Linear Finite-Field Deterministic Networks With Many Sources and One Destination

M. Majid Butt*, Giuseppe Caire†, Ralf R. Müller*
*Institute of Electronics and Telecommunications, Norwegian University of Science and Technology, Trondheim, Norway †Department of Electrical Engineering, University of Southern California, Los Angeles, CA

***Abstract*—We find the capacity region of linear finite-field deterministic networks with many sources and one destination. Nodes in the network are subject to interference and broadcast constraints, specified by the linear finite-field deterministic model. Each node can inject its own information as well as relay other nodes' information. We show that the capacity region coincides with the cut-set region. Also, for a specific case of correlated sources we provide necessary and sufficient conditions for the sources transmissibility. Given the "deterministic model" approximation for the corresponding Gaussian network model, our results may be relevant to wireless sensor networks where the sensing nodes multiplex the relayed data from the other nodes with their own data, and where the goal is to decode all data at a single "collector" node.**

I. Introduction

Wireless sensor networks (WSN) consist of many nodes with sensing, computation, and communication capabilities, sharing a common wireless communication channel. In a typical WSN configuration, a large number of nodes measure possibly correlated data and transmit to a single collector node. This network problem is referred to as the "sensor reachback problem" in [1]. In many applications, nodes are energy-limited and the physical distance between each sensing node and the common destination makes the transmission difficult or (energy wise) expensive. We investigate such a scenario where communicating nodes cooperate with each other and act as relays in order to transport their own data along with the data from the other sensing nodes.

Wireless channels differ from their wired line counterpart in two fundamental aspects. On one hand, the wireless channel is a broadcast (shared) medium and the signal from any transmitter is received by potentially many receivers. This is called *broadcast* constraint. On the other hand, any receiver observes the superposition (linear combination) of signals from possibly many transmitters. This is called *interference* constraint. The simultaneous presence of these two constraints makes a general wireless network quite difficult to analyze. The multiuser Gaussian channel that models a relay network, unfortunately, has so far escaped a sharp general characterization, even in the simplest case of a Gaussian relay network with a single source, single destination and a single relay [2]. The capacities of Gaussian relay channel and certain discrete relay channels are evaluated in [3] and a lower bound to the capacity of general relay channel is presented. In [4], capacity is determined for a Gaussian relay network when the number of relays is asymptotically large.

In [5], a simpler deterministic channel is proposed. While this channel model is significantly simpler to analyze, it is able to capture the key aspects of broadcast and interference constraints. For this model, referred to as the linear finite-field deterministic model, [5] determines the capacity for a general relay network with one source and one destination, as well as the multicast capacity with one source, multiple destinations and common information only. Our contribution in this paper builds heavily on the results and techniques of [5] and can be regarded almost as a trivial extension thereof. Nevertheless, to the best of our knowledge and somehow surprisingly, this simple extension has not been reported before.

We consider the "sensor reachback problem" [1] for a linear finite-field deterministic network with arbitrary topology, a single destination node and independent information at the source nodes. We show that the capacity region for this network is given by the cut-set bound and takes on a very simple and appealing closed-form expression. Also, for a specific sources correlation model, we find necessary and sufficient conditions for the sources transmissibility. This result reminds closely Theorem 1 of [1], with the following main differences: on one hand, the result of [1] is more general since it applies to general correlated discrete sources observed at the sensor nodes and general noisy channels. On the other hand, our result applies to networks with broadcast and interference constraints while the result of [1] requires "orthogonal" channels, i.e., with neither broadcast nor interference constraints.

We expect that the achievability technique for the Gaussian (noisy) relay network proposed in [5] can be generalized to the case of multiple independent sources and a single destination as examined in this paper, so that a scheme that achieves a bounded and fixed gap to the capacity region in the Gaussian case can be found. Also, we believe that a fixed-gap rate-distortion achievable region can be found using independent quantization and Slepian-Wolf binning for the case of correlated Gaussian sources with mean-squared distortion and Gaussian noisy channels, at least for some specific source correlation model (see [6]), especially matched to the discrete correlated source model considered here. This, however, seems to be a far more involved result since event in the standard case of Gaussian/quadratic separated lossy encoding (that corresponds to the case where the communication network reduces to a set of orthogonal links from the sensor nodes to the destination), a general fixed-gap characterization of the rate-distortion region is missing [6].

In this work we limit ourselves to the linear finite-field deterministic model and we leave the fixed-gap achievability for the Gaussian case to future work.

II. Review of the deterministic linear finite-field model

In this section we briefly review the deterministic channel model proposed in [5] and used in this work. The received signal at each node is a deterministic function of the transmitted signal. This model focuses on the signals interaction rather than on the channel noise. In a Gaussian (real) network, a single link from node i to node j with SNR $\mathsf{snr}_{i,j}$ has capacity $C_{i,j} = \frac{1}{2}\log(1+\mathsf{snr}_{i,j}) \approx \log\sqrt{\mathsf{snr}_{i,j}}$. Therefore, approximately, $n_{i,j} = \lceil \log\sqrt{\mathsf{snr}_{i,j}} \rceil$ bits per channel use can be sent reliably. In [5] (see also references therein), the Gaussian channel is replaced by a finite-field deterministic model that reflects the above behavior. Namely, the transmitted signal amplitude is represented through its binary[1] expansion $X = \sum_{\ell=1}^{\infty} B_\ell 2^{-\ell}$ where $B_\ell \in \mathbb{F}_2$. At the receiver, all the input bits such that $\sqrt{\mathsf{snr}_{i,j}}2^{-\ell} > 1$ (i.e., received "above the noise level") are perfectly decoded, while all those such that $\sqrt{\mathsf{snr}_{i,j}}2^{-\ell} \leq 1$ (i.e., received "below the noise level") are completely lost. It follows that only the most significant bits (MSBs) can be reliably decoded, such that the capacity of the deterministic channel is given exactly by $n_{i,j}$ and it is achieved by letting $B_1, \ldots, B_{n_{i,j}}$ i.i.d. Bernoulli-1/2.

A linear finite-field deterministic relay network is defined as a directed acyclic graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ such that the received signal at any node $j \in \mathcal{V}$ is given by

$$\mathbf{y}_j = \sum_{i \in \mathcal{V}:(i,j)\in\mathcal{E}} \mathbf{S}^{q-n_{i,j}} \mathbf{x}_i \tag{1}$$

where $\mathbf{y}_j, \mathbf{x}_i \in \mathbb{F}_2^q$, sum and products are defined over the vector space $\mathbb{F}_2^q$, and where

$$\mathbf{S} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$$

is a "down-shift" matrix. Notice that $n_{i,j} \leq q$ indicates the deterministic channel capacity for the link (i,j) as described before. Without loss of generality, the integer q can be set equal to the maximum of all $\{n_{i,j} : (i,j) \in \mathcal{E}\}$. The broadcast constraint is captured by the fact that the input $\mathbf{x}_i$ for each node i is common to all channels $(i,j) \in \mathcal{E}$.

In the case of single source (denoted by s) single destination (denoted by d), Theorem 4.3 of [5] yields the capacity of linear finite-field deterministic relay networks in the form

$$C = \min_{(\mathcal{S},\mathcal{S}^c)\in\Lambda_d} \operatorname{rank}\{\mathbf{G}_{\mathcal{S},\mathcal{S}^c}\} \tag{2}$$

where Λ_d is the set of cuts $\mathcal{S} \subset \mathcal{V}$, $\Omega^c = \mathcal{V} - \mathcal{S}$ such that $s \in \mathcal{S}$ and $d \in \mathcal{S}^c$, and where $\mathbf{G}_{\mathcal{S},\mathcal{S}^c}$ is the transfer matrix for the cut $(\mathcal{S}, \mathcal{S}^c)$, formally defined as follows. Let $\mathcal{N}(i)$ denote the set of nodes j for which $(i,j) \in \mathcal{E}$ (this is the "fan-out" of node i) and let $\mathcal{P}(j)$ denote the set of nodes i for which $(i,j) \in \mathcal{E}$ (this is the "fan-in" of node j). The transfer matrix $\mathbf{G}_{\mathcal{S},\mathcal{S}^c}$ is defined as the matrix of the linear transformation between the transmitted vectors (channel inputs) of nodes $\beta_{\rm in}(\mathcal{S})$ and the received vectors (channel outputs) of nodes $\beta_{\rm out}(\mathcal{S})$, where the inner and outer boundaries $\beta_{\rm in}(\mathcal{S})$ and $\beta_{\rm out}(\mathcal{S})$ of $\mathcal{S}$ are defined as [7]:

$$\beta_{\rm in}(\mathcal{S}) = \{i \in \mathcal{S} : \mathcal{N}(i) \cap \mathcal{S}^c \neq \emptyset\}$$

and

$$\beta_{\rm out}(\mathcal{S}) = \{j \in \mathcal{S}^c : \mathcal{P}(j) \cap \mathcal{S} \neq \emptyset\}$$

In words: $\beta_{\rm in}(\mathcal{S})$ is the set of nodes of $\mathcal{S}$ with a direct link to nodes in $\mathcal{S}^c$, and $\beta_{\rm out}(\mathcal{S})$ is the set of nodes in $\mathcal{S}^c$ with a direct link from nodes in $\mathcal{S}$.

Going through the proof of Theorem 4.3 in [5] we notice that the "down-shift" structure for the individual channels is irrelevant. In fact, this structure is useful in making the connection between the linear finite-field model and the corresponding Gaussian case. As a matter of fact, if the channel matrices $\mathbf{S}^{q-n_{i,j}}$ in the above model are replaced by general matrices $\mathbf{S}_{i,j} \in \mathbb{F}_2^{q\times q}$, the result (2) still holds.

III. Main result

In a linear finite-field deterministic network defined as above, let $\mathcal{V} = \{1, \ldots, N, d\}$, where node d denotes the common destination and all other nodes $\{1, \ldots, N\}$ have independent information to send to node d. For any integer $T = 1, 2, \ldots$ we let $\mathcal{W}_i = \{1, \ldots, \lceil 2^{TR_i} \rceil\}$ denote the message set of node $i = 1, \ldots, N$. A $(T, R_1, \ldots, R_N)$ code for the network is defined by a sequence of *strictly causal* encoding functions $f_i^{[t]} : \mathcal{W}_i \times \mathbb{F}_2^{q(t-1)} \to \mathbb{F}_2^q$, for $t = 1, \ldots, T$ and $i = 1, \ldots, N$, such that the transmitted signal of node i at (discrete) time t is given by $\mathbf{x}_i[t] = f_i^{[t]}(w_i, \mathbf{y}_i[1], \ldots, \mathbf{y}_i[t-1])$, and by a decoding function $g : \mathbb{F}_2^{Tq} \to \mathcal{W}_1 \times \cdots \mathcal{W}_N$, such that the set of decoded messages is given by $(\widehat{w}_1, \ldots, \widehat{w}_N) = g(\mathbf{y}_d[1], \ldots, \mathbf{y}_d[T])$.

The average probability of error for such code is defined as $P_n(e) = \mathbb{P}((W_1, \ldots, W_N) \neq (\widehat{W}_1, \ldots, \widehat{W}_N)$, where the random variables W_i are independent and uniformly distributed on the corresponding message sets $\mathcal{W}_i$. The rate N-tuple $(R_1, \ldots, R_N)$ is *achievable* if there exists a sequence of $(T, R_1, \ldots, R_N)$-codes with $P_n(e) \to 0$ as $T \to \infty$. The capacity region $\mathcal{C}$ of the network is the closure of the set of all achievable rates. With these definitions, we have:

Theorem 1: The capacity region $\mathcal{C}$ of a linear finite-field deterministic network $(\mathcal{V}, \mathcal{E})$ with independent information at the nodes $\{1, \ldots, N\}$ and a single destination d is given by

$$\sum_{i\in\mathcal{S}} R_i \leq \operatorname{rank}\{\mathbf{G}_{\mathcal{S},\mathcal{S}^c}\}, \quad \forall\, \mathcal{S} \subseteq \{1, \ldots, N\}. \tag{3}$$

Proof: The converse of (3) follows directly from the general cut-set bound and by the fact that, for the linear deterministic network model, uniform i.i.d. inputs maximize all cut-set values at once [5], [7], [8].

For the direct part, we build an augmented network by introducing a virtual source node 0 and by expanding the channel output alphabet of each node $i = \{1, \ldots, N\}$. Let

[1]The generalization to p-ary expansion is trivial. Here we focus on the binary expansion as in [5].

$\{n_{0,i} : i = 1, \dots, N\}$ be arbitrary non-negative integers. The channel output alphabet of node i in the augmented network is given by $\mathbb{F}_2^{q+n_{0,i}}$. The virtual source node 0 has $n_0 = \sum_{i=1}^N n_{0,i}$ input bits, partitioned into N disjoint sets $\mathcal{U}_i$ of cardinality $n_{0,i}$ for $i = 1, \dots, N$, respectively, such that the bits of subset $\mathcal{U}_i$ are sent directly to node i and are received at the top $n_{0,i}$ MSB positions of the expanded channel output alphabet. Fig. 1 shows an example of such network augmentation for a "diamond" network [5].

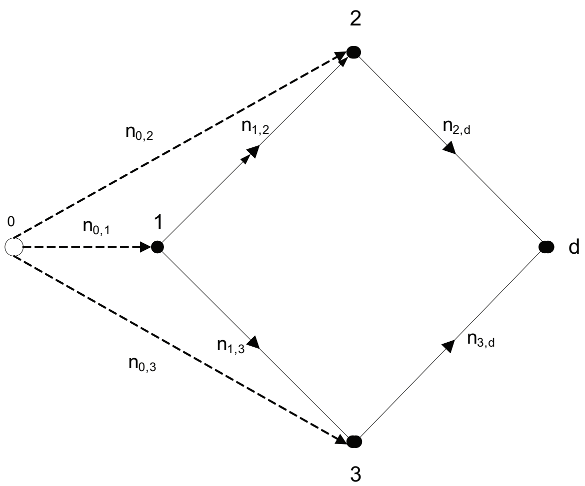

Fig. 1. A diamond network with a source node 1, two relay nodes 2 and 3 and a common destination d is augmented by adding node 0 and virtual links to nodes 1, 2 and 3.

After introducing the virtual source node, the augmented linear finite-field deterministic network belongs to the class studied in [5] with the minor difference that the channel linear transformations are not necessarily limited to "down-shifts". Nevertheless, as we observed before, Theorem 4.3 of [5] still applies. Letting R_0 denote the rate from the virtual source node 0 to the destination node d, we have that all rates R_0 satisfying

$$R_0 \leq \min_{(\Omega_0, \Omega_0^c) \in \Lambda_d} \operatorname{rank}\left\{\mathbf{G}_{\Omega_0, \Omega_0^c}\right\} \tag{4}$$

are achievable, where Λ_d is the set of all cuts (Ω_0, Ω_0^c) of the augmented network such that $0 \in \Omega_0$ and $d \in \Omega_0^c$.

For any such set Ω_0 we have that $\Omega_0 = \mathcal{S} \cup \{0\}$, for some $\mathcal{S} \subseteq \{1, \dots, N\}$. Consequently, we have that $\Omega_0^c = \mathcal{S}^c$, where $\mathcal{S}, \mathcal{S}^c$ are subsets as defined in the statement of Theorem 1. Since the links from 0 to any nodes $i \in \{1, \dots, N\}$ are orthogonal by construction (not subject to any broadcast or interference constraint), we have that $\mathbf{G}_{\Omega_0, \Omega_0^c}$ has a block-diagonal form where a block is given by $\mathbf{G}_{\mathcal{S}, \mathcal{S}^c}$ (the links of the original network, corresponding to the cut (Ω_0, Ω_0^c) via the correspondence $\Omega_0 \leftrightarrow \mathcal{S}$ defined above) and other blocks, denoted by $\mathbf{G}_{0,j}$ for all $j \in \mathcal{S}^c$, have rank $n_{0,j}$, respectively. By construction, there is no direct link between 0 and d so, without loss of generality, we can assume $n_{0,d} = 0$. The general form for $\mathbf{G}_{\Omega_0, \Omega_0^c}$ is

$$\mathbf{G}_{\Omega_0, \Omega_0^c} = \begin{bmatrix} \mathbf{G}_{\mathcal{S}, \mathcal{S}^c} & 0 & \cdots & 0 \\ 0 & \mathbf{G}_{0,i_1} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \mathbf{G}_{0,i_{|\mathcal{S}^c|}} \end{bmatrix}$$

where we have indicated $\mathcal{S}^c = \{i_1, \dots, i_{|\mathcal{S}^c|}\}$. Therefore, we have

$$\operatorname{rank}\left\{\mathbf{G}_{\Omega_0, \Omega_0^c}\right\} = \operatorname{rank}\left\{\mathbf{G}_{\mathcal{S}, \mathcal{S}^c}\right\} + \sum_{j \in \mathcal{S}^c} n_{0,j} \tag{5}$$

In particular, the cut $\Omega_0 = \{0\}$ yields

$$R_0 \leq \sum_{j=1}^N n_{0,j} \tag{6}$$

By letting this inequality hold with equality, and by replacing this into all other inequalities, we obtain the set of inequalities

$$\sum_{i \in \mathcal{S}} n_{0,i} \leq \operatorname{rank}\left\{\mathbf{G}_{\mathcal{S}, \mathcal{S}^c}\right\}, \quad \forall \ \mathcal{S} \subseteq \{1, \dots, N\} \tag{7}$$

where we used the fact that $\sum_{j=1}^N n_{0,j} - \sum_{j \in \mathcal{S}^c} n_{0,j} = \sum_{i \in \mathcal{S}} n_{0,i}$.

Consider now the ensemble of augmented networks for which there exist integers $\{n_{0,i} : i = 1, \dots, N\}$ that satisfy (7). For such networks, the rate $R_0 = \sum_{j=1}^N n_{0,j}$ is achievable (by [5]) and therefore the individual rates $R_i = n_{0,i}$ are achievable by the argument above. Finally, the closure of the convex hull of all individual rate vectors $\mathbf{R} = (n_{0,1}, \dots, n_{0,N})$ of such networks is achievable by time-sharing. It is immediate to see that this convex hull is provided by the inequalities (3).[2]

■

IV. A SPECIFIC EXAMPLE: DIAMOND NETWORK

In this section we work out a simple example and provide an explicit achievability strategy. Consider the "diamond" network shown in Fig. 1, with nodes $\{1, 2, 3, d\}$ and links of capacity $n_{1,2}, n_{1,3}, n_{2,d}$ and $n_{3,d}$. In this case, Theorem 1 yields the capacity region $\mathcal{C}$ given by

$$\begin{aligned} R_1 + R_2 + R_3 &\leq \max\{n_{2,d}, n_{3,d}\} && (8) \\ R_1 + R_2 &\leq n_{2,d} + n_{1,3} && (9) \\ R_1 + R_3 &\leq n_{3,d} + n_{1,2} && (10) \\ R_1 &\leq \max\{n_{1,2}, n_{1,3}\} && (11) \\ R_2 &\leq n_{2,d} && (12) \\ R_3 &\leq n_{3,d} && (13) \end{aligned}$$

Next, we provide simple coding strategies that achieve all relevant vertices of $\mathcal{C}$. Any point $\mathbf{R} \in \mathcal{C}$ can be obtained by suitable time-sharing of the vertices-achieving strategies. There are 24 possible orderings of the individual link capacities $n_{1,2}, n_{1,3}, n_{2,d}$ and $n_{3,d}$. Due to symmetry, the regions for the case $n_{3,d} > n_{2,d}$ will be the mirror image of the regions

[2] Indeed, the inequalities (3) represent the convex relaxation of the integer constraints (6).

for the case $n_{2,d} > n_{3,d}$. Therefore, we shall consider only the cases where $n_{2,d} \geq n_{3,d}$.

The remaining 12 cases have to be discussed individually. For example, let's focus on the case $n_{3,d} \leq n_{1,2} \leq n_{1,3} \leq n_{2,d}$. An example of the network for the choice of the link capacities $n_{3,d} = 1, n_{1,2} = 2, n_{1,3} = 3, n_{2,d} = 4$ is given in Fig. 2. Fig. 3 shows qualitatively the shape of the capacity region in the three possible sub-cases of the link-capacity ordering $n_{3,d} \leq n_{1,2} \leq n_{1,3} \leq n_{2,d}$: case 1) for $n_{1,2} + n_{3,d} < n_{1,3}$; case 2) for $n_{1,2} + n_{3,d} \geq n_{1,3}$, and case 3) for $n_{1,2} + n_{3,d} \geq n_{2,d}$. In all cases, the achievability of the vertices B and C of the region of Fig. 3 is trivial, since these correspond to vertices of the multi-access channel with node 2 and 3 as transmitters and node d as receiver.

Case 1). Vertex A has coordinates $(R_1 = n_{1,2}, R_2 = n_{2,d} - n_{1,2} - n_{3,d}, R_3 = n_{3,d})$ and can be achieved by letting node 1 send $n_{1,2}$ to node 2. Node 2 decodes and forwards these bits after multiplexing its own $n_{2,d} - n_{1,2} - n_{3,d} > 0$ bits in the MSB positions, such that node 3 can send $n_{3,d}$ bits without interference from node 2. Vertex D has coordinates $(R_1 = n_{1,2} + n_{3,d}, R_2 = n_{2,d} - n_{1,2} - n_{3,d}, R_3 = 0)$ and can be achieved by letting node 1 send $n_{1,2} + n_{3,d}$ bits. These can be all decoded by node 3, then node 3 can forward the bottom (least-significant) $n_{3,d}$ bits of node 1 to node d. Node 2 decodes the top (most-significant) $n_{1,2}$ bits from node 1, and forwards them after multiplexing its own bits.

Case 2). Vertices A, D and E have coordinates $(R_1 = n_{1,2}, R_2 = n_{2,d} - n_{1,2} - n_{3,d}, R_3 = n_{3,d})$, $(R_1 = n_{1,3}, R_2 = n_{2,d} - n_{1,3}, R_3 = 0)$ and $(R_1 = n_{1,3}, R_2 = n_{2,d} - n_{1,2} - n_{3,d}, R_3 = n_{1,2} + n_{3,d} - n_{1,3})$, respectively. Vertex A can be achieved in the same way as in Case 1). Vertex D can be achieved by letting node 1 send $n_{1,3}$ bits to node 3. Node 3 decodes and forwards the bottom $n_{3,d}$. Since in this case $n_{1,2} \geq n_{1,3} - n_{3,d}$, node 2 can decode the top $n_{1,3} - n_{3,d}$ bits of node 1, and forwards them to node d after multiplexing its own $n_{2,d} - n_{1,3}$ bits, using its $n_{2,d} - n_{3,d}$ MSBs. Vertex E can be achieved by letting node 1 transmit $n_{1,3}$ bits, where the top $n_{1,2}$ of which are received by node 2. Node 3 forwards the bottom $n_{1,3} - n_{1,2}$ bits of node 1, and multiplex its own $n_{3,d} + n_{1,2} - n_{1,3}$ bits. Node 2 forwards the top $n_{1,2}$ bits from node 1, by multiplexing its own $n_{2,d} - n_{1,2} - n_{3,d}$ bits, transmitting over its $n_{2,d} - n_{3,d}$ MSBs.

Case 3). Vertices A, D and E have coordinates $(R_1 = n_{2,d} - n_{3,d}, R_2 = 0, R_3 = n_{3,d})$, $(R_1 = n_{1,3}, R_2 = n_{2,d} - n_{1,3}, R_3 = 0)$ and $(R_1 = n_{1,3}, R_2 = 0, R_3 = n_{2,d} - n_{1,3})$, respectively. Vertex A can be achieved by letting node 1 send $n_{2,d} - n_{3,d}$ bits to node 2. Since $n_{2,d} - n_{3,d} \leq n_{1,2}$ these can be decoded and forwarded to node d in the MSB positions. Node 3 simply sends $n_{3,d}$ bits to node d without interfering with node 2. Vertex D is achieved by letting node 1 send $n_{1,3}$ bits. The top $n_{1,3} - n_{3,d}$ of these are decoded by node 2 and forwarded together with $n_{2,d} - n_{1,3}$ own bits. The bottom $n_{3,d}$ bits of node 1 are decoded and forwarded by node 3. Finally, vertex E is achieved by letting node 1 send $n_{1,3}$ bits. The bottom $n_{3,d} - n_{2,d} + n_{1,3}$ of these are forwarded by node 3, after multiplexing its own $n_{2,d} - n_{1,3}$ bits. Since $n_{2,d} - n_{3,d} \leq n_{1,2}$, node 2 can decode the top $n_{2,d} - n_{3,d}$ bits from node 1 and forward them to node d using its MSB positions. Other cases follow similarly and the whole capacity region is achieved by decode and forward.

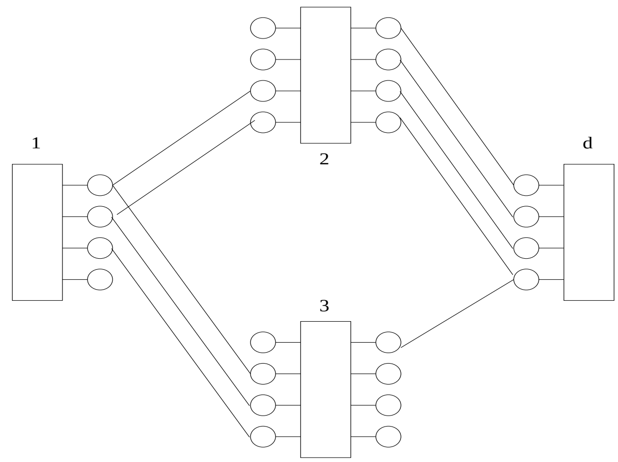

Fig. 2. The configuration of the diamond network in the example (Case (1) in Fig.3)

.

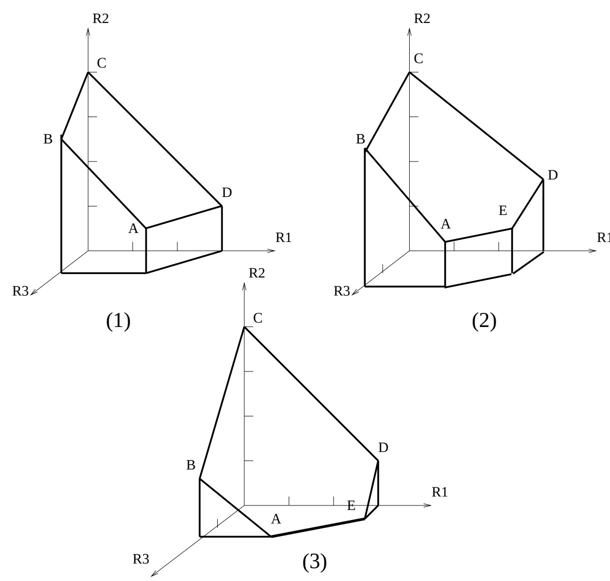

Fig. 3. The capacity region of the diamond network in the example.

V. Transmissibility for correlated sources

Consider the case of a sensor network where the nodes $\{1, \ldots, N\}$ observe samples from a spatially-correlated, i.i.d. in time, discrete vector source $\mathbf{U} = (U_1, \ldots, U_N)$ (see the source model in [1]). The goal is to reproduce the source blocks $\mathbf{u}[1], \ldots, \mathbf{u}[T]$ at the common destination node d. If the source blocks can be recovered at the destination with vanishing probability of error as $T \to \infty$, the vector source is said to be *transmissible*. In the case of a network of orthogonal links with capacities $C_{i,j}$, this problem was solved in [1] and yields the necessary and sufficient transmissibility condition[3]

$$H(U_{\mathcal{S}}|U_{\mathcal{S}^c}) \leq \sum_{(i,j) \in \mathcal{S} \times \mathcal{S}^c} C_{i,j}, \quad \forall\, \mathcal{S} \subseteq \{1, \ldots, N\}. \tag{14}$$

[3]The notation $U_{\mathcal{S}} = \{U_i : i \in \mathcal{S}\}$ is standard.

From the system design viewpoint, the above result yields the optimality of the "separation" approach consisting of the concatenation of Slepian-Wolf coding for the source with routing and single-user channel coding for the network [1].

With the same assumptions and linear finite-field deterministic network defined before, we consider a specific model for the vector source as defined in [6]. Let n_0 be a non-negative integer, and let $\mathbf{V} \in \mathbb{F}_2^{n_0}$ be a random vector of uniform i.i.d. bits. For all $i = 1, \dots, N$, let $\mathcal{U}_i \subseteq \{1, \dots, n_0\}$ and define $U_i \in \mathbb{F}_2^{|\mathcal{U}_i|}$ as the restriction of $\mathbf{V}$ to the components $\{V_\ell : \ell \in \mathcal{U}_i\}$ of $\mathbf{V}$. Then, the correlation model for the source $(U_1, \dots, U_N)$ is reduced to the following "common bits" case: sources U_i and U_j have common part $\{V_\ell : \ell \in \mathcal{U}_i \cap \mathcal{U}_j\}$ while the bits V_ℓ in $\mathcal{U}_i - \mathcal{U}_j$ and in $\mathcal{U}_j - \mathcal{U}_i$ are mutually independent. It follows that $H(U_i|U_j) = |\mathcal{U}_i| - |\mathcal{U}_i \cap \mathcal{U}_j|$.

This source model is somehow "matched" to a correlated source defined over the reals in the following intuitive sense. Consider $N = 2$ and let U_1 and U_2 denote the binary quantization indices resulting from quantizing two correlated random variables $A_1 \in \mathbb{R}$ and $A_2 \in \mathbb{R}$ using "embedded" scalar uniform quantizers with n bits, such that their first m MSBs are identical and their last $n - m$ least significant bits (LSBs) are mutually independent. If A_1, A_2 are marginally uniform and symmetric, U_1 and U_2 are *exactly* obtained by defining $\mathbf{V}$ as above, with $n_0 = 2n - m$ independent bits, and letting U_1 include the m MSBs and the first set of $n - m$ LBSs of $\mathbf{V}$, and U_2 include the same m MSBs and the second set of $n - m$ LBSs of $\mathbf{V}$. This model trivially generalizes to the case of N correlated sources and is related to the Gaussian sources with "tree" dependency considered in [6]. For the source model defined above we have the following simple result:

Theorem 2: The vector source $\mathbf{U} = (U_1, \dots, U_N)$ is transmissible over the linear finite-field deterministic network $(\mathcal{V}, \mathcal{E})$ if and only if

$$H(U_\mathcal{S}|U_{\mathcal{S}^c}) \leq \mathrm{rank}\left\{\mathbf{G}_{\mathcal{S},\mathcal{S}^c}\right\}, \quad \forall\, \mathcal{S} \subseteq \{1, \dots, N\}. \quad (15)$$

Proof: Again, we consider an augmented network with a single source node denoted by 0, with n_0 output bits that we denote by $\mathbf{V}$. As before, subsets $\mathcal{U}_i$ of cardinalities $n_{0,i}$ of these bits are sent to nodes i, respectively. However, differently from before we choose the subsets $\mathcal{U}_i$ to overlap in accordance with the vector source model. For the augmented network, the rate R_0 from the virtual source to the destination d must satisfy (4). In particular, choosing $\Omega_0 = \{0\}$ we get $R_0 \leq n_0$. Generalizing the proof of Theorem 1 to the case of overlapping sets $\{\mathcal{U}_i\}$, we find that for any cut (Ω_0, Ω_0^c) of the augmented network such that $\Omega_0 = \mathcal{S} \cup \{0\}$ and $\Omega_0^c = \mathcal{S}^c$, with $\mathcal{S} \subseteq \{1, \dots, N\}$ we have

$$\mathrm{rank}\left\{\mathbf{G}_{\Omega_0,\Omega_0^c}\right\} = \mathrm{rank}\left\{\mathbf{G}_{\mathcal{S},\mathcal{S}^c}\right\} + \mathrm{rank}\left\{\mathbf{G}_{0,\mathcal{S}^c}\right\}$$

where $\mathbf{G}_{0,\mathcal{S}^c}$ is the linear transformation between the inputs $\mathbf{V}$ and the (augmented) channel outputs of nodes $j \in \mathcal{S}^c$. By construction, the matrix $\mathbf{G}_{0,\mathcal{S}^c}$ is formed by linear independent columns for all bits V_ℓ with $\ell \in \bigcup_{j \in \mathcal{S}^c} \mathcal{U}_j$. Therefore,

$$\mathrm{rank}\left\{\mathbf{G}_{0,\mathcal{S}^c}\right\} = \left| \bigcup_{j \in \mathcal{S}^c} \mathcal{U}_j \right| = H(U_{\mathcal{S}^c})$$

Since $\mathbf{V}$ is uniform i.i.d., we have $R_0 = n_0 = H(\mathbf{V}) = H(\mathbf{U})$. Replacing these equalities into the set of inequalities (4) and using the chain rule of entropy $H(\mathbf{U}) = H(U_\mathcal{S}|U_{\mathcal{S}^c}) + H(U_{\mathcal{S}^c})$ we obtain that the conditions (15) are sufficient for transmissibility. On the other hand, if a source as defined in our model was transmissible, then the set of conditions (15) must hold, otherwise the rate R_0 of the corresponding single-source single destination augmented network would violate (4). Hence, necessity also holds. ∎

VI. Conclusions

In this work we have characterized the capacity region for a linear finite-field deterministic network with independent information at all nodes and a single destination node. In our setup, all nodes may relay information from other nodes as well as inject their own information into the network. This may serve as a simplified model for a large WSN where sensing nodes cooperate with each other to send the collective data towards a single collector node. For a specific model of discrete binary source correlation at the nodes, we have also found necessary and sufficient conditions for the source transmissibility. Albeit restrictive, this correlation model may be useful (e.g., see [6]) as a simple discrete "equivalent" (up to some bounded mean-square distortion penalty) for a spatially-correlated real sources whose components are observed and encoded separately at the network nodes.

Motivated by these results, it is natural to investigate the performance of achievability schemes based on the techniques as in [5] (for independent information) and separated quantization and Slepian-Wolf binning (for lossy transmission of correlated sources) in order to achieve the capacity region or the distortion region of actual WSN, within a bounded performance gap.

References

[1] J. Barros and S. Servetto, "Network Information Flow With Correlated Sources," *IEEE Trans. Inform. Theory*, vol. 52, no. 1, pp. 155–170, January 2006.

[2] E. C. V. d. Meulen, "A survey of multi-way channels in information theory," *IEEE Trans. Inform. Theory*, vol. 23, no. 1, pp. 1–37, January 1977.

[3] T. M. Cover and A. E. Gamal, "Capacity theorems for the relay channel," *IEEE Trans. Inform. Theory*, vol. 25, no. 5, pp. 572–584, September 1979.

[4] M. Gastpar and M. Vetterli, "On the capacity of large Gaussian relay networks," *IEEE Trans. Inform. Theory*, vol. 51, no. 3, pp. 765–779, March 2005.

[5] A. S. Avestimehr, S. N. Diggavi, and D. N. Tse, "Wireless Network Information Flow: A Deterministic Approach," arXiv:0906.5394v2 [cs.IT], Aug. 3, 2009.

[6] Mohammad Ali Maddah-Ali and David N. C. Tse, "Approximating the Rate-Distortion Region of the Distributed Source Coding for Three Jointly Gaussian Tree-Structured Sources," in *IEEE Int. Symp. on Inform. Theory, ISIT 09*, Seoul, Korea, July 2009.

[7] G. Kramer, "Pipelined encoding for deterministic and noisy relay networks," in *2009 Workshop on Network Coding, Theory and Applications, NETCOD09*, Lausanne, Swtizerland, June 2009.

[8] T. M. Cover and J. Thomas, *Elements of Information Theory*, 2nd ed. Wiley and Sons, 2006.