THE PRIME SPECTRA OF RELATIVE STABLE MODULE CATEGORIES

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To Dave Benson on the occasion of his 60th birthday

ABSTRACT. For a finite group G and an arbitrary commutative ring R, Broué has placed a Frobenius exact structure on the category of finitely generated RG-modules by taking the exact sequences to be those that split upon restriction to the trivial subgroup. The corresponding stable category is then tensor triangulated. In this paper we examine the case $R = S/t^n$, where S is a discrete valuation ring having uniformising parameter t. We prove that the prime ideal spectrum (in the sense of Balmer) of this 'relative' version of the stable module category of RG is a disjoint union of n copies of that for kG, where k is the residue field of S.

CONTENTS

1. INTRODUCTION

Let G be a finite group and k a field whose characteristic divides the order of G. One of the main goals in modular representation theory is to try to understand the stable module category $\mathsf{stmod}\, kG$ of kG. The objects in $\mathsf{stmod}\, kG$ are the same as those in the category $\mathsf{mod}\, kG$ of finitely generated kG-modules. If M and N are finitely generated kG-modules, then the morphisms from M to N in $\mathsf{stmod}\, kG$ are elements of the quotient

$$\underline{\mathrm{Hom}}_{kG}(M,N) = \mathrm{Hom}_{kG}(M,N)/\mathrm{PHom}_{kG}(M,N),$$

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where $\mathrm{PHom}_{kG}(M, N)$ denotes the set of kG-module homomorphisms $M \to N$ that factor through some projective kG-module. In this case $\mathsf{mod}\, kG$ is a *Frobenius category*, and so $\mathsf{stmod}\, kG$ is triangulated. The suspension of an object M in $\mathsf{stmod}\, kG$ is defined to be its cosyzygy $\Omega^{-1}M$, the cokernel of the inclusion of M into its injective hull. A distinguished triangle

$$M' \longrightarrow M \longrightarrow M'' \longrightarrow \Omega^{-1}M'$$

in $\mathsf{stmod}\, kG$ is, by definition, induced by a short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

in $\mathsf{mod}\, kG$. Further details may be found in [8], for example. Moreover, equipping $- \otimes_k -$ with the diagonal G-action gives an exact symmetric monoidal structure on $\mathsf{mod}\, kG$. This passes down to $\mathsf{stmod}\, kG$, making it a *tensor triangulated category.*

Now suppose that we replace k with an arbitrary commutative ring R. Can one hope to study the category $\mathsf{mod}\, RG$ by mimicking the above setup? The first obstruction to doing so is the fact that $\mathsf{mod}\, RG$ may no longer be Frobenius, in which case $\mathsf{stmod}\, RG$ would fail to be triangulated in the usual way. Even if $\mathsf{mod}\, RG$ were Frobenius, there would still be no guarantee that tensoring over R would be exact, that is, $\mathsf{stmod}\, RG$ might not be tensor triangulated.

Broué [7] has introduced an alternative exact structure on $\mathsf{mod}\, RG$, which was based on work of Higman [9] and later examined by Benson, Iyengar and Krause [6]. Let

$$\iota \colon R \longrightarrow RG$$

denote the inclusion of the ground ring R and

$$\iota_* \colon \ \mathsf{mod}\, RG \longrightarrow \mathsf{mod}\, R$$

the restriction of scalars functor. An exact sequence of RG-modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is defined to be admissible if its restriction

$$0 \longrightarrow \iota_* M' \longrightarrow \iota_* M \longrightarrow \iota_* M'' \longrightarrow 0$$

is split exact. In this paper we will denote the category of finitely generated RG-modules endowed with this 'relatively split' exact structure by $\mathsf{rel}\, RG$. As shown in [7], $\mathsf{rel}\, RG$ *is* a Frobenius category, so its stable category $\mathsf{strel}\, RG$ is triangulated.

Specifically, the injective/projective objects in $\mathsf{rel}\, RG$ are the direct summands of those RG-modules lying in the essential image of the induction functor

$$\iota^* = RG \otimes_R - \colon \ \mathsf{mod}\, R \longrightarrow \mathsf{mod}\, RG.$$

We shall call such modules *weakly projective.* The suspension of an object M in $\mathsf{strel}\, RG$ is the cokernel ΣM in the R-split short exact sequence

$$0 \longrightarrow M \longrightarrow \iota^*\iota_* M \longrightarrow \Sigma M \longrightarrow 0,$$

where the map $M \to \iota^*\iota_* M = RG \otimes_R M$ is given by $m \mapsto \sum_{g\in G} g \otimes g^{-1}m$. A nice feature of the relative stable category is that, in the special case where $R = k$ is a field, $\mathsf{strel}\, kG$ and $\mathsf{stmod}\, kG$ coincide.

Another motivation for appealing to this relative exact structure is that tensoring over R preserves R-split short exact sequences, so $\mathsf{strel}\, RG$ is tensor triangulated with unit the trivial module R. The relative stable category therefore provides a setting in which one may exploit a monoidal structure to study representations of G over arbitrary commutative rings.

With the relative stable category in mind, we now recall that Balmer [2] has developed a general framework for studying the coarse structures of essentially small tensor triangulated categories in terms of supports, namely the yoga of *tensor triangular geometry*. Let $(\mathcal{K}, \otimes, \mathbf{1})$ be an essentially small tensor triangulated category. A thick subcategory $\mathcal{I}$ of $\mathcal{K}$ is a *tensor ideal* of $\mathcal{K}$ if $\mathcal{K} \otimes \mathcal{I} \subseteq \mathcal{I}$. Continuing the analogy with commutative algebra, Balmer defines a proper tensor ideal $\mathcal{P}$ of $\mathcal{K}$ to be *prime* if whenever x and y are objects in $\mathcal{K}$ satisfying $x \otimes y \in \mathcal{P}$, then $x \in \mathcal{P}$ or $y \in \mathcal{P}$. The collection of prime ideals of $\mathcal{K}$ is called the (*prime ideal*) *spectrum* of $\mathcal{K}$, denoted $\operatorname{Spc}\mathcal{K}$.

For an object $x \in \mathcal{K}$, Balmer defines the *support* of x to be the subset

$$\operatorname{supp}(x) = \{\mathcal{P} \in \operatorname{Spc}\mathcal{K} \mid x \notin \mathcal{P}\}$$

of $\operatorname{Spc}\mathcal{K}$. Such subsets form a basis of closed subsets of a Zariski topology on $\operatorname{Spc}\mathcal{K}$. The topological space $\operatorname{Spc}\mathcal{K}$ and the assignment $x \mapsto \operatorname{supp}(x)$ form something Balmer calls a *universal support data* for $\mathcal{K}$. Roughly speaking, this means that $\operatorname{Spc}\mathcal{K}$ is the best possible abstract setting in which to study support theoretic questions about the category $\mathcal{K}$. For example, one may express the universality of $\operatorname{Spc}\mathcal{K}$ by interpreting it as the space dual to the lattice of (radical) thick tensor ideals. Accordingly, any question about the structure of the lattice of tensor ideals is really a question about $\operatorname{Spc}\mathcal{K}$.

To see all of this in action, we recall a celebrated result of Benson, Carlson and Rickard [5], which states that the thick tensor ideals of $\mathsf{stmod}\, kG$ are in one to one correspondence with the specialisation closed subsets of $\mathsf{Proj}\, H^*(G,k)$, where $H^*(G,k)$ denotes group cohomology with coefficients in k, i.e., $\operatorname{Ext}^*_{kG}(k,k)$. Statements that relate the structure of $\mathsf{stmod}\, kG$ to the geometry of $\mathsf{Proj}\, H^*(G,k)$ occupy the realm of *support theory*. For example, the above correspondence assigns to each thick tensor ideal $\mathcal{I}$ in $\mathsf{stmod}\, kG$ the subset $\bigcup_{M\in\mathcal{I}} V_G(M)$ of $\mathsf{Proj}\, H^*(G,k)$, where $V_G(M)$ is the *support variety* of M. (See [4, Chapter 5] for details.) Viewed in Balmer's framework, this result may be reinterpreted as saying that $(\mathsf{Proj}\, H^*(G,k), V_G(-))$ is the *classifying support data* for $\mathsf{stmod}\, kG$. (See [2, Section 5].) In particular, $\operatorname{Spc}(\mathsf{stmod}\, kG)$ is homeomorphic to $\mathsf{Proj}\, H^*(G,k)$.

We remark that little is known about relative stable categories in general. The goal of this paper is to determine the prime ideal spectrum of $\mathsf{strel}\, RG$ in perhaps the most basic non-trivial case, namely that in which R is the ring $\mathbb{Z}/p^n$, where p is a prime number and $n \geq 0$.

More generally, the proofs go through for $R = S/t^n$ where S is a discrete valuation ring with uniformising parameter t. In that context, our main result is the following (the above case being that of the localisation $S = \mathbb{Z}_{(p)}$ and $t = p$).

Theorem 1.1. *Let S be a discrete valuation ring having residue field k and uniformising parameter t. Setting $R_n = S/t^n$, there is a homeomorphism*

$$\operatorname{Spc}(\mathsf{strel}\, R_n G) \cong \coprod_{i=1}^{n} \operatorname{Spc}(\mathsf{strel}\, kG).$$

In other words, the prime ideal spectrum of $\mathsf{strel}\, R_n G$ decomposes into n disjoint copies of the prime ideal spectrum of $\mathsf{strel}\, kG = \mathsf{stmod}\, kG$.

As mentioned above, the spectrum of $\mathsf{stmod}\, kG$ is known, so the theorem yields a complete description of the spectrum of $\mathsf{strel}\, R_n G$.

This computation is also valuable from the point of view of abstract tensor triangular geometry. There are many examples in which the spectrum has been computed, but they all tend to share a common feature—the tensor triangulated category in question is rigid. Relative stable categories need not be rigid, hence they provide a new family of examples that can be fed back into the abstract theory. For instance, if the spectrum of a rigid tensor triangulated category is a disjoint union of subspaces, then the category itself decomposes into a direct sum of subcategories indexed by those subspaces. However, in our example the relative stable category is indecomposable. The information encoded in the triviality of the topology of the spectrum must therefore manifest in more subtle ways that would be interesting to explore.

2. Notation and preliminary calculations

Let $(S, \mathfrak{m}, k)$ be a *discrete valuation ring*, that is, a local principal ideal domain whose unique maximal ideal is $\mathfrak{m}$ and whose residue field is

$$k = S/\mathfrak{m}.$$

(See [1, Chapter 9].) We denote by t a *uniformising parameter*, i.e., a generator of $\mathfrak{m}$. For a positive integer n we let

$$R_n = S/t^n.$$

Throughout this paper G will denote a finite group. We set

$$A_n = R_n G,$$

the group algebra of G over R_n. Keeping the notation from the introduction, we continue to denote the inclusion of the ground ring by $\iota\colon R_n \hookrightarrow A_n$. For each $i \le n$, the canonical surjection $A_n \to A_i$ gives A_i the structure of an A_n-module. We write Ω_i for the syzygy and Ω_i^{-1} for the cosyzygy, taken with respect to the usual abelian structure in $\mathsf{mod}\, A_i$.

We dedicate the remainder of this section to module theoretic computations that rely on the group structure of G. The rest of the paper depends by and large only on the ring structure of S and may be read almost independently of these results.

We begin with an explicit description of the weakly projective modules in $\mathsf{mod}\, A_n$, that is, the injective/projective objects in $\mathsf{rel}\, A_n$.

Proposition 2.1. *Every weakly projective A_n-module is a direct sum of objects in*

$$\bigcup_{i=1}^{n} \mathsf{proj}\, A_i,$$

where $\mathsf{proj}\, A_i$ *is the full subcategory of finitely generated projective A_i-modules.*

Proof. As mentioned in the introduction, the weakly projective A_n-modules are the direct summands of the modules in

$$\{\iota^* N \mid N \in \mathsf{mod}\, R_n\}.$$

(Recall that $\iota^* N$ is the induced module $A_n \otimes_{R_n} N$.) The indecomposable R_n-modules are of the form $R_i = S/t^i$ for $1 \leq i \leq n$. The result follows by noting that $A_n \otimes_{R_n} R_i = A_i$. $\square$

We remark that since each A_i is R_i-free, every object in $\mathsf{proj}\, A_i$ is also R_i-free.

The main object of focus in the sequel will be the cosyzygy $\Omega_n^{-1} k$, which is the cokernel in the short exact sequence of A_n-modules

$$0 \longrightarrow k \longrightarrow A_n \longrightarrow \Omega_n^{-1} k \longrightarrow 0, \tag{1}$$

the map $k \to A_n$ being given by $1 \mapsto t^{n-1} \sum_{g \in G} g$. We first study the behaviour of $\Omega_n^{-1} k$ under base change.

Lemma 2.2. $\Omega_n^{-1} k \otimes_{R_n} R_{n-1} \cong A_{n-1}$.

Proof. Since the embedding $k \hookrightarrow A_n$ maps into $t^{n-1} A_n$, tensoring it with R_{n-1} yields the zero map. By the right exactness of $- \otimes_{R_n} R_{n-1}$, the sequence (1) gives rise to the exact sequence

$$k \xrightarrow{\;0\;} A_{n-1} \longrightarrow \Omega_n^{-1} k \otimes_{R_n} R_{n-1} \longrightarrow 0.$$

The map $A_{n-1} \to \Omega_n^{-1} k \otimes_{R_n} R_{n-1}$ is therefore an isomorphism. $\square$

The following ingredient will ensure that a special triangle exists in $\mathsf{strel}\, A_n$.

Lemma 2.3. *There exists an R_n-split short exact sequence of A_n-modules*

$$0 \longrightarrow R_{n-1} \longrightarrow \Omega_n^{-1} k \longrightarrow \Omega_n^{-1} R_n \longrightarrow 0. \tag{2}$$

Proof. The submodule $\Omega_n k$ of A_n is the collection of elements $\sum_{g \in G} r_g g$ satisfying

$$\sum_{g \in G} r_g \in t R_n.$$

Consider the surjective A_n-module homomorphism $\phi \colon \Omega_n k \to R_{n-1}$ given by

$$\sum_{g \in G} r_g g \longmapsto \frac{1}{t} \sum_{g \in G} r_g \pmod{t^{n-1}}.$$

The kernel of ϕ is the collection of elements $\sum_{g \in G} r_g g$ satisfying $\sum_{g \in G} r_g = 0$, which we identify with $\Omega_n R_n$. We therefore have a short exact sequence

$$0 \longrightarrow \Omega_n R_n \longrightarrow \Omega_n k \longrightarrow R_{n-1} \longrightarrow 0.$$

Note that $\Omega_n R_n$ is injective as an R_n-module, so this sequence is R_n-split. The sequence (2) is obtained by applying $\mathrm{Hom}_{R_n}(-, R_n)$, which preserves R_n-split exact sequences. $\square$

Our final result of the section will be used later in Section 6 to establish certain orthogonality relations.

Lemma 2.4. *For all* $1 \le i < j \le n$ *we have* $\Omega_i^{-1}k \otimes_{R_n} \Omega_j^{-1}k \cong A_{i-1} \oplus A_i^{\oplus(|G|-1)}$.

Proof. Because t^j annihilates both R_i and R_j, we have

$$\Omega_i^{-1}k \otimes_{R_n} \Omega_j^{-1}k \cong \Omega_i^{-1}k \otimes_{R_j} \Omega_j^{-1}k.$$

We may therefore compute the right hand term in $\mathsf{mod}\, A_j$. Writing $f = \sum_{g \in G} g \in A_j$ and applying $- \otimes_{R_j} \Omega_i^{-1}k$ to the short exact sequence (1) yields the exact sequence

$$k \otimes_{R_j} \Omega_i^{-1}k \xrightarrow{t^{j-1}f \otimes 1} A_j \otimes_{R_j} \Omega_i^{-1}k \longrightarrow \Omega_j^{-1}k \otimes_{R_j} \Omega_i^{-1}k \longrightarrow 0$$

so that $\Omega_j^{-1}k \otimes_{R_j} \Omega_i^{-1}k$ is isomorphic to $A_j \otimes_{R_j} \Omega_i^{-1}k$ modulo the image of $t^{j-1}f \otimes 1$. By Frobenius reciprocity and the sequence (2), one sees that

$$A_j \otimes_{R_j} \Omega_i^{-1}k \cong A_{i-1} \oplus A_i^{\oplus(|G|-1)}. \tag{3}$$

Since $i < j$, t^{j-1} annihilates the right hand term, so the image of $t^{j-1}f \otimes 1$ is zero and

$$\Omega_j^{-1}k \otimes_{R_j} \Omega_i^{-1}k \cong A_{i-1} \oplus A_i^{\oplus(|G|-1)}.$$ $\square$

3. Localisation sequences and splitting of spectra

To begin our discussion of triangulated categories, we now set

$$\mathcal{D}_n = \mathsf{strel}\, A_n,$$

the relative stable module category of A_n. As noted in the introduction, this is a tensor triangulated category with tensor unit

$$\mathbf{1}_n = R_n.$$

As is customary, we denote the suspension in $\mathcal{D}_n$ by Σ, keeping in mind that Σ is not the cosyzygy Ω_n^{-1} in general.

Observe that it is not necessary to specify in which category Σ is suspension. Indeed, if $i \le n$, then the canonical surjection $A_n \to A_i$ induces a fully faithful embedding $\mathcal{D}_i \subseteq \mathcal{D}_n$, because a short exact sequence of A_i-modules is R_i-split if and only if it is R_n-split. Thus the suspension of an object in $\mathcal{D}_i$ will equal that in $\mathcal{D}_n$.

An essential ingredient in proving Theorem 1.1 will be the notion of a semi-orthogonal decomposition of a tensor triangulated category. We recall that a *localisation sequence* or *semi-orthogonal decomposition* is a diagram of exact functors

$$\mathcal{R} \underset{\psi^!}{\overset{\psi_*}{\rightleftarrows}} \mathcal{S} \underset{\phi_*}{\overset{\phi^*}{\rightleftarrows}} \mathcal{T}$$

in which $\psi^!$ is right adjoint to ψ_* and ϕ_* is right adjoint to ϕ^*, the functors ψ_* and ϕ_* are fully faithful and there are equalities

$$\phi_*\mathcal{T} = (\psi_*\mathcal{R})^\perp = \{x \in \mathcal{S} \mid \mathrm{Hom}_{\mathcal{S}}(\psi_*\mathcal{R}, x) = 0\}$$

and

$$\psi_*\mathcal{R} = {}^\perp(\phi_*\mathcal{T}) = \{x \in \mathcal{S} \mid \mathrm{Hom}_{\mathcal{S}}(x, \phi_*\mathcal{T}) = 0\}.$$

For our purposes we also require that $\mathcal{S}$ is an essentially small tensor triangulated category and that $\mathcal{R}$ and $\mathcal{T}$ are tensor ideals of $\mathcal{S}$ under ψ_* and ϕ_*, respectively. In this very special situation one also obtains a decomposition of the prime ideal spectrum of $\mathcal{S}$. If $\mathcal{C}$ is any subcategory of $\mathcal{S}$, we define the *support* of $\mathcal{C}$ to be the specialisation closed subset

$$\mathrm{supp}_{\mathcal{S}}\,\mathcal{C} = \bigcup_{x\in\mathcal{C}} \mathrm{supp}_{\mathcal{S}}\, x$$

of $\mathrm{Spc}\,\mathcal{S}$.

Theorem 3.1 ([6], Theorem A.5)**.** *The subsets* $\mathrm{Spc}\,\mathcal{R} = \mathrm{supp}_{\mathcal{S}}\,\mathcal{R}$ *and* $\mathrm{Spc}\,\mathcal{T} = \mathrm{supp}_{\mathcal{S}}\,\mathcal{T}$ *of* $\mathrm{Spc}\,\mathcal{S}$ *are open and closed, and there is a decomposition*

$$\mathrm{Spc}\,\mathcal{S} = \mathrm{Spc}\,\mathcal{R} \coprod \mathrm{Spc}\,\mathcal{T}.$$

We now return to the categories $\mathcal{D}_i = \mathsf{strel}\,A_i$, $i \geq 1$. As explained in [6, Remark 6.10], each canonical ring epimorphism $\phi_i\colon A_i \to A_{i-1}$ induces a localisation sequence

$$\begin{tikzcd} \mathcal{K}_i \arrow[r, shift left, "{\psi_i}_*"] & \mathcal{D}_i \arrow[l, shift left, "{\psi_i}^!"] \arrow[r, shift left, "{\phi_i}^*"] & \mathcal{D}_{i-1} \arrow[l, shift left, "{\phi_i}_*"] \end{tikzcd}$$

where ${\phi_i}_*$ is restriction of scalars, ${\phi_i}^* = - \otimes \mathbf{1}_{i-1} = - \otimes_{R_i} R_{i-1}$ and

$$\mathcal{K}_i = \ker(- \otimes \mathbf{1}_{i-1}). \tag{4}$$

Note that the functors ${\phi_i}^*$ and ${\psi_i}^!$ are strong monoidal, i.e., they preserve tensor products and units. (See [11, Chapter XI.2] for full details on strong monoidal functors.) It follows that ${\psi_i}_*\mathcal{K}_i$ and ${\phi_i}_*\mathcal{D}_{i-1}$ are thick tensor ideals in $\mathcal{D}_i$. We frequently identify $\mathcal{K}_i$ and $\mathcal{D}_{i-1}$ with their images under these fully faithful embeddings. We also remark in passing that $\mathcal{D}_1$ is just the usual stable module category $\mathsf{stmod}\,kG$ of kG. A slight generalisation of the conclusion of [6, Remark 6.10] is the following result.

Theorem 3.2. *Setting* $\mathcal{K}_1 = \mathcal{D}_1$*, there is a decomposition of spectra*

$$\mathrm{Spc}\,\mathcal{D}_n = \coprod_{i=1}^{n} \mathrm{Spc}\,\mathcal{K}_i.$$

Proof. This follows by induction on n and the appropriate use of Theorem 3.1. □

In order to prove Theorem 1.1, it therefore suffices to show that each $\mathrm{Spc}\,\mathcal{K}_i$ is homeomorphic to $\mathrm{Spc}\,\mathcal{D}_1$. To establish this fact, we first show that each $\mathcal{K}_i$ is equal to the thick tensor ideal

$$\mathsf{thick}^{\otimes}_{\mathcal{D}_i}(\Omega_i^{-1}k)$$

of $\mathcal{D}_i$ generated by $\Omega_i^{-1}k$. We then exhibit a monoidal equivalence between $\mathsf{thick}^{\otimes}_{\mathcal{D}_i}(\Omega_i^{-1}k)$ and $\mathcal{D}_1$.

4. The tensor ideal $\mathsf{thick}^{\otimes}_{\mathcal{D}_n}(W_n)$

To simplify notation somewhat and to emphasise its central role throughout the rest of the paper, we now set

$$W_n = \Omega_n^{-1}k.$$

Lemma 4.1. *We have $\mathcal{K}_n = \mathsf{thick}^{\otimes}_{\mathcal{D}_n}(W_n)$, where $\mathcal{K}_n$ is as defined by (4).*

Proof. By Lemma 2.2 we have $W_n \otimes \mathbf{1}_{n-1} = 0$ in $\mathcal{D}_n$ so that $\mathsf{thick}^{\otimes}_{\mathcal{D}_n}(W_n) \subseteq \mathcal{K}_n$.

Conversely, let $X \in \mathcal{K}_n$. The R_n-split short exact sequence (2) induces a triangle

$$\mathbf{1}_{n-1} \longrightarrow W_n \longrightarrow \Sigma\mathbf{1}_n \longrightarrow$$

in $\mathcal{D}_n$. Since $X \in \mathcal{K}_n$, tensoring this triangle with X yields a triangle of the form

$$0 \longrightarrow X \otimes W_n \longrightarrow \Sigma X \longrightarrow \ .$$

It follows that $\Sigma X \cong X \otimes W_n$ so that $X \in \mathsf{thick}^{\otimes}_{\mathcal{D}_n}(W_n)$. $\square$

Lemma 4.2. *Every object in $\mathcal{K}_n$ is isomorphic in $\mathcal{D}_n$ to an A_n-module whose S-module decomposition contains only summands of the form R_n and R_{n-1}.*

Proof. Let $X \in \mathcal{K}_n$. Viewing X as an A_n-module, write $\iota_* X \cong \bigoplus_{i=1}^n R_i^{\oplus r_i}$. Let

$$U = R_n^{\oplus r_n} \oplus R_{n-1}^{\oplus r_{n-1}} \qquad \text{and} \qquad V = \bigoplus_{i=1}^{n-2} R_i^{\oplus r_i}$$

so that $\iota_* X = U \oplus V$. Note that $t^{n-1}X \subseteq U$ since t^{n-1} annihilates V. We thus have

$$\iota_*(X \otimes_{R_n} R_{n-1}) = (U/t^{n-1}X) \oplus V.$$

By assumption, $X \otimes \mathbf{1}_{n-1} = 0$ in $\mathcal{D}_n$, so $X \otimes_{R_n} R_{n-1}$ is weakly projective. Proposition 2.1 implies that

$$X \otimes_{R_n} R_{n-1} \cong Y \oplus Z,$$

where $Y \in \mathsf{proj}\, A_{n-1}$ and Z is a direct sum of objects in $\bigcup_{i=1}^{n-2} \mathsf{proj}\, A_i$. Comparing S-module structures, we must have

$$\iota_* Y \cong U/t^{n-1}X \qquad \text{and} \qquad \iota_* Z \cong V.$$

These S-module isomorphisms allow us to place A_n-module structures $\widetilde{U/t^{n-1}}X$ and $\widetilde{V}$ on $U/t^{n-1}X$ and V, respectively, through which $X \otimes_{R_n} R_{n-1} = \widetilde{U/t^{n-1}}X \oplus \widetilde{V}$.

Now consider the short exact sequence of A_n-modules

$$0 \longrightarrow X' \longrightarrow X \longrightarrow \widetilde{V} \longrightarrow 0, \tag{5}$$

where the right hand map is the composition

$$X \xrightarrow{\pi} X \otimes_{R_n} R_{n-1} \xrightarrow{\mathrm{pr}_2} \widetilde{V}.$$

Observe that we have $X' = \{m \in X \mid \pi(m) \in \ker \mathrm{pr}_2\} = U$ as abelian groups. Applying ι_* to the sequence (5) therefore yields a split short exact sequence of S-modules

$$0 \longrightarrow U \longrightarrow \iota_* X \longrightarrow V \longrightarrow 0.$$

This means that (5) gives rise to a triangle

$$X' \longrightarrow X \longrightarrow \widetilde{V} \longrightarrow$$

in $\mathcal{D}_n$. Because $\widetilde{V} \cong Z$ is weakly projective in $\mathsf{mod}\, A_n$, we have $\widetilde{V} = 0$ in $\mathcal{D}_n$. It follows that $X' \cong X$ in $\mathcal{D}_n$. $\square$

Lemma 4.3. *We have $\Omega_n \mathcal{K}_n = \mathcal{D}_1$.*

Proof. Let $X \in \mathcal{K}_n$. To prove that $\Omega_n X \in \mathcal{D}_1$, it suffices to show that t annihilates $\Omega_n X$. Employing Lemma 4.2, we assume that the S-module decomposition of $\iota_* X$ is $R_n^{\oplus r} \oplus R_{n-1}^{\oplus s}$ for some non-negative integers r and s. We then have

$$\iota_*(X \otimes_{R_n} R_{n-1}) \cong R_{n-1}^{\oplus(r+s)}. \tag{6}$$

On the other hand, $X \otimes_{R_n} R_{n-1}$ is known to be weakly projective. The decomposition (6) implies that $X \otimes_{R_n} R_{n-1} \in \mathsf{proj}\, A_{n-1}$. We thus have $X \otimes_{R_n} R_{n-1} \cong \bigoplus_j A_{n-1} e_j$ for some idempotents $e_j \in A_{n-1}$. It is known (see [10, Theorem 21.28]) that since the extension of scalars $A_n \to A_{n-1}$ is a surjective ring homomorphism with nilpotent kernel, the e_j lift to idempotents $f_j \in A_n$.

Now consider the projective module $Y = \bigoplus_j A_n f_j$. Let

$$\pi\colon X \longrightarrow X \otimes_{R_n} R_{n-1} \qquad \text{and} \qquad \phi\colon Y \longrightarrow X \otimes_{R_n} R_{n-1}$$

be the A_n-module homomorphisms induced by the extension of scalars from R_n to R_{n-1}. Because π is surjective and Y is projective, ϕ lifts to an A_n-module homomorphism

$$\psi\colon Y \longrightarrow X$$

satisfying $\pi \circ \psi = \phi$. Note that $\psi \otimes_{R_n} R_{n-1}$ is the identity on $X \otimes_{R_n} R_{n-1}$; in particular, it is surjective. It follows by Nakayama's lemma that ψ is surjective. We therefore have a short exact sequence

$$0 \longrightarrow \Omega_n X \longrightarrow Y \xrightarrow{\psi} X \longrightarrow 0.$$

But $\Omega_n X$ is also contained in the kernel of the composition $\pi \circ \psi = \phi$. The kernel of the latter is annihilated by t, hence the same is true of its submodule $\Omega_n X$.

Conversely, let $X \in \mathcal{D}_1$ and consider the short exact sequence

$$0 \longrightarrow X \xrightarrow{\phi} A_n^{\oplus r} \xrightarrow{\psi} \Omega_n^{-1} X \longrightarrow 0$$

defining a cosyzygy of X in $\mathsf{mod}\, A_n$. Since X lies in $\mathcal{D}_1$, we know that $\iota_* X$ is a k-vector space, so the image of ϕ is contained in $t^{n-1} A_n^{\oplus r}$. This means that $\phi \otimes_{R_n} R_{n-1} = 0$, hence $\psi \otimes_{R_n} R_{n-1}$ is an isomorphism. We therefore have $\Omega_n^{-1} X \otimes_{R_n} R_{n-1} \cong A_{n-1}^{\oplus r}$, and the latter is weakly projective. This shows that $\Omega_n^{-1} X$ lies in $\mathcal{K}_n$ as claimed. $\square$

We are now ready for the main theorem of this section.

Theorem 4.4. *The syzygy Ω_n induces an equivalence of triangulated categories*

$$\mathsf{thick}^{\otimes}_{\mathcal{D}_n}(W_n) \cong \mathsf{thick}^{\otimes}_{\mathcal{D}_n}(k) = \mathsf{stmod}\, kG.$$

Proof. We saw in Lemma 4.1 that $\mathsf{thick}^{\otimes}_{\mathcal{D}_n}(W_n) = \mathcal{K}_n$, and we know that $\mathsf{thick}^{\otimes}_{\mathcal{D}_n}(k) = \mathcal{D}_1$. Notice that Ω_n induces a (not necessarily monoidal) exact autoequivalence of $\mathcal{D}_n$. Indeed, it is straightforward to check that Ω_n preserves R_n-split short exact sequences in $\mathsf{mod}\, A_n$. The same is then true of its quasi-inverse Ω_n^{-1}. The restriction of Ω_n to $\mathcal{K}_n$ is therefore an equivalence onto its essential image. By Lemma 4.3, the latter is precisely $\mathcal{D}_1$. $\square$

Although the above equivalence is *not* monoidal in general, the next section will show that $\mathsf{thick}^{\otimes}_{\mathcal{D}_n}(W_n)$ and $\mathsf{stmod}\, kG$ are in fact equivalent as *tensor triangulated* categories.

5. A monoidal equivalence

By the localisation sequences in Section 3, we know that $\mathsf{thick}^{\otimes}_{\mathcal{D}_n}(W_n) = \mathcal{K}_n$ is equal to $\mathcal{D}_n/\mathcal{D}_{n-1}$ as tensor ideals in $\mathcal{D}_n$, so $\mathsf{thick}^{\otimes}_{\mathcal{D}_n}(W_n)$ is tensor triangulated. In this section we exhibit a monoidal exact equivalence between $\mathsf{thick}^{\otimes}_{\mathcal{D}_n}(W_n)$ and $\mathcal{D}_1$, namely the restriction of the functor

$$P\colon \mathcal{D}_n \longrightarrow \mathcal{D}_1$$

induced by multiplication by t^{n-1}.

Specifically, if M is an A_n-module, then as an abelian group, $P(M)$ is defined to be the A_n-submodule $t^{n-1}M$ of M. Identifying A_1 with A_n/t, there is an action of A_1 on $t^{n-1}M$ given by

$$\overline{a}(t^{n-1}m) = t^{n-1}am \qquad \text{for all } a \in A_n.$$

This action is well defined since t annihilates $t^{n-1}M$. If $\phi\colon M \to N$ is a homomorphism of A_n-modules, then so is $\phi|_{t^{n-1}M}\colon t^{n-1}M \to N$. Moreover, for $m \in M$ we have

$$\phi(t^{n-1}m) = t^{n-1}\phi(m) \in t^{n-1}N,$$

hence $\phi|_{t^{n-1}M}$ induces a map $t^{n-1}M \to t^{n-1}N$. We therefore set $P(\phi) = \phi|_{t^{n-1}M}$.

Note that multiplication by t^{n-1} preserves R_n-split short exact sequences in $\mathsf{mod}\, A_n$, so P is exact. We claim that P is monoidal. To see this, let M and M' be A_n-modules and consider the map

$$\phi\colon P(M) \otimes_{R_1} P(M') \longrightarrow P(M \otimes_{R_n} M')$$

given by $t^{n-1}m \otimes t^{n-1}m' \mapsto t^{n-1}(m \otimes m')$. One readily checks that ϕ is a well defined isomorphism of abelian groups and that the action of G commutes with ϕ.

Having established that P is an exact tensor functor, we now let

$$\widetilde{P}\colon \mathsf{thick}^{\otimes}_{\mathcal{D}_n}(W_n) \longrightarrow \mathcal{D}_1$$

denote the restriction of P to $\mathsf{thick}^{\otimes}_{\mathcal{D}_n}(W_n)$. Our strategy in proving that $\widetilde{P}$ is a monoidal equivalence will be to show that the functor

$$F = \Omega_n^{-1}\Omega_1\colon \mathcal{D}_1 \longrightarrow \mathsf{thick}^{\otimes}_{\mathcal{D}_n}(W_n).$$

is a quasi-inverse.

Lemma 5.1. *The composition $PF\colon \mathcal{D}_1 \to \mathcal{D}_1$ is naturally isomorphic to the identity functor.*

Proof. Let X be an object in $\mathcal{D}_1$ and consider the short exact sequence of A_1-modules

$$0 \longrightarrow \Omega_1 X \longrightarrow A_1^{\oplus r} \longrightarrow X \longrightarrow 0$$

defining a syzygy $\Omega_1 X$ in $\mathsf{mod}\, A_1$. A cosyzygy of $\Omega_1 X$ in $\mathsf{mod}\, A_n$ is then obtained via the short exact sequence

$$0 \longrightarrow \Omega_1 X \longrightarrow A_n^{\oplus r} \longrightarrow \Omega_n^{-1}\Omega_1 X \longrightarrow 0$$

and we have a diagram

$$\begin{tikzcd} 0 \arrow[r] & \Omega_1 X \arrow[r] \arrow[d, equal] & A_1^{\oplus r} \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \Omega_1 X \arrow[r] & A_n^{\oplus r} \arrow[r] & FX \arrow[r] & 0, \end{tikzcd}$$

where the right two vertical arrows are those induced by multiplication by t^{n-1}. The right hand arrow therefore identifies X with the submodule PFX. $\square$

Corollary 5.2. *The functor $\widetilde{P}$ is full, essentially surjective and monoidal.*

Proof. The fact that $\widetilde{P}$ is full and essentially surjective follows from Lemmas 5.1 and 4.3, the latter of which implies that the essential image of F is contained in $\mathsf{thick}^{\otimes}_{\mathcal{D}_n}(W_n)$.

By the discussion preceding Lemma 5.1 we know that $P\colon \mathcal{D}_n \to \mathcal{D}_1$ is monoidal, so its restriction $\widetilde{P}\colon \mathsf{thick}^{\otimes}_{\mathcal{D}_n}(W_n) \to \mathcal{D}_1$ respects tensor products. Any such functor that is also essentially surjective will automatically be monoidal. $\square$

Lemma 5.3. *The kernel of $\widetilde{P}$ is trivial.*

Proof. Let X be an object in $\mathsf{thick}^{\otimes}_{\mathcal{D}_n}(W_n)$ with $\widetilde{P}X$ weakly projective. By Lemma 4.2, we may assume that $\iota_* X \cong R_n^{\oplus r} \oplus R_{n-1}^{\oplus s}$ for some non-negative integers r and s. Because $\widetilde{P}X$ is weakly projective, we have $\widetilde{P}X \cong \bigoplus_j A_1 e_j$ for some idempotents e_j in A_1.

The surjection $A_n \to A_1$ given by multiplication by t^{n-1} has kernel tA_n, thus it is isomorphic to the base change homomorphism $A_n \to A_n \otimes_{R_n} k$. As in the proof of Lemma 4.3, there then exist idempotents f_j in A_n satisfying $e_j = t^{n-1} f_j$. Letting $Y = \bigoplus_j A_n f_j$, we obtain a natural embedding $\phi\colon \widetilde{P}X \hookrightarrow Y$ mapping $\widetilde{P}X$ isomorphically onto $t^{n-1}Y$.

Note that since A_n is injective as a module over itself and Y is a direct summand of a free A_n-module, Y is also injective. (Actually, Y is the injective hull of $\widetilde{P}X$.) This, along with the embedding $\widetilde{P}X \hookrightarrow X$, allows us to extend ϕ to a morphism $\psi\colon X \to Y$.

Now let $\widetilde{\psi}$ denote the map of free R_n-modules obtained by restricting ψ to the R_n-free component $R_n^{\oplus r}$ of X. Then $t^{n-1}\widetilde{\psi} = \phi$, hence $\widetilde{\psi}$ is an isomorphism on socles. This shows that Y has rank r as a free R_n-module and that ψ is surjective. Because Y is projective, we may therefore split off a direct summand Y from X and assume that $\iota_* X \cong R_{n-1}^{\oplus s}$. We then have $X \otimes_{R_n} R_{n-1} \cong X$. But X lies in $\mathsf{thick}^{\otimes}_{\mathcal{D}_n}(W_n)$, the kernel of $- \otimes_{R_n} R_{n-1}$, hence $X \cong 0$ in $\mathcal{D}_n$. □

Theorem 5.4. *The functor $\widetilde{P}$ is a monoidal equivalence of triangulated categories.*

Proof. We know that $\widetilde{P}$ is full and essentially surjective by Corollary 5.2, and it has trivial kernel by Lemma 5.3. Appealing to a bit of folklore (see [3, Proposition 3.18]), these conditions are sufficient for $\widetilde{P}$ to be an equivalence. It is monoidal by Corollary 5.2. □

We are now in a position to prove the main result.

Theorem (1.1)**.** *For every positive integer n there is a homeomorphism*

$$\operatorname{Spc} \mathcal{D}_n \cong \coprod_{i=1}^{n} \operatorname{Spc} \mathcal{D}_1.$$

Proof. Setting $\mathcal{K}_1 = \mathcal{D}_1$, Theorem 3.2 tells us that there is a decomposition of spectra

$$\operatorname{Spc} \mathcal{D}_n = \coprod_{i=1}^{n} \operatorname{Spc} \mathcal{K}_i.$$

By Lemma 4.1 we have $\mathcal{K}_i = \mathsf{thick}^{\otimes}_{\mathcal{D}_i}(W_i)$. Theorem 5.4 shows that the functor

$$\widetilde{P}\colon \mathsf{thick}^{\otimes}_{\mathcal{D}_i}(W_i) \longrightarrow \mathcal{D}_1$$

induced by multiplication by t^{i-1} is an equivalence of tensor triangulated categories. Putting this all together, we have $\operatorname{Spc} \mathcal{K}_i \cong \operatorname{Spc} \mathcal{D}_1$ for all $1 \le i \le n$. □

The following corollary summarises the consequences of our results for $\mathcal{D}_n$.

Corollary 5.5. *The relative stable module category $\mathcal{D}_n = \mathsf{strel}\, R_n G$ admits a semi-orthogonal decomposition into n tensor ideals*

$$\mathsf{strel}\, R_n G = (\mathsf{stmod}\, kG, \ldots, \mathsf{stmod}\, kG),$$

where the ith copy embeds as $\mathcal{K}_i = \mathsf{thick}^{\otimes}_{\mathcal{D}_n}(W_i)$.

Proof. This follows from the discussion in Section 3, along with Theorem 5.4 and Lemma 4.1. □

6. An example: cyclic groups of prime order

In this section we provide an explicit description of the spectrum of $\mathsf{strel}\, R_n G$ in the case where the residue field k has prime characteristic p and $G = C_p$, the cyclic group of order p. In particular, we give concrete generators for all of the prime tensor ideals.

The first several results actually hold for any finite group G. We remind the reader that W_i denotes the cosyzygy $\Omega_i^{-1} k$, taken with respect to the usual abelian category structure in $\mathsf{mod}\, A_i = \mathsf{mod}\, R_i G$.

Proposition 6.1. *For any finite group G, the A_n-modules W_i for $i \le n$ generate $\mathcal{D}_n = \mathsf{strel}\, A_n$ as a thick tensor ideal. Any prime tensor ideal in $\operatorname{Spc} \mathcal{D}_n$ contains at least $n-1$ objects in the set $\{W_1, \ldots, W_n\}$.*

Proof. The first statement follows directly from Corollary 5.5. For the second statement, let $\mathcal{P} \in \operatorname{Spc} \mathcal{D}_n$ and suppose that $W_i \notin \mathcal{P}$. By Lemma 2.4 we have

$$W_i \otimes W_j = 0 \in \mathcal{P}.$$

Since $\mathcal{P}$ is prime, this shows that $W_j \in \mathcal{P}$ for all $j \neq i$. □

Motivated by the previous lemma, we now focus our attention on certain thick tensor ideals of $\mathcal{D}_n$. For $1 \le i \le n$, we let

$$\mathcal{P}_{i,n} = \mathsf{thick}^{\otimes}_{\mathcal{D}_n}(\{W_1, \ldots, W_n\} \setminus \{W_i\}).$$

Our goal will be to show that these are precisely the prime tensor ideals in $\mathcal{D}_n$ in the case where G is the cyclic group C_p. (We know from Theorem 1.1 that the spectrum will be a disjoint union of n points.) The following two results hold for any finite group G.

Lemma 6.2. *For all $X \in \mathcal{P}_{i,n}$ we have $X \otimes \mathbf{1}_{n-1} \in \mathcal{P}_{i,n-1}$, i.e.,*

$$\phi_n{}^* \mathcal{P}_{i,n} \subseteq \mathcal{P}_{i,n-1}.$$

Proof. If $i \le n-1$ then $W_i \otimes \mathbf{1}_{n-1} = W_i$, whereas $W_n \otimes \mathbf{1}_{n-1} = 0$ by Lemma 2.2. Hence $\phi_n{}^*$ sends the generators of $\mathcal{P}_{i,n}$ into $\mathcal{P}_{i,n-1}$. Because $\mathcal{P}_{i,n-1}$ is thick and $\phi_n{}^*$ is exact, the lemma follows immediately. (The dubious reader may consult [12, Lemma 3.8].) □

Lemma 6.3. *Each tensor ideal $\mathcal{P}_{i,n}$ is proper in $\mathcal{D}_n$.*

Proof. We fix i and proceed by induction on n. For the base $n = i$ we need to show that

$$\mathcal{P}_{i,i} = \mathsf{thick}^{\otimes}_{\mathcal{D}_i}(W_1, \ldots, W_{i-1})$$

is proper. We saw in Section 3 that the restriction of scalars ϕ_{i*} embeds $\mathcal{D}_{i-1}$ as a proper tensor ideal in $\mathcal{D}_i$. For each $1 \le j \le i-1$ we have $W_j \in \phi_{i*}\mathcal{D}_{i-1}$, so $\mathcal{P}_{i,i}$ is contained in $\phi_{i*}\mathcal{D}_{i-1}$ and $\mathcal{P}_{i,i}$ is proper in $\mathcal{D}_i$. (In fact, $\mathcal{P}_{i,i} = \phi_{i*}\mathcal{D}_{i-1}$ by Proposition 6.1.)

Now let $n > i$ and assume that $\mathcal{P}_{i,n-1}$ is proper in $\mathcal{D}_{n-1}$. For the sake of contradiction, suppose that $\mathcal{P}_{i,n}$ is not proper in $\mathcal{D}_n$ so that it contains the tensor unit $\mathbf{1}_n$. Then $\mathbf{1}_{n-1} = \mathbf{1}_n \otimes \mathbf{1}_{n-1}$ lies in $\mathcal{P}_{i,n-1}$ by Lemma 6.2, a contradiction. □

We are only now forced to specialise to the case $G = C_p$.

Lemma 6.4. *Each $\mathcal{P}_{i,n}$ is a maximal tensor ideal in $\mathcal{D}_n$.*

Proof. We fix i and proceed by induction on n. For the base case $n = i$ we need to show that $\mathsf{thick}^{\otimes}_{\mathcal{D}_i}(W_1, \ldots, W_{i-1}) = \mathcal{D}_{i-1}$ is maximal in $\mathcal{D}_i$. Recall that the thick tensor ideals in $\mathcal{D}_i$ containing $\mathcal{D}_{i-1}$ are in bijection with those in the quotient $\mathcal{D}_i/\mathcal{D}_{i-1}$. By the discussion in Section 3, that quotient is tensor equivalent to $\mathcal{K}_i$, which in turn is equivalent to

$$\mathcal{D}_1 = \mathsf{strel}\, kC_p = \mathsf{stmod}\, kC_p$$

by Theorem 5.4. It is known that the rightmost category has precisely two tensor ideals, namely the zero ideal and the entire category. It follows that the only tensor ideal in $\mathcal{D}_i$ properly containing $\mathcal{D}_{i-1}$ is $\mathcal{D}_i$ itself, so $\mathcal{D}_{i-1}$ is maximal as claimed.

Now let $n > i$, assume that $\mathcal{P}_{i,n-1}$ is maximal in $\mathcal{D}_{n-1}$ and choose $X \notin \mathcal{P}_{i,n}$. Tensoring the short exact sequence of Lemma 2.3 with X produces a triangle

$$X \otimes \mathbf{1}_{n-1} \longrightarrow X \otimes W_n \longrightarrow \Sigma X \longrightarrow$$

in $\mathcal{D}_n$. The middle term lies in $\mathcal{P}_{i,n}$, but the right hand term does not. This implies that $X \otimes \mathbf{1}_{n-1}$ does not lie in $\mathcal{P}_{i,n}$. In particular, it cannot lie in $\mathcal{P}_{i,n-1}$ since the latter is contained in the former. By the inductive hypothesis on maximality, this means that

$$\mathbf{1}_{n-1} \in \mathsf{thick}^{\otimes}_{\mathcal{D}_{n-1}}(\{W_1, \dots, W_{n-1}, X \otimes \mathbf{1}_{n-1}\} \setminus \{W_i\}).$$

Now consider the triangle

$$\mathbf{1}_{n-1} \longrightarrow W_n \longrightarrow \Sigma \mathbf{1}_n \longrightarrow$$

in $\mathcal{D}_n$ induced by the short exact sequence of Lemma 2.3. By the above remarks, the left two terms lie in $\mathsf{thick}^{\otimes}_{\mathcal{D}_n}(\{W_1, \dots, W_n, X\} \setminus \{W_i\})$, whence so does the right hand term. In other words

$$\mathbf{1}_n \in \mathsf{thick}^{\otimes}_{\mathcal{D}_n}(\{W_1, \dots, W_n, X\} \setminus \{W_i\}),$$

proving that $\mathcal{P}_{i.n}$ is maximal. □

It now follows from [2, Proposition 2.3] that each $\mathcal{P}_{i,n}$ is a prime tensor ideal, i.e., gives a point in $\operatorname{Spc} \mathcal{D}_n$.

Lemma 6.5. *Each $\mathcal{P}_{i,n}$ is a minimal prime.*

Proof. Suppose there exists a prime ideal $\mathcal{P}$ properly contained in $\mathcal{P}_{i,n}$. Then there is an object in the set $\{W_1, \dots, W_n\} \setminus \{W_i\}$ not contained in $\mathcal{P}$. Since $\mathcal{P}$ is prime, Proposition 6.1 forces us to have $W_i \in \mathcal{P}$. But this implies that $W_i \in \mathcal{P}_{i,n}$, so $\{W_1, \dots, W_n\} \subseteq \mathcal{P}_{i,n}$. Proposition 6.1 now tells us that $\mathcal{P}_{i,n} = \mathcal{D}_n$, contradicting Lemma 6.3. □

We are now ready to give a direct computation of the spectrum, verifying Theorem 1.1 in this case.

Theorem 6.6. *The prime ideal spectrum of $\mathcal{D}_n = \mathsf{strel}\ R_n C_p$ is a disjoint union of n points.*

Proof. If $\mathcal{P}$ is an element of $\operatorname{Spc} \mathcal{D}_n$, then by Proposition 6.1 there is an integer $1 \leq i \leq n$ such that $\{W_1, \dots, W_n\} \setminus \{W_i\} \subseteq \mathcal{P}$. We then have $\mathcal{P}_{i,n} \subseteq \mathcal{P}$. Since $\mathcal{P}_{i,n}$ is maximal, this implies that $\mathcal{P} = \mathcal{P}_{i,n}$. It follows that $\operatorname{Spc} \mathcal{D}_n = \{\mathcal{P}_{1,n}, \dots, \mathcal{P}_{n,n}\}$ as a set. We now recall from [2, Proposition 2.9] that the closed points in the prime ideal spectrum are precisely the *minimal* primes. Lemma 6.5 therefore informs us that each point $\mathcal{P}_{i,n}$ is closed in the topology of $\operatorname{Spc} \mathcal{D}_n$. □

We now make a few observations based on and related to the theorem; all of the statements are more or less trivial consequences of what we have already done, but are perhaps worth making explicit.

Corollary 6.7. *Each $W_i = \Omega_i^{-1}k$ is supported at a single point, namely*

$$\operatorname{supp}(W_i) = \{\mathcal{P}_{i,n}\}.$$

Proof. Given the computation of the spectrum and the definition of support, we have

$$\operatorname{supp}(W_i) = \{\mathcal{P} \in \operatorname{Spc}\mathcal{D}_n \mid W_i \notin \mathcal{P}\} = \{\mathcal{P}_{i,n}\}. \qquad \square$$

Corollary 6.8. *The base change functor*

$$- \otimes \mathbf{1}_m \colon \mathcal{D}_n \longrightarrow \mathcal{D}_m$$

induces an embedding $\operatorname{Spc}\mathcal{D}_m \hookrightarrow \operatorname{Spc}\mathcal{D}_n$ *with image* $\{\mathcal{P}_{i,n} \mid 1 \le i \le m\}$.

Proof. Letting ϕ^* denote the base change functor, it is a general fact (see [2, Proposition 3.11]) that $\operatorname{Spc}(\phi^*)$ is a homeomorphism onto its image. The latter set consists precisely of those prime ideals that contain the kernel of ϕ^*. It follows by repeated applications of Lemma 4.1 that the kernel of ϕ^* is generated by $\{W_i \mid m+1 \le i \le n\}$. The result follows immediately. $\square$

In the above corollary, one might also consider the corresponding base change functors having source $\mathsf{strel}\, SC_p$. The result remains true, except for the description of the image. By invoking [6, Theorem A.5], one may write

$$\operatorname{Spc}(\mathsf{strel}\, SC_p) \cong Z_n \coprod \operatorname{Spc}\mathcal{D}_n$$

for all $n \ge 1$, where Z_n is the spectrum of the kernel. Thus for any $n \ge 1$ we can find a disjoint union of n points as an open and closed subspace of the spectrum of $\mathsf{strel}\, SC_p$. It would be interesting to fully understand the space $\operatorname{Spc}(\mathsf{strel}\, SC_p)$ and, in particular, how the spectra of the $\mathcal{D}_n$ sit inside of it.

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