

Dimers, crystals and quantum Kostka numbers

Christian Korff

*School of Mathematics and Statistics, University of Glasgow
University Place, Glasgow G12 8QQ, Scotland, UK*

Abstract. We relate the counting of honeycomb dimer configurations on the cylinder to the counting of certain vertices in Kirillov-Reshetikhin crystal graphs. We show that these dimer configurations yield the quantum Kostka numbers of the small quantum cohomology ring of the Grassmannian, i.e. the expansion coefficients when multiplying a Schubert class repeatedly with different Chern classes. This allows one to derive sum rules for Gromov-Witten invariants in terms of dimer configurations.

Keywords: dimers, crystal graphs, quantum cohomology

1 Dimer configurations on the cylinder

This is an extended abstract of results for dimer configurations on the cylinder. Details and proofs will be provided in a separate publication. The problem of counting dimer configurations on various lattices goes back to the works of Kasteleyn [15] and Fisher and Temperley [25]; see e.g. [16, 17] for an introduction and an overview.

Fix two integers $n \geq 3$ and $0 \leq k \leq n$. Consider a hexagonal or honeycomb lattice on a cylinder of circumference n and height k ; see Figure 1 for an example. A *perfect matching* of the lattice is a selection of edges, called *dimers*, such that each vertex of the lattice is occupied by one and only one dimer; see Figure 2 for an example. For simplicity, we shall visualise the cylindrical lattice as a strip of height k in the Euclidean plane and consider only matchings with period n . We number the edges on the top and bottom of the strip consecutively with integers and call the vertical line separating the edges labeled with '0' and '1' the *boundary*; see Figure 1. In what follows we are interested in the following refined counting problem of dimer configurations subject to a number of constraints or boundary conditions.

Firstly, we fix the dimer configurations on the top and bottom edges of the cylinder as shown in Figure 2, in terms of two binary strings $b^{in} = b_n^{in} \dots b_2^{in} b_1^{in}$ and $b^{out} = b_n^{out} \dots b_2^{out} b_1^{out}$, where a 1-letter signals a selected edge or dimer and a 0-letter an unoccupied edge. We require that b^{in} and b^{out} each contain k one-letters, $\sum_{i=1}^n b_i^{in} = \sum_{i=1}^n b_i^{out} = k$. Let $n \geq \ell_k^{in} > \dots > \ell_1^{in} \geq 1$ and $n \geq \ell_k^{out} > \dots > \ell_1^{out} \geq 1$ be the positions of the 1-letters in b^{in}, b^{out} from right to left. Define two partitions μ, ν by defining their parts through the relations $\mu_{k+1-i} + i = \ell_i^{in}$ and $\nu_{k+1-i} + i = \ell_i^{out}$, respectively. The Young diagrams of these partitions fit into a bounding box of height k and width $k' = n - k$. The

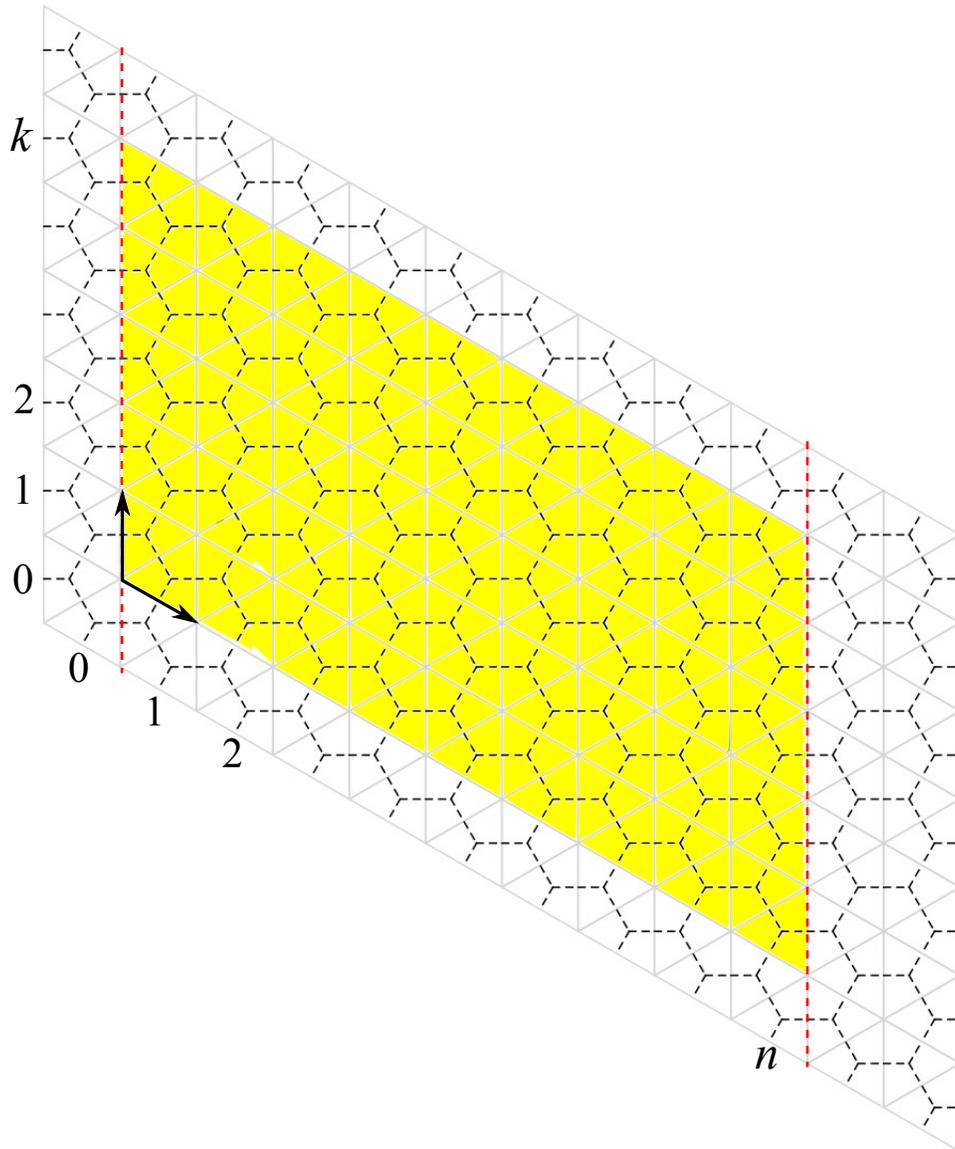


Figure 1: Example of a honeycomb lattice on the cylinder when depicted in the plane. The yellow strip is glued together by identifying the lattice edges which are intersected by the red dotted lines, the boundary. The resulting cylinder has circumference $n = 9$ and height $k = 5$.

implicitly defined map between such partitions μ and binary strings of length n with k one-letters is a well-known bijection $\mu \mapsto b^\mu$ and, therefore, we shall identify both sets denoting them by the same symbol $\Pi_{k,n}$.

Secondly, we also fix the number of horizontal dimers in row i to be λ_i with $i = 1, \dots, k$ and set $|\lambda| = \sum_{i=1}^k \lambda_i$ to be the total number of horizontal dimers occurring in a configuration.

Theorem 1. For given $\lambda, \mu, \nu \in \Pi_{k,n}$ denote by $\Gamma_\lambda(\mu, \nu)$ the set of perfect matchings or dimer configurations subject to the mentioned constraints with $b^{\text{in}} = b^\mu$ and $b^{\text{out}} = b^\nu$.

- (i) If α is any permutation of the parts of λ , then $|\Gamma_\lambda(\mu, \nu)| = |\Gamma_\alpha(\mu, \nu)|$.
- (ii) The number of dimer configurations $|\Gamma_\lambda(\mu, \nu)| = 0$ unless $|\lambda| + |\mu| - |\nu|$ is divisible by the circumference n of the cylinder.
- (iii) If $|\Gamma_\lambda(\mu, \nu)| > 0$ then in each perfect matching precisely

$$d = \frac{|\lambda| + |\mu| - |\nu|}{n} \quad (1.1)$$

horizontal dimers cross the boundary.

We can make a further statement about the possible minimum number of horizontal dimers if we only fix the boundary conditions b^{in} and b^{out} on the bottom and top of the cylinder but leave the number of horizontal dimers in each row arbitrary. For fixed $\mu, \nu \in \Pi_{k,n}$ introduce the integers

$$n_i(\mu) = \sum_{j=n+1-i}^n b_j^\mu \quad (1.2)$$

which are the partial sums of a binary string b^μ . Set

$$d_{\min}(\nu, \mu) := \max_{i \in [n]} \{n_i(\nu) - n_i(\mu)\} \quad (1.3)$$

and denote by $\Gamma(\mu, \nu) = \cup_\alpha \Gamma_\alpha(\mu, \nu)$ with α now a composition of length $\leq k$ with $\alpha_i \leq k' = n - k$. Then we have the following:

Proposition 1. (i) The minimal number of horizontal dimers in any perfect matching $\gamma \in \Gamma(\mu, \nu)$ is given by $|\lambda^{\min}| = nd_{\min} + |\nu| - |\mu|$ and in that configuration precisely d_{\min} dimers are crossing the boundary. (ii) If $\lambda \in \Pi_{k,n}$ in (1.1) is such that $d > d_{\min}(\nu, \mu)$, then $|\Gamma_\lambda(\mu, \nu)| > 0$ if and only if

$$\theta_i(\mu, \nu; d) := d + n_i(\nu) - n_i(\mu) > 0, \quad 1 \leq i \leq n. \quad (1.4)$$

The last constraint translates into the requirement that in each column of the lattice there is at least one horizontal dimer.

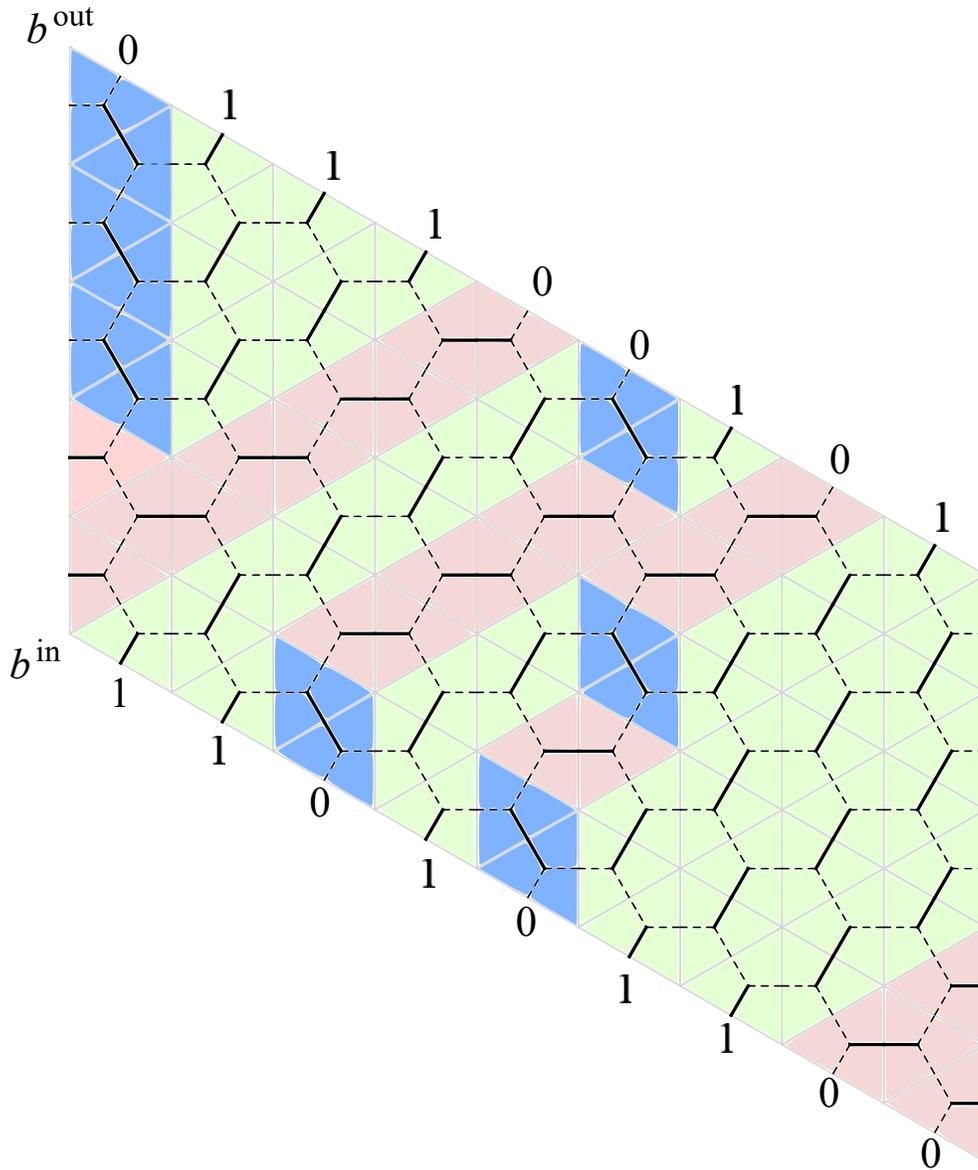


Figure 2: Example of a dimer configuration on a cylinder with circumference $n = 9$ and height $k = 5$. The colouring of the lozenges depicts the bijection to periodic tilings and plane partitions. The boundary conditions on the bottom and top of the cylinder are fixed by two binary strings b^{in} and b^{out} corresponding to the partitions $\mu = (4, 4, 3, 2, 2)$ and $\nu = (3, 3, 3, 1, 0)$. The number of horizontal dimers in each row (bottom to top) is given by $\lambda = (2, 4, 2, 3, 2)$ and there are $d = 2$ dimers crossing the boundary in row 1 and 2.

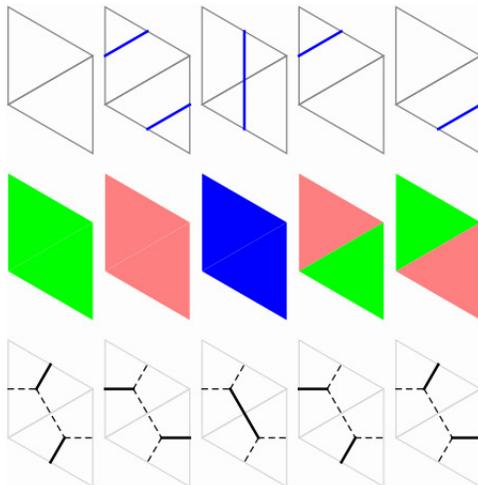


Figure 3: Consider tilings of the plane using the depicted lozenges in each row, such that adjacent tilings match. The bottom row yields the dimer configurations from Figure 2 if lozenges are placed such that dimers connect to dimers. Replacing each dimer lozenge with the corresponding one in the row above we obtain plane partitions and if we use the top row then we obtain domain walls or nonintersecting paths; see Figure 2 for an example of how to map to a plane partition with periodic boundary conditions.

There are known bijections between the discussed dimer configurations and plane partitions or lozenge tilings as well as domain walls (non-intersecting paths) describing the spin configurations of the ground state of the triangular antiferromagnetic Ising model; see Figure 3. The stated results can therefore be reformulated in terms of any of these combinatorial tilings. It is the domain wall picture, see Figure 3, which was discussed in [18] in connection with the small quantum cohomology of the Grassmannian. One then recognises the minimal number of horizontal dimers (1.3) as the minimal power of the deformation parameter q occurring in a product of the small quantum cohomology ring Grassmannian; compare with [28] and [7].

2 Kirillov-Reshetikhin crystals

Kashiwara's crystal bases [14] and their associated coloured, directed graphs, called *crystal graphs* or simply *crystals*, are an important combinatorial tool in representation theory; see e.g. [12] for a textbook and references therein. A crystal graph consists of a set of vertices B , the basis elements, and certain maps $e_i, f_i : B \rightarrow B \sqcup \{\emptyset\}$, $i \in [n]$ called *Kashiwara operators* which define the directed, coloured edges of the graph: there exists an edge $b \rightarrow b'$ of colour i , if and only if, $f_i(b) = b'$ in which case we also must have

$e_i(b') = b$. In particular, there are no multiple edges.

Given a crystal graph and a vertex $b \in B$ one can consider the maximal length of a directed path along edges of a fixed colour i which ends or starts at b ,

$$\varepsilon_i(b) = \max\{p \in \mathbb{Z}_{\geq 0} : e_i^p(b) \neq \emptyset\}, \quad \varphi_i(b) = \max\{p \in \mathbb{Z}_{\geq 0} : f_i^p(b) \neq \emptyset\}. \quad (2.1)$$

The functions (2.1) allow one to introduce the tensor product $B_1 \otimes B_2$ of two crystal graphs B_1, B_2 as the crystal obtained through the following action of the Kashiwara operators $e_i, f_i : B_1 \times B_1 \rightarrow B_1 \times B_2 \sqcup \{\emptyset\}$ on the Cartesian product of the respective vertex sets,

$$e_i(b_1 \otimes b_2) = \begin{cases} e_i(b_1) \otimes b_2, & \varepsilon_i(b_1) > \varphi_i(b_2) \\ b_1 \otimes e_i(b_2), & \text{else} \end{cases} \quad (2.2)$$

$$f_i(b_1 \otimes b_2) = \begin{cases} f_i(b_1) \otimes b_2, & \varepsilon_i(b_1) \geq \varphi_i(b_2) \\ b_1 \otimes f_i(b_2), & \text{else} \end{cases} \quad (2.3)$$

together with the convention $b \otimes \emptyset = \emptyset$ and $\emptyset \otimes b = \emptyset$. Note that there exist different conventions for the definition of the tensor product in the literature, our choice will be suited for the discussion at hand.

Here we are concerned with tensor products

$$B_\lambda := B_{\lambda_1} \otimes \cdots \otimes B_{\lambda_k}, \quad \lambda \in \Pi_{k,n}, \quad (2.4)$$

of the crystal graphs B_r , $0 \leq r < n$ (usually denoted by $B^{r,1}$ in the literature) of certain finite-dimensional level one modules of the affine quantum algebra $U_v(\widehat{\mathfrak{sl}}_n)$, so-called Kirillov-Reshetikhin (KR) modules [6]. The basis elements in B_r are labelled by binary strings b of length n with r -one letters, hence as sets we have $B_r = \Pi_{r,n}$. The basis elements in B_λ are then k -tuples $b^{(1)} \otimes \cdots \otimes b^{(k)}$ of binary strings $b^{(j)}$ which can be efficiently recorded in terms column tableaux, where column j contains the positions $\ell_i^{(j)}$ of 1-letters of $b^{(j)}$.

Not every module of $U_v(\widehat{\mathfrak{sl}}_n)$ possesses a crystal basis, but KR modules are distinguished by the fact that they do and that the associated crystal graphs are *perfect*. That is, tensor products of KR modules again possess crystal bases and their associated crystal graphs are connected; see [12] and references therein.

2.1 Combinatorial R-matrix and dimers

Fix $\lambda \in \Pi_{k,n}$. There exists a unique bijection $R_\lambda : B_{n-k} \otimes B_\lambda \rightarrow B_\lambda \otimes B_{n-k}$, called *combinatorial R-matrix*, which preserves the crystal graph structure and is a (set-theoretical) solution of the Yang-Baxter equation; see e.g. [22]. In addition, the $\widehat{\mathfrak{sl}}_n$ Dynkin diagram automorphism Ω induces another trivial graph isomorphism $\Omega : B_\lambda \rightarrow B_\lambda$ by cyclic permutations of the letters in the binary strings, i.e. $\Omega(b^{(1)} \otimes \cdots \otimes b^{(k)}) = \Omega(b^{(1)}) \otimes \cdots \otimes \Omega(b^{(k)})$

with $\Omega(b^{(j)}) = b_1^{(j)} b_k^{(j)} b_{k-1}^{(j)} \cdots b_2^{(j)}$. It commutes with the combinatorial R -matrix. We now define a particular subset of crystal vertices $b \in B_\lambda$.

For fixed $\nu, \mu \in \Pi_{k,n}$ set

$$\Theta_i = \begin{cases} 1, & \theta_i(\mu, \nu, d) = d + n_i(\nu) - n_i(\mu) > 0 \\ 0, & \text{else} \end{cases},$$

where θ is the same integer vector as in (1.4). Denote by $\mu', \nu' \in \Pi_{n-k,n}$ the conjugate partitions of μ, ν obtained by transposing the respective Young diagrams.

Proposition 2. *Let $b \in B_\lambda$. The following statements are equivalent.*

- (i) $\varphi_i(b) = b_{n-i}^\mu \Theta_i$ and $\varepsilon_i(b) = b_{n-i}^\nu \Theta_{i+1}$ with $i \in \mathbb{Z}_n$.
- (ii) $R_\lambda(b^{\mu'} \otimes \Omega(b)) = b \otimes b^{\nu'}$.

Suppose $d > d_{\min}$. Then property (i) simplifies to

$$\varphi_i(b) = b_{n-i}^\mu \quad \text{and} \quad \varepsilon_i(b) = b_{n-i}^\nu. \quad (2.5)$$

This characterisation of crystal vertices is an affine extension of the one considered by Berenstein and Zelevinsky in [1] when describing Kostka numbers and Littlewood-Richardson coefficients for type A . Their results extend to all finite semi-simple Lie algebras.

Theorem 2. *Denote by $B_\lambda(\mu, \nu)$ the set of crystal graph vertices $b \in B_\lambda$ satisfying the conditions of the previous proposition. There exists a bijective map between the elements in $\Gamma_\lambda(\mu, \nu)$ and $B_\lambda(\mu, \nu)$; see Figure 4 for an example.*

In other words, the i -signatures of the elements in $B_\lambda(\nu, \mu)$ are fixed in terms of the start and end positions of the dimer configurations and any crystal vertex with these i -signatures must be the image of such a dimer configuration.

3 Quantum cohomology and toric Schur functions

Quantum cohomology arose from works of Gepner [8], Intriligator [13], Vafa [26], Witten [27] and since then has been studied extensively in the mathematics literature. The small quantum cohomology ring of the Grassmannian of k -planes in \mathbb{C}^n has the following known presentation [24]

$$qH^*(\text{Gr}_k(\mathbb{C}^n)) \cong \mathbb{Z}[q][e_1, \dots, e_k] / \langle h_{n-k+1}, \dots, h_{n-1}, h_n + q(-1)^k \rangle, \quad (3.1)$$

where $h_r = \det(e_{1-i+j})_{1 \leq i, j \leq r}$ are the Chern classes of the normal vector bundle and the e_i 's are the Chern classes of the canonical bundle. Denote by $\sigma_\lambda = \det(h_{\lambda_i - i + j})_{1 \leq i, j \leq k}$

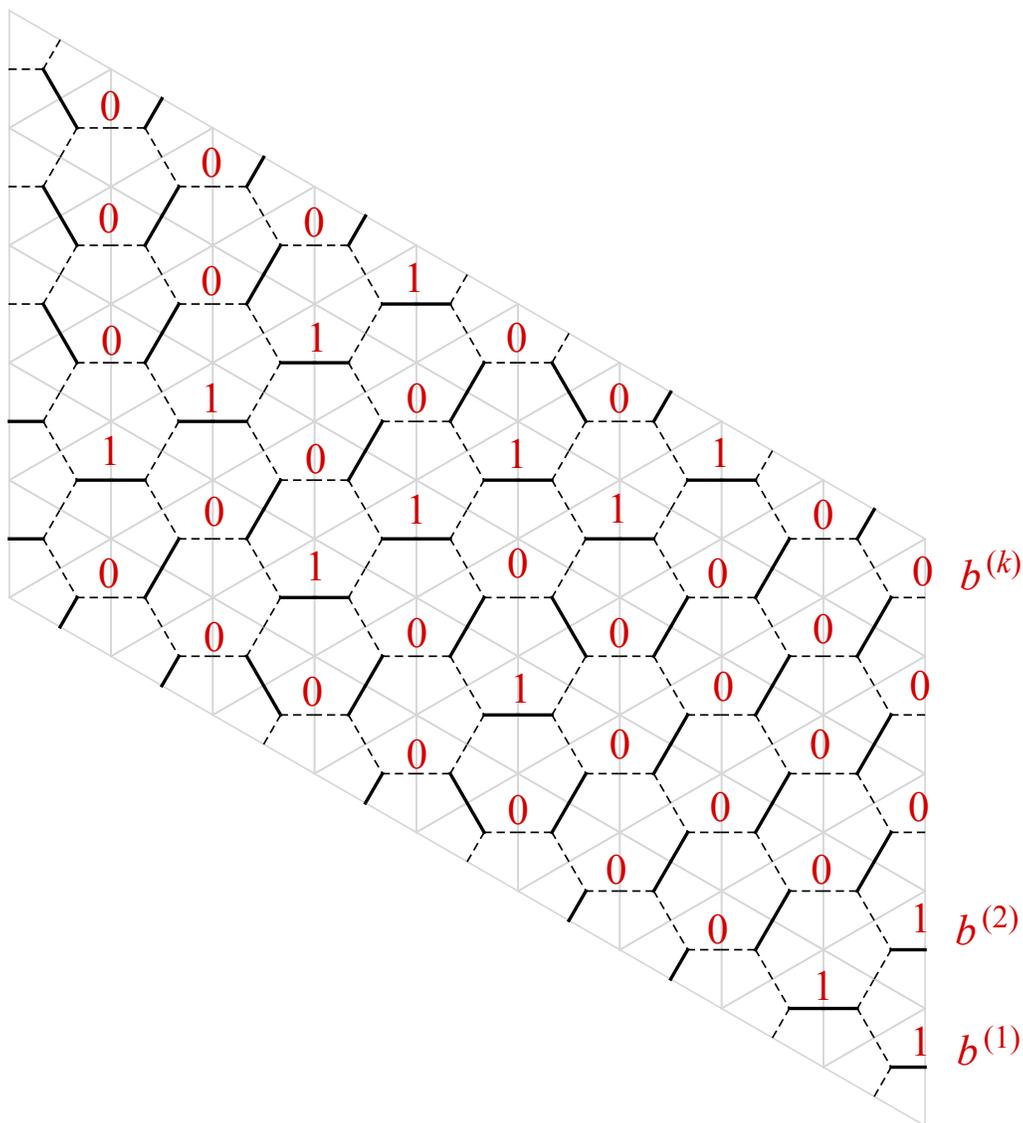


Figure 4: Example of mapping the dimer configuration from [Figure 2](#) onto a vertex in the KR crystal $B_{2,4,2,3,2} = B_2 \otimes B_4 \otimes B_2 \otimes B_3 \otimes B_2$ by recording in each lattice row the occurrence of a horizontal dimer with a 1-letter and otherwise putting down a 0-letter. This results in a $k = 5$ -tuple $b^{(1)} \otimes \cdots \otimes b^{(k)}$ of binary strings which satisfy the conditions from [Proposition 2](#).

the Schubert classes and consider the coefficients in the following product expansion in (3.1),

$$\sigma_\mu * h_{\lambda_1} * \cdots * h_{\lambda_r} = \sum_{\nu \in \Pi_{k,n}} q^d \sigma_\nu K_{\nu/d/\mu,\lambda}, \quad (3.2)$$

which are called *quantum Kostka numbers* [3]. Here the power d , called ‘degree’, is fixed through the equality (1.1), otherwise the coefficient vanishes.

Theorem 3 (Sum rule for quantum Kostka numbers). (i) *The number of possible dimer configurations and crystal vertices fixed by the partitions $\lambda, \mu, \nu \in \Pi_{k,n}$ matches the quantum Kostka number,*

$$K_{\nu/d/\mu,\lambda} = |\Gamma_\lambda(\mu, \nu)| = |B_\lambda(\mu, \nu)|. \quad (3.3)$$

(ii) *Summing over all compositions $\alpha = (\alpha_1, \dots, \alpha_k)$ with $0 \leq \alpha_i \leq n - k$, one obtains the total number $|\Gamma(\mu, \nu)|$ of dimer configurations on the cylinder subject only to the boundary conditions $b^{in} = b^\mu$ and $b^{out} = b^\nu$ on the bottom and top of the cylinder,*

$$|\Gamma(\mu, \nu)| = \sum_\alpha |\Gamma_\alpha(\mu, \nu)| = \sum_{\lambda \in \Pi_{k,n}} K_{\nu/d/\mu,\lambda} \frac{\ell(\lambda)!}{\prod_{i \geq 1} m_i(\lambda)!} \binom{k}{\ell(\lambda)}, \quad (3.4)$$

where $\ell(\lambda)$ is the length of the partition λ and $m_i(\lambda)$ the multiplicity of i in λ .

3.1 Toric Schur functions

The predominant mathematical interest in the ring (3.1) is the computation of the 3-point genus 0 Gromov-Witten invariants $C_{\lambda\mu}^{v,d}$. The latter occur in the product expansion of two Schubert classes

$$\sigma_\mu * \sigma_\lambda = \sum_{d \geq 0, \nu \in (n,k)} q^d C_{\lambda\mu}^{v,d} \sigma_\nu, \quad (3.5)$$

and count rational curves of degree d intersecting three Schubert varieties in general position, which are parametrised by λ, μ, ν ; for details we refer the reader to the literature, e.g. [2], [3], [4], [5] and references therein.

A combinatorial interpretation of Gromov-Witten invariants was given in [23]: one generalises the notion of an ordinary skew Schur function $s_{\nu/\mu}$, where the expansion coefficients in the basis of Schur functions s_λ are given by Littlewood-Richardson coefficients, $c_{\lambda\mu}^{\nu} = C_{\lambda\mu}^{\nu,0}$ to so-called toric Schur functions,

$$s_{\nu/d/\mu}(x_1, \dots, x_k) = \sum_{\lambda \in \Pi_{k,n}} C_{\lambda\mu}^{\nu,d} s_\lambda(x_1, \dots, x_k). \quad (3.6)$$

Postnikov introduced these functions in terms of so-called toric tableaux [23], which are special cases of the cylindric plane partitions considered by Gessel and Krattenthaler in [9]. Here we express them as weighted sums over KR crystals and dimer configurations.

Proposition 3. *Toric Schur functions can be expressed as the following weighted sums,*

$$s_{\nu/d/\mu}(x_1, \dots, x_k) = \sum_{\alpha} |B_{\alpha}(\mu, \nu)| x^{\alpha} = \sum_{\alpha} |\Gamma_{\alpha}(\mu, \nu)| x^{\alpha}, \quad (3.7)$$

where α runs over all compositions which have at most k parts $\alpha_i \leq n - k$.

If the degree (1.1) vanishes, $d = 0$, one has $s_{\nu/0/\mu} = s_{\nu/\mu}$ and in this case one can use the Robinson-Schensted-Knuth correspondence to arrive at the familiar crystal theoretic interpretation of skew Schur functions. This interpretation can be extended to $d = d_{\min}$ using the cyclic \mathbb{Z}_n symmetry of the cylinder manifest in Proposition 2 (ii), similar to the discussion in [20].

From the expansion (3.6) one now arrives at the following:

Theorem 4 (Sum rule for Gromov-Witten invariants). *One has the following alternative sum rule for the total number $|\Gamma(\mu, \nu)|$ of dimer configurations on the cylinder,*

$$|\Gamma(\mu, \nu)| = \sum_{\alpha} |\Gamma_{\alpha}(\mu, \nu)| = \sum_{\lambda \in \Pi_{k,n}} C_{\lambda\mu}^{v,d} \prod_{s \in \lambda} \frac{n + c(s)}{h(s)}, \quad (3.8)$$

where the product runs over all squares $s = \langle i, j \rangle$ in the Young diagram of λ and $c(s) = j - i$ denotes its content and $h(s) = \lambda_i + \lambda'_j - i - j + 1$ its hook length.

A different connection between Gromov-Witten invariants and (combinatorially defined) crystals has been found by Morse and Schilling in [21]. In *loc. cit.* the authors define a crystal structure on particular factorisations of affine permutations and identify the Gromov-Witten invariants of the full flag variety with the number of certain highest weight factorisations of affine permutations. They recover the Gromov-Witten invariants for the Grassmannian as special case of their more general construction [21, Thm 5.16].

In contrast the results here connect the small quantum cohomology ring with the known crystal structure of KR modules of $U_v(\widehat{\mathfrak{sl}}_n)$ and their combinatorial R -matrix. We hope to make the connection between both crystal structures in future work. We believe that such a connection would help with the construction of ‘quantum group structures’, so-called Yang-Baxter algebras which provide maps $qH^*(\text{Gr}_k(\mathbb{C}^n)) \rightarrow qH^*(\text{Gr}_{k\pm 1}(\mathbb{C}^n))$ [19, 18], to general flag varieties; see also [10] for an extension of the discussion to equivariant quantum cohomology and [11] for quantum K-theory.

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