Implications of Incomplete Markets for International Economies

Gurdip Bakshi[a], Mario Cerrato[b], and John Crosby[c]

[a]Smith School of Business, University of Maryland
[b]Glasgow University
[c]Business School, University of Technology, Sydney

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Abstract

We develop a restriction that precludes implausibly high reward-for-risk in incomplete international economies to consider a theoretical problem that characterizes a lower bound on the covariance between stochastic discount factors (SDFs) subject to correct pricing. The problem is analytically solvable and synthesizes domestic and foreign SDFs into spanned and unspanned components. Our novelty is that exchange rate growth need not equal the ratio of SDFs and that the SDF correlations are plausibly lowered. Exploiting the realities of cross-country correlations of macroeconomic quantities, namely, consumption, wealth, dividend growths, and asset returns, our empirical investigation refutes the specification of complete markets. (*JEL* F31, G12, G15)

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This paper formulates a theoretical problem that characterizes a lower bound on the covariance between stochastic discount factors (SDFs) in incomplete international economies, subject to correct pricing. The problem is disciplined by the imposition of a constraint that precludes trading opportunities with unreasonably high reward-for-risk. While recognizing that there is a multitude of SDFs in incomplete markets, the solution to the problem distills the insight that the domestic and foreign SDFs can be partitioned into spanned and unspanned components in such a way that the correlation between the SDFs is lowered in an economically motivated fashion. The developed theoretical links are instrumental to our empirical investigation, which undermines the notion that markets are complete. We further show that the implications of our incomplete markets framework can be consistent with the limited cross-country correlations of consumption growth, dividend growth, and long-term bond returns and also sizable cross-country correlations of wealth growth.

What do we do differently from others? First, we consider a discrete-time international economy with the understanding that there is an infinite number of SDFs in incomplete markets. Second, unlike most others, we do *not* assume that the exchange rate growth is the ratio of the foreign to domestic SDFs. Third, we develop a restriction that precludes "good deals" in international economies with incomplete markets. Specifically, a good deal is the possibility of forming a portfolio, which has an implausibly high reward-for-risk. We show that ruling out an implausibly high reward-for-risk places an upper bound on the dispersion of domestic and foreign SDFs. Fourth, we develop a new theoretical optimization problem that solves for the lowest possible SDF covariances. A core derived feature of our framework is an additive form of the SDFs in which the SDFs are analytically decomposed into two parts: one that is spanned by assets in the international economy, and one that is unspanned. The limited correlation between domestic and foreign SDFs can arise when the country-specific unspanned components are negatively correlated.

Four questions are the centerpiece of our research: Can we reject the baseline specification of complete markets? What are the consequences of incomplete markets for the SDF pairs in international economies? In particular, what are the quantitative attributes of volatilities and correlation between SDFs that are consistent with data on consumption growth, wealth growth, dividend growth, and asset returns and that are consistent with an economically plausible lower

bound on SDF covariances? What does our incomplete markets approach say about the currency volatility puzzle?

A key input to the theoretical problem and empirical assessment is the market incompleteness parameter Θ, which we introduce and use to quantify an upper bound on the dispersion of the SDFs. The identifiable restriction of incomplete markets is the strict positivity of Θ. We motivate an algorithm to estimate Θ that minimizes the discrepancy between the correlations between the SDFs of a candidate model and the corresponding one that is synthesized from our problem, in conjunction with the returns of a set of basis assets. The specific models of SDFs that we use are consistent with Epstein-Zin preferences, as well as power utility over wealth, or consumption, or dividends. We show that our approach is effectively equivalent to aligning the correlation between macroeconomic quantities that are based on wealth growth, or consumption growth, or dividend growth, and under certain assumptions, between the inverse of gross bond returns (thus, consistent with the evolutions of the domestic and foreign yield curves). The algorithm encapsulates the idea that deviations from complete markets cannot be measured from data on asset returns alone. This feature arises not because of choice or theoretical design, but by the nature of the incomplete markets problem in which there is an infinite number of SDFs.

With our new framework for characterizing SDF pairs, there exists a market incompleteness parameter Θ for which values of correlations between SDFs (or international risk sharing), based on asset returns, are in line with model-based SDFs that are based on consumption, wealth, and dividends. Our empirical analysis is backed by evidence from 45 (9) pairs of industrial (emerging) economies. This empirical reconciliation features a universe of basis assets that incorporates the returns of a risk-free bond and an equity index from each half of a country pair, together with the returns of the U.S. 30-year Treasury bond and the world equity index that are common to all country pairs, and incorporates multicurrency and multicountry exposures. The estimation algorithm can be tailored to consider expanded (or reduced) sets of basis assets, and we show that our headline conclusions relating to SDF correlations and the size of the unspanned components remain intact. The specific exercises that we consider include expanding the set of basis assets to include returns of (1) a commodity index and (2) options of G-10 currencies. The common thread is that the baseline specification of complete markets is not empirically supportable.

Additionally, we employ rolling and expanding window schemes to estimate the market incompleteness parameter Θ, shedding light on the properties of the SDFs when the investment opportunity set is changing. Extending our analysis, we match cross-country correlations between long-term bond returns, a data dimension that is complementary to matching cross-country correlations between equity returns. Pointing to the consistency of our conclusions, each part of our investigation is uniform in not supporting the baseline specification of complete markets.

We also elaborate on how our incomplete markets approach can prove useful in interpreting international finance puzzles. For example, the question of low volatility of exchange rate growth (the volatility/risk sharing puzzle) was addressed in Brandt, Cochrane, and Santa-Clara (2006) by arguing that the correlations between log SDFs must be high. In contrast, our tenet of incomplete markets furnishes limited SDF correlations, while producing plausible SDF volatilities, and implies limited correlations between macroeconomic quantities.

Reinforcing our theoretical and empirical analysis, we also glean evidence on trades with prospectively high reward-for-risk and show that such trades do not necessarily persist. Illustrative trading opportunities that we implement and assess include the yen-Icelandic carry, writing out-of-the-money index put options, as well as shorting market volatility, from the perspective of a G-10 currency. In so doing, we draw inferences based on the stationary bootstrap of Politis and Romano (1994), and choose the block size following the procedure of Politis and White (2004).

Our work connects to a body of literature that is at the intersection of consumption risk sharing, asset pricing, exchange rates, and incomplete markets. For example, Lewis (1996) finds some supporting empirical evidence for the idea that the documented low consumption growth correlations and, thus, low international risk sharing, may be explainable by incomplete markets. The work of Bansal and Lundblad (2002), Burnside and Graveline (2014), Corsetti, Dedola, and Leduc (2008), Kim and Schiller (2015), and Zhang (2015) further motivates us to formalize the implications of incomplete markets for limited risk sharing and international puzzles.

While many scholars are beginning to expose the consequences of incomplete markets for studying stylized data features and as a lens through which to view international finance puzzles, the extant approaches are different from ours. In particular, the novelty of our solution is that the unspanned components are analytically determined as a part of our theoretical problem, and their

properties are linked to the degree of market incompleteness. Thus, methodologically, our approach is distinct from Pavlova and Rigobon (2007), Maurer and Tran (2015), and Stathopoulos (2017), in which the goal is not to model the properties of the unspanned components of the domestic and foreign SDFs or their correlations. What our framework adds to the literature is that it can be consistent with the data realities associated with cross-country correlations between macroeconomic quantities, via the channel of incomplete international markets. In our analyses, we also describe how our theoretical approach featuring the additive class of the SDFs is complementary, yet distinct from the work of Backus, Foresi, and Telmer (2001) and Lustig and Verdelhan (2015).

The studies of Colacito and Croce (2011, 2013) consider a two-country asset pricing model with long-run risk and recursive preferences to reconcile the correlation between asset returns and the correlation between consumption growths. Their model, through the mechanism of a high correlation of the long-run components of consumption growth, produces correlated SDFs, while maintaining a low unconditional correlation between the consumption growth of the two countries.

Additionally, Favilukis, Garlappi, and Neamati (2015) employ restrictions in financial trade to induce market incompleteness, and show that their framework could explain stylized facts, including a positive correlation between currency appreciation and consumption growth. Gabaix and Maggiori (2015) also allow for incomplete markets in their work on international trade and exchange rates. Building on prior studies, we develop a framework in which SDFs are not unique, and an incomplete markets setting is at the center of a restriction that precludes implausibly high reward-for-risk trading opportunities. Our objective is to answer the following question: seen through the vantage point of incomplete international economies, what are the consequences of modeling unspanned components of the domestic and foreign SDFs for reconciling the realities of international macroeconomic data?

1 A Framework for Analyzing Incomplete International Economies

This section formulates a theoretical problem that characterizes a lower bound on the covariance between SDFs in incomplete international economies, subject to correct pricing. The problem is disciplined by the imposition of a constraint that precludes trading opportunities with unreasonably high reward-for-risk. The key departure from others is that we can analytically decompose domestic

and foreign SDFs into spanned and unspanned components in such a way that the correlation between the SDFs is lowered in an economically motivated fashion. The solution traits hinge on the properties of the asset return data, while recognizing that there is a multitude of SDFs in incomplete markets.

1.1 The economic environment

We consider a discrete-time economy with two dates, namelyn t, the current time, and $t+1$, the time one period ahead. There are $\mathbb{J}$ (finite or infinite) possible states of the world at time $t+1$. We consider two countries, denoted domestic and foreign, and use a superscript asterisk, that is, *, to denote quantities in the foreign country.

The exchange rate, defined as the number of units of domestic currency per unit of foreign currency, at time t, is denoted by S_t. We assume that asset markets are frictionless. For example, there are no bid-ask spreads and no short-sale constraints.

We assume that there are N assets that can be traded by both domestic and foreign investors and that none of the assets is redundant. $\mathbf{R}_{t+1}$ and $\mathbf{R}^\star_{t+1}$ are the N-dimensional vector of domestic and foreign gross returns. Included within the return vectors $\mathbf{R}_{t+1}$ and $\mathbf{R}^\star_{t+1}$ are risk-free bonds in the domestic and foreign country with gross returns R_f and $R^\star_f$, respectively.

More generally, we include all available asset returns, in either currency, in both $\mathbf{R}_{t+1}$ and $\mathbf{R}^\star_{t+1}$. This means that $\mathbf{R}_{t+1}$ and $\mathbf{R}^\star_{t+1}$ are related by

$$\mathbf{R}_{t+1} = (S_{t+1}/S_t)\,\mathbf{R}^\star_{t+1}. \tag{1}$$

Hence, "domestic" and "foreign" returns refer to the currency in which the return is made, not, for example, to the country in which the equity index is based.

Let m_{t+1} and $m^\star_{t+1}$ denote the domestic and foreign SDFs, respectively. We assume that the first and second moments of all relevant quantities, for example, m_{t+1}, $m^\star_{t+1}$, $\mathbf{R}_{t+1}$, and $\mathbf{R}^\star_{t+1}$, exist and are finite. In particular, $\mathbb{E}_t[m^2_{t+1}] < +\infty$ and $\mathbb{E}_t[(m^\star_{t+1})^2] < +\infty$ and, hence, $\left|\mathbb{E}_t[m_{t+1}\, m^\star_{t+1}]\right| < +\infty$ by the Cauchy-Schwarz inequality, where $\mathbb{E}_t[\,.\,]$ indicates time t conditional expectation. Through-

out, $\mathrm{Var}_t[.]$ and $\mathrm{Cov}_t[.,.]$ denote variance and covariance, respectively. The unconditional counterparts omit the time subscripts, and are, respectively, denoted by $\mathbb{E}[\,.\,]$, $\mathrm{Var}[.]$, and $\mathrm{Cov}[.,.]$.

1.2 Basic implications of incomplete markets

Because m_{t+1} and $m^\star_{t+1}$ price domestic returns $\mathbf{R}_{t+1}$ and foreign returns $\mathbf{R}^\star_{t+1}$, respectively,

$$\mathbb{E}_t[m_{t+1}\,\mathbf{R}_{t+1}] = \mathbf{1} \qquad \text{and} \qquad \mathbb{E}_t[m^\star_{t+1}\,\mathbf{R}^\star_{t+1}] = \mathbf{1}, \tag{2}$$

where $\mathbf{1}$ denotes an N-dimensional vector of ones. However, since $\mathbf{R}_{t+1} = (\frac{S_{t+1}}{S_t})\mathbf{R}^\star_{t+1}$, we also have $\mathbb{E}_t[m_{t+1}(\frac{S_{t+1}}{S_t})\mathbf{R}^\star_{t+1}] = \mathbf{1}$ and isomorphically $\mathbb{E}_t[m^\star_{t+1}\mathbf{R}_{t+1}/(\frac{S_{t+1}}{S_t})] = \mathbf{1}$. Subtracting $\mathbb{E}_t[m^\star_{t+1}\mathbf{R}^\star_{t+1}] = \mathbf{1}$ from $\mathbb{E}_t[m_{t+1}(\frac{S_{t+1}}{S_t})\mathbf{R}^\star_{t+1}] = \mathbf{1}$, and $\mathbb{E}_t[m_{t+1}\mathbf{R}_{t+1}] = \mathbf{1}$ from $\mathbb{E}_t[m^\star_{t+1}\mathbf{R}_{t+1}/(\frac{S_{t+1}}{S_t})] = \mathbf{1}$ implies

$$\mathbb{E}_t[(m_{t+1}(\frac{S_{t+1}}{S_t}) \;-\; m^\star_{t+1})\,\mathbf{R}^\star_{t+1}] = 0 \qquad \text{and} \qquad \mathbb{E}_t[(m_{t+1} \;-\; m^\star_{t+1}/(\frac{S_{t+1}}{S_t}))\,\mathbf{R}_{t+1}] = 0. \tag{3}$$

Because Equation (3) is true for all returns $\mathbf{R}^\star_{t+1}$ and $\mathbf{R}_{t+1}$, it is true for the foreign risk-free return $R^\star_f$ and the domestic risk-free return R_f, which leads to the following implications:

$$\mathbb{E}_t[m_{t+1}(\frac{S_{t+1}}{S_t}) - m^\star_{t+1}] = 0 \qquad \text{and} \qquad \mathbb{E}_t[m_{t+1} - m^\star_{t+1}/(\frac{S_{t+1}}{S_t})] = 0. \tag{4}$$

In light of the link between m_{t+1}, $m^\star_{t+1}$, and $\frac{S_{t+1}}{S_t}$ in Equation (4), we first formalize the following:

Definition 1 (Incomplete markets)

International markets are incomplete if $(m_{t+1}, m^\star_{t+1})$ *satisfying (2)–(4) are not unique.* (5)

Next, consider the (class of) random variables $(\xi_{t+1},\, \xi^\star_{t+1})$, defined by

$$\xi_{t+1} \;\equiv\; m_{t+1}(\frac{S_{t+1}}{S_t}) \;-\; m^\star_{t+1} \qquad \text{and} \qquad \xi^\star_{t+1} \;\equiv\; m_{t+1} \;-\; m^\star_{t+1}/(\frac{S_{t+1}}{S_t}). \tag{6}$$

Definition 2 (Incomplete markets problem) *By choice of* $(m_{t+1},\, m^{\star}_{t+1})$ *solve*

$$\mathbb{E}_t[\xi_{t+1}] = 0, \qquad \mathbb{E}_t[\xi_{t+1}\mathbf{R}^{\star}_{t+1}] = 0, \tag{7}$$

$$\mathbb{E}_t[\xi^{\star}_{t+1}] = 0, \qquad \mathbb{E}_t[\xi^{\star}_{t+1}\mathbf{R}_{t+1}] = 0, \tag{8}$$

where the random variables $(\xi_{t+1},\, \xi^{\star}_{t+1})$ *defined in Equation (6) are* not *unique.*

The incomplete markets problem agrees with a view that there are some states for which no Arrow-Debreu security trades. More formally, with $\text{span}(\mathbf{R}) \equiv \{\boldsymbol{a}'\mathbf{R} : \boldsymbol{a} \in \mathbb{R}^N\}$ denoting the set of possible portfolio returns, it agrees with $\text{span}(\mathbf{R}) \neq \mathbb{R}^{\mathbb{J}}$, with $N < \mathbb{J}$ (e.g., Duffie 1992).

In the case of complete markets, $(\xi_{t+1},\, \xi^{\star}_{t+1})$ satisfying Equations (7) and (8) are unique and identically zero in every state. Correspondingly, there is an Arrow-Debreu security tradeable for every $t+1$ state of the world, which implies, in the absence of arbitrage, that $S_t\, m^{\star}_{t+1} = S_{t+1}\, m_{t+1}$, or equivalently, we obtain the relation (e.g., Backus, Foresi, and Telmer 2001)

$$m_{t+1}(\frac{S_{t+1}}{S_t}) - m^{\star}_{t+1} = 0 \;\text{ in a complete markets setting.} \tag{9}$$

We stress that Equations (3) and (4) always hold, regardless of whether the market is complete or incomplete. The defining attribute of incomplete markets is that some m_{t+1} and $m^{\star}_{t+1}$ satisfy $m_{t+1}(\frac{S_{t+1}}{S_t}) - m^{\star}_{t+1} = 0$, and some do not.[1]

1.3 Rationale for our transformations in incomplete international markets

It simplifies the exposition and analytical characterizations if we define (note $S_{t+1}/S_t > 0$) the N-dimensional vector $\mathbf{Z}_{t+1}$ by

$$\mathbf{Z}_{t+1} \;\equiv\; \mathbf{R}_{t+1}/\sqrt{S_{t+1}/S_t} \;=\; \sqrt{S_{t+1}/S_t}\,\mathbf{R}^{\star}_{t+1}, \;\text{ and} \tag{10}$$

$$y_{t+1} \equiv m_{t+1}\sqrt{S_{t+1}/S_t} \qquad \text{and} \qquad y^{\star}_{t+1} \equiv m^{\star}_{t+1}/\sqrt{S_{t+1}/S_t}. \tag{11}$$

[1] That "$m_{t+1}(\frac{S_{t+1}}{S_t}) - m^{\star}_{t+1}$ need not equal zero" is intuitive, because, in incomplete markets, there are some outcomes for which no Arrow-Debreu security trades, and different investors will place different marginal utility on those outcomes. For example, if a representative agent exists in each country and if, say, the domestic agent is more risk averse than the foreign counterpart, then the former will assign greater marginal utility to unpleasant states. This idea underscores the development of our Result 1, which precludes trading opportunities with unreasonably high reward-for-risk.

The transformations postulated in Equation (11) imply $y_{t+1}\, y^{\star}_{t+1} = m_{t+1}\, m^{\star}_{t+1}$. Additionally, $y_{t+1}\, y^{\star}_{t+1}$ has a bounded expectation, because $\left|\mathbb{E}_t[y_{t+1}\, y^{\star}_{t+1}]\right| = \left|\mathbb{E}_t[m_{t+1}\, m^{\star}_{t+1}]\right| < +\infty$. Furthermore, from Equation (2), the transformations (10) and (11) result in the following restrictions:

$$\mathbb{E}_t[y_{t+1}\, \mathbf{Z}_{t+1}] = \mathbf{1} \qquad \text{and} \qquad \mathbb{E}_t[y^{\star}_{t+1}\, \mathbf{Z}_{t+1}] = \mathbf{1}. \tag{12}$$

Equations (10) and (12) imply that $\mathbf{Z}$ can be interpreted as the gross returns in a hypothetical economy in which $\mathbf{Z}$ represents the geometric average of $\mathbf{R}$ and $\mathbf{R}^{\star}$ (since Equation (10) implies $(\mathbf{R}\mathbf{R}^{\star})^{\frac{1}{2}} = \mathbf{Z}$). Further, y_{t+1} and $y^{\star}_{t+1}$ can be interpreted as SDFs in this hypothetical economy.

What is the rationale for our focus on the transformed variables in Equations (10) and (11)? For one, our transformations are devices that allow cash flow pricing in a *symmetric* fashion, circumventing the need to duplicate calculations in different currency units. For another, potential deviations from $y_{t+1} = y^{\star}_{t+1}$ depict market incompleteness. This is so because

$$y_{t+1} = y^{\star}_{t+1} \;\; \text{implies} \;\; m_{t+1}(\frac{S_{t+1}}{S_t}) - m^{\star}_{t+1} = 0, \;\; \text{which represents complete markets.} \tag{13}$$

Our approach, which allows for "y_{t+1} not equal to $y^{\star}_{t+1}$" or (by substituting from Equation (11)) $m_{t+1}(\frac{S_{t+1}}{S_t}) - m^{\star}_{t+1}$ need not equal zero, is consistent with the notion that there is an infinite number of m_{t+1} and $m^{\star}_{t+1}$ pairs in incomplete markets. We show that our method facilitates an analytical solution that decomposes each SDF into its spanned and unspanned components is tractable for studying the correlation between domestic and foreign SDFs, allowing us to quantify the degree of market incompleteness across country pairs.

1.4 Constraint on reward-for-risk in the international economy

Whether markets are complete or incomplete, domestic and foreign investors agree on the prices of securities in the linear span (i.e., a linear combination) of $\mathbf{R}$ or $\mathbf{R}^{\star}$. In incomplete markets, $m_{t+1}(\frac{S_{t+1}}{S_t}) - m^{\star}_{t+1}$ need not equal zero, y_{t+1} need not equal $y^{\star}_{t+1}$, and the valuations of domestic and foreign investors need not coincide for securities that are *not* in the linear span of $\mathbf{R}$ (or $\mathbf{R}^{\star}$).

How different can those valuations be in incomplete markets? Our approach, broadly speaking, is to ask: By how much can domestic investors and foreign investors disagree on the valuation of

securities outside the linear span of $\mathbf{R}$ (or $\mathbf{R}^\star$) before financial intermediaries would be presented with a "good deal" (i.e., an implausibly high reward-for-risk)?

We state the following result that places an upper bound on the dispersion of the SDFs:

Result 1 (Ruling out implausibly high reward-for-risk) *Suppose y and $y^\star$ satisfy the construction in Equation (11) and markets are incomplete. Then*

$$\mathbb{E}[(y - y^\star)^2] \leq \Theta^2 \tag{14}$$

for some economically and empirically motivated choice of Θ (for $0 \leq \Theta < +\infty$). We will refer to Θ as the market incompleteness parameter.

Proof: See Appendix A. ■

The substantive content of an upper bound $\mathbb{E}[(y - y^\star)^2] \leq \Theta^2$ arises in incomplete markets when y need not equal $y^\star$ and is related to the degree of market incompleteness. Equation (14) places a restriction on y and $y^\star$ and, thus, on the set of admissible m and $m^\star$.

The economic rationale behind Result 1 is as follows. In incomplete markets, domestic and foreign investors will typically place different valuations on securities outside the linear span of $\mathbf{R}$ (or $\mathbf{R}^\star$). This is a situation that financial intermediaries may potentially wish to exploit, and they can do this by creating a synthetic security that offers payoffs outside the linear span of $\mathbf{R}$ (or $\mathbf{R}^\star$).

A financial intermediary that considers the possibility of privately negotiating a contract between itself and the domestic investor faces a trade-off between the reward (which depends on the differences in valuations) and the risks of entering into the privately negotiated contract. If the reward-for-risk faced by the financial intermediary is high, the financial intermediary would, presumably, enter into the privately negotiated contract. The same argument works for the foreign investor.

Our approach derives an upper bound on the reward-for-risk, which has the effect of ruling out a potential privately negotiated contract that is too good to be true (e.g., Cochrane and Saá-Requejo 2000). In other words, we preclude a reward-for-risk that is implausibly high. This leads to an economically motivated bound in Equation (14) on the differences in valuation of securities that are *not* in the linear span of $\mathbf{R}$ (or $\mathbf{R}^\star$). Appendix A delineates the details.

Equation (14) is central to our analysis and can be distinguished from the setting in which $m(\frac{S_{t+1}}{S_t}) - m^\star = 0$, which entails $y - y^\star = 0$, which in turn, imposes $\mathbb{E}[(y - y^\star)^2] = 0$. One may view Θ as quantifying *deviations from market completeness.*

We propose an algorithm to estimate the market incompleteness parameter Θ within the context of our empirical work (Section 2.1). Specifically, given cross-country data on consumption, wealth, dividends, and a broad selection of returns (e.g., bonds, equities, commodities, and currency options), we develop a theory that allows us to ask: can we reject the baseline specification of complete markets, or, in other words, is Θ statistically different from zero?

1.5 Operationalizing the framework when $m(\frac{S_{t+1}}{S_t}) - m^\star$ need not equal zero

We define the correlation between m and $m^\star$ (respectively, international risk sharing) to be $\text{Cov}[m, m^\star]$ divided by the geometric (respectively, arithmetic) average of the variances of m and $m^\star$.

This subsection studies a theory that allows for a multitude of SDFs in incomplete markets and seeks to synthesize m and $m^\star$, which are restricted by a feasible set (outlined shortly) that may or may not be consistent with $m(\frac{S_{t+1}}{S_t}) - m^\star = 0$ and simultaneously offers flexibility in producing patterns of correlation and risk sharing computed based on models and international data on consumption growth, wealth growth, dividend growth, and bond returns.

Thus, we ask the following questions: If we restrict our attention to (m, $m^\star$) pairs, which are economically plausible and consistent with Equations (6), (7), and (8), how realistic is $\text{Cov}\,[m, m^\star]$, or, isomorphically, the correlation between m and $m^\star$ (or risk sharing), based on asset return data? Are the magnitudes aligned with the realities of the data?

In the setting of incomplete markets, there is an infinite number of m and $m^\star$ and, thus, an infinite number of possible values of $\text{Cov}[m, m^\star] = \mathbb{E}[m\, m^\star] - 1/(R_f R_f^\star)$. This leads us to possibly *take infimums*, over m and $m^\star$ of $\mathbb{E}[m\, m^\star]$. Thus, in essence, we are asking what is a plausible, but economically justified, *lower bound* on the covariances (correlations) between SDFs based on available data?

Recalling from Equation (11) that $y \equiv m\sqrt{S_{t+1}/S_t}$ and $y^\star \equiv m^\star/\sqrt{S_{t+1}/S_t}$, we consider the following problem (which, since $y\, y^\star = m\, m^\star$, is equivalent to the objective $\inf_{m,m^\star}\, \mathbb{E}[\, m\, m^\star]$).

Problem 1 *Choose y and $y^\star$ to*

$$\inf_{y,y^\star} \mathbb{E}[\, y\, y^\star], \tag{15}$$

subject to

$$\mathbb{E}[(y-y^\star)^2] \;\leq\; \Theta^2, \qquad \text{(relaxes the restriction that } m(\frac{S_{t+1}}{S_t}) - m^\star = 0) \tag{16}$$

$$\mathbb{E}[y\mathbf{Z}] = \mathbb{E}[y^\star\mathbf{Z}] = \mathbf{1}, \qquad \text{(correct pricing)} \tag{17}$$

$$y \geq 0 \;\text{ and }\; y^\star \geq 0. \qquad \text{(nonnegativity constraints)} \tag{18}$$

In Problem 1, the inequality constraint (16) arises as a consequence of incorporating the incomplete markets assumption in the international economy, whereby $m(\frac{S_{t+1}}{S_t}) - m^\star$ need not equal zero. The equality constraint $\mathbb{E}[y\mathbf{Z}] = \mathbb{E}[y^\star\mathbf{Z}] = \mathbf{1}$ in Equation (17) is equivalent to $\mathbb{E}[m\mathbf{R}] = \mathbf{1}$ and $\mathbb{E}[m^\star\mathbf{R}^\star] = \mathbf{1}$ and enforces that m and $m^\star$ must price the returns $\mathbf{R}$ and $\mathbf{R}^\star$.

We are interested in analyzing what incomplete markets have to say about SDF volatilities, SDF correlations and risk sharing, which we show are linked to the market incompleteness parameter Θ. At the same time, we are interested in inferring marginal utility growth rates that are consistent with the data. The marginal utilities are nonnegative, so, following Hansen and Jagannathan (1991), we focus on nonnegative SDFs in the admissible set. The constraints $y \geq 0$ and $y^\star \geq 0$ in Equation (18) are equivalent to $m \geq 0$ and $m^\star \geq 0$. The restrictions in Problem 1 are imposed unconditionally, which is a weaker condition.

In the objective function (15), we essentially compute a *lower bound* on the value of $\mathbb{E}[m\, m^\star]$ consistent with repricing the returns $\mathbf{R}$ and $\mathbf{R}^\star$, consistent with the absence of arbitrage, and consistent with the upper bound $\mathbb{E}[(y-y^\star)^2] \leq \Theta^2$. Still, the optimization problem could become ill-posed if one could find m and $m^\star$, where the objective is unbounded. Such an outcome is disallowed with our constraints and via $\left|\mathbb{E}[yy^\star]\right| < +\infty$. The solution depends critically on $\mathbb{E}[(y-y^\star)^2] \;\leq\; \Theta^2$.

1.6 Characterizing the spanned and unspanned components of m and $m^\star$

The next result is central to our theoretical and empirical investigation.

Result 2 (Solution to SDFs in incomplete markets) *The solution to Problem 1 is*

$$y = \max(\boldsymbol{\lambda}'\mathbf{Z}, 0) \; + \; \frac{1}{2} d\,\Theta\,\delta \qquad \text{and} \qquad y^\star = \max(\boldsymbol{\lambda}'\mathbf{Z}, 0) \; + \; \frac{1}{2} d^\star\,\Theta\,\delta, \tag{19}$$

where the N-dimensional vector of Lagrange multipliers $\boldsymbol{\lambda} \in \mathbb{R}^N$ solves

$$\min_{\boldsymbol{\lambda} \in \mathbb{R}^N} \quad \mathbb{E}[(\max(\boldsymbol{\lambda}'\mathbf{Z}, 0))^2] \; - \; 2\,\boldsymbol{\lambda}'\mathbf{1}. \tag{20}$$

δ, which depends on $\mathbf{Z}$, can be recovered as

$$\delta = \frac{e_{\mathbf{Z}}}{\sqrt{\mathbb{E}[e_{\mathbf{Z}}^2]}}, \text{ and } \; e_{\mathbf{Z}} = 1 - \mathbb{E}[\mathbf{Z}]' \left(\mathbb{E}[\mathbf{Z}\mathbf{Z}']\right)^{-1} \mathbf{Z}. \tag{21}$$

The constants d and $d^\star$ are

$$\begin{cases} \text{If } d_p \geq 1 \text{ and } d_n \leq -1, \text{ then } d = 1 \text{ and } d^\star = -1, \\ \text{else } d = \min(d_p, -d_n) \text{ and } d^\star = -d, \end{cases} \tag{22}$$

where d_p (respectively, d_n) is the smallest positive value (respectively, least negative value) of $-\max(\boldsymbol{\lambda}'\mathbf{Z}, 0)/\left(\frac{1}{2}\Theta\delta\right)$ across the $\mathbb{J}$ possible states of the world.

Proof: See Appendix B. ■

Our characterization of δ in Equation (21) partitions each SDF into two distinct portions: one that is spanned by the available set of asset returns in international economies, and a second one that is unspanned. The solution for y and $y^\star$ in Result 2 yields $m = y/\sqrt{S_{t+1}/S_t}$ and $m^\star = y^\star\sqrt{S_{t+1}/S_t}$, which facilitates the characterization of the spanned and unspanned components of the following SDFs:

$$m_{t+1} \; = \; \underbrace{m_{\mathbf{Z},t+1}}_{\text{spanned}} \; + \; \underbrace{u_{t+1}}_{\text{unspanned}} \qquad \text{and} \qquad m^\star_{t+1} = \underbrace{m^\star_{\mathbf{Z},t+1}}_{\text{spanned}} \; + \; \underbrace{u^\star_{t+1}}_{\text{unspanned}}, \tag{23}$$

with

$$m_{\mathbf{z}} \equiv \frac{\max(\boldsymbol{\lambda}'\mathbf{Z},0)}{\sqrt{S_{t+1}/S_t}} = \frac{1}{\frac{S_{t+1}}{S_t}} \max(\boldsymbol{\lambda}'\mathbf{R},0), \qquad \text{and} \quad u \equiv \frac{1}{2}\,(d\,\Theta\,\delta)\,\frac{1}{\sqrt{S_{t+1}/S_t}}, \tag{24}$$

$$m^{\star}_{\mathbf{z}} \equiv \max(\boldsymbol{\lambda}'\mathbf{Z},0)\sqrt{S_{t+1}/S_t} = \frac{S_{t+1}}{S_t} \max(\boldsymbol{\lambda}'\mathbf{R}^{\star},0), \quad \text{and} \quad u^{\star} \equiv \frac{1}{2}\,(d^{\star}\,\Theta\,\delta)\,\sqrt{S_{t+1}/S_t}, \tag{25}$$

where $\boldsymbol{\lambda}$ solves (20) and δ is determined via Equation (21). We will establish that $\mathbb{E}[u] = \mathbb{E}[u^{\star}] = 0$.

The extraction of the spanned and unspanned components is consistent with the objective $\inf_{m,m^{\star}} \mathbb{E}[\,m\,m^{\star}]$, consistent with ruling out reward-for-risk perceived to be unacceptable in international economies, and consistent with the nonnegativity of SDFs. We shall refer to the characterization in Equation (23) as the *additive form of the SDFs*.

Our depiction of the spanned and unspanned components of SDFs in Equations (24) and (25) which are consistent with the lower bound on SDF covariances, furnishes the following new insights:

- The properties of u and $u^{\star}$ critically hinge on Θ. For instance, a higher value of Θ has the effect of increasing the volatility of u and $u^{\star}$ and, therefore, of m and $m^{\star}$. In contrast, the volatility of both $m_{\mathbf{z}}$ and $m^{\star}_{\mathbf{z}}$ is invariant to Θ.

- Next, $\mathrm{Cov}\,[u, u^{\star}] = \frac{1}{4} d d^{\star} \Theta^2 \leq 0$ (using the result that $\mathbb{E}[\delta^2] = 1$ from Equation (21) along with $\mathbb{E}[u] = \mathbb{E}[u^{\star}] = 0$), which is provably negative, given our objective $\inf_{d,d^{\star}} \{\frac{1}{4} d\, d^{\star} \Theta^2\}$ and the result (see Equation (22)) that d and $d^{\star}$ are of opposite signs.

These derived attributes to our solution provide the intuition for the ensuing quantitative assessments regarding risk sharing and the correlation between m and $m^{\star}$. For example, a higher Θ can attenuate risk sharing by making $\mathrm{Cov}\,[u, u^{\star}]$ more negative. We additionally note that imposing $\Theta = 0$ translates into $u = u^{\star} = 0$ and is isomorphic to high international risk sharing.[2]

One may be able to garner a better conceptual understanding of our solution mechanism by drawing on the work of others. In Kim and Schiller (2015), for example, the economies are inhabited by both stockholders and nonstockholders. Since the stockholders have access to capital markets,

[2] In complete markets, $m(\frac{S_{t+1}}{S_t}) - m^{\star} = 0$ tightly links exchange rate growth and m and $m^{\star}$. Moreover, Brandt, Cochrane, and Santa-Clara (2006) show that the *minimum variance* m and $m^{\star}$, recovered from asset returns data, also satisfy $m(\frac{S_{t+1}}{S_t}) - m^{\star} = 0$ in an incomplete market. Their analysis further reveals that with $m(\frac{S_{t+1}}{S_t}) - m^{\star} = 0$ imposed, the risk sharing index, based on asset returns (defined in their Equation (2)), is computed to be high, indicating a high degree of international risk sharing, whereas risk sharing based on consumption growth data is low.

they are able to achieve high risk sharing, because the consumption growths of stockholders in the domestic and foreign countries, and thus, marginal utilities are highly correlated. At the same time, market incompleteness is introduced because the nonstockholders can only trade in a bond. Such a friction can hinder the ability of nonstockholders to share adverse economic shocks. The salient outcome is that aggregate consumption (stockholder plus nonstockholder) growth can be moderately correlated, whereas the stockholders consumption growth and SDFs can be sizably correlated. In our paper, we have offered a different framework in which market incompleteness permeates throughout the economy.

The decomposition articulated in Equations (24) and (25) allows us to study the implications of incomplete markets for the behavior of exchange rate growth, as presented in Equation (4):

$$\begin{aligned} 0 &= \mathbb{E}_t[m\,(S_{t+1}/S_t) - m^\star], \\ &= \mathbb{E}_t[\underbrace{\max(\boldsymbol{\lambda}'\mathbf{R},0) - (S_{t+1}/S_t)\max(\boldsymbol{\lambda}'\mathbf{R}^\star,0)}_{=0,\text{ since } \mathbf{R}_{t+1}=(S_{t+1}/S_t)\mathbf{R}^\star_{t+1}} + \frac{\sqrt{S_{t+1}/S_t}}{2} d\,\Theta\,\delta - \frac{\sqrt{S_{t+1}/S_t}}{2} d^\star\,\Theta\,\delta], \quad (26) \\ &= \frac{1}{2}\,(d-d^\star)\,\Theta\,\mathbb{E}_t[\delta\,\sqrt{S_{t+1}/S_t}] \qquad \text{(note that } d-d^\star \neq 0 \text{ and } \Theta \neq 0). \quad (27) \end{aligned}$$

Thus, we obtain $\mathbb{E}_t[u^\star] = 0$, and likewise $0 = \mathbb{E}_t[m - m^\star/(\frac{S_{t+1}}{S_t})]$ implies that $\mathbb{E}_t[u] = 0$.

Overall, the departure from the literature is our treatment that enables the closed-form tractability of the spanned and unspanned components of the SDFs, revealing how market incompleteness parameter Θ impacts the variances of SDFs and their covariances. Drawing on this analysis, we additionally solve a particularly parameterized economy with five states in the Internet Appendix (Section II and Table Internet-I), which synthesizes, in a simplified setting, the various elements of our approach, to study SDF correlations and volatilities under incomplete markets.

Finally, we explore complementarities and conceptual distinctions from the multiplicative wedge approach proposed by Backus, Foresi, and Telmer (2001), and considered further in Lustig and Verdelhan (2015). While the two approaches are complementary, Appendix C shows that the key economic distinction is that the additive form of SDFs derived in our paper are not subsumed within the multiplicative wedge class.

We now move on to study the empirical implications of our incomplete markets framework.

2 What Does Our Approach Tell Us about International Economies?

To assess the extent of market incompleteness and the empirical properties of SDFs in international economies, we employ data on consumption, wealth, dividends, risk-free bonds, long-term bonds, and broad-based equity indexes for 10 countries (i.e., 45 country pairs), namely Australia (AUD), New Zealand (NZD), United Kingdom (STG), France (FRA), Canada (CAD), United States (USD), the Netherlands (NLG), Germany (GER), Japan (JPY), and Switzerland (SWI). The sample period is January 1975 to December 2015 (492 monthly observations), and the use of such data is in line with others, including Griffin and Karolyi (1998), Bansal and Lundblad (2002), Pavlova and Rigobon (2007), Bekaert, Hodrick, and Zhang (2009), Bekaert et al. (2011), Asness, Moskowitz, and Pedersen (2013), Colacito and Croce (2011, 2013), and Karolyi and Wu (forthcoming). In addition, we construct currency option returns for each of the G-10 currencies.

All nominal returns are converted into real returns by adjusting by expost realized inflation. The sources of the data are described below:

Inflation: Country-specific inflation data are from Datastream and are CPI data. For the United Kingdom and France, we splice the CPI data with retail price index data for the periods 1975 to 1988 and 1975 to 1989, respectively.

Exchange rates: The spot exchange rate data for all country pairs is the midpoint of the bid and ask quotes (from Datastream). The exchange rates for France, Germany, and the Netherlands from January 1999 (the introduction of the euro) onward are taken to be the relevant fixed conversion rate to the euro (e.g., DM 1.95583 = 1 euro).

Equity index returns: The equity index returns data, including that for the world equity index, is MSCI (from Datastream), and we employ total returns (including dividends). MSCI data are not available for New Zealand prior to 1988, so we use returns data supplied by Martin Lally and Alastair Marsden for the period 1975 to 1987.

Risk-free bonds: Risk-free bond prices are constructed from LIBOR quotes as $1/(1+\tau\,\text{LIBOR})$, where τ is the day count fraction, that is, $\tau = 1/12$ for monthly. When LIBOR is not available, we use the nearest substitute, such as 30-day bank bill rates (which are money market rates).

Long-term bond returns: To construct the monthly gross returns of long-term bonds, we use the benchmark government bond index with a constant maturity of 10 years (from Datastream). Additionally, the data on the returns of U.S. 30-year Treasury bond is from Wharton Research Data Services (WRDS).

Consumption: We use annual real consumption data from World Development Indicators.

Dividends: Monthly dividends are calculated from equity indexes as their total value minus their capital value. The *annual* real dividends is the sum of monthly real dividends.

Commodities: S&P commodity index data (in U.S. dollars) is from Datastream (ticker: OFCL).

Currency options: We construct returns of options differentiated by moneyness: 10-delta put, 25-delta put, at-the-money put, at-the-money call, 25-delta call, and 10-delta call, for each of the G-10 currencies, with the U.S. dollar as the domestic currency. These data are from Thompson-Reuters, and Section III of the Internet Appendix provides the details. The currency options span payoffs ranging from crashes in foreign currencies (puts) to crashes in the U.S. dollar (calls).

When computing baseline results for the volatilities and correlations of SDFs that are supported in our incomplete markets framework, for example, for Australia and Japan, the gross return vector $\mathbf{R}_{t+1}$ includes *real* returns on six assets, namely on the Australian risk-free bond, on the Australian equity index, on the Japanese risk-free bond, on the Japanese equity index, on the U.S. 30-year Treasury bond, and on the MSCI world equity index, all denominated in Australian dollars, while $\mathbf{R}^{\star}_{t+1}$ includes returns on the same six assets, but in this case all the *real* returns are denominated in Japanese yen.

We also investigate the effect of enhancing (or reducing) the dimensionality of $\mathbf{R}_{t+1}$ (or $\mathbf{R}^{\star}_{t+1}$) on the properties of $(m_{t+1},\ m^{\star}_{t+1})$ pairs. Our choice of six primitive test assets, which embed multi-currency and multi-country equity and bond exposures, provides a way for studying market incompleteness in international economies and for analyzing the impact of changes in the investment opportunity set.

2.1 Identification and admissible values of Θ

At the front and center of our theory is the feature that a higher market incompleteness parameter Θ is associated with a greater volatility of the unspanned components of the SDFs. In this regard,

the constraint $\mathbb{E}[(y-y^\star)^2] \leq \Theta^2$, in Equation (14), is pivotal to our characterizations in incomplete markets, but it leaves open the question of how to identify and estimate Θ.

2.1.1 Motivating an algorithm for identifying Θ. Our identification strategy for identifying Θ involves the consideration of two crucial theoretical objects as well as salient and pervasive attributes of international macroeconomic data on consumption, wealth, dividends, and an array of long-term bond returns and equity returns.

First, to establish the reasonableness of Θ, we exploit the closed-form tractability of $\mathrm{Var}[m_{t+1}]$, which we compute via Equations (23) and (24) as

$$\mathrm{Var}\left[m_{t+1}\right] \approx \underbrace{\mathrm{Var}[m_{\mathbf{z},t+1}]}_{\text{independent of } \Theta} + \underbrace{\mathbb{E}[u_{t+1}^2]}_{\text{varies with } \Theta} \quad (\text{since } \mathbb{E}[u_{t+1}]=0,\ \mathrm{Cov}[m_{\mathbf{z},t+1}, u_{t+1}] \approx 0). \tag{28}$$

Second, departures from complete markets can translate into less correlated SDFs as well as mitigated risk sharing, like in

$$\rho_{m,m^\star} \equiv \frac{\mathrm{Cov}[m_{t+1}, m_{t+1}^\star]}{(\mathrm{Var}[m_{t+1}]\,\mathrm{Var}[m_{t+1}^\star])^{1/2}} \approx \underbrace{\frac{\mathrm{Cov}[m_{\mathbf{z},t+1}, m_{\mathbf{z},t+1}^\star] + \mathrm{Cov}[u_{t+1}, u_{t+1}^\star]}{\sqrt{(\mathrm{Var}[m_{\mathbf{z},t+1}] + \mathrm{Var}[u_{t+1}])(\mathrm{Var}[m_{\mathbf{z},t+1}^\star] + \mathrm{Var}[u_{t+1}^\star])}}}_{\text{varies with } \Theta}, \tag{29}$$

$$\mathrm{RSI} \equiv \frac{\mathrm{Cov}[m_{t+1}, m_{t+1}^\star]}{(\mathrm{Var}[m_{t+1}] + \mathrm{Var}[m_{t+1}^\star])/2} \approx \frac{\mathrm{Cov}[m_{\mathbf{z},t+1}, m_{\mathbf{z},t+1}^\star] + \mathrm{Cov}[u_{t+1}, u_{t+1}^\star]}{(\mathrm{Var}[m_{\mathbf{z},t+1}] + \mathrm{Var}[u_{t+1}] + \mathrm{Var}[m_{\mathbf{z},t+1}^\star] + \mathrm{Var}[u_{t+1}^\star])/2}. \tag{30}$$

Here, $\rho_{m,m^\star}$ is the correlation between m and $m^\star$, whereas RSI defines a risk sharing index.

The insight to garner is that there is a trade-off between the volatility of the SDFs and the correlation $\rho_{m,m^\star}$ (or the RSI), namely, a higher Θ increases the volatility of the unspanned components of the SDFs but lowers the covariance and, typically, lowers the correlation between the SDFs.

Our theory argues that a financial intermediary would have an incentive to privately negotiate contracts with investors that exploit the fact that domestic and foreign investors disagree on the valuations of securities outside the linear span of $\mathbf{R}_{t+1}$ (or $\mathbf{R}_{t+1}^\star$), if the potential reward-for-risk were to be high. The domestic and foreign SDFs become less correlated with larger discrepancy between the valuations of securities outside the linear span of $\mathbf{R}_{t+1}$ (or $\mathbf{R}_{t+1}^\star$).

In contrast, a theory with $\Theta = 0$ implies $m_{t+1}(\frac{S_{t+1}}{S_t}) = m^{\star}_{t+1}$, and domestic and foreign investors would place identical valuation on all Arrow-Debreu securities regardless of whether they trade. It also runs counter to the intuition that, if consumption growths are imperfectly correlated, then unspanned states that are relatively unpleasant (favorable) for domestic (foreign) investors would result in domestic investors placing greater marginal utility on them than foreign investors.

Third, germane to our identification strategy is the consensus that correlations between consumption growth and between dividend growth in industrialized countries are not high, while those for wealth growth are sizable, as shown by Bansal and Lundblad (2002) and Colacito and Croce (2011, 2013). Exploiting this link, together with an analysis of SDF volatilities and correlations, allows us to identify admissible values of Θ. Thus, our aim is to investigate whether the identified Θ for a country pair is compatible with the dimensions of consumption, wealth, and dividend data, along with the returns of a set of basis assets. A body of literature has sought to reconcile these aspects of the data, as laid out in Bansal and Lundblad (2002), Lewis (1996), Colacito and Croce (2011, 2013), Gavazzoni, Sambalaibat, and Telmer (2013), Colacito et al. (2015), Stathopoulos (2017), and Zhang (2015), among others.

Our identification strategy leads us to consider alternative specifications of the SDF, in which $R_{c,t+1}$, $R_{w,t+1}$, and $R_{d,t+1}$ denote, respectively, consumption growth (i.e., $\frac{c_{t+1}}{c_t}$), wealth growth (which we surrogate using real equity index returns), and dividend growth. Our treatment of the SDF specification with dividend growth is consistent with, among others, Wachter (2013).

$$\text{SDFs} = \begin{cases} m_{t+1} = \beta^{\theta}\, R_{c,t+1}^{-\theta/\psi}\, R_{w,t+1}^{\theta-1} \;\text{ and }\; m^{\star}_{t+1} = \beta^{\theta}\, R_{c^{\star},t+1}^{-\theta/\psi}\, R_{w^{\star},t+1}^{\theta-1}, \\ \qquad\qquad \text{where } \theta = \frac{1-\gamma}{1-\frac{1}{\psi}}, \;\; \text{(Epstein-Zin preferences)} \\ m_{t+1} = \beta R_{w,t+1}^{-\alpha} \text{ and } \; m^{\star}_{t+1} = \beta\, R_{w^{\star},t+1}^{-\alpha} \;\; \text{(Power utility over wealth)} \\ m_{t+1} = \beta R_{c,t+1}^{-\alpha} \text{ and } \; m^{\star}_{t+1} = \beta\, R_{c^{\star},t+1}^{-\alpha} \;\; \text{(Power utility over consumption)} \\ m_{t+1} = \beta R_{d,t+1}^{-\kappa} \text{ and } \; m^{\star}_{t+1} = \beta\, R_{d^{\star},t+1}^{-\kappa} \;\; \text{(Power utility over dividends)} \end{cases} \tag{31}$$

These SDF specifications define wealth-based, consumption-based, and dividend-based $\rho_{m,m^{\star}}$ and RSI and consequently address essential matters from the standpoint of our empirical conclusions.

In their analysis, Brandt, Cochrane, and Santa-Clara (2006) are, in effect, equating risk sharing with $\rho_{m,m^\star}$ (or RSI). As Colacito and Croce (2011) and Burnside and Graveline (2014) emphasize, risk sharing in international economies is not necessarily the same concept as the correlation between m and $m^\star$. For example, if there is consumption home bias, SDFs may be less than perfectly correlated, even if risk sharing is perfect. We study the implications of using both $\rho_{m,m^\star}$ and RSI to connect to a broad swath of the literature on international finance.

Building on the above discussions, we consider the following algorithm to identify Θ.

1. Start with a trial value of, for example, $\Theta = 0.01$ (i.e., $m_{t+1}(\frac{S_{t+1}}{S_t}) - m^\star_{t+1} \approx 0$, close to the baseline of complete markets), and solve Problem 1 using the returns $\mathbf{R}_{t+1}$ ($\mathbf{R}^\star_{t+1}$) of a set of basis assets. The output is ($\boldsymbol{\lambda}$, δ, d, $d^\star$) and, hence, m and $m^\star$.

2. Compute $\rho_{m,m^\star}$ (or RSI) based on the asset market view in Equation (29) and compare it with its counterpart computed based on a candidate model of SDF in Equation (31).

3. Iterate over the choice of Θ to minimize the discrepancy between the model-based $\rho_{m,m^\star}$ (or RSI), and the corresponding one constructed from asset returns.

The proposed methodology to identify Θ for country-specific pairs is, in part, an acknowledgement that Θ cannot be directly computed from asset return data alone, unless a stand is taken on the size of the unspanned components (i.e., $\mathrm{Var}[u]/\mathrm{Var}[m]$ or $\mathrm{Var}[u^\star]/\mathrm{Var}[m^\star]$). Hampering identification from the mean equation, we further note from Equation (27) that $\Theta\, \mathbb{E}_t[\delta \sqrt{S_{t+1}/S_t}] = 0$.

Both $\rho_{m,m^\star}$ and RSI appear to be attractive quantities for estimation purposes. First, both quantities are dimensionless. Second, we recognize that joint payouts contingent on country-specific equity (or bond) indexes do not exist. Hence, our methodology for estimating Θ is a practical one.

In the spirit of Hansen and Jagannathan (1991), we inquire whether the volatilities of the SDF pairs (m_{t+1}, $m^\star_{t+1}$), synthesized using the Θ estimates, are plausible, while being consistent with the empirical regularities of consumption-based (or wealth-based or dividend-based) $\rho_{m,m^\star}$ and RSI, and consistent with a lower bound on the covariance between the SDFs. Bansal and Lundblad (2002) deem wealth-based correlations to be a fundamental object in their quantifications. Additionally, we compute SDF correlations, and address the currency volatility puzzle in an incomplete markets setting. In so doing, we strive to bridge some empirical realities of international consumption,

wealth, dividend, and asset return data and learn about the structure of theoretically supportable $(m_{t+1}, m^{\star}_{t+1})$ pairs.

2.1.2 Discussion and rationale for the estimates of Θ. Operationalizing our algorithm, Table 1 presents a snapshot of the Θ estimates across all 45 country pairs (there are $\frac{1}{2} \times 10 \times 9 = 45$ pairs arising from the 10 countries), when $\mathbf{R}_{t+1}$ ($\mathbf{R}^{\star}_{t+1}$) contains *six* assets. We note that implementation with Epstein-Zin preferences requires inputs for γ, ψ, and β, which are taken from Colacito and Croce (2011, table 1). In contrast, implementation with other SDF specifications is, to first-order, free of parameterizations.[3]

With six assets, and our algorithm based on matching $\rho_{m,m^{\star}}$ and an SDF specification based on Epstein-Zin preferences (respectively, power utility over consumption), we observe values of Θ that have an average of 0.39 (0.68), a standard deviation of 0.09 (0.20), and 5th and 95th percentile values of 0.26 (0.45) and 0.54 (0.99), respectively. The estimates of Θ based on matching RSI are virtually identical to those from matching $\rho_{m,m^{\star}}$, which stems from the feature that the model-based $\text{Var}[m] \approx \text{Var}[m^{\star}]$.

Intuitively, $\Theta > 0$ implies that $\text{Var}[m_{t+1}] > \text{Var}[m_{\mathbf{z},t+1}]$ from Equation (28), where the latter represents the Hansen and Jagannathan (1991) lower volatility bound with nonnegativity. The additional key observation is that the estimates of Θ obtained with Epstein-Zin preferences are close to those obtained from using power utility over wealth but lower than those using power utility over dividends and lower than those using power utility over consumption. In our context, the quantitative conclusions resting on the SDFs of the Epstein-Zin preferences are important in light of the work of Colacito and Croce (2011), Bansal and Shaliastovich (2013), Colacito et al. (2015), and Zviadadze (2017), who show that recursive preferences stand out in reconciling prominent features of the data as well as international asset pricing puzzles.

Table 2 presents our estimates of Θ (matching $\rho_{m,m^{\star}}$) across each of the 45 country pairs using Epstein-Zin preferences in Equation (31). To establish the reported 95% lower and upper confidence intervals on Θ displayed in brackets, we randomly select, with replacement, raw asset returns and

[3]To show this, assume the same α in each economy (like in Backus and Smith 1993; Brandt, Cochrane, and Santa-Clara 2006). Consider the approximation $(c_{t+1}/c_t)^{-\alpha} = \exp(-\alpha \log(c_{t+1}/c_t)) \approx 1 - \alpha \log(c_{t+1}/c_t)$, so $\text{Var}[(c_{t+1}/c_t)^{-\alpha}] \approx \alpha^2 \text{Var}\left[\log(c_{t+1}/c_t)\right]$, $\text{Var}[(c^{\star}_{t+1}/c^{\star}_t)^{-\alpha}] \approx \alpha^2 \text{Var}\left[\log(c^{\star}_{t+1}/c^{\star}_t)\right]$, and $\text{Cov}[(c_{t+1}/c_t)^{-\alpha}, (c^{\star}_{t+1}/c^{\star}_t)^{-\alpha}] \approx \alpha^2 \text{Cov}[\log(c_{t+1}/c_t), \log(c^{\star}_{t+1}/c^{\star}_t)]$. Thus, the $\rho_{m,m^{\star}}$ and RSI can both be computed from the time-series of two log consumption growths, as α^2 cancels in the numerator and denominator.

recompute $\mathbf{Z}$. We solve again for $\boldsymbol{\lambda}$ via Equation (20), δ via Equation (21), and then (d, $d^\star$) via Equation (22). Then, we recompute m, $m^\star$, and consequently $\rho_{m,m^\star}$ in Equation (29). Finally, we recompute Θ using the algorithm in Section 2.1.1. We perform 5,000 bootstrap trials.

A particular observation is that our procedure is not tilted toward either low or high values of Θ with considerable cross-sectional dispersion. Pertinent to our methodology's rationale, the solution of $\Theta = 0$ is never attained and the average value of Θ is statistically distinct from zero. We reach this conclusion from three perspectives. First, we regress the 45 values of Θ onto a constant. The constant is positive and we reject the hypothesis of zero average Θ with a two-sided p-value of 0.00. Next, the two-sided p-values from a t-test, allowing for unequal variance, favors the same conclusion. Third, the 95% confidence intervals for Θ never traverse zero.

From Table 2 (maintaining the focus on results with Epstein-Zin preferences), we can make some other observations. For example, the AUD/NZD pair has the lowest estimated Θ of 0.25, whereas GER/SWI, FRA/GER, and NLG/GER manifest below average Θ estimates of 0.26, 0.26, and 0.27, respectively. On the other hand, there are some country pairs that manifest above average Θ estimates, notably, JPY/SWI, NZD/STG, and USD/JPY, and all exhibit Θ values exceeding 0.53.

How sensible are the values of Θ? Cochrane and Saá-Requejo (2000, p. 82) suggest eliminating good deals by ruling out Sharpe ratios greater than twice the Sharpe ratio available on a broad-based equity index. Their choice is not directly observable but is instead based on introspection and common sense. Taking our cue from them, we bridge implementation and theory by considering empirical analogs. For example, if one computes the reward-for-risk as the ratio of the mean to standard deviation of foreign equity index returns over and above domestic equity index returns (the risky benchmark) across all currency pairs, the most favorable annualized reward-for-risk is 0.26 (NLG over and above JPY; details in Table Internet-II). Twice (using the analog of Cochrane and Saá-Requejo 2000) this figure is 0.52. In Section 3.5, we expand on this analysis and explore investment opportunities with relatively high Sharpe ratios that may be alluring to investors but are potentially short-lived.

The message is that there exist values of Θ not equal to zero (thus, refuting the notion of complete markets), such that one can reconcile the evidence on consumption-based, wealth-based,

and dividend-based correlations and risk sharing with their counterparts computed from asset return data. Further, the values of Θ are, from several different angles, economically motivated and anchored around sensible benchmarks.

2.2 Gauging empirical plausibility: Volatilities and correlations among SDFs

With nonzero values of Θ, the first important question is: how do the SDF volatilities in our incomplete markets setting compare to the minimum second-moment SDF volatilities often used in the literature? Next, an equally relevant question is: what are the derived magnitudes of the correlation between the SDFs? Third, what is the size of the unspanned components of the SDFs? Fourth, and finally, can our characterization of $(m_{t+1},\ m^{\star}_{t+1})$ pairs help to improve the understanding of economic phenomena, as seen from the vantage point of the currency volatility puzzle?

To address the aforementioned questions, we compute the volatilities of m and $m^{\star}$, denoted by $\sigma[m]$ and $\sigma[m^{\star}]$ (via Equation (28)), together with the pairwise correlation between m and $m^{\star}$ (i.e., $\rho_{m,m^{\star}}$ via Equation (29)). The results for each of the 45 country pairs are reported in Table 2, and the snapshot is in Table 3.

In order to interpret these numbers, we consider the conceptually important benchmark of $\Theta = 0.01$, which corresponds to a situation in which $\frac{\mathrm{Var}[u]}{\mathrm{Var}[m]}$ and $\frac{\mathrm{Var}[u^{\star}]}{\mathrm{Var}[m^{\star}]}$ are virtually zero.

	Avg.	SD	Min.	Max.	5th	25th	50th	75th	95th
$\sigma[m]\big\vert_{\Theta=0.01}$	43	7	24	53	33	37	44	49	51
$\sigma[m^{\star}]\big\vert_{\Theta=0.01}$	45	6	30	54	36	39	45	50	52
$\rho_{m,m^{\star}}\big\vert_{\Theta=0.01}$	0.96	0.02	0.86	1.0	0.93	0.95	0.97	0.98	1.00

The values of $\rho_{m,m^{\star}}\big\vert_{\Theta=0.01}$ are between 0.86 and 1.00, establishing that, *in the absence of an unspanned* component of SDFs, the SDF correlations across the 45 pairs are universally high.

Our quantitative evaluation, reported in Tables 2 and 3, zeros in on two aspects of the economic environment. First, using the estimated Θ increases the volatility of m and $m^{\star}$, enabling concrete insights about the nature of the unspanned component of SDFs that are supportable in our system of incomplete markets. Second, our analysis quantifies the values of $\rho_{m,m^{\star}}$ that are compatible

with the lower bound on $\text{Cov}[m, m^\star]$, that is, $\inf_{m,m^\star} \mathbb{E}[m\, m^\star]$. In particular, the decline in $\rho_{m,m^\star}$ is pronounced, consistent with $\text{Cov}[u, u^\star]$ becoming more negative. The average $\rho_{m,m^\star}$ is 0.66 (the 5th and 95th percentile values are 0.46 and 0.85, respectively), which contrasts the average value of 0.96 (the 5th and 95th percentile values are 0.93 and 1.00, respectively) when $\Theta = 0.01$.

Additionally, our framework provides guidance on the size of the country-specific unspanned components, computed as $\frac{\text{Var}[u]}{\text{Var}[m]}$ and $\frac{\text{Var}[u^\star]}{\text{Var}[m^\star]}$, thus, taking us beyond Hansen and Jagannathan (1991), who provide insights about the spanned components of the SDFs. Our exercises elicit the observation that the size of $\frac{\text{Var}[u]}{\text{Var}[m]}$ has a minimum value of 0.08 and a maximum value of 0.28, with an average of 0.17 (as presented in panel C of Table 3). The overall implication is that relaxing $\Theta = 0$ (or allowing $m(\frac{S_{t+1}}{S_t}) - m^\star \neq 0$), translates into less correlated m and $m^\star$ pairs and sizable $\frac{\text{Var}[u]}{\text{Var}[m]}$ and $\frac{\text{Var}[u^\star]}{\text{Var}[m^\star]}$, culminating into a consistency between the correlation computed from consumption and wealth data versus the one inferred from asset return data.

Expanding on these themes, the average values of $\sigma[m]$ and $\sigma[m^\star]$, consistent with our lower bound on covariances, across the 45 country pairs, are 48% and 51%. These values are higher than the corresponding values when $\Theta = 0.01$ - as they must be (by Equation (28)) - but the new insight is that none of the SDF volatilities is so high as to be implausible. Specifically, the reported values of $\sigma[m]$ and $\sigma[m^\star]$ are not out of line with textbook benchmarks (e.g., Cochrane 2005, p. 456).[4]

Finally, what is the impact of expanding (reducing) the number of primitive assets in $\mathbf{R}_{t+1}$ ($\mathbf{R}^\star_{t+1}$)? We experimented with several choices, and our conclusions do not change. For example, we increased the dimensionality of $\mathbf{R}_{t+1}$ ($\mathbf{R}^\star_{t+1}$) to *seven* by augmenting the set of asset returns to include the S&P commodity index. At the heart of our finding, as seen from Table 3, is that the distribution of $\frac{\text{Var}[u]}{\text{Var}[m]}$ barely changed with only a small effect on the average value of Θ.

3 Complementary Empirical Evidence

Complementing our theoretical and empirical inquiry, we provide further evidence from six angles.

[4]Table Internet-III presents the counterpart to Table 2 that uses nominal quantities in the estimations. Focusing on power utility over nominal wealth growth (see Equation (31)), the results show that our conclusions regarding the estimates of Θ and the size of the unspanned components are robust to inflation considerations.

3.1 Is the baseline of complete markets rejected with currency option returns?

Our results so far are predicated on using primitive assets in the estimation of Θ. Departing from complete markets, we have endeavored to construct $(m_{t+1}, m^{\star}_{t+1})$ pairs by solving an optimization problem that minimizes the covariance between the domestic and foreign SDFs. Doing so rules out trades with unacceptably large reward-for-risk and enforces correct pricing of primitive assets and the nonnegativity of SDFs, while being consistent with correlations between macroeconomic quantities.

The featured $\mathbf{R}_{t+1}$ ($\mathbf{R}^{\star}_{t+1}$) with six primitive assets prompts us to study the implications of including convex payoffs. To do so, we consider an additional asset class in the form of currency volatility, which we mimic using returns of currency straddles, 25-delta currency strangles, and 10-delta currency strangles (e.g., Kosowski and Nefti 2015, chapter 11). For example, the gross return of a 25-delta strangle is computed as $(\max(S_{t+1}-K,0)+\max(K-S_{t+1},0))/(C_t[K]+P_t[K])$, where S_t/K (K/S_t) corresponds to the moneyness of a 25-delta call (put), and the call (put) price is denoted by $C_t[K]$ (respectively, $P_t[K]$). The results from traded currency volatility, with convex payoffs, can provide an additional perspective on assessing deviations from market completeness.

Our results in Table 4, based on returns of straddles and strangles across all G-10 currencies, affirm several features. Specifically, the Θ estimates are still far above zero (the not reported bootstrap evidence implies that the hypothesis of $\Theta = 0$ is rejected for all 45 pairs). Although including options helps to mitigate market incompleteness, the proportion of SDF variance attributable to the unspanned component is not reduced to zero. Moreover, incorporating the returns of an options combination strategy increases the volatility of SDFs (to about 80%). To guide intuition regarding these findings, Internet Appendix (Section II.C) explores the impact of correctly pricing assets with convex payoffs. This example economy, which isolates the effect of convex payoffs, illustrates that the SDF pairs become more correlated and exhibit higher volatility.

3.2 How does changing the investment opportunity sets affect Θ?

This subsection compares the information about the properties of SDFs contained in $\mathbf{R}_{t+1}$ ($\mathbf{R}^{\star}_{t+1}$) measured over different sample periods, in the context of our incomplete markets international economy. The work of Karolyi and Wu (forthcoming) studies the impact of changing investment

opportunity sets in global equity markets, depicting in particular its impact on the pricing of risks. Our motivation is that the return moments of the set of spanning assets are potentially time-varying, allowing us to distinguish the influences of changing investment opportunity sets. Central to our methodology is the question of whether $\Theta = 0$ is ever attained.

Recognizing the difficulty of modeling empirically the conditioning information available to investors, we consider both a rolling window scheme and an expanding window scheme. For example, we fix the length of the evaluation sample to 240 months in the rolling window scheme. To draw inferences about the properties of $(m_{t+1},\ m^{\star}_{t+1})$ pairs and the extent of deviations from market completeness, we use the algorithm of Section 2.1.1 to estimate Θ by matching the correlation of monthly wealth growth. Here we focus on wealth growth because, first, the values of Θ estimated with wealth growth are almost identical to those from Epstein-Zin (compare panels A and B of Table 1), and, second, wealth growth (unlike consumption growth) data are available at a monthly frequency. Equity-based wealth growth correlations convey important links in a system of economies, as articulated by, among others, Bansal and Lundblad (2002) and Zhang (2015).

The first rolling sample corresponds to returns $\mathbf{R}_{t+1}$ ($\mathbf{R}^{\star}_{t+1}$) constructed over 1975:01 to 1994:12. Discarding the initial 60 months and moving forward 60 months while maintaining the window of 240 months, we obtain another estimate of Θ over 1980:01 to 1999:12. This procedure is continued for 1985:01 to 2004:12, 1990:01 to 2009:12, and 1995:01 to 2014:12. The expanding window scheme provides an alternative (in estimating Θ) by adjusting the length of estimation window (in our case, a minimum (maximum) of 240 (480) months).

Table 5 reports the Θ estimates and enumerates the properties of $(m_{t+1},\ m^{\star}_{t+1})$ pairs based on all 45 country pairs using the rolling (respectively, expanding) scheme in panel A (panel B). While the Θ estimates and the volatility of the SDF pairs exhibit some variation, the notable observation is that the sizes of the unspanned components are fairly invariant over the rolling samples. Our results show that a higher volatility of the SDFs is associated with a higher Θ, and the estimates imply imperfect wealth growth correlations.

The further inference to draw is that international risk sharing or, more precisely, the SDF correlation implied by wealth growth (i.e., real equity index returns), has tended to increase a bit over the last 40 years. For example, the correlations $\rho_{m,m^{\star}}$ are always higher in the later samples

(i.e., 1990:01 to 2009:12 and 1995:01 to 2014:12) than in the earlier samples (i.e., 1975:01 to 1994:12 and 1980:01 to 1999:12). The documented increase in correlations is not accompanied by dramatic shifts in Θ estimates (all are between 0.36 and 0.58), although slightly higher values of $\rho_{m,m^\star}$ naturally translate, on average, into slightly lower values of Θ.

Just as Θ estimates are confined to a range through time, so too are SDF volatilities, with the latter a little higher than average in the rolling sample 1980:01 to 1999:12. One possible explanation is that this time period roughly coincides with the period when markets were rising the most (for example, the annualized Sharpe ratio on the MSCI world equity index over the period 1980:01 to 1999:12 was 0.56, the highest of the five rolling sample periods), and, hence, the reward-for-risk available to international investors would have been higher. The Hansen and Jagannathan (1991) bound shows that *minimum second-moment* SDF volatilities should be higher when the reward-for-risk is higher. Our analysis, working in incomplete international economies, synthesizes the volatilities of SDFs that incorporate unspanned components (i.e., are not minimum second-moment SDFs), but our results in Table 5 point in the same direction: higher reward-for-risk (in the form of higher estimates of Θ) may go hand in hand with higher SDF volatilities.

3.3 Using bond return correlations to identify Θ

If one recognizes that the length of the period for the (one period) SDFs m_{t+1} and $m^\star_{t+1}$ can be arbitrary, for instance, a month or n months, then the n-month bond prices admit the characterization $B_{t,n} \equiv \mathbb{E}_t[m_{t+n}]$ and $B^\star_{t,n} \equiv \mathbb{E}_t[m^\star_{t+n}]$. However, in light of the additive form of the SDFs in Equations (23)–(25) and $\mathbb{E}_t[u^\star] = 0$, $\mathbb{E}_t[u] = 0$, the bond prices, and therefore bond yields, would depend only on the spanned components of the SDFs but are detached from the unspanned components (and, thus from Θ), which prevents identification from the correlation of bond yields.

To provide alternative estimates of Θ that build on our results based on consumption, dividends, and equity data, we exploit a result in Alvarez and Jermann (2005) and Hansen and Scheinkman (2009) that uniquely decomposes the SDF into a martingale component and a component that is

the inverse of the return of a long-term discount bond. Under the Alvarez and Jermann (2005) setup, and assuming that the martingale component of the SDF is unity, we can express

$$m_{t+1} = \frac{1}{R_{t+1,\infty}} \qquad \text{and} \qquad m^{\star}_{t+1} = \frac{1}{R^{\star}_{t+1,\infty}}, \tag{32}$$

where $R_{t+1,\infty} = \lim_{n\to\infty} R_{t+1,n}$ is the return of a n-month domestic discount bond in the limit of large n, and likewise for $R^{\star}_{t+1,\infty}$. Going from theory to implementation, we employ the returns of a bond with a constant maturity of 120 months and consequently match the correlation $\frac{\mathrm{Cov}[\frac{1}{R_{t+1,\infty}}, \frac{1}{R^{\star}_{t+1,\infty}}]}{(\mathrm{Var}[\frac{1}{R_{t+1,\infty}}]\,\mathrm{Var}[\frac{1}{R^{\star}_{t+1,\infty}}])^{1/2}}$ in our estimation algorithm.

What do we learn when correlation between long-term bond returns is matched (in essence, achieving consistency with the comovement in the evolution of the yield curves)? Imperative to Table 6 is a finding that Θ estimates are slightly *higher* than their values using the SDFs based on power utility over wealth in panel B of Table 1, but they are lower than those from power utility over consumption and power utility over dividends. These Θ estimates generate SDF volatilities $\sigma[m]$ and $\sigma[m^{\star}]$ and sizes of the unspanned components that are slightly higher but imply reduced SDF correlations. Further, the results from Table 6 manifest slight departures from those under Epstein-Zin preferences, even though our estimates are guided by the constancy of the martingale components of SDFs in Equation (32).

Easing robustness concerns, each of our estimation approaches agrees in suggesting that Θ is strictly positive.

3.4 Implications of incomplete markets and framing of volatility puzzle

We now consider the volatility/risk sharing puzzle, and ask whether this puzzle is amenable to reconciliation through a route in which markets are deemed to be incomplete (in line with our evidence that indicates Θ is strictly positive). The volatility/risk sharing puzzle, as framed and exposed in Brandt, Cochrane, and Santa-Clara (2006), is that when $\log(\frac{S_{t+1}}{S_t}) = \log(m^{\star}_{t+1}) - \log(m_{t+1})$ holds *state-by-state in complete markets*, one can rationalize the relatively low volatility of (log) exchange rate growth observed in the data only if $\log(m_{t+1})$ and $\log(m^{\star}_{t+1})$ are highly correlated or if risk sharing defined in Equation (30) is high.

Matters are conceptually different in incomplete markets. Some SDFs satisfy $m_{t+1}(\frac{S_{t+1}}{S_t}) - m^\star_{t+1} = 0$ or $m_{t+1} - m^\star_{t+1}/(\frac{S_{t+1}}{S_t}) = 0$, while others do not. In particular, under $\mathbb{E}_t[m_{t+1}(\frac{S_{t+1}}{S_t}) - m^\star_{t+1}] = 0$, there is no longer a tractable functional relationship that allows one to take logs and apply the operations of expectation or variances. Instead, we take the exchange rate dynamics, along with other asset returns, as fixed by the data, and solve for the SDFs. Thus, there is no unique mapping between the triplet $(m_{t+1}, m^\star_{t+1}, \frac{S_{t+1}}{S_t})$ in incomplete markets, suggesting that the volatility puzzle may not be well posed.

To examine empirical plausibility, and at the same time elaborate on the consistency of our approach with the variance of exchange rate growth, we first propose a result that holds under all martingale measures, regardless of whether markets are complete or incomplete.

Let $\mu^{\text{fx}}_t \equiv \mathbb{E}_t[r^{\text{fx}}_{t+1}]$ with $r^{\text{fx}}_{t+1} \equiv \frac{S_{t+1}}{S_t} - 1$, where $F_t \equiv S_t R_f / R^\star_f$ is the forward exchange rate. Define $\text{vrp}^{\text{fx}}_{t+1} \equiv -\text{Cov}_t[\frac{m_{t+1}}{\mathbb{E}_t[m_{t+1}]}, (r^{\text{fx}}_{t+1} - \mathbb{E}_t[r^{\text{fx}}_{t+1}])^2] + (\mathbb{E}_t[r^{\text{fx}}_{t+1}] - (\frac{F_t}{S_t} - 1))^2$ as the currency variance risk premium (see Equation (D7)). We now state the following:

Result 3 *Each m_{t+1} satisfies*

$$\underbrace{\mathbb{E}_t[(r^{\text{fx}}_{t+1} - \mathbb{E}_t[r^{\text{fx}}_{t+1}])^2]}_{\text{currency variance}} \;=\; \text{vrp}^{\text{fx}}_{t+1} \;+\; \underbrace{\mathbb{E}_t[\frac{m_{t+1}}{\mathbb{E}_t[m_{t+1}]}(r^{\text{fx}}_{t+1})^2] - (\frac{F_t}{S_t} - 1)^2}_{\text{risk-neutral currency variance}}. \tag{33}$$

Deviations from market completeness, Θ, can be relevant for the variance risk premium ($\text{vrp}^{\text{fx}}_{t+1}$) whenever $\text{Cov}_t[u_{t+1}, (r^{\text{fx}}_{t+1} - \mathbb{E}_t[r^{\text{fx}}_{t+1}])^2] = \frac{1}{2} d\,\Theta \mathbb{E}_t[\delta\,(S_{t+1}/S_t)^{3/2}] \neq 0$.

Proof: See Appendix D. ■

With m_{t+1} synthesized from Result 2 (and focusing on Epstein-Zin preferences in our algorithm to estimate Θ), we compute, for each currency pair, the model risk-neutral variance $\text{vq}^2 \equiv \mathbb{E}[\frac{m_{t+1}}{\mathbb{E}[m_{t+1}]}(r^{\text{fx}}_{t+1})^2] - (\mathbb{E}[\frac{m_{t+1}}{\mathbb{E}[m_{t+1}]} r^{\text{fx}}_{t+1}])^2$, recognizing $\mathbb{E}_t[\frac{m_{t+1}}{\mathbb{E}_t[m_{t+1}]} r^{\text{fx}}_{t+1}] = \frac{F_t}{S_t} - 1$. We also compute the data-based currency variance as $\text{vp}^2 \equiv \mathbb{E}[(r^{\text{fx}}_{t+1} - \mathbb{E}[r^{\text{fx}}_{t+1}])^2]$. Then we convert Equation (33) into a testing equation by considering the cross-sectional regression

$$100 \times \log(\text{vp}^2_i / \text{vq}^2_i) \;=\; \Omega_0 \;+\; e_i, \quad \text{for } i = 1, \ldots, \mathcal{I}. \tag{34}$$

The coefficient Ω_0 can be interpreted as the unconditional (log) variance risk premium, and we test the null hypothesis of $\Omega_0 = 0$ versus the alternative of $\Omega_0 < 0$. The advantage of considering the log relative $\log(\mathrm{vp}_i^2/\mathrm{vq}_i^2)$ is that the average (log) volatility risk premium is half of the (log) variance risk premium.

Table 7 reports the OLS regression results with standard errors corrected for heteroscedasticity (for each of USD, JPY, and SWI as the domestic currency against the remaining nine currencies). Two conclusions can be drawn from our analysis. First, as reflected in a small estimate of Ω_0, the average risk-neutral currency variances generated in our incomplete markets setting appear anchored to the average unconditional currency variances. Second, the finding $\Omega_0 < 0$ points to the presence of a negative (average) currency variance risk premium of -1.73%, and is statistically significant. The latter is consistent with the evidence from currency options markets (e.g., Ammann and Buesser 2013, table 1). Taken together, the SDFs constructed based on Result 2 uncover a finding that the average currency variance risk premium implied by the model is negative, even though we did not incorporate returns of currency options in their construction.

3.5 Identifying good deals that do not persist

The centerpiece of our framework is the constraint $\mathbb{E}[(y - y^\star)^2] \leq \Theta^2$, which suggests that one should not be concerned with trades in international economies that are outrageously good. In this subsection, we explore two related questions: Could one identify opportunities that are sufficiently salient to induce trade based on a relatively high reward-for-risk? Do such trading opportunities display a propensity to fall out of favor with investors?

To this purpose, we showcase three strategies that have a global element and that seek to exploit opportunities perceived to be highly profitable. To establish statistical significance of average excess returns and of Sharpe ratios, we report the 95% lower and upper confidence intervals based on the stationary bootstrap of Politis and Romano (1994), where the block size is based on Politis and White (2004).

3.5.1 Yen-Icelandic carry trade. Prominently featured in the financial press, the icelandic carry trade was meant to exploit a large interest rate differential and is described as follows:

"This phenomenon was known as the search for yield and resulted in a flow of funds from Tokyo to Reykjavik. Iceland had relatively high interest rates, so investors borrowed heavily in Japanese yen and bought Icelandic bonds. Money flooded into Iceland and its big banks borrowed $120bn on the international markets — six times the size of the country's GDP.... The financial crash has put paid to Iceland's get rich quick scheme — known as the yen carry trade — and left Iceland saddled with debts it has no hope of paying without impoverishing its people for decades to come." (Elliott 2009)

Unpacking the above, we consider a U.S. investor who uses Japanese yen as the funding currency and Icelandic krona as the investment currency. Following Backus, Foresi, and Telmer (2001), Burnside, Eichenbaum, Kleshchelski, and Rebelo (2011), Lustig, Roussanov, and Verdelhan (2011), and Bakshi and Panayotov (2013, equations (1) and (2)), the excess return of the strategy (committing half a dollar to each leg) is

$$\mathrm{er}_{t+1} \equiv \frac{1}{2}\underbrace{\left(\frac{S_{t+1}^{\mathrm{bid},\$|\mathrm{krona}}}{F_t^{\mathrm{ask},\$|\mathrm{krona}}} - 1\right)}_{\text{long leg}} + \frac{1}{2}\underbrace{\left(1 - \frac{S_{t+1}^{\mathrm{ask},\$|¥}}{F_t^{\mathrm{bid},\$|¥}}\right)}_{\text{short leg}}. \tag{35}$$

In Equation (35), S_{t+1}^{bid} (S_{t+1}^{ask}) is the bid (ask) spot exchange rate at the end of month $t+1$ and F_t^{bid} (F_t^{ask}) is the one-month forward exchange rate with the U.S. dollar as the domestic currency.

Table 8 shows that prior to the financial crisis (2004:03 to 2007:12; 2004:03 is the earliest date the krona forward rates are available), the yen-Icelandic carry trade thrived with an annualized average er_{t+1} of 7.41% and an associated Sharpe ratio of 1.13. The reported 95% bootstrap confidence intervals for average er_{t+1} do not straddle zero, and, hence, the average er_{t+1} is reliably positive.

The carry trade soured thereafter, as the Icelandic krona devalued from 64 to 130 krona per U.S. dollar, eroding its profitably, with an average er_{t+1} of -11.26% over 2008:01 to 2011:12. While the trade reverted to profitability over 2012:01 to 2015:12, with statistically significant Sharpe ratio of 0.93, the average er_{t+1} over 2008:01 to 2015:12 is negative but statistically insignificant.

3.5.2 Reward-for-risk of writing out-of-the-money index puts by global investors. Next, we study the reward-for-risk inherent in writing deep out-of-the-money S&P 500 index puts, initiated from the perspective of a G-10 currency, over the sample period of 1990:01 to 2015:12.

The excess returns of writing a put, denoted by $\mathrm{er}_{t+1}[K]$, with strike price $K \approx A_t \times e^{-0.05}$, index level A_t, put premium $P_t[K]$, and U.S. interest rate r_t^{usd}, is (see Internet Appendix (Section IV))

$$\mathrm{er}_{t+1}[K] \quad = \quad \frac{\frac{-\max(K-A_{t+1},0)}{S_{t+1}^{\mathrm{bid}}}}{\frac{\mathcal{C}_t^{\bullet}}{S_t^{\mathrm{bid}}}} + \frac{(1+r_t^{\mathrm{usd}})P_t[K]\frac{1}{F_t^{\mathrm{ask}}}}{\frac{\mathcal{C}_t^{\bullet}}{S_t^{\mathrm{bid}}}}, \tag{36}$$

which incorporates a role for collateral $\mathcal{C}_t^{\bullet} \equiv P_t[K] + \max(0.1K, 0.15A_t - \max(A_t - K, 0))$, as required by the CBOE (2000, p. 22), and as considered by Jurek and Stafford (2015, Section II.A). Our calculations assume that the investor provides collateral, equivalent to $\mathcal{C}_t^{\bullet}/S_t^{\mathrm{bid}}$ in a G-10 currency (e.g., the Australian dollar). Hence, $\mathrm{er}_{t+1}[K]$ is measured in the same G-10 currency.

Table 9 shows that prospects for selling equity tail events were favorable from 1990:01 to 2007:12, resulting in (annualized) average excess returns between 33.3% to 37.9% and Sharpe ratios between 0.72 to 1.10. In the vein of our view that promising investment opportunities are not assured, the trade languished over the next four years over 2008:01 to 2011:12 with statistically insignificant excess returns.

In the absence of large equity market movements to the downside over 2012:01 to 2015:12, the trade yielded high average excess returns (about 42%), low standard deviations (about 14%), and enviable Sharpe ratios (about 3.0). Thus, our evidence indicates that the changing macroeconomic landscape can offer investible opportunities with sizable reward-for-risk, which prospectively appear, abruptly disappear, and even reemerge.

3.5.3 The short volatility trade initiated from the perspective of a G-10 currency. The short volatility trade provides a way to extract the risk premium paid by investors that dislike equity market volatility. The popularity of the short trade also stems from the contango feature prevalent in VIX futures. Our focus is on documenting the excess returns of shorting volatility when the trade is initiated from the perspective of a G-10 currency.

The mechanics of the trade involves an investor who commits to a fully collateralized position, equivalent to V_t/S_t^{bid} in a G-10 currency (e.g., the Australian dollar), where V_t corresponds to the

VIX futures price in U.S. dollars at the end of month t. It can be shown that the monthly excess return, denoted by er_{t+1}, can be computed as (see Online Appendix V)

$$\text{er}_{t+1} = \begin{cases} -\{\frac{V_{t+1}}{V_t} - 1\}\frac{S_t^{\text{bid}}}{S_{t+1}^{\text{ask}}} & \text{If } V_{t+1} < V_t, \\ -\{\frac{V_{t+1}}{V_t} - 1\}\frac{S_t^{\text{bid}}}{S_{t+1}^{\text{bid}}} & \text{If } V_{t+1} > V_t. \end{cases} \tag{37}$$

The specific VIX futures contract we employ is the one with the second-nearest maturity and avoids rollover positions. The bifurcated nature of the excess returns reflects the feature that U.S. dollars must be bought (sold) in the event of a loss (gain) upon closing the futures contract.

Our results in Table 10 illustrate that the trade was profitable across all the G-10 currencies over the 2004:03 to 2007:12 sample period (the VIX futures started in March 2004, and the data are from the CBOE historical files, the details of which are in the note in Table 10). For example, the trade initiated in Japanese yen has an annualized average er_{t+1} of 53.56%, lower and upper bootstrap confidence intervals that do not contain zero, and a sample Sharpe ratio of 1.33. In contrast, our exercises indicate that the trade failed to produce statistically significant er_{t+1} and Sharpe ratios over subsequent samples of 2008:01 to 2015:12, 2008:01 to 2011:12, and 2012:01 to 2015:12, suggesting that the attractiveness of this trade was short-lived.

3.6 Evidence on market incompleteness from emerging economies

In this subsection, we return to the issue of market incompleteness. However, now we step outside of the set of industrialized economies and examine the data from emerging economies.

We focus on the following sample of nine emerging economies starting in January 1995 (for which we could collect data on inflation, equity market indices, and interest rates): BRR (Brazil), CHP (Chile), COP (Colombia), MXP (Mexico), MYR (Malaysia), PHP (Philippines), PLZ (Poland), THB (Thailand), and TWD (Taiwan). These economies exhibit varying degrees of capital controls, development of financial markets, and impediments to cross-border investments.

The setting of emerging economies can be revealing about two questions: Is the market incompleteness parameter, Θ, higher (or lower), on average, for emerging economies? Are emerging economies associated with a larger (or smaller) size of the unspanned component of the SDF?

To answer these questions, we use the United States as the benchmark foreign economy and consider an algorithm to estimate Θ that minimizes the discrepancy between model and observed $\rho_{m,m^\star}$, when (m, $m^\star$) are based on power utility over real wealth. To ease comparisons with industrialized economies, we stay with six assets: the U.S. risk-free bond, the U.S. MSCI equity index (total return), the risk-free bond and the MSCI equity index (total return) of the relevant emerging economy, the U.S. 30-year Treasury bond, and the MSCI world equity index (total return).

Table 11 reports the estimates of Θ, lower and upper bootstrap confidence intervals, the correlation between the SDFs ($\rho_{m,m^\star}$), the size of the unspanned component of the SDF ($\frac{\mathrm{Var}[u]}{\mathrm{Var}[m]}$), and the SDF volatility. The central finding is that the estimates of Θ and the size of the unspanned components are 1.6 times as high, on average, in emerging economies in comparison to the industrialized counterparts (AUD, NZD, STG, FRA, CAD, NLG, GER, JPY, SFR, all against the United States; sample period matched). For example, BRR/USD (respectively, MYR/USD, PHP/USD) manifests a Θ of 0.68 (1.05, 0.74), and the size of the unspanned component of 0.13 (respectively, 0.36, 0.31). Thus, our sample of emerging economies reflects more pronounced levels of market incompleteness.

4 Conclusions

We present a framework for characterizing domestic and foreign stochastic discount factor (SDF) pairs in international economies and incomplete markets, with novel ingredients. Importantly, we do not assume that exchange rate growth equals the ratio of SDFs. Moreover, we develop a restriction that precludes "good deals" in international economies with incomplete markets. We show that ruling out good deals — the possibility to form a portfolio with an implausibly high reward-for-risk — places an upper bound on the dispersion of the domestic and foreign SDFs. A derived feature of our model is an additive form of the SDFs in which the SDFs are analytically decomposed into their spanned and unspanned components, and a limited correlation between the domestic and foreign SDFs can arise when the country-specific unspanned components are negatively correlated.

At the core of our analysis is the market incompleteness parameter Θ that quantifies the upper bound on the dispersion of the SDFs. We consider an algorithm to estimate the market incompleteness parameter from data on consumption growth, wealth growth, dividend growth, in conjunction

with asset return data. The resultant framework of incomplete markets is both tractable and versatile: It can be aligned to patterns of limited international risk sharing, accommodates a relatively large difference between the unspanned components of the SDFs, and realistically models the volatility of the SDFs and their correlations.

Additionally, we show that our incomplete markets approach is useful for thinking about international finance puzzles. For example, we show that it is possible to generate limited correlations between the domestic and foreign SDFs. This feature of our model merits attention in the context of the volatility/risk sharing puzzle, and offers differentiation from Brandt, Cochrane, and Santa-Clara (2006), who argue that the correlations between the SDFs and the level of international risk sharing, imputed from asset return data, must be high.

Our empirical investigation undermines the notion that markets are complete. Moreover, our paper supports the view that understanding market incompleteness is crucial for both theorists and empiricists. For theoreticians, it provides a way forward for the specification and estimation of SDFs in international economies, and for the enrichment of international macro-finance models. For empiricists, it offers a lens through which to view a range of economic phenomena and, in particular, to dissect the mechanism by which risks are shared – or not shared – across international borders.

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A Appendix A: Proof of Result 1 that $\mathbb{E}[(y - y^\star)^2] \leq \Theta^2$

We derive a restriction of the form $\mathbb{E}[(y - y^\star)^2] \leq \Theta^2$ for some constant Θ, where $y = m\sqrt{S_{t+1}/S_t}$ and $y^\star = m^\star/\sqrt{S_{t+1}/S_t}$. In incomplete international economies, this restriction is equivalent to placing an economically motivated bound on the differences in valuation of securities that are *not* in the linear span of $\mathbf{R}$ (or $\mathbf{R}^\star$).

In the treatment that follows, we sometimes omit time subscripts for brevity.

To prove Result 1, we consider a candidate solution (project y and $y^\star$ onto the space of gross returns $\mathbf{Z} = \mathbf{R}/\sqrt{S_{t+1}/S_t} = \sqrt{S_{t+1}/S_t}\,\mathbf{R}^\star$):

$$y = y_{\mathbf{z}} \;+\; \frac{1}{2}\,q_0\,\delta \quad \text{and} \quad y^\star = y_{\mathbf{z}} \;+\; \frac{1}{2}\,q_0^\star\,\delta, \tag{A1}$$

where q_0 and $q_0^\star$ are constant scalars and $y_{\mathbf{z}}$ and δ are random variables that satisfy

$$\mathbb{E}_t[y_{\mathbf{z}}\,\mathbf{Z}] = 1, \quad \mathbb{E}_t[y_{\mathbf{z}}\,\delta] = 0, \quad \mathbb{E}_t[\delta\,\mathbf{Z}] = 0 \;\text{ for each element of } \mathbf{Z}, \quad \text{and} \quad \mathbb{E}_t[\delta^2] = 1. \tag{A2}$$

The decomposition in Equation (A1) breaks y and $y^\star$ into two components. The first component, $y_{\mathbf{z}} = \mathbf{1}'\left(\mathbb{E}_t[\mathbf{Z}\mathbf{Z}']\right)^{-1}\mathbf{Z}$, can be interpreted as the minimum second-moment SDF in the hypothetical economy in which gross returns are $\mathbf{Z}$. The second components, $\frac{1}{2}q_0\,\delta$ and $\frac{1}{2}q_0^\star\,\delta$, are orthogonal to $\mathbf{Z}$, where δ is normalized to have a second-moment equal to unity.

To illustrate that domestic and foreign investors disagree, in general, when $q_0 \neq q_0^\star$, on their valuations of securities outside the linear span of $\mathbf{R}$ (or $\mathbf{R}^\star$), consider, for example, a synthetic security on the product of two random variables

$$\text{that pays } \delta\,\sqrt{S_{t+1}/S_t} \text{ units of domestic currency, at time } t+1. \tag{A3}$$

This would be privately valued, in domestic currency, at time t, at

$$\mathbb{E}_t[m\,\delta\,\sqrt{S_{t+1}/S_t}] = \mathbb{E}_t[y\,\delta] = \frac{1}{2}q_0 \text{ by domestic investors, and at} \tag{A4}$$

$$\mathbb{E}_t[m^\star\,\delta\,(\sqrt{S_{t+1}/S_t})\frac{S_t}{S_{t+1}}] = \mathbb{E}_t[y^\star\,\delta] = \frac{1}{2}q_0^\star \text{ by foreign investors.} \tag{A5}$$

If q_0 were to equal $q_0^\star$, these valuations would be the same, but $q_0 = q_0^\star$ also implies (from Equation (A1)) that (1) $y = y^\star$ and hence, (2) $m(\frac{S_{t+1}}{S_t}) - m^\star = 0$.

The discrepancy between valuations of the synthetic security is greater when $|q_0 - q_0^\star|$ is larger, which implies that $|y - y^\star|$ is larger and $|m(\frac{S_{t+1}}{S_t}) - m^\star|$ is larger. This is a situation that financial intermediaries may, potentially, wish to exploit, and they can do this by creating a synthetic security that offers payoffs outside the linear span of $\mathbf{R}$ (or $\mathbf{R}^\star$).

The larger $|q_0 - q_0^\star|$ is, the greater is the potential profit for financial intermediaries. Hence, we study the consequences of a class of "good deals" characterized by

$$q_0 \;=\; -\,q_0^\star \;\equiv\; q \;\neq\; 0. \tag{A6}$$

Specifically, Equation (A6), in conjunction with Equation (A1), translates into two restrictions on y and $y^\star$:

$$\mathbb{E}_t[y - y^\star] \;=\; \frac{1}{2}(q_0 - q_0^\star)\,\mathbb{E}_t[\delta] \;=\; q\,\mathbb{E}_t[\delta], \tag{A7}$$

$$\mathbb{E}_t[(y - y^\star)^2] \;=\; \frac{1}{4}(q_0 - q_0^\star)^2\,\mathbb{E}_t[\delta^2] \;=\; q^2. \tag{A8}$$

We use Equations (A7) and (A8) to rule out implausibly high reward-for-risk strategies in the international economy with incomplete markets.

Consider now a financial intermediary that evaluates the *possibility* of privately negotiating a contract between itself and a domestic investor. We assume that the possible opportunity to enter into this private contract does not materially alter m, $m^\star$, $\mathbf{R}$, or $\mathbf{R}^\star$.

If entered into, the private contract with the domestic investor would require the investor to buy a synthetic security with payoff $x[\mathbf{Z}, S_{t+1}]$ in units of domestic currency, at time $t+1$, from the financial intermediary, where (again, we consider a security with a product payoff)

$$x[\mathbf{Z}, S_{t+1}] \;=\; \sqrt{S_{t+1}/S_t}\,\boldsymbol{w}'\mathbf{Z} \;+\; \delta\,\sqrt{S_{t+1}}. \tag{A9}$$

Here, $\boldsymbol{w}$ is an N-dimensional vector of portfolio weights in the traded assets assumed to be of the form

$$\boldsymbol{w} \;=\; -\frac{1}{2}(q\sqrt{S_t})\boldsymbol{v}, \qquad \text{where we postulate} \quad \boldsymbol{v}'\mathbf{1} = 1. \tag{A10}$$

The domestic investor computes the value, at time t, in units of domestic currency, of this synthetic security as

$$\begin{aligned}
\mathbb{E}_t\left[m\left(\sqrt{S_{t+1}/S_t}\,\boldsymbol{w}'\mathbf{Z} + \delta\,\sqrt{S_{t+1}}\right)\right] &= \mathbb{E}_t[m\left(\boldsymbol{w}'\mathbf{R}\right)] \;+\; \mathbb{E}_t[(y_{\mathbf{z}} + \frac{1}{2}q\,\delta)\delta\sqrt{S_t}], \\
&= \boldsymbol{w}'\mathbf{1} \;+\; \frac{1}{2}q\sqrt{S_t}, \\
&= 0. \qquad \text{(since } \boldsymbol{w} = -\frac{1}{2}(q\sqrt{S_t})\boldsymbol{v}, \;\; \text{from (A10))}
\end{aligned} \tag{A11}$$

Thus, the cash flow postulated in Equation (A9) can be synthesized at zero cost (i.e., at time t, the domestic investor values the time $t+1$ cash flow $x[\mathbf{Z}, S_{t+1}]$ at exactly zero).

If the financial intermediary were to enter into this private contract, it would have a short exposure to the cash flow $x[\mathbf{Z}, S_{t+1}]$ at time $t+1$. Substituting $\boldsymbol{w} = -\frac{1}{2}(q\sqrt{S_t})\boldsymbol{v}$ into Equation (A9),

$$X \;\equiv\; -x[\mathbf{Z}, S_{t+1}] = \frac{1}{2}(q\,\sqrt{S_t})\boldsymbol{v}'\mathbf{R} \;-\; \delta\,\sqrt{S_{t+1}}, \qquad \text{(in units of domestic currency)} \tag{A12}$$

$$= \frac{1}{2}q\,\boldsymbol{v}'\mathbf{Z} \;-\; \delta, \qquad \text{(in currency units of the hypothetical economy, i.e.,} \sqrt{S_{t+1}}) \tag{A13}$$

where in moving from Equation (A12) to (A13), we have divided by $\sqrt{S_{t+1}}$ because, since there are, at time $t+1$, S_{t+1} units of domestic currency per unit of foreign currency, there are $\sqrt{S_{t+1}}$ units of domestic currency per unit of currency of the hypothetical economy (using Equation (10)).

The financial intermediary faces a trade-off between the risks and rewards inherent in the cash flow X. We evaluate this trade-off in the currency units of the hypothetical economy to emphasize symmetry (without repeating the cash flow calculations in different currency units). Then

$$\mathbb{E}_t[X] \;=\; \frac{1}{2}q\,\mathbb{E}_t[\boldsymbol{v}'\mathbf{Z}] \;-\; \mathbb{E}_t[\delta], \tag{A14}$$

$$\begin{aligned}
\mathbb{E}_t[(X - \mathbb{E}_t[X])^2] &= \mathrm{Var}_t[\frac{1}{2}q\,\boldsymbol{v}'\mathbf{Z}] + \mathbb{E}_t[\delta^2] - (\mathbb{E}_t[\delta])^2 \;-\; \underbrace{q\,\mathbb{E}_t[\boldsymbol{v}'\mathbf{Z}\,\delta]}_{=0,\text{ from Eq. (A2)}} \;+\; q\,\mathbb{E}_t[\boldsymbol{v}'\mathbf{Z}]\,\mathbb{E}_t[\delta], \\
&= \mathrm{Var}_t[\frac{1}{2}q\,\boldsymbol{v}'\mathbf{Z}] + \mathbb{E}_t[\delta^2] - (\mathbb{E}_t[\delta])^2 + q\,\mathbb{E}_t[\boldsymbol{v}'\mathbf{Z}]\,\mathbb{E}_t[\delta].
\end{aligned} \tag{A15}$$

In the Internet Appendix (Section I), we show two results. First, $|\mathbb{E}_t[y - y^\star]|$ is always bounded above by an easily computable quantity that we argue will, in practice, be small. Second, in the special case that m, $m^\star$, and S_{t+1}/S_t are lognormally distributed, $\mathbb{E}_t[y - y^\star]$ is identically equal to zero. Lognormality will only be an approximation to reality, but both results suggest that $|\mathbb{E}_t[y - y^\star]|$ will not be far from zero. From Equation (A7), this implies $\mathbb{E}_t[\delta] \approx 0$, given that $q \neq 0$.

Using the approximation $\mathbb{E}_t[\delta] \approx 0$ in Equation (A14), the *expected payoff*, denoted by EP, to the financial intermediary can be approximated as

$$\begin{aligned} \text{EP} &\equiv \frac{1}{2}\, q\, \mathbb{E}_t[\boldsymbol{v}'\mathbf{Z}], \\ &= \frac{1}{2}\, \mathbb{E}_t[\boldsymbol{v}'\mathbf{Z}]\, \sqrt{\mathbb{E}_t[(y - y^\star)^2]}, \end{aligned} \tag{A16}$$

where we have substituted for q, using Equation (A8). The expression for EP in Equation (A16) is a measure of the potential reward to the financial intermediary. Analogously, by Equation (A15),

$$\text{Var}_t[X] - \text{Var}_t[\frac{1}{2} q\, \boldsymbol{v}'\mathbf{Z}] \approx \mathbb{E}_t[\delta^2] = 1. \tag{A17}$$

In other words, the incremental variance of X over and above that of the payoff $\frac{1}{2} q\, \boldsymbol{v}'\mathbf{Z}$ is unity.

The quantity $\text{EP}/\sqrt{\underbrace{\text{Var}_t[X] - \text{Var}_t[\frac{1}{2} q\, \boldsymbol{v}'\mathbf{Z}]}_{=1}} = \text{EP}$ is a measure of the reward-for-risk potentially available to the financial intermediary. It is equal to (or analogous to, because definitions in the literature vary) what is variously termed (e.g., Sharpe 1982; Roll 1992; Grinold and Kahn 2007) the "information ratio" or the "appraisal ratio" in that the reward is an excess return (the strategy has zero initial cost by Equation (A11)) and the risk is measured as the square root of the incremental variance over and above that of a risky benchmark. In our setting, this risky benchmark is the portfolio with return $\frac{1}{2} q\, \boldsymbol{v}'\mathbf{Z}$. By construction, this incremental variance is unity (by Equation (A17)).

If the reward-for-risk EP were high enough, a financial intermediary would have an incentive to privately negotiate contracts with investors that exploit the fact that domestic and foreign investors disagree on the valuations of securities outside the linear span of $\mathbf{R}$ (or $\mathbf{R}^\star$).

We therefore place an upper bound on $\mathrm{EP}/\sqrt{\mathrm{Var}_t[X] - \mathrm{Var}_t[\frac{1}{2} q\, \boldsymbol{v}'\mathbf{Z}]}$, which has the effect of ruling out a potential contract that is too good to be true (e.g., Cochrane and Saá-Requejo (2000)).

Specifically, for some $\widehat{\Theta}$, satisfying $0 \leq \widehat{\Theta} < +\infty$, the reward-for-risk is bounded:

$$|\mathrm{EP}| = \left| \frac{1}{2}\, \mathbb{E}_t[\boldsymbol{v}'\mathbf{Z}]\, \sqrt{\mathbb{E}_t[(y - y^\star)^2]} \right| \;\leq\; \widehat{\Theta}. \tag{A18}$$

Alternatively, defining Θ by $\mathbb{E}_t[v'\mathbf{Z}]\,\Theta \equiv 2\,\widehat{\Theta}$, substituting it into Equation (A18), and henceforth, for brevity, dropping the subscript t from the expectation operator, we exclude good deals in the international economy by placing an upper bound Θ as shown in Equation (14) of Result 1. ■

B Appendix B: Proof of Result 2

Based on Equations (15)–(18) of Problem 1, we consider solutions for $y = m\sqrt{S_{t+1}/S_t}$ and $y^\star = m^\star/\sqrt{S_{t+1}/S_t}$ of the form

$$y = y_{\mathbf{z}} \;+\; \frac{1}{2} d\,\Theta\,\delta \quad \text{and} \quad y^\star = y_{\mathbf{z}} \;+\; \frac{1}{2} d^\star\,\Theta\,\delta, \tag{B1}$$

where $y_{\mathbf{z}} \geq 0$, δ, d, and $d^\star$ are yet to be determined (the conjectured solution inherits the form in Equation (A1), but with $q_0 = d\,\Theta$ and $q_0^\star = d^\star\,\Theta$).

For now, d and $d^\star$ are constant scalars satisfying

$$|d - d^\star| \leq 2. \tag{B2}$$

This restriction follows because $y - y^\star = \frac{1}{2}(d - d^\star)\,\Theta\,\delta$, $\mathbb{E}[(y - y^\star)^2] \leq \Theta^2$, and $\mathbb{E}[\delta^2] = 1$ (see Equation (B4)).

The constraint $\mathbb{E}[y\mathbf{Z}] = \mathbb{E}[y^\star\mathbf{Z}] = \mathbf{1}$ in Equation (17) holds, provided

$$\mathbb{E}[y_{\mathbf{z}}\mathbf{Z}] = \mathbf{1}. \tag{B3}$$

Furthermore, the random variables $y_{\mathbf{z}}$ and δ satisfy

$$\mathbb{E}[\delta\,\mathbf{Z}] = 0 \;\text{ for each element of } \mathbf{Z}, \qquad \mathbb{E}[y_{\mathbf{z}}\,\delta] = 0, \qquad \text{and} \qquad \mathbb{E}[\delta^2] = 1. \tag{B4}$$

Because $y - y^\star = \frac{1}{2}\,(d - d^\star)\,\Theta\delta$, the constraint $\mathbb{E}[(y - y^\star)^2] \leq \Theta^2$ is automatically satisfied.

With the conjectured forms of y and $y^\star$ in Equation (B1), we additionally have

$$\mathbb{E}[m\,m^\star] \;=\; \mathbb{E}[y\,y^\star] \;=\; \mathbb{E}[y_{\mathbf{z}}^2] \;+\; \frac{1}{4} d\,d^\star\Theta^2. \tag{B5}$$

Hence, the infimum $\inf_{y,y^\star}\,\mathbb{E}[y\,y^\star]$ in Problem 1 separates into two distinct problems:

$$\inf_{y_{\mathbf{z}}}\{\mathbb{E}[y_{\mathbf{z}}^2]\} \;+\; \inf_{d,d^\star}\{\frac{1}{4} d\,d^\star\Theta^2\}, \tag{B6}$$

subject to $\mathbb{E}[y_{\mathbf{z}}\mathbf{Z}] = \mathbf{1}$, $y_{\mathbf{z}} \geq 0$, $y \geq 0$, and $y^\star \geq 0$.

Exploiting this feature of the solution, we sequentially solve for $y_{\mathbf{z}}$, then for δ and, finally, for d and $d^\star$.

We first determine $y_{\mathbf{z}}$ by solving

$$\inf_{y_{\mathbf{z}}}\,\mathbb{E}[y_{\mathbf{z}}^2] \quad \text{such that} \quad \mathbb{E}[y_{\mathbf{z}}\,\mathbf{Z}] = \mathbf{1}, \quad y_{\mathbf{z}} \geq 0. \tag{B7}$$

Here, $y_{\mathbf{z}}$ can be interpreted as the minimum second-moment SDF with nonnegativity in the hypothetical economy in which gross returns are $\mathbf{Z}$.

We introduce an N-dimensional vector of Lagrange multipliers $\boldsymbol{\lambda} \in \mathbb{R}^N$. Then the solution to the problem in Equation (B7) is the solution to

$$\max_{\boldsymbol{\lambda}\in\mathbb{R}^N}\;\{\inf_{y_{\mathbf{z}}\geq 0}\{\mathbb{E}[y_{\mathbf{z}}^2] \;-\; 2\boldsymbol{\lambda}^{'}\,(\mathbb{E}[y_{\mathbf{z}}\,\mathbf{Z}] - \mathbf{1})\}\}. \tag{B8}$$

The first-order condition implies $0 = 2y_{\mathbf{z}} - 2\boldsymbol{\lambda}^{'}\mathbf{Z}$. Both the first-order condition and the constraint $y_{\mathbf{z}} \geq 0$ will be satisfied if

$$y_{\mathbf{z}} \;=\; \max(\boldsymbol{\lambda}^{'}\mathbf{Z}, 0). \tag{B9}$$

Substituting $y_{\mathbf{z}}$ from Equation (B9) into Equation (B8), $\boldsymbol{\lambda}$ solves

$$\max_{\boldsymbol{\lambda}\in\mathbb{R}^N} \ \{2\boldsymbol{\lambda}'\mathbf{1} - \mathbb{E}[(\max(\boldsymbol{\lambda}'\mathbf{Z},0))^2]\}, \tag{B10}$$

which yields the equivalent *minimization* problem to be solved in Equation (20).

Next, to solve for δ (see Equation (B4)), we note that δ is proportional to $e_{\mathbf{z}}$, where $e_{\mathbf{z}}$ is the residual from the projection of one onto the space of returns $\mathbf{Z}$. Hence, $e_{\mathbf{z}}$ can be computed from the ordinary least-squares regression formula.

Then δ is obtained by multiplicatively scaling $e_{\mathbf{z}}$ in such a way that $\mathbb{E}[\delta^2] = 1$. More formally,

$$e_{\mathbf{z}} = 1 - \mathbb{E}[\mathbf{Z}]' \left(\mathbb{E}[\mathbf{Z}\mathbf{Z}']\right)^{-1} \mathbf{Z}, \quad \text{and then} \quad \delta = e_{\mathbf{z}}/\sqrt{\mathbb{E}[e_{\mathbf{z}}^2]}. \tag{B11}$$

In the degenerate case that $e_{\mathbf{z}} = 0$ in every state, we set $\delta = 0$.

Finally, we solve for d and $d^\star$. The second part of Equation (B6) minimizes $\frac{1}{4} d\, d^\star \Theta^2$, and the minimum requires that d and $d^\star$ be of opposite signs. Hence, without loss of generality, we assume $d \geq 0$, $d^\star \leq 0$.

The solution of Problem 1 must also accommodate $y = y_{\mathbf{z}} + \frac{1}{2} d\Theta\delta \geq 0$ and $y^\star = y_{\mathbf{z}} + \frac{1}{2} d^\star\Theta\delta \geq 0$, as well as the constraint $|d - d^\star| \leq 2$.

Let d_p (respectively, d_n) be the smallest positive value (respectively, largest negative, i.e., least negative value) of $-y_{\mathbf{z}}/\left(\frac{1}{2}\Theta\delta\right)$ across the $\mathbb{J}$ possible states of the world. Then $y \geq 0$, and $y^\star \geq 0$ requires that $d \leq d_p$ and $d_n \leq d^\star$.

The proof of Result 2 is complete. ■

C Appendix C: Exploring Complementarities with Other Approaches

In this appendix, we explore complementarities and conceptual distinctions from the multiplicative wedge approach proposed by Backus, Foresi, and Telmer (2001) and further considered in Lustig and Verdelhan (2015). The central question we address is: is our solution for $(m_{t+1},\ m^\star_{t+1})$ in Equation (23) subsumed within the multiplicative wedge class?

To offer our rationale, consider Lustig and Verdelhan (2015). In their work, an econometrician commits to a model of "base case" SDFs. The foreign base case SDF is then "perturbed" by a multiplicative wedge $e^{\eta_{t+1}}$, satisfying $m_{t+1}\left(\frac{S_{t+1}}{S_t}\right) = m^{\star}_{t+1}e^{\eta_{t+1}}$. In analogy, our base case SDFs are $m_{\mathbf{z},t+1} = \frac{y_{\mathbf{z}}}{\sqrt{S_{t+1}/S_t}}$ (domestic) and $m^{\star}_{\mathbf{z},t+1} = y_{\mathbf{z}}\sqrt{S_{t+1}/S_t}$ (foreign), defined in Equations (24) and (25), since complete markets would correspond to the case when the only (i.e., unique) SDFs satisfying (6)–(8) were of this form. Recall that $y_{\mathbf{z}}$ is interpretable as a minimum second-moment SDF.

Our analysis leads to consideration of domestic and foreign SDFs of the form $m = y/\sqrt{S_{t+1}/S_t}$ and $m^{\star} = y^{\star}\sqrt{S_{t+1}/S_t}$, where y and $y^{\star}$ are of the form $y = y_{\mathbf{z}} + \frac{1}{2}\,d\,\Theta\,\delta$ and $y^{\star} = y_{\mathbf{z}} + \frac{1}{2}d^{\star}\,\Theta\,\delta$ (see Equation (B1)). Given the symmetric construction, it is clear that one would need to "perturb" not only the base case foreign SDF, but also the base case domestic SDF.

However, even if one were to consider candidate perturbed SDFs of the form $m_{t+1}e^{-h\eta_{t+1}}$ and $m^{\star}_{t+1}e^{(1-h)\eta_{t+1}}$ (consistent with $m_{t+1}(\frac{S_{t+1}}{S_t}) = m^{\star}_{t+1}e^{\eta_{t+1}}$), for some constant scalar h, one still cannot obtain the additive SDFs proposed in our paper.

To obtain such SDFs, one would need $e^{-h\eta_{t+1}}\,m_{\mathbf{z},t+1} = \left(y_{\mathbf{z}} + \frac{1}{2}\,d\,\Theta\,\delta\right)/\sqrt{S_{t+1}/S_t}$ and, simultaneously, $e^{(1-h)\eta_{t+1}}\,m^{\star}_{\mathbf{z},t+1} = \left(y_{\mathbf{z}} + \frac{1}{2}d^{\star}\,\Theta\,\delta\right)\sqrt{S_{t+1}/S_t}$. Or, equivalently,

$$e^{(1-h)\eta_{t+1}} = \left(\frac{1}{y_{\mathbf{z}}}(y_{\mathbf{z}} + \frac{1}{2}\,d\,\Theta\,\delta)\right)^{-(1-h)/h} = \frac{1}{y_{\mathbf{z}}}\left(y_{\mathbf{z}} + \frac{1}{2}d^{\star}\,\Theta\,\delta\right). \tag{C1}$$

But the right-hand side equality is not mathematically feasible except when $d\,\Theta = d^{\star}\,\Theta = 0$ (which leads back to the base case SDFs).

In particular, with base case SDFs of the form $m_{\mathbf{z},t+1} \equiv \frac{y_{\mathbf{z}}}{\sqrt{S_{t+1}/S_t}}$ and $m^{\star}_{\mathbf{z},t+1} \equiv y_{\mathbf{z}}\sqrt{S_{t+1}/S_t}$, the relation $m_{t+1}(\frac{S_{t+1}}{S_t}) = m^{\star}_{t+1}e^{\eta_{t+1}}$ implies $\frac{y_{\mathbf{z}}}{\sqrt{S_{t+1}/S_t}}(\frac{S_{t+1}}{S_t}) = y_{\mathbf{z}}\sqrt{S_{t+1}/S_t}e^{\eta_{t+1}}$ or $1 = e^{\eta_{t+1}}$ or $\eta_{t+1} = 0$.

Thus, not all SDFs — and certainly not the SDFs that we synthesize — can be tailored to be in line with the multiplicative wedge paradigm.

Hence, our analysis wards off a possible misconception that multiplicative wedges subsume our additive form of the SDFs. The additive form of the SDFs is implicit in the constructions of Hansen and Jagannathan (1991). ■

D Appendix D: Proof of Result 3

For brevity of Equation presentation, let

$$\mu_t^{\text{fx}} \equiv \mathbb{E}_t[r_{t+1}^{\text{fx}}], \quad \text{where} \quad r_{t+1}^{\text{fx}} \equiv \frac{S_{t+1}}{S_t} - 1. \tag{D1}$$

Define the risk-neutral (martingale) measure such that the expectation $\mathbb{E}_t^Q[.]$ satisfies

$$\mathbb{E}_t^Q[(r_{t+1}^{\text{fx}})^2] \;=\; \mathbb{E}_t[\frac{m_{t+1}}{\mathbb{E}_t[m_{t+1}]}(r_{t+1}^{\text{fx}})^2] \;\text{ and} \tag{D2}$$

$$\mathbb{E}_t^Q[r_{t+1}^{\text{fx}}] \;=\; \mathbb{E}_t[\frac{m_{t+1}}{\mathbb{E}_t[m_{t+1}]}r_{t+1}^{\text{fx}}] \;=\; \frac{F_t}{S_t} - 1. \tag{D3}$$

The risk-neutral mean in Equation (D3) follows, since $\mathbb{E}_t[m_{t+1}(\frac{S_{t+1}}{S_t} - \frac{F_t}{S_t})] = 0$.

Using the covariance operator, we obtain

$$\mathbb{E}_t[m_{t+1}(r_{t+1}^{\text{fx}} - \mu_t^{\text{fx}})^2] = \text{Cov}_t[m_{t+1}, (r_{t+1}^{\text{fx}} - \mu_t^{\text{fx}})^2] \;+\; \mathbb{E}_t[m_{t+1}]\,\mathbb{E}_t[(r_{t+1}^{\text{fx}} - \mu_t^{\text{fx}})^2]. \tag{D4}$$

Rearranging and simplifying, we obtain

$$\mathbb{E}_t[(r_{t+1}^{\text{fx}} - \mu_t^{\text{fx}})^2] - \mathbb{E}_t^Q[(r_{t+1}^{\text{fx}} - \mu_t^{\text{fx}})^2] \;=\; -\frac{1}{\mathbb{E}_t[m_{t+1}]}\text{Cov}_t[m_{t+1}, (r_{t+1}^{\text{fx}} - \mu_t^{\text{fx}})^2]. \tag{D5}$$

We may reexpress the left-hand side of Equation (D5) by adding and subtracting the risk-neutral mean $\frac{F_t}{S_t} - 1$ in the second term as

$$\begin{aligned} \mathbb{E}_t[(r_{t+1}^{\text{fx}} - \mu_t^{\text{fx}})^2] - \mathbb{E}_t^Q[(r_{t+1}^{\text{fx}} - (\frac{F_t}{S_t} - 1) - \{\mu_t^{\text{fx}} - (\frac{F_t}{S_t} - 1)\})^2] \;&= \\ -\text{Cov}_t[\frac{m_{t+1}}{\mathbb{E}_t[m_{t+1}]}, (r_{t+1}^{\text{fx}} - \mu_t^{\text{fx}})^2]. \end{aligned} \tag{D6}$$

The expression for the currency variance risk premium is (because $\mathbb{E}_t^Q[r_{t+1}^{\text{fx}}] = \frac{F_t}{S_t} - 1$)

$$\underbrace{\mathbb{E}_t[(r_{t+1}^{\text{fx}} - \mu_t^{\text{fx}})^2] - \mathbb{E}_t^Q[(r_{t+1}^{\text{fx}} - (\frac{F_t}{S_t} - 1))^2]}_{\text{Currency variance risk premium}} = \{\mu_t^{\text{fx}} - (\frac{F_t}{S_t} - 1)\}^2 - \text{Cov}_t[\frac{m_{t+1}}{\mathbb{E}_t[m_{t+1}]}, (r_{t+1}^{\text{fx}} - \mu_t^{\text{fx}})^2]. \tag{D7}$$

We have the proof of Result 3, noting that $\mathbb{E}_t^Q[(r_{t+1}^{\text{fx}} - (\frac{F_t}{S_t} - 1))^2] = \mathbb{E}_t^Q[(r_{t+1}^{\text{fx}})^2] - (\frac{F_t}{S_t} - 1)^2$.

Additionally, the additive form of the SDF implies $m_{t+1} = m_{\mathbf{z},t+1} + u_{t+1}$, where $u_{t+1} = \frac{1}{2} d\,\Theta\,\delta \frac{1}{\sqrt{S_{t+1}/S_t}}$ from Equation (24). Moreover, $(r^{\text{fx}}_{t+1} - \mu^{\text{fx}}_t)^2 = (r^{\text{fx}}_{t+1})^2 + (\mu^{\text{fx}}_t)^2 - 2(\mu^{\text{fx}}_t) r^{\text{fx}}_{t+1}$. Thus, the currency variance risk premium depends on the market incompleteness parameter Θ via

$$\begin{aligned} \text{Cov}_t[u_{t+1}, (r^{\text{fx}}_{t+1} - \mu^{\text{fx}}_t)^2] &= \frac{1}{2} d\,\Theta\,\mathbb{E}_t[\delta \frac{1}{\sqrt{S_{t+1}/S_t}} (S_{t+1}/S_t)^2], & \text{(D8)} \\ &= \frac{1}{2} d\,\Theta\,\mathbb{E}_t[\delta\,(S_{t+1}/S_t)^{3/2}], & \text{(D9)} \end{aligned}$$

since $\mathbb{E}_t[\delta \sqrt{S_{t+1}/S_t}] = 0$.

The remaining step is to show that $\mathbb{E}_t[r^{\text{fx}}_{t+1}] - (\frac{F_t}{S_t} - 1)$ is independent of Θ. From the pricing of the foreign currency risk-free return $(S_{t+1}/S_t)\,R^\star_f$ it holds that $\mathbb{E}_t[m_{t+1}\,(S_{t+1}/S_t)\,R^\star_f] = 1$. Exploiting $m_{t+1} = m_{\mathbf{z},t+1} + u_{t+1}$, and using the orthogonality condition $\mathbb{E}_t[\delta \sqrt{S_{t+1}/S_t}] = 0$, we deduce $\mathbb{E}_t[m_{\mathbf{z},t+1}\,(S_{t+1}/S_t)] = 1/R^\star_f$. Then, using the definition of covariance, we finally obtain $\mathbb{E}_t[1 + r^{\text{fx}}_{t+1}] = R_f/R^\star_f - R_f\,\text{Cov}_t[m_{\mathbf{z},t+1}, r^{\text{fx}}_{t+1}]$. ■

Table 1: **Estimates of market incompleteness parameter Θ across all 45 country pairs**

The estimation of Θ is performed for each of the 45 country pairs of industrialized countries. The countries in our sample are Australia (AUD), New Zealand (NZD), United Kingdom (STG), France (FRA), Canada (CAD), United States (USD), the Netherlands (NLG), Germany (GER), Japan (JPY), and Switzerland (SWI). Reported are the mean, the standard deviation (SD), the minimum, the maximum, and the percentiles of the Θ estimates. We use the following algorithm to estimate Θ. Start with a trial value, for example, of $\Theta = 0.01$, and solve Problem 1. The output is ($\boldsymbol{\lambda}$, δ, d, $d^\star$) and, hence, m and $m^\star$. Next, we compute $\rho_{m,m^\star}$ (or the RSI in Equation (30)) based on the asset market view in Equation (29) and compare it with its counterpart computed based on a candidate model of SDF in Equation (31). Finally, we iterate over the choice of Θ to minimize the discrepancy between the model-based $\rho_{m,m^\star}$ (or RSI), and the corresponding one constructed from asset returns. We consider different sets of asset returns, allowing for multicurrency and multicountry exposures. Consider the Australia versus Japan (AUD/JPY) country pair, in which the gross return vector $\mathbf{R}_{t+1}$ contains six assets:

$$\underbrace{\mathbf{R}_{t+1}}_{6\times 1} = \begin{cases} \text{Return of the Australian risk-free bond (in AUD)} \\ \text{Return of the Australian equity index (in AUD)} \\ \text{Return of the Japanese risk-free bond (in AUD)} \\ \text{Return of the Japanese equity index (in AUD)} \\ \text{Return of the U.S. 30-year Treasury bond (in AUD)} \\ \text{Return of the MSCI world equity index (in AUD).} \end{cases}$$

Symmetrically, $\mathbf{R}^\star_{t+1} = \mathbf{R}_{t+1}/(\frac{S_{t+1}}{S_t})$ contains the same set of gross returns denominated in Japanese yen, where $\frac{S_{t+1}}{S_t}$ is the exchange rate growth with the Japanese yen as the reference currency. The sample period considered is January 1975 to December 2015 (492 observations). Following Colacito and Croce (2011, Table 1), we choose the following Epstein-Zin preference parameters: $\gamma = 4.25$, $\psi = 2$, $\beta = 0.998$. "CORR" is the pairwise (i.e., cross-sectional) correlation coefficient among the 45 Θ estimates based on matching correlation and those based on matching RSI. The final column shows the p-values from the t-test (assuming unequal variance) that the Θ obtained from correlation and RSI are equal.

					Percentiles						p-val.
	Mean	SD	Min.	Max.	5th	25th	50th	75th	95th	CORR	(t-test)
A. When the correlation and RSI are based on Epstein-Zin preferences											
$\rho_{m,m^\star}$	0.39	0.09	0.25	0.64	0.26	0.32	0.38	0.44	0.53		
RSI	0.40	0.09	0.26	0.64	0.28	0.34	0.40	0.46	0.55		
										0.99	.47
B. When the correlation and RSI are based on power utility over wealth											
$\rho_{m,m^\star}$	0.38	0.09	0.25	0.63	0.26	0.31	0.38	0.43	0.52		
RSI	0.40	0.09	0.26	0.63	0.28	0.33	0.40	0.45	0.54		
										0.99	.47
C. When the correlation and RSI are based on power utility over consumption											
$\rho_{m,m^\star}$	0.68	0.20	0.35	1.25	0.45	0.54	0.65	0.76	0.99		
RSI	0.69	0.19	0.35	1.17	0.45	0.55	0.66	0.80	0.99		
										0.99	.83
D. When the RSI and correlation are based on power utility over dividends											
$\rho_{m,m^\star}$	0.52	0.09	0.30	0.70	0.38	0.46	0.51	0.58	0.68		
RSI	0.52	0.09	0.30	0.70	0.38	0.47	0.51	0.58	0.68		
										0.99	.94

Table 2: **Estimates of Θ, SDF volatilities, correlation between SDFs, and the size of the unspanned component of the SDFs, when $\mathbf{R}_{t+1}$ ($\mathbf{R}^{\star}_{t+1}$) contains six assets**

Reported are the Θ estimates, the SDF volatilities, the correlation between the SDFs, and the size of the unspanned components of the SDFs for each of the 45 country pairs. To estimate Θ, we consider an algorithm that minimizes the discrepancy between $\rho_{m,m^{\star}}$ that is constructed based on *Epstein-Zin preferences* and the corresponding one constructed from asset returns (as detailed in the note to Table 1). The results are displayed in the order of decreasing average interest-rates (so Australia (AUD) has the highest average interest-rates, while Switzerland (SWI) has the lowest average interest-rates). The 95% lower and upper bootstrap confidence intervals are in brackets.

	A		*B. Properties of the domestic and foreign SDFs*				
	Θ	Bootstrap 95% CI	$\sigma[m]$	$\sigma[m^{\star}]$	$\rho_{m,m^{\star}}$	$\frac{\mathrm{Var}[u]}{\mathrm{Var}[m]}$	$\frac{\mathrm{Var}[u^{\star}]}{\mathrm{Var}[m^{\star}]}$
AUD/NZD	0.25	[0.16 0.38]	34	37	0.72	0.16	0.12
AUD/STG	0.47	[0.31 0.59]	50	53	0.55	0.20	0.22
AUD/FRA	0.29	[0.19 0.39]	43	45	0.74	0.11	0.11
AUD/CAD	0.28	[0.18 0.40]	41	44	0.76	0.12	0.11
AUD/USD	0.46	[0.29 0.58]	52	56	0.62	0.19	0.19
AUD/NLG	0.41	[0.29 0.48]	54	57	0.71	0.14	0.14
AUD/GER	0.42	[0.31 0.51]	46	50	0.60	0.20	0.19
AUD/JPY	0.45	[0.37 0.56]	44	49	0.49	0.25	0.23
AUD/SWI	0.38	[0.26 0.46]	45	52	0.67	0.18	0.15
NZD/STG	0.56	[0.29 0.80]	51	53	0.40	0.29	0.30
NZD/FRA	0.30	[0.21 0.39]	34	38	0.61	0.19	0.17
NZD/CAD	0.43	[0.22 0.63]	41	47	0.52	0.25	0.25
NZD/USD	0.50	[0.28 0.66]	47	53	0.48	0.28	0.26
NZD/NLG	0.41	[0.29 0.49]	45	48	0.58	0.20	0.20
NZD/GER	0.37	[0.31 0.47]	39	44	0.57	0.22	0.20
NZD/JPY	0.40	[0.32 0.56]	40	44	0.50	0.24	0.22
NZD/SWI	0.50	[0.37 0.59]	54	57	0.56	0.21	0.21
STG/FRA	0.40	[0.27 0.52]	49	50	0.65	0.16	0.17
STG/CAD	0.47	[0.22 0.65]	45	49	0.50	0.27	0.23
STG/USD	0.41	[0.19 0.55]	58	59	0.69	0.15	0.15
STG/NLG	0.33	[0.20 0.40]	55	56	0.81	0.09	0.09
STG/GER	0.33	[0.22 0.42]	41	46	0.70	0.17	0.14
STG/JPY	0.48	[0.29 0.65]	37	44	0.46	0.31	0.25
STG/SWI	0.42	[0.31 0.50]	57	59	0.72	0.13	0.13
FRA/CAD	0.31	[0.20 0.40]	45	47	0.75	0.12	0.11
FRA/USD	0.37	[0.21 0.49]	48	51	0.70	0.15	0.14
FRA/NLG	0.30	[0.20 0.37]	55	56	0.85	0.08	0.08
FRA/GER	0.26	[0.13 0.35]	48	50	0.85	0.08	0.07
FRA/JPY	0.35	[0.24 0.50]	43	45	0.64	0.17	0.17
FRA/SWI	0.32	[0.18 0.42]	53	55	0.81	0.10	0.09
CAD/USD	0.33	[0.12 0.48]	40	45	0.69	0.18	0.15
CAD/NLG	0.34	[0.21 0.43]	51	53	0.76	0.12	0.11
CAD/GER	0.39	[0.26 0.51]	49	50	0.67	0.16	0.15
CAD/JPY	0.44	[0.31 0.54]	49	51	0.58	0.19	0.20
CAD/SWI	0.34	[0.18 0.43]	41	46	0.66	0.18	0.14
USD/NLG	0.35	[0.22 0.44]	59	59	0.80	0.09	0.09
USD/GER	0.32	[0.20 0.42]	48	46	0.74	0.11	0.12
USD/JPY	0.54	[0.35 0.67]	55	55	0.50	0.23	0.25
USD/SWI	0.38	[0.24 0.46]	55	56	0.75	0.12	0.11
NLG/GER	0.27	[0.16 0.35]	49	51	0.85	0.08	0.07
NLG/JPY	0.50	[0.36 0.60]	57	58	0.60	0.19	0.20
NLG/SWI	0.31	[0.19 0.37]	58	59	0.86	0.07	0.07
GER/JPY	0.41	[0.24 0.63]	46	46	0.56	0.20	0.21
GER/SWI	0.26	[0.13 0.34]	53	54	0.88	0.06	0.06
JPY/SWI	0.64	[0.46 0.75]	61	61	0.43	0.28	0.27

Table 3: **A snapshot of SDF volatilities, correlation between SDFs, and the size of the unspanned components**

Reported is a snapshot of the following quantities based on the estimates of Θ across each of the 45 country pairs:

$\sigma[m]$ (respectively, $\sigma[m^\star]$)=the annualized volatilities of the domestic (foreign) SDFs,

$\rho_{m,m^\star}$=the correlation between the domestic and foreign SDFs,

$\frac{\mathrm{Var}[u]}{\mathrm{Var}[m]}$ ($\frac{\mathrm{Var}[u^\star]}{\mathrm{Var}[m^\star]}$)=the size of the unspanned components of the domestic (foreign) SDFs.

To estimate Θ, we consider an algorithm that minimizes the discrepancy between $\rho_{m,m^\star}$ that is constructed based on *Epstein-Zin preferences* and the corresponding one constructed from asset returns (as detailed in the note to Table 1). $\mathbf{R}_{t+1}$ with *six assets*, in the case of AUD/JPY, contains the returns of (1) the Australian risk-free bond, (2) the Australian equity index, (3) the Japanese risk-free bond, (4) the Japanese equity index, (5) the U.S. 30-year Treasury bond, and (6) the MSCI world index, in which each of the returns are denominated in Australian dollars, whereas $\mathbf{R}^\star_{t+1}$ contains the same set of assets denominated in Japanese yen. The $\mathbf{R}_{t+1}$ ($\mathbf{R}^\star_{t+1}$) with *seven assets* is an augmented version of the six asset case with S&P commodity index returns, whereas the one with *four assets* is a restricted counterpart of the one with six assets that does not include the U.S. 30-year Treasury bond and the MSCI world equity index as common assets across all the 45 country pairs. The sample period considered is January 1975 to December 2015 (492 observations).

						Percentiles				
		Mean	SD	Min.	Max.	5th	25th	50th	75th	95th
A. SDF volatilities										
$\sigma[m]$	Six assets	48	7	34	61	38	43	48	54	58
	Seven assets	49	6	35	59	38	45	49	54	57
	Four assets	47	7	27	59	36	40	48	52	55
$\sigma[m^\star]$	Six assets	51	6	37	61	44	46	51	56	59
	Seven assets	51	6	40	59	42	47	53	56	59
	Four assets	49	7	32	58	39	43	50	53	56
B. Correlation between the SDFs										
$\rho_{m,m^\star}$	Six assets	0.66	0.13	0.40	0.88	0.46	0.56	0.67	0.75	0.85
	Seven assets	0.66	0.13	0.40	0.88	0.46	0.56	0.66	0.75	0.85
	Four assets	0.66	0.13	0.40	0.88	0.46	0.56	0.67	0.74	0.85
C. Size of the unspanned components of SDFs										
$\frac{\mathrm{Var}[u]}{\mathrm{Var}[m]}$	Six assets	0.17	0.07	0.06	0.31	0.08	0.12	0.17	0.20	0.28
	Seven assets	0.17	0.07	0.06	0.30	0.07	0.11	0.16	0.21	0.28
	Four assets	0.16	0.06	0.06	0.28	0.08	0.11	0.16	0.20	0.26
$\frac{\mathrm{Var}[u^\star]}{\mathrm{Var}[m^\star]}$	Six assets	0.16	0.06	0.06	0.30	0.07	0.11	0.15	0.21	0.26
	Seven assets	0.16	0.06	0.06	0.30	0.07	0.11	0.15	0.21	0.26
	Four assets	0.16	0.06	0.06	0.30	0.07	0.11	0.15	0.21	0.26

Table 4: **Market incompleteness parameter Θ and properties of SDFs based on G-10 currency options**

To estimate Θ, we consider an algorithm that minimizes the discrepancy between the correlation $\rho_{m,m^\star}$ that is constructed based on *Epstein-Zin preferences* and the corresponding one constructed from asset returns (as detailed in Section 2.1.1 and the note in Table 1). Augmented to the baseline set of six assets are the returns of (1) a currency straddle, (2) a 25-delta currency strangle, and (3) a 10-delta currency strangle. We first construct the returns of straddles and strangles for a G-10 currency pair and then take the equally weighted average across all nine currencies with U.S. dollar as the domestic currency. The sample period for the G-10 currency options is 2008:01 to 2015:12. We report a snapshot of Θ, $\sigma[m]$, $\sigma[m^\star]$, $\rho_{m,m^\star}$, $\frac{\mathrm{Var}[u]}{\mathrm{Var}[m]}$ and $\frac{\mathrm{Var}[u^\star]}{\mathrm{Var}[m^\star]}$. The baseline six assets in $\mathbf{R}_{t+1}$, in the case of AUD/JPY, contains the returns of (1) the Australian risk-free bond, (2) the Australian equity index, (3) the Japanese risk-free bond, (4) the Japanese equity index, (5) the U.S. 30-year Treasury bond, and (6) the MSCI world index, in which each of the returns is denominated in Australian dollars, whereas $\mathbf{R}^\star_{t+1}$ contains the same set of assets denominated in Japanese yen.

		Mean	SD	Min.	Max.
Θ	Six assets + currency straddle	0.66	0.15	0.28	0.99
$\sigma[m]$	Six assets + currency straddle	83	14	39	110
$\sigma[m^\star]$	Six assets + currency straddle	84	13	38	109
$\rho_{m,m^\star}$	Six assets + currency straddle	0.66	0.13	0.40	0.88
$\frac{\mathrm{Var}[u]}{\mathrm{Var}[m]}$	Six assets + currency straddle	0.17	0.07	0.05	0.33
$\frac{\mathrm{Var}[u^\star]}{\mathrm{Var}[m^\star]}$	Six assets + currency straddle	0.17	0.06	0.06	0.30
Θ	Six assets + 25-delta currency strangle	0.66	0.15	0.32	1.10
$\sigma[m]$	Six assets + 25-delta currency strangle	82	13	44	107
$\sigma[m^\star]$	Six assets + 25-delta currency strangle	83	13	43	109
$\rho_{m,m^\star}$	Six assets + 25-delta currency strangle	0.66	0.13	0.40	0.88
$\frac{\mathrm{Var}[u]}{\mathrm{Var}[m]}$	Six assets + 25-delta currency strangle	0.17	0.07	0.05	0.33
$\frac{\mathrm{Var}[u^\star]}{\mathrm{Var}[m^\star]}$	Six assets + 25-delta currency strangle	0.17	0.06	0.06	0.30
Θ	Six assets + 10-delta currency strangle	0.63	0.15	0.24	0.99
$\sigma[m]$	Six assets + 10-delta currency strangle	79	14	34	108
$\sigma[m^\star]$	Six assets + 10-delta currency strangle	80	14	33	106
$\rho_{m,m^\star}$	Six assets + 10-delta currency strangle	0.66	0.13	0.40	0.88
$\frac{\mathrm{Var}[u]}{\mathrm{Var}[m]}$	Six assets + 10-delta currency strangle	0.17	0.07	0.05	0.34
$\frac{\mathrm{Var}[u^\star]}{\mathrm{Var}[m^\star]}$	Six assets + 10-delta currency strangle	0.17	0.06	0.06	0.30

Table 5: **Assessing variation in the estimates of Θ using rolling and expanding window schemes**

We report results using both rolling and expanding window schemes. All results in panel A are based on a rolling window with a length of 240 months. We start the first sample in 1975:01 and end in 1994:12 (i.e., January 1975 to December 1994). Then we move five years forward (60 months), while keeping the length of the rolling window fixed at 240 months. Thus, the additional sample periods of our estimation are 1980:01 to 1999:12, 1985:01 to 2004:12, 1990:01 to 2009:12, and 1995:01 to 2014:12. In each sample period, we use the following algorithm to estimate Θ. Start with a trial value, for example, of $\Theta = 0.01$, and solve Problem 1. The output is $(\boldsymbol{\lambda}, \delta, d, d^\star)$ and, hence, m and $m^\star$. Next, we compute $\rho_{m,m^\star}$ based on the asset market view in Equation (29) and compare it with its counterpart computed based on wealth growth in Equation (31). Finally, we iterate over the choice of Θ to minimize the discrepancy between the model-based $\rho_{m,m^\star}$, and the corresponding one constructed from asset returns. Panel B reports the counterpart results when the rolling window scheme is substituted with the expanding window scheme. Under the expanding scheme, the length of the window successively increases by 60 months.

		A. Rolling scheme, all 45 country pairs					*B. Expanding scheme, all 45 country pairs*				
Start		1975:01	1980:01	1985:01	1990:01	1995:01	1975:01	1975:01	1975:01	1975:01	1975:01
End		1994:12	1999:01	2004:01	2009:12	2014:12	1994:12	1999:01	2004:01	2009:12	2014:12
Window		240	240	240	240	240	240	300	360	420	480
Θ	Mean	0.47	0.58	0.44	0.46	0.53	0.47	0.58	0.41	0.36	0.39
	SD	0.12	0.13	0.16	0.15	0.16	0.12	0.13	0.09	0.08	0.09
	Min.	0.25	0.36	0.20	0.13	0.28	0.25	0.33	0.25	0.22	0.24
	Max.	0.75	0.90	0.94	0.84	0.94	0.75	0.87	0.62	0.59	0.64
$\rho_{m,m^\star}$	Mean	0.45	0.50	0.55	0.64	0.66	0.45	0.47	0.49	0.53	0.53
	SD	0.14	0.14	0.16	0.13	0.14	0.14	0.14	0.15	0.14	0.14
	Min.	0.19	0.26	0.26	0.44	0.40	0.19	0.23	0.24	0.29	0.29
	Max.	0.70	0.75	0.79	0.87	0.90	0.70	0.72	0.74	0.76	0.75
$\sigma[m]$	Mean	59	73	57	58	66	59	73	52	46	49
	SD	8	11	17	13	12	8	8	6	7	6
	Min.	46	46	20	15	41	46	50	37	28	33
	Max.	76	94	97	81	93	76	89	62	58	61
$\sigma[m^\star]$	Mean	60	73	57	58	66	60	73	53	46	50
	SD	8	11	17	13	12	8	8	6	7	6
	Min.	46	46	20	15	40	46	50	37	27	34
	Max.	77	95	100	82	93	77	89	62	58	61
$\frac{\mathrm{Var}[u]}{\mathrm{Var}[m]}$	Mean	0.16	0.17	0.16	0.16	0.17	0.16	0.17	0.16	0.16	0.16
	SD	0.06	0.06	0.06	0.06	0.06	0.06	0.06	0.06	0.06	0.06
	Min.	0.06	0.06	0.06	0.06	0.06	0.06	0.06	0.06	0.06	0.06
	Max.	0.30	0.31	0.29	0.29	0.31	0.30	0.30	0.30	0.30	0.30
$\frac{\mathrm{Var}[u^\star]}{\mathrm{Var}[m^\star]}$	Mean	0.16	0.17	0.16	0.16	0.17	0.16	0.17	0.16	0.16	0.16
	SD	0.06	0.06	0.06	0.06	0.07	0.06	0.07	0.06	0.06	0.06
	Min.	0.06	0.06	0.06	0.06	0.06	0.06	0.06	0.05	0.06	0.06
	Max.	0.32	0.32	0.32	0.29	0.34	0.32	0.33	0.32	0.29	0.31

Table 6: **A snapshot of SDF volatilities, correlation between SDFs, and the size of the unspanned components, when the SDFs are inverse of the return of the long-term bond**

This table differs from Tables 1 and 3 in that the SDFs take the form $m_{t+1} = \frac{1}{R_{t+1,\infty}}$ and $m^{\star}_{t+1} = \frac{1}{R^{\star}_{t+1,\infty}}$, where $R_{t+1,\infty} = \lim_{n\to\infty} R_{t+1,n}$ is the return of a n-month domestic discount bond for a large n, and, likewise, for $R^{\star}_{t+1,\infty}$. Reported is a snapshot of the following quantities based on the estimates of Θ for each of the 45 country pairs:

$\sigma[m]$ (respectively, $\sigma[m^{\star}]$)=the annualized volatilities of the domestic (foreign) SDFs,

$\rho_{m,m^{\star}}$=the correlation between the domestic and foreign SDFs,

$\frac{\mathrm{Var}[u]}{\mathrm{Var}[m]}$ ($\frac{\mathrm{Var}[u^{\star}]}{\mathrm{Var}[m^{\star}]}$)=the size of the unspanned components of the domestic (foreign) SDFs.

$\mathbf{R}_{t+1}$ with *six assets*, in the case of AUD/JPY, contains the returns of (1) the Australian risk-free bond, (2) the Australian equity index, (3) the Japanese risk-free bond, (4) the Japanese equity index, (5) the U.S. 30-year Treasury bond, and (6) the MSCI world index, in which each of the returns is denominated in Australian dollars, whereas $\mathbf{R}^{\star}_{t+1}$ contains the same set of assets denominated in Japanese yen. The earliest start date is 1989:01 and the end date is 2015:12. We use returns of bonds with a constant maturity of 120 months, which is extracted from Datastream.

						Percentiles				
		Mean	SD	Min.	Max.	5th	25th	50th	75th	95th
Θ	Six assets	0.44	0.13	0.13	0.76	0.21	0.37	0.43	0.51	0.64
$\sigma[m]$	Six assets	49	7	34	65	39	44	48	55	61
$\sigma[m^{\star}]$	Six assets	52	6	36	64	44	48	51	57	62
$\rho_{m,m^{\star}}$	Six assets	0.59	0.18	0.24	0.96	0.29	0.46	0.61	0.70	0.88
$\frac{\mathrm{Var}[u]}{\mathrm{Var}[m]}$	Six assets	0.21	0.09	0.02	0.43	0.06	0.14	0.20	0.27	0.35
$\frac{\mathrm{Var}[u^{\star}]}{\mathrm{Var}[m^{\star}]}$	Six assets	0.20	0.09	0.02	0.36	0.06	0.14	0.19	0.24	0.35

Table 7: **Incomplete international markets and the volatility puzzle**

Reported results are based on the following cross-sectional regression:

$$100 \times \log(\mathrm{vp}_i^2/\mathrm{vq}_i^2) \;=\; \Omega_0 \;+\; e_i, \qquad \text{for } i = 1, \ldots, \mathcal{I},$$

where we compute vp_i^2 as the data-based unconditional variance of a currency pair and vq_i^2 as the unconditional model risk-neutral variance $\mathbb{E}[\frac{m_{t+1}}{\mathbb{E}[m_{t+1}]}(r_{t+1}^{\mathrm{fx}})^2] - (\mathbb{E}[\frac{m_{t+1}}{\mathbb{E}[m_{t+1}]}r_{t+1}^{\mathrm{fx}}])^2$ (recognizing $\mathbb{E}_t[\frac{m_{t+1}}{\mathbb{E}_t[m_{t+1}]}r_{t+1}^{\mathrm{fx}}] = \frac{F_t}{S_t} - 1$), with $r_{t+1}^{\mathrm{fx}} \equiv \frac{S_{t+1}}{S_t} - 1$. The model risk-neutral variances are based on SDFs synthesized from Result 2, and we focus on Epstein-Zin preferences in our algorithm to estimate Θ. We use $\mathbf{R}_{t+1}$ with *six assets*, which in the case of USD/JPY, contains the returns of (1) the U.S. risk-free bond, (2) the U.S. equity index, (3) the Japanese risk-free bond, (4) the Japanese equity index, (5) the U.S. 30-year Treasury bond, and (6) the MSCI world index, where each of the returns are denominated in U.S. dollars, whereas $\mathbf{R}_{t+1}^{\star}$ contains the same set of assets denominated in Japanese yen. The estimation is least squares, with heteroscedasticity-consistent (White) standard errors. The 27 currency pairs used in the cross-sectional regression correspond to USD, JPY, and SWI as the domestic currency against the remaining nine currencies.

	Coefficient Ω_0	Standard error	t-statistic
All 27 country pairs	-1.73%	(0.332)	[-5.21]

Table 8: **High reward-for-risk trades that potentially do not persist: Icelandic carry trade**

We consider the carry trade from the perspective of a U.S. investor who uses the Icelandic krona as the investment currency and the Japanese yen as the funding currency. The payoff of the long leg of the carry trade is computed as

$$\text{carry payoff}_{t+1}^{\text{long leg}} \equiv \frac{S_{t+1}^{\text{bid},\$|\text{krona}}}{F_t^{\text{ask},\$|\text{krona}}} - 1,$$

and the payoff of the short leg of the carry trade is computed as

$$\text{carry payoff}_{t+1}^{\text{short leg}} \equiv 1 \; - \; \frac{S_{t+1}^{\text{ask},\$|¥}}{F_t^{\text{bid},\$|¥}}.$$

The excess returns of the carry trade is the sum of the long and short legs, and we scale each leg of the trade to $\$\frac{1}{2}$ (to maintain a one dollar total commitment each month). S_{t+1}^{bid} (S_{t+1}^{ask}) is the bid (ask) spot exchange rate at the end of month $t+1$, and F_t^{bid} (F_t^{ask}) is the one-month forward exchange rate at the end of month t, with the U.S. dollar as the domestic currency. The bid and ask spot and forward data are from Bloomberg, and the earliest date of the availability of the forward rates is 2004:03 (i.e., March 2004). We report the average annualized excess returns of carry and its 95% confidence interval based on a stationary bootstrap (denoted as lower bootstrap CI and upper bootstrap CI) of Politis and Romano (1994) with 10,000 bootstrap iterations, in which the block size is based on the algorithm of Politis and White (2004), the annualized monthly standard deviation (SD), and the annualized sample Sharpe ratio and its 95% bootstrap confidence interval.

	Samples			
Start	2004:03	2008:01	2008:01	2012:01
End	2007:12	2015:12	2011:12	2015:12
Mean of excess return	7.41	-3.03	-11.26	5.20
Lower bootstrap CI	0.30	-9.40	-22.40	0.20
Upper bootstrap CI	13.60	3.60	0.40	11.0
SD of excess return	6.55	9.47	11.78	5.56
Sharpe ratio	1.13	-0.32	-0.96	0.93
Lower bootstrap CI	0.05	-0.89	-1.75	0.04
Upper bootstrap CI	2.33	0.45	0.04	2.00

Table 9: **Reward-for-risk of writing 5% out-of-the-money put options from the perspective of a G-10 currency**

We compute the excess returns of writing 5% out-of-the-money puts on the S&P 500 index from the perspective of a G-10 currency (over contract expiration cycles of 28 days) as $\mathrm{er}_{t+1} = \frac{\frac{-\max(K-A_{t+1},0)}{S_{t+1}^{\mathrm{bid}}}}{\frac{\mathcal{C}_t^{\bullet}}{S_t^{\mathrm{bid}}}} + \frac{(1+r_t^{\mathrm{usd}})P_t[K]\frac{1}{F_t^{\mathrm{ask}}}}{\frac{\mathcal{C}_t^{\bullet}}{S_t^{\mathrm{bid}}}}$, where A_t is the level of the S&P 500 index, $K \approx e^{-0.05}A_t$ is the underlying strike price, and r_t^{usd} is the maturity-matched U.S. interest rate. The collateral $\mathcal{C}_t^{\bullet} \equiv P_t[K] + \max(0.1K, 0.15A_t - \max(A_t - K, 0))$ is as required by the CBOE (2000, p. 22). We assume that the investor commits to $\mathcal{C}_t^{\bullet}/S_t^{\mathrm{bid}}$ in a G-10 currency. S_t^{bid} (respectively, S_t^{ask}) is the spot bid (ask) at the end of month t. The G-10 currencies are U.S. dollar (US), Australian dollar (AD), British pound (BP), Canadian dollar (CD), euro (EUR), Japanese yen (JY), New Zealand dollar (NZ), Norwegian krone (NK), Swiss franc (SF), and Swedish krona (SK). We use 28-day expiration cycle data on S&P 500 index options, which is constructed from the daily record of option prices across all strikes and maturities and purchased from the CBOE. The sample period considered is 1990:01 to 2015:12.

	US	AD	BP	CD	EUR	JY	NZ	NK	SF	SK
				A. 1990:01 to 2007:12 sample period						
Mean of excess return	37.5	36.6	37.8	37.2	33.3	37.5	36.7	37.7	37.9	37.2
Lower bootstrap CI	19.8	17.9	20.7	18.6	-1.0	19.7	17.9	20.1	20.9	19.0
Upper bootstrap CI	51.6	51.1	52.0	51.5	58.8	51.4	51.2	52.0	51.6	51.8
SD of excess return	34.7	36.5	34.3	35.2	46.2	34.3	36.5	34.4	33.8	35.5
Sharpe ratio	1.08	1.00	1.10	1.06	0.72	1.09	1.01	1.10	1.12	1.05
Lower bootstrap CI	0.42	0.35	0.44	0.38	-0.01	0.43	0.35	0.43	0.46	0.38
Upper bootstrap CI	2.48	2.39	2.50	2.47	2.39	2.47	2.40	2.53	2.51	2.51
				B. 2004:03 to 2007:12 sample period						
Mean of excess return	28.1	27.2	27.8	28.0	27.9	28.5	26.8	27.8	28.0	27.7
Lower bootstrap CI	8.6	5.5	6.3	7.6	7.7	10.0	4.1	6.9	8.2	7.1
Upper bootstrap CI	43.4	43.2	43.2	43.3	43.5	43.0	43.2	43.5	43.3	43.5
SD of excess return	18.7	20.4	19.3	18.9	19.1	17.9	21.2	19.3	18.7	19.4
Sharpe ratio	1.50	1.34	1.44	1.48	1.46	1.60	1.26	1.44	1.50	1.43
Lower bootstrap CI	0.30	0.17	0.21	0.26	0.26	0.37	0.12	0.24	0.28	0.24
Upper bootstrap CI	6.20	6.25	6.24	6.27	6.19	6.26	6.23	6.26	6.36	6.19
				C. 2008:01 to 2011:12 sample period						
Mean of excess return	7.5	-4.4	-0.4	-1.0	1.7	12.4	-2.8	-2.3	3.6	-1.3
Lower bootstrap CI	-90.6	-120.0	-107.5	-108.1	-103.3	-84.0	-112.5	-117.1	-100.4	-111.6
Upper bootstrap CI	110.2	107.4	108.9	109.1	109.2	107.6	108.1	108.7	107.6	108.7
SD of excess return	106.7	120.2	115.4	115.8	113.1	101.8	117.9	118.2	111.0	116.6
Sharpe ratio	0.07	-0.04	0.00	-0.01	0.02	0.12	-0.02	-0.02	0.03	-0.01
Lower bootstrap CI	-0.62	-0.71	-0.69	-0.68	-0.67	-0.60	-0.70	-0.71	-0.66	-0.69
Upper bootstrap CI	2.70	2.55	2.65	2.62	2.63	2.68	2.58	2.60	2.59	2.61
				D. 2012:01 to 2015:12 sample period						
Mean of excess return	42.1	42.2	42.0	42.0	42.0	42.1	42.1	42.0	42.0	42.0
Lower bootstrap CI	27.4	27.6	27.2	27.2	27.3	27.3	27.3	27.3	27.0	27.4
Upper bootstrap CI	53.5	53.5	53.4	53.4	53.7	53.5	53.3	53.2	53.1	52.9
SD of excess return	13.6	13.6	13.7	13.7	13.6	13.6	13.7	13.8	13.6	13.6
Sharpe ratio	3.09	3.09	3.08	3.06	3.08	3.11	3.08	3.05	3.08	3.08
Lower bootstrap CI	1.73	1.72	1.68	1.70	1.70	1.74	1.68	1.70	1.66	1.71
Upper bootstrap CI	8.02	8.10	7.99	8.13	7.96	8.13	7.89	8.11	8.09	8.09

Table 10: **Reward-for-risk of shorting volatility by global investors**

Let V_t be the level of VIX futures. We compute excess returns of a fully collateralized futures position from the perspective of each of the G-10 currency investor as

$$\mathrm{er}_{t+1} = \begin{cases} -\{\frac{V_{t+1}}{V_t} - 1\}\frac{S_t^{\mathrm{bid}}}{S_{t+1}^{\mathrm{ask}}} & \text{If } V_{t+1} < V_t, \\ -\{\frac{V_{t+1}}{V_t} - 1\}\frac{S_t^{\mathrm{bid}}}{S_{t+1}^{\mathrm{bid}}} & \text{If } V_{t+1} > V_t. \end{cases}$$

We select the second-nearest maturity futures contract to avoid rolling over the front-month futures contract. The VIX futures data are from CBOE historical files (i.e., the 12 contracts with tickers VX/F, VX/G, VX/H, VX/J, VX/K, VX/M, VX/N, VX/Q, VX/U, VX/V, VX/X, and VX/Z). In all calculations, S_t^{bid} (respectively, S_t^{ask}) is the spot bid (ask) at the end of month t. The G-10 currencies are U.S. dollar (US), Australian dollar (AD), British pound (BP), Canadian dollar (CD), euro (EUR), Japanese yen (JY), New Zealand dollar (NZ), Norwegian krone (NK), Swiss franc (SF), and Swedish krona (SK). The sample period considered is 2004:03 to 2015:12.

	US	AD	BP	CD	EUR	JY	NZ	NK	SF	SK
A. 2004:03 to 2007:12 sample period										
Mean of excess return	52.3	51.5	52.7	51.4	52.5	53.6	51.5	52.4	52.8	51.9
Lower bootstrap CI	5.2	5.3	6.7	5.2	6.1	8.2	6.1	6.3	8.9	7.6
Upper bootstrap CI	93.4	91.3	92.8	92.2	91.6	93.2	91.5	92.6	93.8	92.4
SD of excess return	40.7	40.7	40.5	40.6	40.40	40.3	40.6	40.5	40.4	40.4
Sharpe ratio	1.29	1.26	1.30	1.27	1.30	1.33	1.27	1.29	1.31	1.29
Lower bootstrap CI	0.11	0.11	0.14	0.12	0.13	0.18	0.13	0.14	0.19	0.16
Upper bootstrap CI	2.99	2.95	2.99	2.98	2.94	3.01	2.96	3.02	3.03	3.03
B. 2008:01 to 2015:12 sample period										
Mean of excess return	35.8	29.6	32.9	31.5	32.3	37.6	29.6	30.9	33.1	31.5
Lower bootstrap CI	-12.8	-21.6	-17.2	-19.6	-16.7	-9.8	-20.5	-20.2	-16.9	-20.2
Upper bootstrap CI	81.1	77.2	79.6	78.1	79.4	81.9	76.9	79.1	79.0	78.2
SD of excess return	68.1	72.3	70.3	70.9	70.2	66.7	71.4	71.3	69.3	70.5
Sharpe ratio	0.53	0.41	0.47	0.44	0.46	0.56	0.42	0.43	0.48	0.45
Lower bootstrap CI	-0.16	-0.24	-0.20	-0.23	-0.19	-0.12	-0.23	-0.23	-0.20	-0.23
Upper bootstrap CI	1.61	1.49	1.56	1.51	1.56	1.60	1.47	1.52	1.53	1.52
C. 2008:01 to 2011:12 sample period										
Mean of excess return	16.37	6.14	12.12	9.96	10.66	18.16	7.88	8.94	11.79	9.15
Lower bootstrap CI	-59.98	-77.41	-68.67	-70.35	-71.81	-56.39	-77.69	-75.42	-66.41	-74.03
Upper bootstrap CI	85.75	80.03	84.46	82.73	81.67	85.71	80.49	82.89	81.54	81.11
SD of excess return	74.89	81.45	78.36	79.22	78.57	72.39	80.07	79.92	76.77	79.04
Sharpe ratio	0.22	0.08	0.15	0.13	0.14	0.25	0.10	0.11	0.15	0.12
Lower bootstrap CI	-0.63	-0.72	-0.67	-0.69	-0.69	-0.61	-0.73	-0.71	-0.68	-0.71
Upper bootstrap CI	1.77	1.68	1.79	1.74	1.69	1.81	1.70	1.69	1.70	1.71
D. 2012:01 to 2015:12 sample period										
Mean of excess return	55.29	53.11	53.66	53.05	53.87	57.01	51.40	53.00	54.45	53.81
Lower bootstrap CI	-12.24	-12.94	-14.10	-14.02	-13.35	-9.59	-16.91	-15.64	-13.55	-14.12
Upper bootstrap CI	105.30	104.81	104.49	104.88	103.89	106.64	102.97	103.95	105.66	103.95
SD of excess return	60.94	61.99	61.41	61.65	60.97	60.67	61.70	61.75	61.06	60.97
Sharpe ratio	0.91	0.86	0.87	0.86	0.88	0.94	0.83	0.86	0.89	0.88
Lower bootstrap CI	-0.16	-0.17	-0.19	-0.18	-0.18	-0.13	-0.23	-0.21	-0.18	-0.19
Upper bootstrap CI	2.44	2.39	2.39	2.43	2.40	2.46	2.38	2.41	2.42	2.38

Table 11: **Results based on a sample of emerging economies**

This table provides evidence from a sample of nine emerging economies for which we can construct complete data: BRR (Brazil), CHP (Chile), COP (Colombia), MXP (Mexico), MYR (Malaysia), PHP (Philippines), PLZ (Poland), THB (Thailand), and TWD (Taiwan). The foreign economy here is the United States. The data are from Datastream and starts in January 1995. To estimate Θ, we minimize the discrepancy between $\rho_{m,m^\star}$ based on the asset market view in Equation (29) and its counterpart computed based on real wealth growth in Equation (31). We report the estimates of Θ and its bootstrap-based lower and upper confidence intervals, the SDF volatility, the correlation between the SDFs ($\rho_{m,m^\star}$), and the size of the unspanned component of the SDF ($\frac{\mathrm{Var}[u]}{\mathrm{Var}[m]}$). Reported also are the cross-sectional mean across all the emerging economies (shown under the column "Average"). For comparison, we report the corresponding mean across the industrialized economies (AUD, NZD, STG, FRA, CAD, NLG, GER, JPY, and SFR, each against the United States). We employ six assets: the U.S. risk-free bond, the U.S. MSCI equity index (total return), the risk-free bond and the MSCI equity index (total return) of the relevant emerging economy, the U.S. 30-year Treasury bond, and the MSCI world equity index (total return). All returns are real (computed from nominal returns and adjusting by realized inflation).

	Emerging economies (the United States is the foreign economy)									Average
	BRR	CHP	COP	MXP	MYR	PHP	PLZ	THB	TWD	Emerging
										⌊Industrialized⌋
Θ	0.68	1.27	1.10	0.66	1.05	0.74	0.94	0.64	0.83	0.88
Lower bootstrap CI	0.43	0.89	0.91	0.44	0.67	0.53	0.70	0.41	0.55	⌊0.54⌋
Upper bootstrap CI	0.86	1.54	1.38	0.84	1.56	0.88	1.16	0.91	1.10	
Size of the unspanned component	0.13	0.29	0.35	0.16	0.36	0.31	0.23	0.24	0.24	0.26 ⌊0.16⌋
SDF correlations	0.62	0.49	0.23	0.62	0.37	0.45	0.49	0.45	0.52	0.47 ⌊0.67⌋
SDF volatility	106	123	94	80	95	67	101	67	86	91 ⌊66⌋

Implications of Incomplete Markets for International Economies

Gurdip Bakshi Mario Cerrato John Crosby

Internet Appendix: Not for Publication

Abstract

This internet appendix provides additional theoretical elaboration and data documentation.

Section I shows that $|\mathbb{E}[y - y^\star]|$ should, in practice, be very small, and $\mathbb{E}[y - y^\star]$ is identically zero when m, $m^\star$ and S_{t+1}/S_t are lognormally distributed (two results which support our analysis in Appendix A).

The thrust of Section II is an example economy in which $m\,(S_{t+1}/S_t) - m^\star$ need not be zero in each state of the world and the domestic country has low interest-rates, while the foreign country has high interest-rates. Section II.A describes the parameterized economy with five states and presents the results on the properties of $(m, m^\star)$ in Table Internet-I. Section II.B builds on the preceding analysis and cross-checks the solution of Equation (15) in a simulation setting. Section II.C shows the impact of adding an asset with convex payoffs (the returns of an options combination strategy).

Section III outlines the construction of the currency options data (used in Section 3.1), while Sections IV and V, respectively, provide the intermediate steps for the calculation of excess returns of writing out-of-the-money puts and shorting market volatility from the perspective of a G-10 currency.

Table Internet-III reports results when *nominal* (as opposed to real) returns are used and is the nominal counterpart to Table 2 (and, thus, contextualizes the discussions in footnote 4).

I $|\mathbb{E}[y - y^\star]|$ should, in practice, be very small and proof that $\mathbb{E}[y - y^\star] = 0$ when m, $m^\star$ and S_{t+1}/S_t are lognormally distributed

Our objective is to show that (i) $|\mathbb{E}[y - y^\star]|$ should, in practice, be very small, and (ii) $\mathbb{E}[y - y^\star]$ is identically zero when m, $m^\star$ and S_{t+1}/S_t are lognormally distributed.

We focus first on the case in which $\log(m)$, $\log(m^\star)$, and $\log(S_{t+1}/S_t)$ are jointly normally distributed, and prove an exact result. We have

$$\mathbb{E}[m] = 1/R_f, \qquad \mathbb{E}[m^\star] = 1/R_f^\star, \qquad \text{and} \qquad \mathbb{E}[S_{t+1}/S_t] \equiv \exp(\mu_s), \tag{IA1}$$

and the variances are $\mathrm{Var}[\log(m)]$, $\mathrm{Var}[\log(m^\star)]$, and $\mathrm{Var}[\log(S_{t+1}/S_t)] \equiv \sigma_s^2$, respectively.

Using results on moment generating functions, $\mathbb{E}[(S_{t+1}/S_t)^{1/2}] = \exp\left(\frac{1}{2}\mu_s - \frac{1}{8}\sigma_s^2\right)$ and $\mathbb{E}[(S_{t+1}/S_t)^{-1/2}] = \exp\left(-\frac{1}{2}\mu_s + \frac{3}{8}\sigma_s^2\right)$. Equation (4), namely, $\mathbb{E}_t[m(\frac{S_{t+1}}{S_t}) - m^\star] = 0$, implies

$$\frac{1}{R_f^\star} = \frac{1}{R_f}\exp\left(\mu_s + \mathrm{Cov}\left[\log(m), \log(S_{t+1}/S_t)\right]\right), \tag{IA2}$$

while $\mathbb{E}[m - m^\star/(\frac{S_{t+1}}{S_t})] = 0$ implies (i.e., Balakrishnan and Lai (2009, equation (11.69), page 526)):

$$\frac{1}{R_f} = \frac{1}{R_f^\star}\exp\left(-\mu_s + \sigma_s^2 - \mathrm{Cov}\left[\log(m^\star), \log(S_{t+1}/S_t)\right]\right). \tag{IA3}$$

Further,

$$\begin{aligned}
\mathbb{E}[y] = \mathbb{E}[m(S_{t+1}/S_t)^{1/2}] &= (1/R_f)\exp\left(\frac{1}{2}\mu_s + \frac{1}{2}\mathrm{Cov}\left[\log(m), \log(S_{t+1}/S_t)\right] - \frac{1}{8}\sigma_s^2\right), \\
&= 1/\sqrt{R_f R_f^\star}\,\exp\left(-\frac{1}{8}\sigma_s^2\right), \qquad \text{(using (IA2))} \qquad \text{(IA4)} \\
\mathbb{E}[y^\star] = \mathbb{E}[m^\star(S_{t+1}/S_t)^{-1/2}] &= \left(1/R_f^\star\right)\exp\left(-\frac{1}{2}\mu_s - \frac{1}{2}\mathrm{Cov}\left[\log(m^\star), \log(S_{t+1}/S_t)\right] + \frac{3}{8}\sigma_s^2\right) \\
&= 1/\sqrt{R_f R_f^\star}\,\exp\left(-\frac{1}{8}\sigma_s^2\right). \qquad \text{(using (IA3))} \qquad \text{(IA5)}
\end{aligned}$$

Thus, $\mathbb{E}[y - y^\star] = 0$ and our assertion is proved. ■

Next, we show $|\mathbb{E}[y - y^\star]| \approx 0$, in general. With $y = m\sqrt{S_{t+1}/S_t}$ and $y^\star = m^\star/\sqrt{S_{t+1}/S_t}$, the analog to equation (4) is

$$\mathbb{E}[(y - y^\star)\,(S_{t+1}/S_t)^{1/2}] = 0 \qquad \text{and, likewise,} \qquad \mathbb{E}[(y - y^\star)\,\frac{1}{(S_{t+1}/S_t)^{1/2}}] = 0. \tag{IA6}$$

More generally, defining $g[\phi] \equiv (\phi\, S_{t+1}/S_t)^{1/2} + 1/(\phi\, S_{t+1}/S_t)^{1/2}$, for any ϕ, satisfying $0 < \phi < +\infty$, we have

$$\mathbb{E}[(y - y^\star)(\phi\, S_{t+1}/S_t)^{1/2} + (y - y^\star)/(\phi\, S_{t+1}/S_t)^{1/2}] \;\equiv\; \mathbb{E}[(y - y^\star)\, g[\phi]] \;=\; 0. \tag{IA7}$$

Hence, $\text{Cov}\,[y - y^\star, g[\phi]] = -\mathbb{E}[y - y^\star]\mathbb{E}[g[\phi]]$. Thus, $|\mathbb{E}[y - y^\star]|^2(\mathbb{E}[g[\phi]])^2 \leq \text{Var}\,[y - y^\star]\,\text{Var}\,[g[\phi]]$. Or,

$$|\mathbb{E}[y - y^\star]|^2\,\{(\mathbb{E}[g[\phi]])^2 - \text{Var}\,[g[\phi]]\} \leq \mathbb{E}[(y - y^\star)^2]\text{Var}\,[g[\phi]]\,. \tag{IA8}$$

We further note that $\text{Var}\,[g[\phi]] = \mathbb{E}[((\phi\, S_{t+1}/S_t) + 1/(\phi\, S_{t+1}/S_t)) - 2] - \mathbb{E}[g[\phi] - 2]\,\mathbb{E}[g[\phi] + 2]$.

In practice, for $\phi \approx 1$, $\text{Var}\,[g[\phi]]$ may be very small. For example, empirical data suggests S_{t+1}/S_t is close to a martingale, so $\mathbb{E}[S_{t+1}/S_t] \approx 1$. Moreover, S_{t+1}/S_t has relatively low volatility, suggesting $\mathbb{E}[1/(S_{t+1}/S_t)]$ may also be close to unity. Hence, $\mathbb{E}[((S_{t+1}/S_t) + 1/(S_{t+1}/S_t)) - 2] \approx 0$, $\mathbb{E}[g[1]] \approx 2$, and $\mathbb{E}[g[1] - 2] = \mathbb{E}[(S_{t+1}/S_t)^{1/2} + 1/(S_{t+1}/S_t)^{1/2} - 2] \approx 0$, and, thus, $\text{Var}\,[g\,(1)]$ should typically be close to zero. Hence, in practice, equation (IA8), evaluated in the special case of $\phi = 1$, should imply a tight bound on $|\mathbb{E}[y - y^\star]|$. ■

II An incomplete markets model parameterized by five states

In Table Internet-I, we present a parameterized two-country economy with five states of the world, featuring that $m\,(S_{t+1}/S_t) - m^\star$ need not be zero in each state.

The economy is constructed to capture some relevant features. First, the domestic country supports a low risk-free interest-rate (say, Japan), whereas the foreign country (say, Australia) a high risk-free interest-rate. Second, the returns of the risky asset (e.g., equity) display positive correlation.

A Properties of the international economy

Our objective is to illustrate the solution technique and highlight the volatilities and correlation between m and $m^\star$. We further show that the problem is well-posed with a finite objective and well-defined Lagrange multipliers, and the solution supports $m > 0$ and $m^\star > 0$ over a wide range of values of Θ.

We compute $e_{\mathbf{z}}$ by solving equation (B11) and then d and $d^\star$. The solution method for computing e_{z} is to regress one on $\mathbf{Z}$. With $y_{\mathbf{z}}$, δ, d, and $d^\star$, we compute y and $y^\star$ using equation (B1). We verify our solution and check if $\mathbb{E}[\delta\mathbf{Z}] = 0$, for each element of $\mathbf{Z}$. In accordance with Appendix A and the Internet Appendix (Section I), we verify that $\mathbb{E}[y - y^\star]$ is not far from zero.

With the computed values of y and $y^\star$, we obtain m and $m^\star$ across the five states using equation (11). Prompted by the specifics of our solution, we compute (i) $\sqrt{\mathrm{Var}[m]}$ and $\sqrt{\mathrm{Var}[m^\star]}$, and (ii) the correlation $\rho_{m,m^\star}$.

The question is: If the market incompleteness parameter Θ were to be assumed close to zero, how would the properties of $(m,\, m^\star)$ change by ruling out "good deals," that is, $\mathbb{E}[(y - y^\star)^2] \leq \Theta^2$ in comparison with $\mathbb{E}[(y - y^\star)^2] \approx 0$. As seen, the economy supports a lower correlation $\rho_{m,m^\star}$ but with the added effect of raising the volatility of m and $m^\star$. We also check our solution by directly minimizing the objective in (15), subject to the constraints in (16)–(18).

In summary, we construct a discrete-time, five-state economy in which $m\,(S_{t+1}/S_t) - m^\star$ need not be zero and the domestic (foreign) country has low (high) risk-free interest-rates. We show that the nonnegativity constraints do not bind, and m and $m^\star$ are strictly positive in each state. Crucially, increasing the market incompleteness parameter, Θ, reduces the correlation between m and $m^\star$, while increasing the volatility of m and $m^\star$. ■

B Cross-check of the solution by simulating return realizations

In this exercise, we simulate return realizations 492 times (matching the monthly data in Table 1) from the distribution in Panel A of Table Internet-I. Setting $\Theta = 0.6$, we compute y and $y^\star$ and, hence, m and $m^\star$ (by minimizing the objective in equation (15)). The following results obtain:

	Properties of y, $y^\star$, m, and $m^\star$ in the simulation							
	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	Mean	$\sigma[m]$	$\rho_{m,m^\star}$
	(States of the world)						($\sigma[m^\star]$)	
y	2.0501	1.0072	1.2317	0.3493	0.1873	0.975	-	
$y^\star$	1.8712	1.7298	0.6038	0.9002	0.6203	0.975	-	
m	2.1610	1.0676	1.2317	0.3316	0.1786	0.9901	51	
$m^\star$	1.7752	1.6319	0.6038	0.9484	0.6506	0.9615	(44)	
								0.19

We note that (within rounding errors) $\mathbb{E}[m_{t+1}\,\mathbf{R}_{t+1}] = \mathbf{1}$, and $\mathbb{E}[m^\star_{t+1}\,\mathbf{R}^\star_{t+1}] = \mathbf{1}$. Moreover, since the constraints $y \geq 0$ and $y^\star \geq 0$ do not bind (i.e., the constraints hold with strict inequality), the constraint $\mathbb{E}[(y-y^\star)^2] \leq \Theta^2$ must hold with *equality*. In other words, it must be true, and we verify, that $\mathbb{E}[(y-y^\star)^2] = \Theta^2 = 0.6^2$. This simulation exercise cross-validates our solution. ■

C Impact of adding convex payoffs in the form of option returns

One may inquire: What is the impact of adding an asset with convex payoffs? To provide an answer, we stay in the setting of Table Internet-I, but construct the return of a long position in a call and a put on the exchange rate (as in Section 3.1). The return of the options combination strategy in state ω is $(\max(S[\omega]-K,0)+\max(K-S[\omega],0))/0.082$. Here, $K=0.97$ is the strike price and 0.082 is the total cost of the call and put position. We remove the foreign risky asset to preserve the incomplete market setting with number of assets less than the number of states.

The big picture from the results reported in Panel C of Table Internet-I (focusing on the case when $\Theta = 0.4$) is that, first, in adding option returns, the correlation $\rho_{m,m^\star}$ rises from 0.55 (in Panel B) to 0.77. Second, the international economy supports a higher volatility of the SDFs. ■

III Details of the construction of currency options data

The currency market convention is to quote option data as 10 delta, 25 delta, and at-the-money put or call volatilities. Quoted implied volatilities are available for G-10 currencies and correspond to European options of constant maturity and constant deltas. We employ the following notation:

τ: Remaining time to expiration of options and forwards, and equals 30/360;

$\sigma_t[\Delta_C]$ or $\sigma_t[\Delta_P]$: Implied volatilities at time t, where Δ_C or Δ_P takes a value of 10, 25 or at-the-money (ATM);

$F^{j,i}_{t,t+\tau} \equiv S^{j,i}_t \frac{e^{-r^j\tau}}{e^{-r^i\tau}}$: τ period forward price of one unit of currency j (the foreign currency) in terms of currency i (the domestic currency);

r^i_t and r^j_t: τ-period matched (net) risk-free rates in the two economies.

Let K_{ATM}, K_{Δ_C}, and K_{Δ_P} be the strike prices for the respective call and put options. The following conversion formulas are used to extract the strike prices (e.g., Wystup (2006)):

$$K_{\text{ATM}} = F^{j,i}_{t,t+\tau} \exp\left(\frac{1}{2}\sigma_t[\text{ATM}]^2\tau\right), \tag{IA9}$$

$$K_{\Delta_C} = F^{j,i}_{t,t+\tau} \exp\left(\frac{1}{2}\sigma^2_t[\Delta_C]\tau - \sigma_t[\Delta_C]\,\sqrt{\tau}\,N^{-1}\left[\exp(r^j_t\,\tau)\,\Delta_C\right]\right), \quad \text{and} \tag{IA10}$$

$$K_{\Delta_P} = F^{j,i}_{t,t+\tau} \exp\left(\frac{1}{2}\sigma^2_t[\Delta_P]\tau + \sigma_t[\Delta_P]\,\sqrt{\tau}\,N^{-1}\left[-\exp(r^j_t\,\tau)\,\Delta_P\right]\right). \tag{IA11}$$

Next, we construct the put and call option prices using the Garman-Kolhagen formula as

$$C_t[K] = e^{-r^i_t\tau}\,(F^{j,i}_{t,t+\tau}\,\mathcal{N}[d_1] \;-\; K\,\mathcal{N}[d_2]), \tag{IA12}$$

$$P_t[K] = e^{-r^i_t\tau}\,(K\,\mathcal{N}[-d_2]) \;-\; F^{j,i}_{t,t+\tau}\,\mathcal{N}[-d_1]), \tag{IA13}$$

where

$$d_1 \equiv \frac{1}{\sqrt{\tau}\,\sigma_t[K]}\,\log(F^{j,i}_{t,t+\tau}/K) \;+\; \frac{1}{2}\,\sigma_t[K] \qquad \text{and} \qquad d_2 \equiv d_1 - \sqrt{\tau}\,\sigma_t[K]. \tag{IA14}$$

Finally, we construct the gross returns of straddles and strangles. ■

IV Expected excess returns of writing equity index puts

To streamline the presentation, we adopt the following conventions in our notation:

A_t= level of the S&P 500 index;

K= strike price of put option on the S&P 500 index;

$P_t[K]$= put option price with strike price K;

r_t^{usd}= maturity-matched net interest rate on a deposit in U.S. dollars;

S_t^{bid} (F_t^{bid})= Bid spot (ask forward) exchange rate at the beginning of expiration cycle t;

$\mathcal{C}_t^{\bullet} = P_t[K] + \max(0.1\,K, 0.15A_t - \max(A_t - K, 0))$ is the collateral required by the CBOE.

The investor commits a capital of $\mathcal{C}_t^{\bullet}/S_t^{\text{bid}}$ in a G-10 currency to fund the CBOE collateral (i.e., buying the U.S. dollar). The gross return in a G-10 currency is

$$1 + r_{t+1}^{\text{put}} = \frac{1}{\frac{\mathcal{C}_t^{\bullet}}{S_t^{\text{bid}}}} \left(\frac{-\max(K - A_{t+1}, 0)}{S_{t+1}^{\text{bid}}} + (1 + r_t^{\text{usd}})(\mathcal{C}_t^{\bullet} + P_t[K]) \frac{1}{F_t^{\text{ask}}} \right). \tag{IA15}$$

Using the covered parity condition, the excess return is consequently

$$\text{er}_{t+1} = \frac{1}{\frac{\mathcal{C}_t^{\bullet}}{S_t^{\text{bid}}}} \left(\frac{-\max(K - A_{t+1}, 0)}{S_{t+1}^{\text{bid}}} + (1 + r_t^{\text{usd}}) P_t[K] \frac{1}{F_t^{\text{ask}}} \right). \tag{IA16}$$

The excess return in equation (IA16) is featured in our calculations in Table 9. ■

V Expected excess returns of shorting equity volatility

To streamline presentation, we adopt the following conventions in our notation:

V_t= level of the VIX futures;

r_t^{usd}= maturity-matched net interest rate on a deposit in U.S. dollars.

We have

$$1 + r_{t+1}^{\text{short}} = \begin{cases} -\{\frac{V_{t+1}}{V_t} - 1\} \frac{S_t^{\text{bid}}}{S_{t+1}^{\text{ask}}} + \frac{(1+r_t^{\text{usd}}) V_t \frac{1}{F_t^{\text{ask}}}}{\frac{V_t}{S_t^{\text{bid}}}} & \text{If } V_{t+1} < V_t, \\ -\{\frac{V_{t+1}}{V_t} - 1\} \frac{S_t^{\text{bid}}}{S_{t+1}^{\text{bid}}} + \frac{(1+r_t^{\text{usd}}) V_t \frac{1}{F_t^{\text{ask}}}}{\frac{V_t}{S_t^{\text{bid}}}} & \text{If } V_{t+1} > V_t. \end{cases} \tag{IA17}$$

Therefore,

$$1 + r_{t+1}^{\text{short}} = \begin{cases} -\{\frac{V_{t+1}}{V_t} - 1\}\frac{S_t^{\text{bid}}}{S_{t+1}^{\text{ask}}} + (1 + r_t^{\text{G10 currency}}) & \text{If } V_{t+1} < V_t, \\ -\{\frac{V_{t+1}}{V_t} - 1\}\frac{S_t^{\text{bid}}}{S_{t+1}^{\text{bid}}} + (1 + r_t^{\text{G10 currency}}) & \text{If } V_{t+1} > V_t. \end{cases} \tag{IA18}$$

In this light, the excess return, $\text{er}_{t+1} = r_{t+1}^{\text{short}} - r_t^{\text{G10 currency}}$, is

$$\text{er}_{t+1} = \begin{cases} -\{\frac{V_{t+1}}{V_t} - 1\}\frac{S_t^{\text{bid}}}{S_{t+1}^{\text{ask}}} & \text{If } V_{t+1} < V_t, \\ -\{\frac{V_{t+1}}{V_t} - 1\}\frac{S_t^{\text{bid}}}{S_{t+1}^{\text{bid}}} & \text{If } V_{t+1} > V_t. \end{cases} \tag{IA19}$$

The excess return in equation (IA19) is featured in our calculations in Table 10. ■

Table Internet-I: **Properties of m and $m^\star$ in an example economy with five states and where $m\,(S_{t+1}/S_t) - m^\star \neq 0$**

In our calculations, the domestic country is Japan (with a low risk-free interest-rate), and the foreign country is Australia (with a high risk-free interest-rate). The exchange rate growth $\frac{S_{t+1}}{S_t}$ is denominated in ¥|AD. We compute $e_{\mathbf{z}}$ by solving equation (B11) and then d and $d^\star$. The computer code for obtaining the solution by minimizing the objective in (15), subject to the constraints in Equations (16)–(18), is available from the authors (in C++ and in an Excel spreadsheet setting). $\sigma[m]$ and $\sigma[m^\star]$ are the volatilities of m and $m^\star$, and $\rho_{m,m^\star}$ is the correlation between m and $m^\star$. Panel C considers four assets: domestic and foreign risk-free bonds, the domestic risky asset, and options combination strategy on the foreign exchange. The return of the options combination strategy in state ω is $(\max(S[\omega] - K, 0) + \max(K - S[\omega], 0))/0.082$, where $K = 0.97$ is the strike price and 0.082 is the total cost of the call and put position on the foreign exchange.

Panel A: Parametrization of the economy

	States of the world				
	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
Probability	0.07	0.20	0.45	0.21	0.07
Risk-free (domestic)	1.01	1.01	1.01	1.01	1.01
Risk-free (foreign)	1.04	1.04	1.04	1.04	1.04
Risky (domestic)	0.87	1.01	1.04	1.05	1.08
Risky (foreign)	0.87	1.01	1.08	1.04	1.42
Exchange rate growth	0.90	0.89	1.00	1.11	1.10

Panel B: Properties of m and $m^\star$ obtained by varying Θ

	States of the world							
	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	Mean	$\sigma[m]$ ($\sigma[m^\star]$)	$\rho_{m,m^\star}$
$\Theta = 0.05$								
m	1.9887	1.3949	0.9410	0.5724	0.4036	0.9901	41	
$m^\star$	1.7763	1.2975	0.8889	0.6841	0.4858	0.9615	(32)	
								0.99
$\Theta = 0.40$								
m	2.0415	1.1745	1.1232	0.4188	0.2706	0.9901	44	
$m^\star$	1.7288	1.4937	0.7068	0.8547	0.6321	0.9615	(37)	
								0.55
$\Theta = 0.60$								
m	2.0716	1.0485	1.2273	0.3309	0.1946	0.9901	49	
$m^\star$	1.7017	1.6058	0.6027	0.9522	0.7157	0.9615	(43)	
								0.19

Panel C: Impact of adding convex payoffs (return of options)

$\Theta = 0.40$								
m	2.0095	1.9293	0.4443	0.7555	1.4988	0.9901	64	
$m^\star$	1.5660	1.7888	0.4705	1.2550	0.2700	0.9615	(57)	
								0.77

Table Internet-II: **Reward-for-risk of a strategy in which the domestic investor takes a position in the foreign equity index**

This table reports the excess returns of a strategy in which the domestic investor takes a position in the foreign equity index, and the domestic equity index is the risky benchmark. Specifically, the excess return is

$$\mathrm{er}_{t+1}^{\mathrm{equity}} \equiv \left(\frac{S_{t+1}}{S_t}\right) R_{t+1}^{\mathrm{equity}*} - R_{t+1}^{\mathrm{equity}},$$

where $R_{t+1}^{\mathrm{equity}}$ and $R_{t+1}^{\mathrm{equity}*}$, respectively, are the gross returns of the domestic and foreign equity index, measured in their own currencies. We compute the (absolute) reward-for-risk of this strategy as

$$\frac{\left|\mathbb{E}[\mathrm{er}_{t+1}^{\mathrm{equity}}]\right|}{\sqrt{\mathrm{Var}[\mathrm{er}_{t+1}^{\mathrm{equity}}]}}.$$

To clarify the reward-for-risk reported below, the currency of the domestic investor is specified by the rows, while the currency of the foreign investor is specified by the first column. As an example of how to read the entries in the table, 0.26 is the reward-for-risk for a Japanese (JPY) investor investing in the Netherlands (NLG) equity index. The sample period is January 1975 to December 2015, and the reward-for-risks are reported as annualized decimals.

	Domestic investor									
	AUD	NZD	STG	FRA	CAD	USD	NLG	GER	JPY	SWI
Foreign										
AUD		0.14	0.01	0.06	0.13	0.10	0.01	0.10	0.20	0.06
NZD	0.10		0.12	0.06	0.01	0.03	0.13	0.02	0.09	0.07
STG	0.08	0.17		0.09	0.17	0.15	0.01	0.13	0.24	0.10
FRA	0.01	0.11	0.06		0.09	0.07	0.10	0.06	0.18	0.01
CAD	0.08	0.06	0.12	0.04		0.01	0.14	0.01	0.13	0.04
USD	0.04	0.09	0.10	0.02	0.05		0.14	0.03	0.16	0.02
NLG	0.08	0.19	0.03	0.10	0.19	0.19		0.18	0.26	0.12
GER	0.03	0.07	0.10	0.05	0.04	0.02	0.18		0.14	0.05
JPY	0.12	0.02	0.18	0.12	0.06	0.09	0.21	0.08		0.15
SWI	0.02	0.13	0.05	0.01	0.11	0.09	0.10	0.07	0.20	

Table Internet-III: **Estimates of Θ, when $\mathbf{R}_{t+1}$ ($\mathbf{R}^{\star}_{t+1}$) contains six assets but are *nominal* (as opposed to real) returns**

In this table, we focus on nominal quantities (this table can be viewed as the nominal counterpart to Table 2). Reported are the Θ estimates, the nominal SDF volatilities, the correlation between the nominal SDFs, and the size of the unspanned components of the SDFs for each of the 45 country pairs. To estimate Θ, we consider an algorithm that minimizes the discrepancy between $\rho_{m,m^{\star}}$ that is constructed based on power utility over nominal wealth growth and the corresponding one constructed from nominal asset returns.

	Panel A		*Panel B: Properties of the domestic and foreign nominal SDFs*				
	Θ	Bootstrap 95% CI	$\sigma[m]$	$\sigma[m^{\star}]$	$\rho_{m,m^{\star}}$	$\frac{\mathrm{Var}[u]}{\mathrm{Var}[m]}$	$\frac{\mathrm{Var}[u^{\star}]}{\mathrm{Var}[m^{\star}]}$
AUD/NZD	0.23	[0.15 0.34]	32	34	0.72	0.15	0.12
AUD/STG	0.44	[0.29 0.55]	49	52	0.58	0.18	0.20
AUD/FRA	0.29	[0.19 0.38]	42	45	0.75	0.11	0.11
AUD/CAD	0.33	[0.21 0.46]	47	50	0.76	0.12	0.11
AUD/USD	0.43	[0.28 0.55]	51	55	0.65	0.18	0.17
AUD/NLG	0.31	[0.22 0.36]	38	47	0.73	0.17	0.13
AUD/GER	0.42	[0.31 0.52]	46	50	0.59	0.20	0.19
AUD/JPY	0.41	[0.34 0.51]	43	48	0.55	0.22	0.20
AUD/SWI	0.43	[0.30 0.52]	50	56	0.65	0.18	0.16
NZD/STG	0.43	[0.22 0.62]	41	44	0.44	0.27	0.26
NZD/FRA	0.34	[0.24 0.44]	41	43	0.63	0.16	0.17
NZD/CAD	0.42	[0.22 0.63]	42	47	0.53	0.23	0.24
NZD/USD	0.54	[0.30 0.71]	53	56	0.49	0.25	0.26
NZD/NLG	0.45	[0.32 0.53]	44	49	0.60	0.20	0.19
NZD/GER	0.41	[0.33 0.51]	44	47	0.57	0.21	0.20
NZD/JPY	0.36	[0.29 0.51]	39	43	0.56	0.21	0.19
NZD/SWI	0.42	[0.31 0.49]	43	49	0.55	0.23	0.20
STG/FRA	0.41	[0.28 0.54]	49	50	0.63	0.17	0.18
STG/CAD	0.54	[0.25 0.75]	56	56	0.52	0.23	0.24
STG/USD	0.45	[0.21 0.61]	57	59	0.68	0.16	0.15
STG/NLG	0.35	[0.22 0.43]	55	57	0.79	0.10	0.10
STG/GER	0.39	[0.25 0.48]	44	49	0.65	0.19	0.17
STG/JPY	0.52	[0.31 0.70]	51	53	0.48	0.24	0.25
STG/SWI	0.45	[0.33 0.53]	57	59	0.68	0.15	0.15
FRA/CAD	0.29	[0.19 0.38]	41	42	0.75	0.11	0.11
FRA/USD	0.36	[0.20 0.47]	46	50	0.71	0.16	0.14
FRA/NLG	0.31	[0.21 0.38]	57	59	0.85	0.08	0.08
FRA/GER	0.28	[0.13 0.37]	49	52	0.85	0.08	0.08
FRA/JPY	0.33	[0.23 0.48]	42	44	0.67	0.15	0.15
FRA/SWI	0.33	[0.18 0.43]	52	55	0.80	0.10	0.10
CAD/USD	0.33	[0.12 0.48]	42	47	0.72	0.16	0.14
CAD/NLG	0.34	[0.21 0.43]	51	52	0.76	0.11	0.11
CAD/GER	0.41	[0.27 0.53]	50	51	0.65	0.17	0.16
CAD/JPY	0.42	[0.29 0.52]	49	51	0.61	0.17	0.18
CAD/SWI	0.48	[0.25 0.62]	58	59	0.64	0.17	0.17
USD/NLG	0.33	[0.20 0.41]	58	56	0.81	0.08	0.08
USD/GER	0.36	[0.22 0.47]	51	51	0.73	0.12	0.12
USD/JPY	0.46	[0.30 0.57]	48	49	0.53	0.21	0.22
USD/SWI	0.41	[0.26 0.50]	60	61	0.74	0.12	0.11
NLG/GER	0.30	[0.18 0.39]	55	55	0.85	0.07	0.07
NLG/JPY	0.49	[0.35 0.59]	57	57	0.60	0.18	0.20
NLG/SWI	0.29	[0.18 0.36]	58	59	0.87	0.06	0.07
GER/JPY	0.43	[0.25 0.65]	47	48	0.56	0.20	0.22
GER/SWI	0.26	[0.13 0.33]	53	54	0.88	0.06	0.06
JPY/SWI	0.54	[0.38 0.63]	51	52	0.44	0.28	0.26