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**N* resonances from KΛ amplitudes in sliced bins in energy**

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**Abstract.** The two reactions γp → K⁺Λ and π⁻p → K⁰Λ are analyzed to determine the leading photoproduction multipoles and the pion-induced partial wave amplitudes in slices of the invariant mass. The multipoles and the partial-wave amplitudes are simultaneously fitted in a multichannel Laurent+Pietarinen model (L+P model), which determines the poles in the complex energy plane on the second Riemann sheet close to the physical axes. The results from the L+P fit are compared with the results of an energy-dependent fit based on the Bonn-Gatchina (BnGa) approach. The study confirms the existence of several poles due to nucleon resonances in the region at about 1.9 GeV with quantum numbers J^P = 1/2^+, 3/2^+, 1/2^−, 3/2^−, 5/2^−.

**1 Introduction**

The nucleon and its excited states are the simplest systems in which the non-abelian character of strong interactions is manifest. Three quarks is the minimum quark content of any baryon, and these three quarks carry the three fundamental colour charges of Quantum Chromodynamics (QCD), and combine to a colourless baryon. At present it is, however, impossible to calculate the spectrum of excited states from first principles, even though considerable progress in lattice gauge calculations has been achieved [1]. Models are therefore necessary when data are to be compared to predictions.

Quark models predict a rich excitation spectrum of the nucleon [2–6]. In quark models, the resonances are classified in shells according to the energy levels of the harmonic oscillator. The shell structure of the excitations is still seen in the data and reproduced in lattice calculations [1]. The first excitation shell is predicted to house five N\(^*\) and two Δ\(^*\) resonances with negative-parity; all of them are firmly established. The second excitation shell contains missing resonances: 22 resonances (14 N\(^*\)'s and 8 Δ\(^*\)'s) are predicted but 15 only are found in the mass range below 2100 MeV, and just 10 of them (5 N\(^*\)'s and 5 Δ\(^*\)'s) are considered as established, with three or four stars in the notation of the Particle Data Group [7]. Thus 9 N\(^*\)'s in the mass region between 1700 MeV and 2100 MeV are predicted to exist which are unobserved or the evidence for their existence is only fair or even poor. This deficit is known as the problem of the missing resonances [8,9]. The search for missing resonances is one of the major aims of a number of experiments in which the interaction of a photon beam in the GeV energy range with a hydrogen and deuterium target is studied.

In πN elastic and charge exchange scattering, the excited states may have isospin I = 1/2 (N\(^*\)) and I = 3/2 (Δ\(^*\)). A large amount of data was analyzed by the groups at Karlsruhe-Helsinki (KH84) [10], Carnegie-Mellon (CM) [11] and at GWU [12]. The 1850 - 2100 MeV mass region – where the missing resonances of the second excitation shell are predicted in most constituent quark models (see, e.g., [2–6,13,14]) – is dominated by the production of Δ\(^*\) resonances with spin-parity J^P = 1/2 ±, 3/2 ±, 5/2 ±, 7/2 ±; nucleon resonances are difficult to establish in this mass range due to the overwhelming background from Δ\(^*\) resonances.

The production of Λ hyperons in pion and photo-induced reactions, in contrast to πN elastic scattering, is ideally suited to search for new nucleon resonances and to confirm resonances that are not yet well established (see, e.g., [13,14] and refs. therein). Due to isospin conservation in strong interactions, only N\(^*\) resonances decay into AK final states, there are no isospin I = 3/2 contributions. Second, the Λ → Nπ weak-interaction decay reveals the polarization P of the Λ. Thus, the recoil polarization is measurable. In πN elastic scattering, the equivalent target polarization, also called P, requires the use of a polarized target. In photoproduction, a third advantage emerges: the process is not suppressed even when the πN coupling...
constants of $N^*$ resonances in the second excitation shell are small [13,14]. Photoproduction may hence reveal the existence of $N^*$ resonances coupling to $\pi N$ only weakly. Indeed, a number of new resonances has been reported (or have been upgraded in the star rating) from a combined analysis of a large number of pion and photo-produced reactions [15]. Some of the “new” resonances had been observed before [10,11,16–18] or were confirmed in later analyses [19–21]. The evidence for the existence of the new states stemmed from energy-dependent fits to the data using the BnGa approach [22–25]. The reaction $\gamma p \to K^+\Lambda$ proved to be particularly useful [26].

The ultimate aim of experiments is to provide sufficient information that the data can be decomposed into partial waves or multipoles of defined and unique spin-parity. It can either be done through constructing an explicit theoretical model, or as we present here, through the reconstruction of partial-wave amplitudes and of multipoles in a truncated partial wave analysis. Limiting the partial wave series to low orbital angular momenta allows us to overcome issues with the still relatively large errors in the measurements of observable quantities.

The main goal of this paper is to test if $N^*$ resonances in the fourth resonance region can be confirmed definitely from a fit to multipoles driving the extraction of partial waves with defined spin-parity, and to extract their properties. This is done in two ways: i.) In a standard way where a theoretical model is constructed. Its free parameters are estimated by fitting to the experimental data set base, and the partial waves of the final solution are analytically continued into the complex energy plane to obtain poles. ii.) In a way which does not depend on detailed model assumptions by using the Laurent+Pietarinen (L+P) method where a solution of the theoretical model is replaced by a most general analytic function consisting of a number poles and branch-cuts, which is embodied by a fast converging power series in a conformal variable. This variable is generated by a conformal mapping of the complex energy plane onto a unit circle. The first Riemann sheet is mapped to the outside of the unit circle, and the second Riemann sheet – where the poles are located – into the inside of the unit circle. In method ii.), poles are extracted by fitting to the single-energy partial wave decomposition, as opposed to a direct global fit to the data.

Method i.), coupled-channel energy-dependent fits, exploits the full statistical potential of the data. The effect of couplings to various other final states like $N\pi, N\eta, \Sigma K, \Delta\pi$, etc. is taken into account exactly as well as all correlations between the different amplitudes. However, all partial waves need to be determined in one single fit, and it is difficult to verify the uniqueness of the results. In method ii.) we use single channel L+P fits (SC L+P) where each channel is fitted individually, and multi-channel L+P fits (MC L+P) where two or more channels are fitted simultaneously. The main advantage of the model-independent approach is that we can fit one partial wave at a time, and that we avoid any dependence on the quality of the model. The drawback is that you first have to extract partial waves, and this procedure depends on the choice of higher partial waves, introducing some model dependence.

2 Construction of $K\Lambda$ amplitudes in slices of their invariant mass

2.1 The partial wave amplitudes for $\pi^- p \to K^0\Lambda$

2.1.1 Formalism

The differential cross sections $d\sigma/d\Omega$ for the reaction $\pi^- p \to K^0\Lambda$ receives contributions from a spin-non-flip and a spin-flip amplitude, $f$ and $g$, according to the relation

$$\frac{d\sigma}{d\Omega} = \frac{k}{q}(|f|^2 + |g|^2), \quad (1)$$

where $q$ and $k$ are the initial and final meson momenta respectively in the centre of mass frame [10]. Both amplitudes depend on the invariant mass $W$ and $z = \cos \theta$, with $\theta$ being the scattering angle. The two amplitudes can be expanded into partial wave amplitudes $A_l^j$

$$f(W, z) = \frac{1}{\sqrt{qk}} \sum_{l=0}^{L} \left[(l+1)A_l^+(W) + lA_l^-(W)\right]P_l(z) ,$$

$$g(W, z) = \frac{1}{\sqrt{qk}} \sin \theta \sum_{l=1}^{L} \left[A_l^+(W) - A_l^-(W)\right]P_l^j(z) , \quad (2)$$

where $P_l(z)$ are the Legendre polynomials. $J = |l \pm 1/2|$ is the total spin of the state. The sign in the relation for $J$ defines the sign in $A_l^j$.

The $A \to N\pi$ decay can be used to determine the decay asymmetry with respect to the scattering plane, called recoil asymmetry $P$. Assuming that the target nucleon is fully polarized, $P$ can be defined as

$$(1 \pm P) \frac{d\sigma}{d\Omega} = |f \pm ig|^2. \quad (3)$$

When the target proton is polarized longitudinally (along the pion beam line), the spin transfer from proton to $A$ yields the spin rotation angle $\beta$.

$$\beta = arg \left(\frac{f - ig}{f + ig}\right) = \tan^{-1} \left(\frac{-2Re(f^*g)}{|f|^2 - |g|^2}\right). \quad (4)$$

It is defined as $\beta = \arctan (-R/A)$, where $A$ and $R$ are the polarization components in direction of the $A$ and its orthogonal component in the scattering plane. $R$ and $A$ are given by

$$R = \frac{2Re(f^*g)}{|f|^2 + |g|^2}, \quad A = \frac{|f|^2 - |g|^2}{|f|^2 + |g|^2}. \quad (5)$$

The polarization variables are constrained by the relation

$$P^2 + A^2 + R^2 = 1. \quad (6)$$
Fig. 1. Differential cross sections $d\sigma/d\Omega$, $A$ recoil polarization $P$, and spin rotation angle $\beta$ for the reaction $\pi^- p \to K^0 A$ from ANL75 (blue) [27] and RAL (black) [28–30]. Note that a few differential cross sections from [27] fall into a single energy window. The $\beta$ is 360-degree cyclic which leads to additional data points shown by empty circles. The solid (black) line corresponds the $L + P$ fit, the dashed (red) line corresponds the fit from which the amplitudes of Fig. 2 are deduced, the dotted (green) line corresponds to BnGa 2011-02 fit.
2.1.2 Fits to the data

Data on the reaction $\pi^- p \to K^0 \Lambda$ were taken in Chicago [27] and at the NIMROD accelerator at the Rutherford Laboratory [28–30]. From these data, the partial wave amplitudes $A_k^\pm$ defined in eqn. (2) should be derived.

A detailed study showed that the data require angular momenta up to $l = 3$ or even $l = 4$ but do not have the precision to determine all partial wave amplitudes [31]. Therefore we try to determine at least the low-$l$ amplitudes, in particular $A_{-11}^r (= S_{11})$, $A_{+11}^r (= P_{11})$, $A_{+13}^+ (= P_{13})$, leading to $J^P = 1/2^-$, $1/2^+$, and $3/2^+$. The higher partial waves, those above $A_{11}^-$, $A_0^+$, $A_1^+$, are taken from our current BuGa fit.

Figure 1 shows the data. The solid curves represent the final BuGa fit. It reproduces the data with a $\chi^2/N_{\text{data}} = 570/916$. The number of free parameters is 75.

The fit returns the real and imaginary parts of amplitudes for the $S_{11}$, $P_{11}$, and $P_{13}$ partial waves. The $S_{11}$ and $P_{11}$ amplitudes are shown in Fig. 2, the $P_{13}$ amplitude in Ref. [31] only (since it could not be fit with the L+P method). The solid line represents the L+P fit described below, and the energy-dependent solution BuGa2011-02 is shown as error band. Note that the higher partial waves are constrained fixed to the BuGa solution, while the other lower amplitudes are free to adopt any values.

2.2 The multipoles for $\gamma p \to K^+ \Lambda$

2.2.1 Formalism

The amplitude for the reaction $\gamma p \to K^+ \Lambda$ can be written in the form

$$ A = \omega^* J_\mu \varepsilon_\mu \omega', $$

where $\omega'$ and $\omega$ are spinors representing the baryon in the initial and final state, $J_\mu$ is the electromagnetic current of the nucleon, and $\varepsilon_\mu$ characterizes the polarization of the photon. The amplitude can be expanded into four invariant (CGLN) amplitudes $F_i$ [32]

$$ J_\mu = iF_1 \sigma_\mu + F_2 \frac{\langle \sigma q \rangle}{|q||q|} \varepsilon_{\mu j} \sigma_i k_j + iF_3 \frac{\langle \sigma k \rangle}{|q|} \eta_\mu + iF_4 \frac{\langle \sigma q \rangle}{q^2} \eta_\mu, \tag{8} $$

where $q$ is the momentum of the $\Lambda$ hyperon in the final state, $k$ is the momentum of the nucleon in the initial state, calculated in the center-of-mass system of the reaction, and $\sigma_i$ are the Pauli matrices. These four functions $F_i$ are functions of the invariant mass and of $z$ with $z = (kq)/(|k||q|) = \cos \theta$ and $\theta$ as the scattering angle. A determination of these four amplitudes requires the measurement with sufficient accuracy of at least eight well chosen observables [33–35,69,37]. For each slice in energy and angle one phase remains undetermined. It needs to be fixed from other sources. In $\pi^+ p$ elastic scattering, the phase can be determined from the (calculable) Coulomb interference. In hyperon production, one could try to fix the phase to the phase of $t$-channel Kaon exchange. Once the $F_i$ functions are known for each energy and angle, the results of all experiments can be predicted.

The relations between the $F_i$ functions and the observables can be found, e.g., in [37]. For convenience, we give the expressions for the observables used in the fits.

The differential cross section $d\sigma/d\Omega$ and the single polarization observables, the beam asymmetry $\Sigma$, the recoil asymmetry $P$, and the target asymmetry $T$, are given by

$$ \frac{d\sigma}{d\Omega} = \frac{k}{q} I - \frac{k}{q} \Re [F_1 F_1^* + F_2 F_2^* - 2z F_2 F_1^* + \sin^2(\theta) \left( F_3 F_3^* + F_4 F_4^* + 2F_4 F_4^* + 2F_3 F_2^* + 2z F_3 F_3^* \right)]. \tag{9a} $$

$$ \Sigma I = -\frac{\sin^2(\theta)}{2} \times \Re [F_2 F_3^* + F_4 F_4^* + 2F_3 F_3^* + 2F_3 F_2^* + 2z F_3 F_3^*], \tag{9b} $$

$$ P I = \sin(\theta) \Im [2F_2 F_3 + F_3 F_3^* + z F_3 F_4] F_1 + \tag{9c} $$

$$ T I = \sin(\theta) \Im [F_2 F_3 - F_4 F_4^*] + z (F_1 F_4 - F_2 F_3) - \sin^2(\theta) F_3 F_4 \tag{9d}. $$

The double polarization observables $O_{x'}, O_{y'}$, $(C_{x'}, C_{y'})$, define the spin transfer from linearly (circularly) polarized photons to the $\Lambda$ hyperon where the $x'$ axis is given by the meson direction. This is referred to as the primed frame. Experimentally, the data on the spin transfer from polarized photons to the $\Lambda$ hyperon are sometimes presented in an unprimed frame, in which the photon momentum is chosen as reference axis. Observables in the two frames
are related by a simple rotation:

\[ C_z = \sin(\theta)C_{z'} + \cos(\theta)C_{z''}, \]
\[ C_{z'} = \cos(\theta)C_z - \sin(\theta)C_{z''}, \]

with similar relations holding for the quantities \(O_x'\) and \(O_x''\).

The double polarization observables \(O_{x'}, O_{z'}\) can be written as

\[ (9f) \quad O_{x'} = \sin(\theta)Im[F_2F_3' - F_1F_4'] + z[F_2F_3' - F_1F_4'], \]
\[ (9g) \quad O_{z'} = -\sin^2(\theta)Im[F_1F_3' + F_2F_4'], \]
\[ (9h) \quad C_x' = \sin(\theta)Re[F_2F_3' - F_1F_4'] + F_2F_3' - F_1F_4' + \]
\[ z[F_2F_3' - F_1F_4'], \]
\[ (9i) \quad C_{z'} = Re[-2F_1F_3' + z(F_1F_3' + F_2F_2')] - \]
\[ \sin^2(\theta)(F_1F_3' + F_2F_2'). \]

When the \(F_i\)'s are known with sufficient statistical accuracy they can be expanded – for each slice in energy – into power series using Legendre polynomials \(P_L(z)\) and their derivatives:

\[ F_1(W, z) = \sum_{L=0}^{\infty} [LM_{L+} + E_{L+}]P_{L+1}(z) + \]
\[ [(L + 1)M_{L-} + E_{L-}]P_{L-1}(z), \quad (6a) \]
\[ F_2(W, z) = \sum_{L=1}^{\infty} [(L + 1)M_{L+} + LM_{L-}]P_{L}'(z), \quad (6b) \]
\[ F_3(W, z) = \sum_{L=1}^{\infty} [E_{L+} - M_{L+}]P_{L+1}'(z) + \]
\[ [E_{L-} + M_{L-}]P_{L-1}'(z), \quad (6c) \]
\[ F_4(W, z) = \sum_{L=2}^{\infty} [M_{L+} - E_{L+} - M_{L-} - E_{L-}]P_{L}'(z). \quad (6d) \]

Here, \(L\) corresponds to the orbital angular momentum in the \(K^+A\) system. \(W\) is the total energy, \(P_L(z)\) are Legendre polynomials with, and \(E_{L\pm}\) and \(M_{L\pm}\) are electric and magnetic multipoles describing transitions to states with \(J = L \pm 1/2\). \(M_{0+}\) or \(E_{1+}\) multipoles do not exist. Processes due to meson exchanges in the channel may provide significant contributions to the reaction. They may demand high-order multipoles. The minimal \(L\) required to describe the data can be determined by polynomial expansions of the data [38]. A direct approach is to insert the \(F_i\) functions (eqns. 6) into the expressions for the observables (eqns. 9a and 9f) and to truncate the expansion at an appropriate value of \(L\) [39]. The observables are now functions of the invariant mass and the scattering angle, and the fit parameters are the electric and magnetic multipoles. In this method, the number of observables required to get the full information might be reduced if the number of contributing higher partial waves is not too big. But still, high precision is mandatory for the expansion.

\[ \text{Fig. 3. (Color online) High-}L\text{- multipoles from BnGa fits to a large body of pion and photo-induced reactions. These multipoles are imposed in the fits to the data Figs. 4-7. Show real and imaginary part of the multipole, \(M_{2+}\) and \(E_{2+}\) and \(M_{3+}\), \(E_{3+}\), \(M_{4+}\), \(E_{4+}\), \(M_{5+}\), \(E_{5+}\). The solid (black) line is the real part of the multipole, dash (blue) line is the imaginary part of the multipole. The multipoles are given in mfm (milli-femi=attometer).} \]

2.2.2 Fits to the data

From the results of the BnGa analysis we expect that in the energy range considered here the \(E_{0+}, M_{1-}\), and \(E_{1+}\) yield the largest contributions, followed by \(M_{1-}, E_{2-}\), and \(M_{2-}\). The \(E_{2+}, M_{2+}, M_{3-}, E_{3-}, M_{3+}, E_{3+}, M_{4-}, E_{4-}\) all contribute with increasingly smaller importance, higher multipoles become negligible. First fits showed that it is not possible, given the statistical and systematic accuracy of the data, to determine all significant partial waves. Due to strong correlations between the parameters, the errors
Fig. 4. (Color online) Fit to the data on $d\sigma/dt$: $[41]$, $P$ $[41]$, and $\Sigma$, $T$, $O_\Sigma$, $O_x$ $[43]$ for $\gamma p \to K^+A$ reaction for the mass range from 1710 to 1850 MeV. The solid (black) line corresponds the $L + P$ fit, the dashed (red) line corresponds to fit used to determine the multipoles of Fig. 8., the dotted (green) line corresponds to BuGa fit.
Fig. 5. (Color online) Fit to the data on $\sigma/\Omega$: [41], $P$ [41], and $\Sigma, T, O_x, O_z$ [43] for $\gamma p \rightarrow K^+\Lambda$ reaction for the mass range from 1850 to 1990 MeV. The solid (black) line corresponds the $L + P$ fit, the dashed (red) line corresponds to fit used to determine the multipoles of Fig. 8., the dotted (green) line corresponds to BhGa fit.
Fig. 6. (Color online) Fit to the data on $d\sigma/d\Omega$: [41], $P$ [41], and $\Sigma$, $T$, $O_x$, $O_y$ [43] for $\gamma p \rightarrow K^+\Lambda$ reaction for the mass range from 1990 to 2130 MeV. The solid (black) line corresponds to the $L + P$ fit, the dashed (red) line corresponds to the fit used to determine the multipoles of Fig. 8, the dotted (green) line corresponds to BnGa fit.
ME errors increased only slightly when the multipoles were fitted free; the higher multipoles were constrained to the BnGa results. These multipoles are shown in Fig. 3. Reasonably small errors were obtained when the four multipoles \( E_{0+}, M_{1-}, E_{1+}, \text{and } M_{1+} \) were fitted. The errors increased only slightly when the multipoles \( E_{2-}, M_{2-}, \text{and } E_{2+} \) were fitted in addition but constrained to the BnGa solution by a penalty function,

\[
\chi^2_{\text{pen}} = \sum_{\alpha} \left( \frac{M_{\alpha} - \hat{M}_{\alpha}}{\delta M_{\alpha}} \right)^2 + \sum_{\alpha} \left( \frac{E_{\alpha} - \hat{E}_{\alpha}}{\delta E_{\alpha}} \right)^2
\]

where \( \hat{E}_{\alpha} \) and \( \hat{M}_{\alpha} \) are the electric and magnetic multipoles from solution with \( E_{2-}, M_{2-}, \text{and } E_{2+} \) fitted free; \( \delta E_{\alpha}, \delta M_{\alpha} \) are the multipole errors.

The reaction \( \gamma p \rightarrow K^+ A \) has been studied extensively by the CLAS collaboration. The early measurement of the differential cross sections \( d\sigma/d\Omega \) [40] was later superseded by a new measurement reporting the differential cross sections and the recoil polarization [41]. The spin transfer from circularly polarized photons to the final-state \( A \) hyperon, the quantities \( C_x \) and \( C_z \), were reported in [42]. The polarization observables \( \Sigma, T, O_x, O_z \) have been determined recently [43]. The data are shown in Figs. 4-6. The data are used to determine the photoproduction multipoles in a truncated partial wave analysis.

The final result for the multipoles are shown in Fig. 8. Strong variations are observed. The imaginary parts of all multipoles, except \( M_{2-} \) and \( E_{2+} \), show threshold enhancements due to \( N(1650)/2^- \), \( N(1710)/2^+ \), \( N(1720)/3^2+ \), \( N(1700)/3^2- \), \( N(1720)/3^2+ \). Further structures are clearly seen at about 1900 MeV in the \( E_{0+}, M_{1-}, E_{1+}, M_{1+}, E_{2-}, M_{2-}, E_{2+} \) multipoles.

These structures emerge reliably when the multipole series is truncated, and only few multipoles are fitted freely. In Fig. 9 we show the results from one of our tests. In this case, the seven largest multipoles, \( E_{0+}, M_{1-}, E_{1+}, M_{1+}, E_{2-}, M_{2-}, \text{and } E_{2+} \) were all left free. In several mass bins, the resulting multipoles show an erratic behavior; the results become unstable. Likewise, it was important to include the multipoles with large orbital angular momenta. Even though they are individually all small, neglecting them (by assuming that they are identically zero) leads to
Fig. 8. (Color online) Real and imaginary parts of the multipoles derived from fits to the data shown in Figs. 4-6. The multipoles $E_{0+}, E_{1+}, M_{1-}$, and $M_{1+}$ are fitted freely, the $E_{2-}, E_{2+}$, and $M_{2-}$ are forced by a penalty function to stay close to the BnGa fit. Higher-order multipoles are taken from the BnGa fit. The triangles (blue) mark the real part of the multipole, circles (red) mark the imaginary part of the multipole. Solid lines are L+P fits, dashed lines represent the BnGa fit. The grey (blue/red) areas denote the systematic errors derived from the spread when different BnGa solutions were used to determine the multipoles. The multipoles are given in mfm (milli-fermi=attometer).

Fig. 9. (Color online) Real and imaginary parts of the $E_{0+}, E_{1+}, M_{1-}, M_{1+}, E_{2-}, E_{2+}$, and $M_{2-}$. These multipoles were fitted freely. Higher-order multipoles are taken from the BnGa fit. The triangles (blue) mark the real part of the multipole, circles (red) mark the imaginary part of the multipole. Solid and dash lines represent the BnGa fit. The multipoles are given in mfm (milli-fermi=attometer).
to biased results. Furthermore, these multipoles fix the overall phase.

Sandorfi, Hoblit, Kamano, and Lee [37] have reconstructed the photoproduction amplitudes for the reaction $\gamma p \rightarrow K^+\Lambda$. For the high partial waves, they used the Born amplitudes. Partly, they fitted all waves with $L \leq 3$ freely and determined the phases as differences to the $E_{0+}$ phase. In other fits, they had the $E_{0+}$ phase free and fitted all waves with $L \leq 2$. The resulting multipoles showed a wide spread. They concluded that a very significant increase in solid-angle coverage and statistics is required when all partial waves up to $L = 3$ are to be determined.

### 3 BnGa fits to the data

The BnGa partial wave analysis uses a $K$ matrix formalism to fit data on pion and photo-induced reactions to extract the leading singularities of the scattering or production processes. The formalism is described in detail in a series of publications [22–25]. Here we briefly outline the dynamical part of the method.

The pion induced reaction $\pi^- p \rightarrow K^0\Lambda$ from the initial state $i = \pi^- p$ to the final state $f = K^0\Lambda$ is described by a partial wave amplitude $A_{ij}^{(\beta)}$. It is given by a $K$-matrix which incorporates a summation of resonant and non-resonant terms in the form

$$A_{ij}^{(\beta)} = \sqrt{\rho_i} \sum_a K_{ia}^{(\beta)} \left( I - i \rho K^{(\beta)} \right)^{-1}_{aj} \sqrt{\rho_j}. \quad (8)$$

The multi-index $\beta$ denotes the quantum numbers of the partial wave, it is suppressed in the following. The factor $\rho$ represents the phase space matrix to all allowed intermediate states, $\rho_i, \rho_j$ are the phase space factors for the initial and the final state. The $K$ matrix parametrizes resonances and background contributions:

$$K_{ab} = \sum \alpha \frac{g_{\alpha}^a g_{\alpha}^b}{M_{\alpha}^2 - s} + f_{ab}. \quad (9)$$

Here $g_{\alpha}^a$ are coupling constants of the pole $\alpha$ to the initial and the final state. The background terms describe non-resonant transitions from the initial to the final state.

For photoproduction reactions, we use the helicity ($h$)-dependent amplitude for photoproduction $a_b^h$ of the final state $b$ [48]

$$a_b^h = P_b^h (I - i \rho K)^{-1}_{ab} \quad \text{where} \quad (10)$$

$$P_b^h = \sum \alpha \frac{A_b^{\alpha} g_{\alpha}^b}{M_{\alpha}^2 - s} + F_b. \quad (11)$$

$A_b^\alpha$ is the photo-coupling of a pole $\alpha$ and $F_b$ a non-resonant transition. The helicity amplitudes $A_b^{1/2}, A_b^{3/2}$ are defined as residues of the helicity-dependent amplitude at the pole position and are complex numbers [44].

In most partial waves, a constant background term is sufficient to achieve a good fit. Only the background in the meson-baryon $S$-wave required a more complicated form:

$$f_{ab} = \frac{(a + b \sqrt{s})}{(s - s_0)}. \quad (12)$$

Further background contributions are obtained from the reggeized exchange of vector mesons [22] in the form

$$A = g(t)R(\xi, \nu, t) \quad \text{where} \quad (13)$$

$$R(\xi, \nu, t) = \frac{1 + \xi \exp(-\nu \alpha(t))}{\sin(\nu \alpha(t))} \left( \frac{\nu}{\nu_0} \right)^{\alpha(t)}.$$

Here, $g(t) = g_0 \exp(-bt)$ represents a vertex function and a form factor. $\alpha(t)$ describes the trajectory, $\nu = \frac{1}{2}(s - u)$, $\nu_0$ is a normalization factor, and $\xi$ the signature of the trajectory. Pion and and Pomeron exchange both have a positive signature and therefore [22]:

$$R(\xi, \nu, t) = \frac{e^{-\frac{\nu}{2} \alpha(t)}}{\sin\left(\frac{\nu}{2} \alpha(t)\right)} \left( \frac{\nu}{\nu_0} \right)^{\alpha(t)}. \quad (14)$$

Additional $\Gamma$-functions eliminate the poles at $t < 0$:

$$\sin\left(\frac{\pi}{2} \alpha(t)\right) \rightarrow \sin\left(\frac{\pi}{2} \alpha(t)\right) \Gamma\left(\frac{\alpha(t)}{2}\right). \quad (15)$$

where the Kaon trajectory is parametrized as $\alpha(t) = -0.25 + 0.85t$, with $t$ given in GeV$^2$.

The data on partial wave amplitudes (Fig. 2) and on the photoproduction multipoles (Fig. 8) were included in the data base of the BnGa partial wave analysis. The data are fitted jointly with data on $N\eta$, $AK$, $\Sigma K$, $N\pi^0\pi^0$, and $N\pi^0$ from both photo- and pion-induced reactions. Thus inelasticities in the meson-baryon system are constrained by real data. A list of the data used for the fit can be found in [15,45–47] and on our website (pwa.hiskp.uni-bonn.de). In Fig. 2, the systematic errors define the error band; in Fig. 8, the systematic error of the real (imaginary) part of the amplitudes is shown a grey (red/blue) histogram at the bottom (top) line. The systematic errors are derived by a variation of the model space by adding further resonances with different spin-parities when the data are fitted.

### 4 The Laurent-Pietarinen expansion

#### 4.1 Formalism

The main task of the single channel Laurent-Pietarinen expansion ($SC_{L+P}$) is extracting pole positions from given partial waves for one reaction. The driving concept behind the method is to replace an elaborate theoretical model by a local power-series representation of partial wave amplitudes [55]. The complexity of a partial-wave analysis model is thus replaced by much simpler model-independent expansion which just exploits analyticity and unitarity. The $L+P$ approach separates pole and regular
part in the form of a Mittag-Leffler expansion\textsuperscript{1}, and instead of modeling the regular part using some physical model it uses the conformal-mapping-generated, rapidly converging power series with well defined analytic properties called a Pietarinen expansion\textsuperscript{2} to represent it effectively. So, the method replaces the regular part calculated in a model with the simplest analytic function which has correct analytic properties of the analyzed partial wave (multipole), and fits the given input. In such an approach the model dependence is minimized, and is reduced to the choice of the number and location of L+P branch-points used in the model. The method is applicable to both, theoretical and experimental input, and represents the first reliable procedure to extract pole positions directly from experimental data, with minimal model bias. The L+P expansion based on the Pietarinen expansion is used in some former papers in the analysis of pion-nucleon scattering by Esco Pietarinen [52,53], introduced by Ciulli and Fisher [50,51], was described in detail and used in pion-nucleon scattering by Esco Pietarinen [52,53], and named as a Pietarinen expansion by G. Höhler in [10].

\textsuperscript{1} Mittag-Leffler expansion [54] is the generalization of a Laurent expansion to a more-than-one pole situation. From now on, for simplicity, we will simply refer to this as a Laurent expansion.

\textsuperscript{2} A conformal mapping expansion of this particular type was introduced by Ciulli and Fisher [50,51], was described in detail and used in pion-nucleon scattering by Esco Pietarinen [52,53], and named as a Pietarinen expansion by G. Höhler in [10].

In this paper we fit partial wave data; the discrete data points coming from a semi-constrained single energy fit of $K\Lambda$ photo-production data, which is obtained in a way that the partial waves with $L > 2$ are fixed to Bonn-Gatchina energy dependent partial waves, and lower ones are allowed to be free. We perform a multichannel fit ($MC_{L+P}$) when possible by including single energy data from $\pi N \rightarrow K\Lambda$ process, and we fit both multipoles for the same angular momentum at the same time in the coupled-multipole fit ($CM_{L+P}$). The regular background part is represented by three Pietarinen expansion series, all free parameters are fitted. The first Pietarinen expansion with branch-point $x_P$ is restricted to an unphysical energy range and represents all left-hand cut contributions. The next two Pietarinen expansions describe the background in the physical range with branch-points $x_Q$ and $x_R$ respecting the analytic properties of the analyzed partial wave. The second branch-point is mostly fixed to the elastic channel branch-point, the third one is either fixed to the dominant channel threshold, or left free. Thus, only rather general physical assumptions about the analytic properties are made like the number of poles and the number and the position of branch-points, and the simplest analytic function with a set of poles and branch-points is constructed.

In the compilation of our results we show the results of four fits: a) the $BnGa$ coupled channel fit to the complete data base including the energy independent solutions for $\pi^- p \rightarrow K^0\Lambda$ and $\gamma p \rightarrow K^+\Lambda$ presented here; b) a single-channel $L+P$ fit to the energy independent solution for $\pi^- p \rightarrow K^0\Lambda$ ($SC_{L+P}^{\pi N,K\Lambda}$); c) a single-channel $L+P$ fit to the energy independent solution for $\gamma p \rightarrow K^+\Lambda$ ($SC_{L+P}^{\gamma N,K\Lambda}$); and d) a multi-channel $L+P$ fit to the energy independent solution for $\pi^- p \rightarrow K^0\Lambda$ and $\gamma p \rightarrow K^+\Lambda$ ($CC_{L+P}^{\pi N,K\Lambda}$).

\textbf{4.2 L+P Fits}

\textbf{4.2.1 $J^P = 1/2^-$-wave}

We have fitted the $J^P = 1/2^-$ partial wave from the energy independent amplitude for the reaction $\pi^- p \rightarrow K^0\Lambda$ in a $SC_{L+P}^{\pi N,K\Lambda}$ fit. A $\chi^2 = 2.45$ was obtained for the 28 data points with 23 parameters. We needed two poles, one at 1667 MeV and second one at 1910 MeV. Due to the low-statistics of the data, the results from the single-channel fit show large errors.

The 48 data points on the $E_{0+}$ multipole from $\gamma p \rightarrow K^+\Lambda$ required only one pole close to 1900 MeV. The strong peak at low mass of the imaginary part of the $E_{0+}$ multipole is reproduced by the function $Y^\alpha(W)$ with a branching point at the $K^+\Lambda$ threshold. Note that the lowest mass bin for the $E_{0+}$ multipole starts at 1700 MeV, significantly above the $N(1650)1/2^-$ mass. The data were described with a $\chi^2 = 0.48$ and 19 parameters in a $SC_{L+P}^{\gamma N,K\Lambda}$ fit. Compared to the pion-induced reaction, the errors on the
higher-mass resonance (at 1900 MeV) are considerably reduced.

The common fit to both data sets (with 76 data points) used two poles, the fit resulted in a $\chi^2 = 0.86$ for 37 parameters. The results are shown in Table 2 and Figs. 8 and 10.

The real part of the pole positions of the $N(1650)1/2^-$ resonance are nicely consistent when the three values are compared, the imaginary part is likely too narrow in the L+P fit. The magnitudes of the inelastic pole residue are consistent at the $2\sigma$ level when the BuGa and CC L+P fits are compared. The phases, however, seem to be inconsistent.

The $N(1895)1/2^-$ pole positions are well defined with acceptable errors and consistent when the four analyses are compared, only the single-channel L+P fit to photoproduction data returns a slightly too narrow width. All four analyses yield compatible magnitudes of the inelastic pole residues, the phases disagree at the $2\sigma$ level. The magnitudes and the phases of the $E0+$ multipole determined by the BuGa fit agree well with the values of the L+P fits within the rather large uncertainties. Note that the errors in the CC L+P and BuGa fits have different origins: The L+P errors are of a statistical nature, the BuGa errors are derived from the spread of results of a variety of different fits. Both approaches establish the need for $N(1650)1/2^-$ and unquestionably require $N(1895)1/2^-$.  

### 4.2.2 $JP=1/2^+$-wave

We have fitted the $J^P = 1/2^+$-wave using the $P_{11}$ energy independent amplitude for the $\pi^- p \to K^0 \Lambda$ reaction and the $M_{1-}$ multipole from $\gamma p \to K^+ \Lambda$. The first data set $\pi^- p \to K^0 \Lambda$ required two poles. The first pole was located near 1700 MeV, the second one was found near 2100 MeV even though with large error bars: the admitted range covers masses from $\sim 1790$ to $\sim 2375$ MeV. The photoproduction data required only one pole close to 1900 MeV. The CC L+P fit to both data sets was performed with two poles.

### Table 1. Formulae defining the Laurent+Pietarinen (L+P) expansion.

$$T^a(W) = \sum_{j=1}^{N_{pole}} x_j^a + y_j^a \frac{W}{W_j^2 - W^2} + \sum_{k=0}^{N} c_k^a X^a(W)^k + \sum_{l=0}^{L} d_l^a Y^a(W)^l + \sum_{m=0}^{M} e_m^a Z^a(W)^m$$

$$X^a(W) = \frac{\alpha^a - \sqrt{x_R^a - W^2}}{\alpha^a + \sqrt{x_R^a - W^2}}$$

$$D_{dp}^a = \frac{1}{2N_W^a - N_{par}^a} \left\{ \left( \frac{\text{Re} T^a(W^{(i)}) - \text{Re} T^{a,exp}(W^{(i)})}{\text{Err}^\text{Re}_{i,a}} \right)^2 + \left( \frac{\text{Im} T^a(W^{(i)}) - \text{Im} T^{a,exp}(W^{(i)})}{\text{Err}^\text{Im}_{i,a}} \right)^2 \right\} + \mathcal{P}^a$$

$$\mathcal{P}^a = \lambda_0^a \sum_{k=0}^{K} (c_k^a)^2 k^3 + \lambda_1^a \sum_{l=1}^{L} (d_l^a)^2 l^3 + \lambda_2^a \sum_{m=1}^{M} (e_m^a)^2 m^3$$

<table>
<thead>
<tr>
<th>$a$</th>
<th>channel index</th>
<th>$N_{pole}$</th>
<th>number of poles</th>
<th>$W_j, W \in \mathbb{C}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_j^a, y_j^a, c_k^a, d_l^a, e_m^a, \alpha^a, \beta^a, \gamma^a, \ldots \in \mathbb{R}$</td>
<td>( N^a )</td>
<td>number of energies in channel ( a )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$K^a, L^a, M^a \ldots \in \mathbb{N}$</td>
<td>number of Pietarinen coefficients in channel ( a )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$D_{dp}^a$, discrepancy function in channel ( a )</td>
<td>$N^a_{par}$</td>
<td>number of fitting parameters in channel ( a )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mathcal{P}^a$, Pietarinen penalty function</td>
<td>$\lambda_0^a, \lambda_1^a, \lambda_2^a$</td>
<td>Pietarinen weighting factors</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_p^a, x_{Q}^a, x_R^a \in \mathbb{R}$ (or $\in \mathbb{C}$).</td>
<td>$\text{Err}^\text{Re}_{i,a}$, minimization error of real and imaginary part respectively.</td>
<td></td>
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</tr>
</tbody>
</table>

### Fig. 10. (Color online) Real and imaginary parts of the $J^P = 1/2^-$: $S_{11}$ partial wave amplitude, and the $E_{0+}$ multipole. The grey band represents the allowed range of BuGa solutions, the dashed curve the single channel fits $SC_{L+P}$ and $SC_{L+P}$, and the solid curves the L+P fit to both data sets.

### Fig. 11. (Color online) Real and imaginary parts of the $J^P = 1/2^+$: $P_{11}$ partial wave amplitude, and the $M_{1-}$ multipole. The grey band represents the allowed range of BuGa solutions, the dashed curve the single channel fits $SC_{L+P}$ and $SC_{L+P}$, and the solid curves the L+P fit to both data sets.
The results are shown in Table 2 and Figs. 8 and 11. The 28 data points for \( \pi^- p \to K^0 A \) were fitted with 23 parameters and a \( \chi^2 \) of 0.67. The 48 data points on the \( M_{1-} \) multipole were described with a \( \chi^2 = 0.366 \) and 19 parameters. The fit to both data sets resulted in a \( \chi^2 = 0.505 \) for 41 parameter. Both approaches, the BuGa and CC L+P fit, establish the need for \( N(1710)1/2^+ \), and unquestionably require \( N(1880)1/2^+ \).

The \( N(1710)1/2^+ \) mass is consistent in the CC L+P and the BuGa fits, its width tends to be smaller in the CC L+P fit, see Tables 2 but the difference is 1.7\( \sigma \) only. The magnitudes of the inelastic residue for this resonance have large error bars in the L+P fits and cover zero, we give upper limits only. The limits are compatible with the BuGa result. In spite of the large errors in the magnitudes, the phases are consistent.
The masses of the $N(1880)^{1/2^+}$ resonance from the BnGa and CC L+P fits are compatible but not the widths. The inelastic residues disagree slightly. Both, the single-channel SC L+P and the coupled-channel CC L+P fit, agree that the $N(1880)^{1/2^+}$ width should be smaller than \(\sim 40\) MeV while BnGa finds a normal hadronic width. However, we have performed a CC L+P fit imposing a mass of 150 MeV. When the result of the CC L+P fit is compared to the observables with 674 data points (Figs. 4 to 7), the fit deteriorates only minimally, the \(\chi^2\) increases by 4.5 units. We conclude that the \(N(1880)^{1/2^+}\) resonance is definitely confirmed in this nearly model-independent analysis and that it has a normal hadronic width. The magnitudes of the inelastic residues and of the \(M_{1^-}\) multipole agree reasonably well, the phases of the inelastic residues are again inconsistent while the \(M_{1^-}\) multipole phases agree well within their uncertainties.

4.2.4 \(J^P = 3/2^-\)-wave

The \(J^P = 3/2^-\)-wave was not derived from the pion induced reaction \(\pi^- p \rightarrow K^0\Lambda\), so the two photoproduction multipoles \(E_2^-\) and \(M_2^-\) were fitted simultaneously in the coupled-multipoles CM L+P mode (CM\(\chi_{L+P}\)). The CM L+P fit required only one pole close to 1900 MeV, no \(N(1720)^{3/2^+}\) was needed. Due to the presence of important thresholds (\(\Sigma K, N(1520)^{3/2^-}\pi, N(1535)^{1/2^-}\pi\)), the \(N(1720)^{3/2^+}\) resonance has a rather complicated pole structure, and we refrain from discussing this resonance here. The fit to the 96 data points in the two data sets is shown in Fig. 8. The fit returned a \(\chi^2 = 0.42\) for 35 parameters. The results are shown in Table 3. The poles from the L+P and BnGa fits are fully consistent. We conclude that \(N(1900)^{3/2^+}\) is definitely confirmed in this nearly model-independent analysis.

5 Comparison to other groups

Figure 12 shows the real and imaginary parts of low-\(L\) partial-wave amplitudes from Refs. [31] and [61]. The amplitudes are similar in magnitude but differ in their shape. The JiBo fit does not contain \(N(1895)^{1/2^-}\), the third resonance in the \(J^P = 1/2^-\) wave that is confirmed here and in a recent analysis of \(\gamma p \rightarrow \eta, \eta' p\) [62]. Both the analysis in Ref. [61] and this work, introduce \(N(1710)^{1/2^-}\) - a resonance not needed in Ref. [12] - but here we find evidence for an additional resonance in this partial wave, \(N(1880)^{1/2^-}\). Thus the differences in the partial-wave amplitudes are to be expected.

There is a large number of papers devoted to partial wave analyses of the reaction \(\gamma p \rightarrow K^+\Lambda\). We discuss here only recently published papers which include at least one measurement of a double polarization variable.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figures/fig12.png}
\caption{Real and imaginary part of the (dimensionless) \(S_{11}\) and \(P_{11}\) waves [31]. The energy-dependent solution BnGa2011-02 is shown as error band. The solid curve represents a L+P fit. The dashed (green) curve is given by the solution JiBo2015-B of the JiBo group [61]. The BnGa and JiBo groups use a different sign convention. The JiBo amplitudes are shown with an inverted sign.}
\end{figure}
The Gent group proposes a methodology based on Bayesian inference to determine those resonances which contribute to $\gamma p \rightarrow K^+A$ [63,64]. They try different groups of 11 resonances and find that the fit with $N(1535)1/2^+$, $N(1560)1/2^-$, $N(1680)5/2^+$, $N(1720)3/2^+$, $N(1875)3/2^-$, $N(1880)1/2^+$, $N(1900)3/2^+$, and $N(2000)5/2^+$ has the highest evidence.

In a similar model, Skoupil and Bydžovský [65] use alternatively 15 or 16 resonances. They confirm the findings of the Gent group but report evidence that $N(1880)1/2^+$ should be replaced by $N(1860)5/2^+$.

A number of groups have analyzed pion or photo-induced reactions with a Kaon and a $\Lambda$ hyperon in the final state. Wu, Xie, and Chen [59] studied the reaction $\pi^- p \rightarrow K^0\Lambda$ up to $W = 1.76$ GeV in an isobar model; the isobars include hyperon exchanges in the $u$-channel and $K^+$ exchange in the $t$-channel. The leading $s$-channel contributions were found to be due $N(1535)1/2^-$, $N(1650)1/2^-$ and $N(1720)3/2^+$ formation. Xiao, Ouyang, Wang, and Zhong [60] studied the mass range below 1.8 GeV and emphasize the leading role of $N(1535)1/2^-$ and $N(1650)1/2^-$. The Jülich-Bonn (JüBo) group [61] described the data on $\pi^- p \rightarrow K^0\Lambda$ simultaneously with other pion-induced reactions in an analytic, unitary, coupled-channel approach. SU(3) flavor symmetry was used to relate both the $t$- and the $u$-channel exchanges. The authors fit the available data (see Fig. 1); all resonances found in the GWU analysis [12] were introduced in the fit and four further ones.

Mart, Clympton and Arifi [66,67] take into account the set of resonances used in the BnGa analysis [15]. They find that spin-5/2 resonances play an important role and have to be taken into account. In their best fit, the authors use 17 $N^*$ resonances. The three resonances $N(1650)1/2^-$, $N(1720)3/2^+$, and $N(1900)3/2^+$ provide the most important contributions.

In Fig. 13, the photoproduction multipoles from the BnGa analysis and those of Skoupil and Bydžovský [65] and of Mart, Clympton and Arifi [66] are compared. There is no much similarity even though partly the same resonances are used. But possibly, this is not too surprising. In a comparison of the best studied process, $\gamma p \rightarrow \pi N$, significant differences were observed in the multipoles obtained by the BuGa, JiBo, and GWU groups [68] even though all three groups were capable of describing the data reasonably well. However, new data enforced a considerable reduction of the spread of the three results. In any case, the comparison demonstrates that further work is needed before the $\gamma p \rightarrow K^+\Lambda$ reaction can be considered as well understood.
6 Summary

For a long time it has been anticipated that photoproduction experiments will provide measurements that are sufficient in number and statistical accuracy to construct the four complex amplitudes governing the photoproduction of an octet baryon and a pseudoscalar meson. A determination of these four amplitudes requires the measurement with sufficient accuracy of at least eight carefully selected observables [69], and one phase still remains undetermined. Alternatively, the multipoles driving the excitation of specific partial waves can be deduced from the data in a truncated partial wave analysis.

In this paper, we have performed such a truncated partial wave analysis of the reaction $\gamma p \rightarrow K^+\Lambda$. The CLAS experiments studied this reaction and reported data on the differential cross section $d\sigma/d\Omega$, on the polarization observables $P$, $T$ and $\Sigma$, and on the spin correlation parameters $O_2$, $O_3$, $C_2$, $C_3$. The data cover the resonance region from 1.71 to 2.13 GeV, mostly in 20 MeV wide bins. Thus at the moment, these data offer the best chance to perform a truncated partial wave analysis.

In a first step, we determined the number of multipoles that can be deduced from the data. When the number of free multipoles is increased in the energy-independent analysis, the errors in the determination of the multipoles increases, and one has to balance precision on the one hand and the number of multipoles on the other hand. It turned out that only the four largest multipoles, $E_0^+, M_1^+, E_1^+, M_1^−$, can be determined without constraints when a good precision of the multipoles is required. In addition, three further multipoles, $E_2^+, M_2^+, E_1^−$, could be derived from the data when a penalty function forced the fit not to deviate too much from an energy dependent solution. In addition to the photoproduction multipoles, we also used partial wave amplitudes for the reaction $\pi^-p \rightarrow K^0\Lambda$ which had been determined earlier.

The energy-dependent solution was found within the BunGa approach. In this approach, a large number of data on pion and photo-induced reaction is fitted in a coupled channel analysis. The data base includes $N\pi$, $N\eta$, $\Lambda K$, $\Sigma K$, $N\pi\pi$, and $N\pi\eta$ final states and, in an iterative procedure, the partial wave amplitudes and photoproduction multipoles derived here. The higher photoproduction multipoles that could not be determined in the fits to the CLAS data were kept fixed to multipoles from the BunGa analysis.

All multipoles considered here, $E_0^+, M_1^+, E_1^+, M_1^−$, $E_2^+, M_2^+, E_1^−$, are fitted within a Laurent-Pietarinen expansion. This expansion exploits the analytic structure of the S-matrix. In the vicinity of a resonance position (and reasonably close to the real axis), the photoproduction amplitude is determined by poles and the opening of thresholds. When this analytic structure is imposed, fits to the photoproduction multipoles and partial wave amplitudes require no further dynamical input, the fits do not impose any model bias. The Laurent-Pietarinen fits were performed to the photoproduction multipoles, to the partial wave amplitudes from the $\pi^- \rightarrow K^0\Lambda$ reaction, and to both in a coupled channel fit. The results are then compared to those from the BunGa fit.

The two resonances $N(1895)1/2^−$ and $N(1900)3/2^+$ are firmly established. The results on their masses, widths, and other properties agree well. Also the $N(1880)1/2^+$ resonance is definitely required but there remains the question of the width: within the Laurent-Pietarinen expansion, its width is 40 MeV or less while its width within the BunGa approach is about 150 MeV. The statistical significance of the narrow width is however very small.

The two resonances $N(1875)3/2^−$ and $N(2060)5/2^−$ are derived from photoproduction multipoles which are constrained to follow the BunGa solution. In the $J^P = 3/2^−$ partial wave, BunGa finds two poles; in the Laurent-Pietarinen fit, only one pole is observed at a mass in between the two BunGa poles. The BunGa and Laurent-Pietarinen results on $N(2060)5/2^−$ are nicely consistent.

Summarizing, we can claim that several resonances found in the BunGa energy-dependent multichannel analysis are confirmed by fits based on a Laurent-Pietarinen expansion with a minimal model dependence.

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