irQ}\| = 1$ for any hermitian projection $Q$. However, $\|I - 2K\| \geq \sqrt{2}$ for $(I - 2K) \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $\left\| \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \right\| = \sqrt{2}$ while $\left\| \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\| = 2$. 

(In fact, $\|I - 2K\| = \|\kappa_{-1}\| = \sqrt{2}$: see §5 below.) □

[AF] showed, also explicitly, that $\|\exp(3\pi i K/2)\| > 1$.

\section{Ultrahermitian projections
}

Consider the following two properties that may hold for a projection $E$ on a Banach space $X$. Note that they are symmetrical in $E$ and its complement $\overline{E}$ ($= I - E$). First,

\begin{equation}
(U1) \quad \|Ex\| \|E'\phi\| + \|E\overline{x}\| \|\overline{E'}\phi\| \leq \|x\| \|\phi\|
\end{equation}

for $x \in X$, $\phi \in X'$: and, second,

\begin{equation}
(U2) \quad \|EAE + E\overline{E}E\| \leq 1
\end{equation}
for any contractions $A, B \in L(X)$.

Hermitian projections on Hilbert spaces have both these properties, as is easy to check. The present author showed, see [S], that the properties (U1) and (U2) are equivalent, and introduced the term \textit{ultrahermitian} for a projection that has either [and so both] of them.

Ultrahermitian projections are automatically hermitian [S, Theorem 4.3] and the product of two hermitian projections of which one is ultrahermitian must be hermitian [S, Corollary 4.8]. Hence

\textbf{Theorem 3.2} The left and right multiplication operators $L_P$ and $R_P$, though hermitian, are not ultrahermitian.

\section{The numerical range $V(K)$}

By Lemma 2.2 this is the convex set of all

$$\varpi_{\lambda, \alpha} := \langle K u^* x \lambda v^*, v f(\alpha) u \rangle$$

$$= \text{tr} ([P u^* x \lambda v^* P] [v f(\alpha) u])$$

$$= \text{tr} ([P u^* x \lambda v^* P] [P v f(\alpha) u P])$$

$$= (u^* x \lambda v^*)_{(1,1)} (v f(\alpha) u)_{(1,1)}$$

where $u, v$ are arbitrary unitaries, $0 \leq \lambda \leq 1$, and $\alpha \left\{ \begin{array}{ll} \in [0, 1] & \lambda = 1 \\ = 1 & 0 \leq \lambda < 1 \end{array} \right.$

As a full set of unitaries we may take

$$u := \omega_0 \begin{bmatrix} c & \omega_2 s \\ \omega_1 s & -\omega_1 \omega_2 c \end{bmatrix} \quad \text{and} \quad v := w_0 \begin{bmatrix} C & w_2 S \\ w_1 S & -w_1 w_2 C \end{bmatrix}$$

with $|\omega_k| = 1$, $c = \cos \theta$, $s = \sin \theta$, $(0 \leq \theta \leq \pi/2)$, and $|w_k| = 1$, $C = \cos \varphi$, $S = \sin \varphi$, $(0 \leq \varphi \leq \pi/2)$. Compute:

$$Pu^* x \lambda v^* P = \omega_0 w_0 \begin{bmatrix} cC + \lambda \omega_1 w_2 sS & 0 \\ 0 & 0 \end{bmatrix}$$

$$P v f(\alpha) u P = \omega_0 w_0 \begin{bmatrix} \alpha cC + (1 - \alpha) \omega_1 w_2 sS & 0 \\ 0 & 0 \end{bmatrix}$$

So

$$\varpi_{\lambda, \alpha} = \alpha c^2 C^2 + \lambda (1 - \alpha) s^2 S^2 + [\alpha \lambda \omega_1 w_2 + (1 - \alpha) \omega_1 w_2] c C S S$$

$$= \begin{cases} c^2 C^2 + \lambda \omega_1 w_2 c C S S & 0 \leq \lambda < 1^* \\ \alpha [c^2 C^2 + \omega_1 w_2 c C S S] + (1 - \alpha) [s^2 S^2 + \omega_1 w_2 c C S S] & \lambda = 1 \end{cases}$$

(* — also for $\lambda = 1$ — put $\alpha = 1$ in the following line.)

Replace $\omega_1 w_2$ by $\omega$. The points $\varpi_{\lambda, 1}$, i.e

$$c^2 C^2 + \lambda \omega c C S S \quad (0 \leq \lambda \leq 1)$$

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form the closed discs
\[ D(\theta, \varphi) := \{ \cos^2 \theta \cos^2 \varphi + \zeta \cos \theta \cos \varphi \sin \theta \sin \varphi : |\zeta| \leq 1 \} \]
with boundaries as in Figure 1. This demonstrates

**Theorem 4.1**
\[ V(K) = \bigcup_{0 \leq \theta \leq \pi/2} \bigcup_{0 \leq \varphi \leq \pi/2} D(\theta, \varphi). \]

**Remark 4.2** Since \(-\frac{1}{8} \in V(K)\) we see that \(\|I - 2K\| \geq |V(I - 2K)| = \frac{5}{4}\), so, again, \(K\) cannot be hermitian.

![Figure 1: \(\{\cos^2 \theta \cos^2 \varphi + \omega \cos \theta \cos \varphi \sin \theta \sin \varphi : |\omega| = 1\}\)](image)

**Lemma 4.3 (Cosine-geometric mean)** Given \(\theta, \varphi\) in the first quadrant define their cosine-geometric mean
\[ \psi := \cos^{-1} \sqrt{\cos \theta \cos \varphi}. \]
Then the disc \(D(\theta, \varphi)\) lies concentrically inside the disc
\[ D(\psi, \psi) = \{ \cos^4 \psi + \zeta \cos^2 \psi \sin^2 \psi : |\zeta| \leq 1 \}. \]

**Proof.** Just check that \(\sin \theta \sin \varphi = \cos(\theta - \varphi) - \cos^2 \psi \leq 1 - \cos^2 \psi = \sin^2 \psi\). \(\square\)

Next, for \(0 < \alpha < 1\), the points \(\varpi_{1, \alpha}\) of the numerical range \(\text{ie} \alpha[c^2C^2 + \varpi cCsS] + (1 - \alpha)[s^2S^2 + \omega cCsS] \) lie in the convex hull of \(D(\psi, \psi)\) and \(D(\tilde{\psi}, \tilde{\psi})\), where \(\tilde{\psi}\) is the cosine-geometric mean of \(\frac{\pi}{2} - \theta\) and \(\frac{\pi}{2} - \varphi\). Thus

**Theorem 4.4**
\[ V(K) = \bigcup_{0 \leq \theta \leq \pi/2} \bigcup_{0 \leq \varphi \leq \pi/2} D(\theta, \varphi) = \bigcup_{0 \leq \psi \leq \pi/2} D(\psi, \psi). \]

The circles \(\partial D(\theta, \varphi)\) and \(\partial D(\psi, \psi)\) lie as shown in Figure 2; and \(V(K)\), the union of the discs \(D(\psi, \psi)\), is as in Figure 3.
The envelope and cusp

The circumference of the disc $D(\psi, \psi)$ is [setting $\gamma = \cos^2 \psi$]

$$(x - \gamma^2)^2 + y^2 = \gamma^2 (1 - \gamma)^2 = \gamma^2 - 2\gamma^3 + \gamma^4.$$  

To find the envelope of the $D(\psi, \psi)$ solve this equation simultaneously with its $\gamma$-derivative

$$2(x - \gamma^2)(-2\gamma) = 2\gamma - 6\gamma^2 + 4\gamma^3$$

to get

$$2x = 3\gamma - 1$$  
$$2y = \pm(1 - \gamma)\{4\gamma - 1\}^{\frac{1}{2}}$$

for $\frac{1}{4} \leq \gamma \leq 1$. 

Figure 2: $\partial D(\theta, \varphi)$ (red) & $\partial D(\psi, \psi)$ (blue)

Figure 3: $V(K) = \bigcup_{0 \leq \theta \leq \pi/2} D(\theta, \theta)$

Figure 4: The cusp angle
5 The map \( \kappa_m \) and its norm \( (m \in \mathbb{C}) \)

The map \( \kappa_m \) is defined as
\[
\kappa_m := I + (m - 1)K : L(\mathbb{C}^2) \to L(\mathbb{C}^2) : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} ma & b \\ c & d \end{bmatrix} .
\]

As a first estimate \( \| \kappa_m \| \geq 1 \) and \( \| \kappa_m \| \geq |m| \).

Since \( \kappa_m \) attains its norm on the unit ball of \( L(\mathbb{C}^2) \), the convex hull of the unitaries (the Russo-Dye theorem [BD, §38]), we next examine the values \( \| \kappa_m u \| \) for unitary \( u \). It will be more convenient to work with the expression \( 2 \| \kappa_m u \|^2 \).

With \( c = \cos \theta, s = \sin \theta \), and \( 0 \leq \theta \leq \pi/2 \), consider a typical unitary
\[
u := \nu(c) = \omega_0 \begin{bmatrix} c & \omega_2 s \\ \omega_1 s & -\omega_1 \omega_2 c \end{bmatrix}
\]
where \( \omega_1 \) and \( \omega_2 \) are arbitrary unimodular complex numbers. Calculate:
\[
\sigma(\kappa_m u)^2 = 2 + (|m|^2 - 1)c^2
\]
\[
\rho(\kappa_m u)^4 = c^4 \left\{ 4|m - 1|^2 + [(|m|^2 - 1)^2 - 4|m - 1|^2]c^2 \right\}
\]
\[
F_m(c) := 2 \| \kappa_m u \|^2 = \sigma(\kappa_m u)^2 + \rho(\kappa_m u)^2
\]
\[
= 2 + (|m|^2 - 1)c^2 + c \left\{ 4|m - 1|^2 + [(|m|^2 - 1)^2 - 4|m - 1|^2]c^2 \right\}^{1/2}
\]

The \( \omega_1 \) and \( \omega_2 \) are now seen to be irrelevant, so, without loss of generality, take \( \omega_1 = \omega_2 = 1 \).

Put
\[
\Gamma := 4|m - 1|^2 - (|m|^2 - 1)^2.
\]

Then
\[
F_m(c) = 2 + (|m|^2 - 1)c^2 + c \left\{ 4|m - 1|^2 - \Gamma c^2 \right\}^{1/2}.
\]

Note that
\[
F_m(0) = 2,
\]
\[
F_m(1) = 2 + |m|^2 - 1 + \left\{ (|m|^2 - 1)^2 \right\}^{1/2},
\]
\[
= 2 \max\{1, |m|^2\} \geq F_m(0).
\]
Thus \[ ||\kappa_m|| = \max\{1, |m|\} \]
when \( F_m \) has no turning point in \([0, 1]\).

**The cardioid \( \Gamma = 0 \)**

The locus \( \Gamma = 0 \), that is, \(|m|^2 - 1 = 2|m - 1|\), is the **cardioid** shown in Figure 6.

![Figure 6: \(|m|^2 - 1 = 2|m - 1|\)](image)

In plane polar coordinates \((r, \phi)\) the equation is \(8r \cos \phi = 3 + 6r^2 - r^4\).

**Outside the cardioid \( \Gamma = 0 \)**

The function \( F_m(c) \) certainly increases on \([0, 1]\) if \( \Gamma \leq 0 \) (which forces \(|m| \geq 1\)) so \( ||\kappa_m|| = \max\{1, |m|\} = |m| \) outside the cardioid.

**Inside the cardioid \( \Gamma = 0 \)**

To find turning points differentiate with respect to \(c\):

\[
F'_m(c) = 2(|m|^2 - 1)c + \left\{ 4|m - 1|^2 - \Gamma c^2 \right\}^{\frac{1}{2}} \\
- \Gamma c^2 \left\{ 4|m - 1|^2 - \Gamma c^2 \right\}^{-\frac{1}{2}} \\
= 2(|m|^2 - 1)c + 2\left\{ 2|m - 1|^2 - \Gamma c^2 \right\} \left\{ 4|m - 1|^2 - \Gamma c^2 \right\}^{-\frac{1}{2}}
\]

Setting \( F'_m(c) = 0 \) and squaring [so possibly introducing spurious solutions] leads to the equation

\[ \Gamma c^4 - 4|m - 1|^2 c^2 + |m - 1|^2 = 0 \]

for \( c^2 \).

Note that if \(|m| = 1 \) [leaving \( m = 1 \) aside] the equation reduces to \((1 - 2c^2)^2 = 0\), and therefore \( \kappa_m \) attains its norm at \( \left[ \begin{array}{c} 1 \\ 1 \end{array} \right] \), independently of \( \arg m \).

Otherwise the discriminant is

\[
\Delta = (2|m - 1|^2)^2 - [4|m - 1|^2 - (|m|^2 - 1)^2] |m - 1|^2 \\
= |m - 1|^2 (|m|^2 - 1)^2 > 0
\]
and the candidate solutions are
\[ c_\pm^2 = \frac{2|m-1|^2 \pm |m-1|(|m|^2 - 1)}{2|m-1| - (|m|^2 - 1)} \]
\[ = \frac{|m-1|}{2|m-1| \mp (|m|^2 - 1)} > 0 \]

It is straightforward to check that
\[ 4|m-1|^2 - \Gamma c_\pm^2 = \frac{|m-1|^2}{c_\pm^2}, \]
\[ 2|m-1|^2 - \Gamma c_\pm^2 = \mp |m-1|(|m|^2 - 1). \]

Thus
\[ F_m'(c_\pm) = 2(|m|^2 - 1)c + 2\{2|m-1|^2 - \Gamma c_\pm^2\} \left\{4|m-1|^2 - \Gamma c_\pm^2\right\}^{\frac{1}{2}} \]
\[ = 2c_\pm \{[|m|^2 - 1] \mp [|m|^2 - 1]\}, \]

which shows that \( c_+ \) alone is a possible turning point for \( F_m \): but does \( c_+ \) lie in \([0,1]\)?

The condition for this is that \( |m-1| \leq 2|m-1| - (|m|^2 - 1) \) ie that
\[ |m|^2 - 1 \leq |m-1|. \]

**The cardioidoid** \( ||m|^2 - 1| = |m-1| \)

The ‘edge locus’ \( ||m|^2 - 1| = |m-1| \), which, for lack of another name I shall call a **cardioidoid**, bounds the blue region in Figure 7.

\[ \text{Figure 7: The cardioidoid} \]

In plane polar coordinates it has equation \( 2r \cos \phi = 3r^2 - r^4 \).

However, the set \( |m|^2 - 1 \leq |m-1| \) includes the unit disc too: I refer to this set as the **filled cardioidoid**.
Inside the filled cardioidoid

Suppose that \( m \) lies inside the filled cardioidoid, so that \( c_+ \in [0, 1] \).

Then

\[
F_m(c_+) = \cdots = 2 \frac{(|m - 1| + 1)^2 - |m|^2}{2|m - 1| + 1 - |m|^2}.
\]

Next

\[
F_m(c_+) - 2 = \frac{2 |m - 1|^2}{2|m - 1| + 1 - |m|^2} \geq 0
\]

and

\[
F_m(c_+) - 2|m|^2 = \frac{2(|m|^2 - 1 - |m - 1|)^2}{2|m - 1| + 1 - |m|^2} \geq 0
\]

so

\[
F_m(c_+) \geq F_m(1) \geq F_m(0).
\]

Therefore

\[
||\kappa_m||^2 = \frac{(|m - 1| + 1)^2 - |m|^2}{2|m - 1| + 1 - |m|^2}
\]

for \( m \) inside the filled cardioidoid. When \( m \) is real, within these limits, this expression reduces to \( \frac{4}{3 + m} \).

To sum up:

**Theorem 5.1**

\[
||\kappa_m|| = \begin{cases}
|m| & \text{outside} \\
\sqrt{\frac{(|m-1|+1)^2-|m|^2}{2|m-1|+1-|m|^2}} & \text{inside} \\
\sqrt{\frac{4}{3+m}} & \text{on real axis inside}
\end{cases}
\]

the filled cardioidoid.
Graph of $\|\kappa_m\|$ for $m$ real

For real $m$ inside the filled cardioid, i.e. $-2 \leq m \leq 1$, we have

$$\|\kappa_m\| = \sqrt{\frac{4}{3 + m}}.$$  

The graph of norm $\kappa_m$ is shown in Figure 9.

Figure 9: $\|\kappa_m\|$ is continuous for all $m$ but is not differentiable at 1, even as a function of a real variable

6 An inequality

The inequalities

$$\|\kappa_mA\| \leq \|\kappa_m\| \|A\|$$

(for complex $2 \times 2$ matrices $A$) are hardly transparent when written out explicitly. However, for $m = 0$, the simplest case, we have $\|I - K\| = \|\kappa_0\| = 2/\sqrt{3}$ so, for any real numbers $a, b, c, d$, we have

$$3 \left(b^2 + c^2 + d^2 + \sqrt{(b^2 + c^2 + d^2)^2 - 4b^2c^2}\right)$$

$$\leq 4 \left(a^2 + b^2 + c^2 + d^2 + \sqrt{(a^2 + b^2 + c^2 + d^2)^2 - 4(ad \pm bc)^2}\right)$$

or, on rewriting,

$$3 \left(b^2 + c^2 + d^2 + \sqrt{[(b - c)^2 + d^2][(b + c)^2 + d^2]}\right)$$

$$\leq 4 \left(a^2 + b^2 + c^2 + d^2 + \sqrt{[(a - d)^2 + (b \mp c)^2][(a + d)^2 + (b \pm c)^2]}\right).$$

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