Resource allocation with partial responsibilities for initial endowments*

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Abstract

This paper studies a resource allocation problem in which each individual is responsible but in general only partially for his initial endowment. We consider pure-exchange economies with initial endowments but we do not assume the individual rationality axiom, taking that the society consists of citizens who cannot opt out from it. We characterize a class of allocation rules which are parametrized by income redistribution codes. In particular, we characterize a one-parameter family of income redistribution codes, in which one extreme corresponds to the case that everybody is 100% responsible for his initial endowment and the other extreme corresponds to the case that nobody is responsible for his endowment at all.

JEL Classification: D50, D60, D70

1 Introduction

This paper studies a resource allocation problem in which each individual is responsible but in general only partially for his initial endowment.

Existing normative and axiomatic studies on resource allocation assume either that everybody is 100% responsible for her initial endowment, or that any such notion of individual responsibility or entitlement is irrelevant. The former line of thought assumes that nobody should get worse off than her initial endowment (individual rationality) and characterizes the Walras rule (see for example Hurwicz (1979), Gevers (1986), Thomson (1979), Nagahisa (1991), Nagahisa and Suh (1995)). This line of thought imagines that individuals are entirely responsible for their endowment. In contrast, the latter line of research imagines individuals are not at all responsible for their endowment. This is reflected in the assumption that only the aggregate endowment matters; the additional imposition of equity axioms typically characterizes the Walras rule starting from equal division (see for example Nagahisa and Suh (1995), Thomson (1988), Thomson and Zhou (1993)).

In this paper, we imagine that a social planner need not restrict herself to either of these two ideas; she may have a different idea of how responsible each agent is for her endowment. To this end, we study a family of rules that allows flexibility in the degree of responsibility. Our main contribution is to derive a parametric family of such rules, which form compromises between the previously discussed families. In order to meaningfully discuss a parametric notion of responsibility, our notion of rule should, at the very least, be "single-valued" in very simple cases. We discuss this idea below.

We consider pure-exchange economies with initial endowments but we do not assume the individual rationality axiom. Instead, we allow that endowments can be redistributed across the agents; as would happen in modern economies with income redistribution. To this end, we imagine that all agents in society are necessary participants to the rule; willing or not. In principle, individuals may try to hide their wealth or endowments, but this is outside the scope of the model we discuss here.

Our main contribution is an axiomatization of a family of rules, in which the notion of responsibility takes a parametric form, which is uniquely identified. In view of the second welfare theorem, the social planner can pick a desired efficient allocation by suitable income redistribution. Our characterization pins down a class of such income redistribution codes.

[1]

In general, understanding what type of income redistribution code has been used is a difficult problem. Individuals' preferences are diverse, and complementarity between different goods can be quite complex. Hence, simply observing preferences and an efficient allocation, it is impossible in general to "back out" or reverse engineer which type of redistribution has taken place. There is, however, one case in which we would be able to uniquely pin this down: the cases in which all individuals have identical and linear preferences.

In this case, any allocation is efficient and only the only decision to be made is redistributive. For such preference profiles, there is a natural measure of welfare that is common to everybody: the income evaluation of consumption measured by the common marginal rate of substitution.

On the subdomain of identical linear preferences, we impose an axiom that the welfare level for each individual should be pinned down uniquely in such cases of purely redistribution problems. In a sense, this is a minimal criterion required for a rule to be considered a "solution," and reflects a positive view that an income redistribution and taxation codes are immutable. This is a mild requirement, in the sense that it is required to apply only to the small domain of identical linear preferences, and it does not exclude any reasonably resolute solution at a conceptual level.

Together with other standard axioms, we characterize a class of allocation rules which are parametrized by income distribution codes. The class is a natural compromise between the Walrasian solution and the Walrasian solution from equal division. In particular, we characterize a one-parameter family of income redistribution codes, in which one extreme corresponds to the case that each individual is 100% responsible for his initial endowment and the other extreme corresponds to the case that everybody is not responsible for his endowment at all.

[1] Another idea is to consider axioms *indexed* by certain parameters describing the degree of responsibility, which is introduced a priori as a part of the setting. In the current paper we focus on a calibration argument, in which such parameters are not presupposed in the outset but endogenously obtained as a part of characterization.

2 The model and axioms

We consider pure exchange economies in which there are n individuals and l goods. Assume $n \geq 3$. The consumption space for each individual is $\mathbb{R}^l_+$, where consumptions are taken to be column vectors. Each individual generically denoted by $i = 1, \cdots, n$ has initial endowment $e_i \in \mathbb{R}^l_{++}$ which is taken to be variable. Let

$$e = \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} \in \mathbb{R}^{nl}_{++}$$

denote an entire profile of initial endowment vectors.[2]

Let $\mathcal{R}$ denote the set of preferences over $\mathbb{R}^l_+$ which are complete, transitive, continuous, convex and strongly monotone.

Let $\mathcal{D} \subset \mathcal{R}^n$ be the domain of preferences to be considered. Given any profile $R = (R_1, \cdots, R_n) \in \mathcal{D}$, the object R_i denotes individual i's weak preference relation, P_i denotes the corresponding strict preference relation, and I_i denotes the corresponding indifference relation.

We assume that $\mathcal{D}$ contains the subdomain of identical linear preferences, denoted $\mathcal{D}_{IL}$, in which any $R = (R_1, \cdots, R_n) \in \mathcal{D}_{IL}$ satisfies

$$x_i R_i y_i \iff p x_i \geq p y_i$$

for all $i = 1, \cdots, n$ with respect to some common row vector $p \in \Delta^\circ$, where $\Delta = \left\{ p \in \mathbb{R}^l_+ : \sum_{k=1}^l p_k = 1 \right\}$ and $\Delta^\circ = \Delta \cap \mathbb{R}^l_{++}$. Every element $R \in \mathcal{D}_{IL}$ is identified with the corresponding normal vector $p \in \Delta^\circ$ that is common across individuals.

The subdomain $\mathcal{D}_{IL}$ plays an important role in our argument. It is the domain in which any allocation is efficient and only distributive properties of allocations are the issue, and we propose some axioms for such class of *pure redistribution problems.*

We now define the feasibility correspondence and social choice correspondence.

[2] More realistically, it will be reasonable to consider that some individuals do not have strictly positive endowment vectors, while the social sum is still strictly positive. Allowing individual endowment vectors to be not strictly positive leads to a well-known problem that Walrasian solution and its relatives may be empty. As our central issue is calibrating income redistribution, we rule out this problem from consideration.

Definition 1 The *feasibility correspondence* $F : \mathbb{R}^{nl}_{++} \twoheadrightarrow \mathbb{R}^{nl}_{+}$ is defined by

$$F(e) = \left\{ x \in \mathbb{R}^{nl}_{+} : \sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} e_i \right\}$$

for all $e \in \mathbb{R}^{nl}_{++}$.

Definition 2 *Social choice correspondence* is a correspondence $\varphi : \mathcal{D} \times \mathbb{R}^{nl}_{++} \twoheadrightarrow \mathbb{R}^{nl}_{+}$ such that

$$x \in F(e)$$

and

$$x_i \neq \mathbf{0}$$

for all $(R, e) \in \mathcal{D} \times \mathbb{R}^{nl}_{++}$, $x \in \varphi(R, e)$ and $i = 1 \cdots, n$.

Remark 1 Below we will invoke some axioms which are necessary and sufficient for Nash implementability in certain kinds of mechanisms, where the planner does not know the individuals' preferences. What about the case that the planner does not know their endowments? Hurwicz, Maskin, and Postlewaite (1994) show that excluding zero consumption vectors is sufficient for Nash implementability via a mechanism in which the individual may destroy (but cannot exaggerate) their endowments.

For this reason, we take the condition that nobody should receive zero consumption vector as a part of the definition of the social choice correspondence throughout.

The individual rationality condition is still required, though, for Nash implementability via a mechanism in which the individuals may withhold their endowments.

We consider the following axioms on the social choice correspondence. First one states that the recommended allocations should not be Pareto-dominated.

Efficiency: For all $(R, e) \in \mathcal{D} \times \mathbb{R}^{nl}_{++}$ and $x \in \varphi(R, e)$ there is no $x' \in F(e)$ such that $x'_i R_i x_i$ for all $i = 1, \cdots, n$ and $x'_i P_i x_i$ for at least one i.

Definition 3 The *Pareto correspondence* is a correspondence $P : \mathcal{D} \times \mathbb{R}^{nl}_{++} \twoheadrightarrow \mathbb{R}^{nl}_{++}$ such that $P(R, e)$ is the set of efficient allocations for each $(R, e) \in \mathcal{D} \times \mathbb{R}^{nl}_{++}$.

Second axioms states that if there is an allocation which is Pareto-indifferent to the allocation already recommended then it should not be excluded.

Non-Discrimination: For all $(R, e) \in \mathcal{D} \times \mathbb{R}^{nl}_{++}$, $x \in \varphi(R, e)$ and $x' \in F(e)$, if $x'_i I_i x_i$ for all $i = 1, \cdots, n$, then $x' \in \varphi(R, e)$.

Dropping the individual rationality axiom leaves it unspecified how much individuals are responsible for their initial endowments. We argue that it is rather *calibrated* as a part of pinning down the solution. It is not an obvious question if such calibration works, as it is not obvious how much one's endowment is valuable for the society, because in general individuals' preferences are diverse and complementarity and substitution between goods are complex.

Therefore we focus on the cases in which we can unambiguously define how much an individual contributes to the society. In particular, we focus on the cases that all individuals have an identical and linear preference and hence any allocation is efficient and only the distributive properties of allocation are the issue. Within such subdomain there is a natural measure of welfare that is common to everybody, income evaluation of consumptions measured by the common marginal rate of substitution.

The axiom below states that welfare level for each individual should be pinned down uniquely in such pure redistribution problems.

Welfare Uniqueness under Identical Linear Preferences: For all $(R, e) \in \mathcal{D}_{IL} \times \mathbb{R}^{nl}_{++}$, and $x, x' \in \varphi(R, e)$, it holds $x'_i I_i x_i$ for all $i = 1, \cdots, n$.

It is imposed for a positive reason rather than normative, as it is a minimally necessary condition for a social choice rule to be indeed a "solution." This is rather a very mild requirement, in the sense that it is required to apply only to the small domain of identical linear preferences, and it does not exclude any reasonable solution at a conceptual level.

Indeed it is weaker than the standard individual rationality axiom, which states that nobody should get worse off than his initial endowment.

Individual Rationality: For all $(R, e) \in \mathcal{D} \times \mathbb{R}^{nl}_{++}$, and $x \in \varphi(R, e)$, $x_i R_i e_i$ for all $i = 1, \cdots, n$.

Lemma 1 Individual Rationality implies Welfare Uniqueness under Identical Linear Preferences.

Proof. For all $(R, e) \in \mathcal{D}_{IL} \times \mathbb{R}^{nl}_{++}$, where $p \in \Delta^\circ$ denotes the normal vector corresponding to R, pick any $x \in \varphi(R, e)$. IR requires $px_i \geq pe_i$ for all $i = 1, \cdots, n$. If there is some i with $px_i > pe_i$, under the feasibility constraint there is j with $pe_j > px_j$, which violates IR. Therefore $px_i = pe_i$ for all $i = 1, \cdots, n$, which implies welfare uniqueness. ■

Next two axioms are about informational efficiency and implementability which are standard in the literature. The Maskin monotonicity axiom is known to be necessary for implementability in Nash equilibria and also sufficient for it in the current setting, because the no veto power condition is vacuously met under $n \geq 3$ (Maskin (1999)).

Maskin Monotonicity: For all $(R, e), (R', e) \in \mathcal{D} \times \mathbb{R}^{nl}_{++}$ and $x \in \varphi(R, e)$, if

$$x_i R_i y_i \implies x_i R'_i y_i$$

for all $i = 1, \cdots, n$ and $y_i \in \left\{ z_i \in \mathbb{R}^l_+ : \exists z_{-i} \in \mathbb{R}^{(n-1)l}_+,\ (z_i, z_{-i}) \in F(e) \right\}$, then $x \in \varphi(R', e)$.

The following Gevers monotonicity axiom is weaker than Maskin Monotonicity (Gevers (1986)).

Gevers Monotonicity: For all $(R, e), (R', e) \in \mathcal{D} \times \mathbb{R}^{nl}_{++}$ and $x \in \varphi(R, e)$, if

$$x_i R_i y_i \implies x_i R'_i y_i$$

for all $i = 1, \cdots, n$ and $y_i \in \mathbb{R}^l_+$, then $x \in \varphi(R', e)$.

The above axioms characterize a class of market-based mechanisms which are indexed by income distribution codes.

Definition 4 An *income distribution code* is a function $t : \Delta^\circ \times \mathbb{R}^{nl}_{++} \to \mathbb{R}^n_{++}$ such that

$$\sum_{i=1}^{n} t_i(p, e) = p \sum_{i=1}^{n} e_i$$

hold for all $(p, e) \in \Delta^\circ \times \mathbb{R}^{nl}_{++}$.

Note that we are making the restriction that all individuals should receive positive income, in order to be consistent with the assumption that nobody should receive zero consumption vector.

Here is the definition of the Walras rule with income distribution code.

Definition 5 *Walras rule with income distribution code* t is a correspondence $W_t : \mathcal{D} \times \mathbb{R}^{nl}_{++} \twoheadrightarrow \mathbb{R}^{nl}_{+}$ such that for all $(R, e) \in \mathcal{D} \times \mathbb{R}^{nl}_{++}$ an allocation $x \in \mathbb{R}^{nl}_{+}$ belongs to $W_t(R, e)$ if and only if $x \in F(e)$ and there exists $p \in \Delta^{\circ}$ such that for all $i = 1, \cdots, n$ it holds

$$px_i \quad \leq \quad t_i(p, e)$$

and it holds

$$px'_i \leq t_i(p, e) \quad \Longrightarrow \quad x_i R_i x'_i$$

for all $x'_i \in \mathbb{R}^{l}_{+}$.

Likewise we can define the constrained version of the above, in which possible individual deviation is limited to socially feasible consumptions.

Definition 6 *Constrained Walras rule with income distribution code* t is a correspondence $CW_t : \mathcal{D} \times \mathbb{R}^{nl}_{++} \twoheadrightarrow \mathbb{R}^{nl}_{+}$ such that for all $(R, e) \in \mathcal{D} \times \mathbb{R}^{nl}_{++}$ an allocation $x \in \mathbb{R}^{nl}_{+}$ belongs to $CW_t(R, e)$ if and only if $x \in F(e)$ and there exists $p \in \Delta^{\circ}$ such that for all $i = 1, \cdots, n$ it holds

$$px_i \quad \leq \quad t_i(p, e)$$

and it holds

$$px'_i \leq t_i(p, e) \quad \Longrightarrow \quad x_i R_i x'_i$$

for all $x'_i \in \{z_i \in \mathbb{R}^{l}_{+} : \exists z_{-i} \in \mathbb{R}^{(n-1)l}_{+},\ (z_i, z_{-i}) \in F(e)\}$.

The following lemma is straightforward.

Lemma 2 $W_t \subset CW_t \subset P$ for any income distribution code t.

The lemma below characterizes the Walras rule with income distribution code in the domain of identical linear preferences.

Lemma 3 If a social choice correspondence φ satisfies Efficiency, Welfare Uniqueness under Identical Linear Preferences and Non-Discrimination, then there is an income distribution code t such that $\varphi(R, e) = W_t(R, e)$ for all $(R, e) \in \mathcal{D}_{IL} \times \mathbb{R}^{nl}_{++}$.

Proof. Define the income distribution code t as follows: For $(p,e) \in \Delta^\circ \times \mathbb{R}^{nl}_{++}$ and $i = 1, \cdots, n$, let $t_i(p,e) = px_i$ for $x \in \varphi(R,e)$, where $R \in \mathcal{D}_{IL}$ is the profile of identical linear preferences which has p as the common normal vector.

By Welfare Uniqueness under Identical Linear Preferences, t is well-defined, that is t does not depend on the choice of $x \in \varphi(R,e)$.

By Efficiency, $\sum_{i=1}^n t_i(p,e) = p\sum_{i=1}^n e_i$ is met,

By definition we have $\varphi(R,e) \subset W_t(R,e)$. By Non-Discrimination we have $\varphi(R,e) \supset W_t(R,e)$. ■

Because $W_t(R,e) = CW_t(R,e)$ for all $(R,e) \in \mathcal{D}_{IL} \times \mathbb{R}^{nl}_{++}$, we also have

Lemma 4 If a social choice correspondence φ satisfies Welfare Uniqueness under Identical Linear Preferences and Non-Discrimination, then there is an income distribution code t such that $\varphi(R,e) = CW_t(R,e)$ for all $(R,e) \in \mathcal{D}_{IL} \times \mathbb{R}^{nl}_{++}$.

Here is our first main result.

Theorem 1 Assume $\mathcal{D} = \mathcal{R}^n$. If a social choice correspondence φ satisfies Efficiency, Welfare Uniqueness under Identical Linear Preferences, Non-Discrimination and Gevers Monotonicity then there is an income distribution code t such that $\varphi(R,e) \supset W_t(R,e)$ for all $(R,e) \in \mathcal{D} \times \mathbb{R}^{nl}_{++}$ and equality holds on $\mathcal{D}_{IL} \times \mathbb{R}^{nl}_{++}$.

Proof. For any $(R,e) \in \mathcal{D} \times \mathbb{R}^{nl}_{++}$, pick any $x \in W_t(R,e)$. Let p be the equilibrium price vector. Let $R^* \in \mathcal{D}_{IL}$ be the profile of identical linear preferences having p as the common normal vector.

Then $x \in W_t(R^*,e)$ and by the previous lemma we have $x \in \varphi(R^*,e)$. By Gevers Monotonicity we obtain $x \in \varphi(R,e)$. ■

Replacing Gevers Monotonicity by Maskin Monotonicity characterizes the constrained Walras rule with income distribution code.

Theorem 2 Assume $\mathcal{D} = \mathcal{R}^n$. If a social choice correspondence φ satisfies Efficiency, Welfare Uniqueness under Identical Linear Preferences, Non-Discrimination and Maskin Monotonicity then there is an income distribution code t such that $\varphi(R,e) \supset CW_t(R,e)$ for all $(R,e) \in \mathcal{D} \times \mathbb{R}^{nl}_{++}$ and equality holds on $\mathcal{D}_{IL} \times \mathbb{R}^{nl}_{++}$.

Proof. For any $(R, e) \in \mathcal{D} \times \mathbb{R}^{nl}_{++}$, pick any $x \in CW_t(R, e)$. Let p be the equilibrium price vector. Let $R^* \in \mathcal{D}_{IL}$ be the profile of identical linear preferences having p as the common normal vector.

Then $x \in CW_t(R^*, e)$ and by the previous lemma we have $x \in \varphi(R^*, e)$. By Maskin Monotonicity we obtain $x \in \varphi(R, e)$. ■

3 Domain of smooth or identical linear preferences

Let $\mathcal{S} \subset \mathcal{R}$ be the set of strongly monotone and strictly convex preferences which are smooth on $\mathbb{R}^l_{++}$ and satisfy the boundary condition, i.e., $\{z' \in \mathbb{R}^l_+ : zIz'\} \subset \mathbb{R}^l_{++}$ for all $R \in \mathcal{S}$ and for all $z \in \mathbb{R}^l_{++}$. See Debreu (1972) and Mas-Colell (1985) for the detailed mathematical definition of smooth preferences.

For each individual i, given $R_i \in \mathcal{S}$ and its differentiable representation u_i, the marginal rate of substitution of good k for m at $x_i \in \mathbb{R}^l_{++}$ for individual i is given by

$$MRS_{km}(x_i, R_i) = \frac{\frac{\partial u_i(x_i)}{\partial x_{im}}}{\frac{\partial u_i(x_i)}{\partial x_{ik}}}$$

Here we consider the domain of smooth or identical linear preferences $\mathcal{D} = \mathcal{S}^n \cup \mathcal{D}_{IL}$, where $\mathcal{D}_{IL}$ is viewed as consisting of limit points of profiles in $\mathcal{S}^n$.

The following local independence axiom (Nagahisa (1991)) is an informational efficiency condition which states that only marginal rates of substitution should matter. When $n \geq 3$ it is known to be necessary and sufficiency condition for implementability in Nash equilibria via "economic" mechanism, where messages to be submitted take the form of prices and quantities (see Dutta, Sen and Vohra (1994)).

Local Independence: For all $(R, e), (R', e) \in (\mathcal{S}^n \cup \mathcal{D}_{IL}) \times \mathbb{R}^{nl}_{++}$ and $x \in \mathbb{R}^{nl}_{++}$, if $MRS_{km}(x_i, R_i) = MRS_{km}(x_i, R'_i)$ for all $i = 1, \cdots, n$ and $k, m = 1, \cdots, l$, then $x \in \varphi(R, e)$ if and only if $x \in \varphi(R', e)$.

Lemma 5 Under Efficiency, Local Independence implies Gevers Monotonicity on $\mathcal{D} = \mathcal{S}^n \cup \mathcal{D}_{IL}$.

Proof. Pick any $(R, e), (R', e) \in (\mathcal{S}^n \cup \mathcal{D}_{IL}) \times \mathbb{R}^{nl}_{++}$ and $x \in \varphi(R, e)$. Suppose it holds that

$$x_i R_i y_i \implies x_i R'_i y_i$$

for all $i = 1, \cdots, n$ and $y_i \in \mathbb{R}^l_+$.

There are two cases

Case 1: $R \in \mathcal{D}_{IL}$

If $R' \in \mathcal{D}_{IL}$ we have $R' = R$, which implies $x \in \varphi(R', e)$. Otherwise, Efficiency implies that x is an interior allocation.

Then, by the nature of smooth preferences it must be that $MRS_{km}(x_i, R_i) = MRS_{km}(x_i, R'_i)$ for all $i = 1, \cdots, n$ and $k, m = 1, \cdots, l$, otherwise the indifference surfaces cross. By Local Independence we have $x \in \varphi(R', e)$.

Case 2: $R \in \mathcal{S}^n$

By Efficiency x is an interior allocation. Then we follow the same argument as above.

■

Together with other axioms the local independence axiom implies that the social choice correspondence must be a selection of the Walrasian correspondence with some income distribution code.

Theorem 3 Assume $\mathcal{D} = \mathcal{S}^n \cup \mathcal{D}_{IL}$. If a social choice correspondence φ satisfies Efficiency, Welfare Uniqueness under Identical Linear Preferences and Local Independence, then there is an income distribution code t such that $\varphi(R, e) \subset W_t(R, e)$ for all $(R, e) \in \mathcal{D} \times \mathbb{R}^{nl}_{++}$ and equality holds on $\mathcal{D}_{IL} \times \mathbb{R}^{nl}_{++}$.

Proof. For any $(R, e) \in \mathcal{S}^n \times \mathbb{R}^{nl}_{++}$, pick any $x \in \varphi(R, e)$. By Efficiency and the smoothness of preference x must be an interior allocation and there is a unique supporting vector p. Let $R^* \in \mathcal{D}_{IL}$ be the profile of identical linear preferences having p as the common normal vector.

By Local Independence we have $x \in \varphi(R^*, e)$. Since $\varphi(R^*, e) = W_t(R^*, e)$ as $R^* \in \mathcal{D}_{IL}$, we have $x \in W_t(R^*, e)$. By the property of W_t, we obtain $x \in W_t(R, e)$. ■

Because Local Independence implies Gevers Monotonicity on the domain of smooth or identical linear preferences we have a full characterization of the Walras rule with income distribution code.

Theorem 4 Assume $\mathcal{D} = \mathcal{S}^n \cup \mathcal{D}_{IL}$. Then a social choice correspondence φ satisfies Efficiency, Welfare Uniqueness under Identical Linear Preferences, Non-Discrimination and Local Independence if and only if there is an income distribution code t such that $\varphi = W_t$.

Proof. It is straightforward to see that W_t satisfies all the axioms. So we prove sufficiency of the axioms.

Because Local Independence implies Gevers Monotonicity, we can apply the same argument as in Theorem 1 and obtain $\varphi \supset W_t$.

On the other hand, $\varphi \subset W_t$ follows from Theorem 3. ■

4 Comparative properties

Now we investigate comparative properties of the allocation rule. Walrasian solution and its relatives are known to have tricky comparative properties when preferences are diverse. Transfer paradox and violation of resource monotonicity are typical examples of this, where having larger endowments may hurt of the individual and others, since having more resources may change how goods are substituted with each other.

Thus we consider comparative properties on the domain of identical linear preferences, in which such properties are fairly intuitive. Note again that imposing conditions to meet only in the small domain of identical linear preferences is a mild and less demanding requirement, rather than doing so on the whole domain of preferences.

First we impose an axiom that having larger endowments should not hurt anybody.

Restricted Endowment Monotonicity: For all $R \in \mathcal{D}_{IL}$ and $e, e' \in \mathbb{R}^{nl}_{++}$, if $e'_i R_i e_i$ for all $i = 1, \cdots, n$ then for all $x \in \varphi(R, e)$ and $x' \in \varphi(R, e')$ it holds $x'_i R_i x_i$ for all $i = 1, \cdots, n$.

Restricted Endowment Monotonicity characterizes the class of income distribution codes in which only the lists of individual incomes should matter.

Lemma 6 Suppose $\varphi = W_t$ holds for some income distribution code t on $\mathcal{D}_{IL} \times \mathbb{R}^{nl}_{++}$. Then φ satisfies Restricted Endowment Monotonicity additionally if and only if it holds

$$pe_j \geq pe'_j \ \forall j = 1, \cdots, n \implies t_i(p, e) \geq t_i(p, e').$$

for all $i = 1, \cdots, n$, for all $p \in \Delta^\circ$ and $e, e' \in \mathbb{R}^{nl}_{++}$.

Next axiom states that allocations are linear in initial endowment vectors when all individuals have identical and linear preferences. Consider that there are two different resource allocation problems with different initial endowment vectors and that the society is deciding allocation after consolidating the two problems. Then, when all individuals have identical and linear preferences the order of such consolidation should not matter.

Restricted Endowment Linearity: For all $R \in \mathcal{D}_{IL}$ and $e, e' \in \mathbb{R}^{nl}_{++}$, for all $\alpha, \beta > 0$, for all $x \in \varphi(R, e)$ and $x' \in \varphi(R, e')$, it holds $\alpha x + \beta x' \in \varphi(R, \alpha e + \beta e')$.

Restricted Endowment Linearity characterizes the class of income distribution codes which are linear in initial endowment vectors. The following lemma is immediate.

Lemma 7 Suppose $\varphi = W_t$ holds for some income distribution code t on $\mathcal{D}_{IL} \times \mathbb{R}^{nl}_{++}$. Then φ satisfies Restricted Endowment Linearity additionally if and only if

$$t_i(p, \alpha e + \beta e') = \alpha t_i(p, e) + \beta t_i(p, e')$$

for all $p \in \Delta^\circ$ and $i = 1, \cdots, n$, and for all $e, e' \in \mathbb{R}^{nl}_{++}$ and $\alpha, \beta > 0$.

To explain the next axiom, let us introduce a mixture operation over the domain of identical linear preferences. Given $R, R' \in \mathcal{D}_{IL}$, let p, p' and be the common normal vectors corresponding to them respectively. Then, for $\alpha \in [0, 1]$, let $\alpha R \oplus (1 - \alpha)R' \equiv (\alpha R_1 \oplus (1-\alpha)R'_1, \cdots, \alpha R_n \oplus (1-\alpha)R'_n) \in \mathcal{D}_{IL}$ be the profile of identical linear preferences given by

$$x_i(\alpha R_i \oplus (1 - \alpha)R'_i)y_i \iff (\alpha p + (1 - \alpha)p')x_i \geq (\alpha p + (1 - \alpha)p')y_i$$

for all i.

To illustrate, imagine that two societies merge so that household i in one society forms a joint household with its counterpart household i in the latter, where the proportion of merging is the same across household identities. As all the households in both societies have linear preferences merging has no complementarity effect. As all the households in each society before merging have identical preferences it is natural that the merging with common proportion does not change welfare weights put over the households. Hence the welfare levels for the society consisting of the merged households are simply the corresponding mixture of the welfare levels given in the original societies before merging. This is what is said by the axiom below.

Restricted Preference Linearity: For all $R, R' \in \mathcal{D}_{IL}$ and $e \in \mathbb{R}^{nl}_{++}$, for all $\alpha \in [0,1]$, for all $x \in \varphi(R,e)$, $x' \in \varphi(R',e)$ and $\widetilde{x} \in \varphi(\alpha R \oplus (1-\alpha)R', e)$, it holds

$$(\alpha p + (1-\alpha)p')\widetilde{x}_i = \alpha p x_i + (1-\alpha)p' x'_i,$$

where p and p' are the common normal vectors corresponding to R and R', respectively.

To motivate further, imagine a situation that there are l states of the world and the society is deciding how to allocate state-contingent consumptions, in which with probability α the society consists of risk neutral households all with an identical belief p and with probability $1-\alpha$ the society consists of risk neutral households all with an identical belief p'. Then, from an ex-ante viewpoint the society is viewed as consisting of risk neutral households with an identical belief $\alpha p + (1-\alpha)p'$. Then the axioms states that the household should receive the ex-ante expected values of welfare.

Restricted Preference Linearity characterizes the class of income distribution codes which are linear in price vectors. The following lemma is immediate.

Lemma 8 Suppose $\varphi = W_t$ holds for some income distribution code t on $\mathcal{D}_{IL} \times \mathbb{R}^{nl}_{++}$. Then φ satisfies Restricted Preference Linearity additionally if and only if

$$t_i(\alpha p + (1-\alpha)p', e) = \alpha t_i(p,e) + (1-\alpha)t_i(p',e)$$

for all $e \in \mathbb{R}^{nl}_{++}$ and $i = 1, \cdots, n$, and for all $p, p' \in \Delta^\circ$ and $\alpha \in [0,1]$.

The last axiom is a weaker version of the anonymity axiom, which will not need explanation.

Restricted Anonymity: For all $(R,e) \in \mathcal{D}_{IL} \times \mathbb{R}^{nl}_{++}$ and any permutation $\pi : \{1, \cdots, n\} \to \{1, \cdots, n\}$, $x \in \varphi(R,e)$ if and only if $x^\pi \in \varphi(R^\pi, e^\pi)$, where $z^\pi_i = z_{\pi^{-1}(i)}$ for each $i = 1, \cdots, n$ for any object z with n-entries.

The following lemma is immediate.

Lemma 9 Suppose $\varphi = W_t$ holds for some income distribution code t on $\mathcal{D}_{IL} \times \mathbb{R}^{nl}_{++}$. Then φ satisfies Anonymity additionally if and only if for all $(p,e) \in \Delta^\circ \times \mathbb{R}^{nl}_{++}$ and any permutation $\pi : \{1, \cdots, n\} \to \{1, \cdots, n\}$ it holds

$$t(p, e^\pi) = (t(p,e))^\pi$$

for all $(p,e) \in \Delta^\circ \times \mathbb{R}^{nl}_{++}$.

The above four additional axioms characterize a one-parameter family of income distribution codes.

Theorem 5 Suppose $\varphi = W_t$ holds for some income distribution code t on $\mathcal{D}_{IL} \times \mathbb{R}^{nl}_{++}$. Then φ satisfies Restricted Endowment Monotonicity, Restricted Endowment Linearity, Restricted Preference Linearity and Restricted Anonymity additionally if and only if there exists a constant $\lambda \in [0,1]$ such that

$$t_i(p,e) = \lambda p e_i + (1-\lambda) p \overline{e}_{-i}$$

for all $(p,e) \in \Delta^\circ \times \mathbb{R}^{nl}_{++}$ and $i = 1, \cdots, n$, where $\overline{e}_{-i} = \frac{1}{n-1}\sum_{j \neq i} e_j$.

Proof. Fix any $i = 1, \cdots, n$.

From Lemma 7 and 8, we can extend t_i to a bi-linear mapping over $\mathbb{R}^l \times \mathbb{R}^{nl}$ as follows.

First, define $t_i^* : \mathbb{R}^l_{++} \times \mathbb{R}^{nl}_{++} \to \mathbb{R}_{++}$ by

$$t_i^*(p,e) = \frac{1}{\alpha(p)} t_i(\alpha(p)p, e)$$

where $\alpha(p) > 0$ is such that $\alpha(p)p \in \Delta^\circ$.

Then, from Lemma 8, for all $p, p' \in \mathbb{R}^l_{++}$ and $\beta, \beta' > 0$, we have

$$\begin{aligned}
\beta t_i^*(p,e) + \beta' t_i^*(p',e) &= \frac{\beta}{\alpha(p)} t_i(\alpha(p)p,e) + \frac{\beta'}{\alpha(p')} t_i(\alpha(p')p',e) \\
&= \left(\frac{\beta}{\alpha(p)} + \frac{\beta'}{\alpha(p')}\right)\left(\frac{\frac{\beta}{\alpha(p)}}{\frac{\beta}{\alpha(p)} + \frac{\beta'}{\alpha(p')}} t_i(\alpha(p)p,e) + \frac{\frac{\beta'}{\alpha(p')}}{\frac{\beta}{\alpha(p)} + \frac{\beta'}{\alpha(p')}} t_i(\alpha(p')p',e)\right) \\
&= \left(\frac{\beta}{\alpha(p)} + \frac{\beta'}{\alpha(p')}\right) t_i\left(\frac{\frac{\beta}{\alpha(p)}}{\frac{\beta}{\alpha(p)} + \frac{\beta'}{\alpha(p')}} \alpha(p)p + \frac{\frac{\beta'}{\alpha(p')}}{\frac{\beta}{\alpha(p)} + \frac{\beta'}{\alpha(p')}} \alpha(p')p', e\right) \\
&= \left(\frac{\beta}{\alpha(p)} + \frac{\beta'}{\alpha(p')}\right) t_i\left(\frac{\beta p + \beta' p'}{\frac{\beta}{\alpha(p)} + \frac{\beta'}{\alpha(p')}}, e\right) \\
&= t_i^*(\beta p + \beta' p', e),
\end{aligned}$$

where $\alpha(p), \alpha(p') > 0$ are such that $\alpha(p)p, \alpha(p')p' \in \Delta^\circ$.

From Lemma 7, for all $e, e' \in \mathbb{R}^{nl}_{++}$ and $\beta, \beta' > 0$ it holds

$$\begin{aligned}
t_i^*(p, \beta e + \beta' e') &= \frac{1}{\alpha(p)} t_i(\alpha(p)p, \beta e + \beta' e') \\
&= \frac{\beta}{\alpha(p)} t_i(\alpha(p)p, e) + \frac{\beta'}{\alpha(p)} t_i(\alpha(p)p, e') \\
&= \beta t_i^*(p,e) + \beta' t_i^*(p,e')
\end{aligned}$$

where $\alpha(p) > 0$ is such that $\alpha(p)p \in \Delta^\circ$.

Second, define $t_i^{**} : \mathbb{R}^l \times \mathbb{R}^{nl}_{++} \to \mathbb{R}$ by

$$t_i^{**}(q, e) = t_i^*(p, e) - t_i^*(p', e),$$

where $p, p' \in \mathbb{R}^l_{++}$ are such that $q = p - p'$.

Note that

$$t_i^{**}(\mathbf{0}, e) = t_i^*(p, e) - t_i^*(p, e) = 0$$

and

$$t_i^{**}(-p, e) = t_i^*(p, e) - t_i^*(2p, e) = t_i^*(p, e) - 2t_i^*(p, e) = -t_i^*(p, e) = -t_i^{**}(p, e).$$

From the property shown in the previous step, for all $q, q' \in \mathbb{R}^l$ and $\beta, \beta' \in \mathbb{R}$, we have

$$t_i^{**}(\beta q + \beta' q', e) = \beta t_i^{**}(q, e) + \beta' t_i^{**}(q', e).$$

Again it maintains the property that

$$t_i^{**}(q, \beta e + \beta' e') = \beta t_i^{**}(q, e) + \beta' t_i^{**}(q, e')$$

holds for all $e, e' \in \mathbb{R}^{nl}_{++}$ and $\beta, \beta' > 0$.

Finally, define $t_i^{***} : \mathbb{R}^l \times \mathbb{R}^{nl} \to \mathbb{R}$ by

$$t_i^{***}(q, a) = t_i^{**}(q, e) - t_i^{**}(q, e'),$$

where $e, e' \in \mathbb{R}^{nl}_{++}$ are such that $a = e - e'$.

Note that

$$t_i^{***}(q, \mathbf{0}) = t_i^{**}(q, e) - t_i^{**}(q, e) = 0$$

and

$$t_i^{***}(q, -e) = t_i^{**}(q, e) - t_i^{**}(q, 2e) = t_i^{**}(q, e) - 2t_i^{**}(q, e) = -t_i^{***}(q, e).$$

Then, for all $q \in \mathbb{R}^l$, for all $a, a' \in \mathbb{R}^{nl}$ and $\beta, \beta' \in \mathbb{R}$ it holds

$$t_i^{***}(q, \beta a + \beta' a') = \beta t_i^{***}(q, a) + \beta' t_i^{***}(q, a').$$

Again it maintains the property that

$$t_i^{***}(\beta q + \beta' q', a) = \beta t_i^{***}(q, a) + \beta' t_i^{***}(q', a)$$

holds for all $q, q' \in \mathbb{R}^l$ and $\beta, \beta' \in \mathbb{R}$.

Because t_i^* is a homogeneous extension of t_i over to $\mathbb{R}^l_{++} \times \mathbb{R}^{nl}_{++}$, from Restricted Endowment Monotonicity it holds that

$$qa_j \geq qa'_j \ \forall j = 1, \cdots, n \implies t_i^*(q, a) \geq t_i^*(q, a')$$

for all $i = 1, \cdots, n$.

Because t_i^{***} is a continuous extension of t_i^*, this property is extended to $\mathbb{R}^l_+ \times \mathbb{R}^{nl}_+$ at least. Note also that the monotonicity property and linearity imply that

$$qa_j = 0 \ \forall j = 1, \cdots, n \implies t_i^{***}(q, a) = 0$$

for all $(q, a) \in \mathbb{R}^l_+ \times \mathbb{R}^{nl}_+$, because it holds $qa_j = q(2a_j) = 0$ for all $j = 1, \cdots, n$ let's say, and hence $t_i^{***}(q, a) = t_i^{***}(q, 2a) = 2t_i^{***}(q, a)$, implying $t_i^{***}(q, a) = 0$.

Finally, from its definition, t^{***} inherits the property following Restricted Anonymity:

$$t^{***}(q, a^\pi) = (t^{***}(q, a))^\pi$$

hold for all $q \in \mathbb{R}^l$ and $a, a' \in \mathbb{R}^{nl}$, and for all permutation $\pi : \{1, \cdots, n\} \to \{1, \cdots, n\}$.

For each $h = 1, \cdots, l$, let $v^h \in \mathbb{R}^l_+$ denote the vector having 1 as its h-th entry and 0 for else. For each $j = 1, \cdots, n$ and $k = 1, \cdots, l$, let $w^{j,k} \in \mathbb{R}^{nl}_+$ denote the vector having 1 as its (j, k)-th entry and 0 for else. That is, when $w^{j,k}_{\mu,\eta}$ denotes the (μ, η)-th entry of vector $w^{j,k}$, where μ goes from 1 to n and η goes from 1 to l, it holds $w^{j,k}_{\mu,\eta} = 1$ only when $\mu = j$ and $\eta = k$ and $w^{j,k}_{\mu,\eta} = 0$ otherwise.

For $i = 1, \cdots, n$, $h = 1, \cdots, l$ and (j, k) with $j = 1, \cdots, n$, $k = 1, \cdots, l$, let

$$T_{i,h,(j,k)} = t_i^{***}(v^h, w^{j,k}).$$

Then by the bilinear property shown above we obtain

$$\begin{aligned} t_i^{***}(q, a) &= t_i^{***}\left(\sum_{h=1}^{l} q_h v^h, \sum_{j=1}^{n}\sum_{k=1}^{l} a_{j,k} w^{j,k}\right) \\ &= \sum_{h=1}^{l} q_h t_i^{***}\left(v^h, \sum_{j=1}^{n}\sum_{k=1}^{l} a_{j,k} w^{j,k}\right) \\ &= \sum_{h=1}^{l}\sum_{j=1}^{n}\sum_{k=1}^{l} q_h a_{j,k} t_i^{***}\left(v^h, w^{j,k}\right) \\ &= \sum_{h=1}^{l}\sum_{j=1}^{n}\sum_{k=1}^{l} T_{i,h,(j,k)} q_h a_{j,k} \end{aligned}$$

Note that $\sum_{\eta=1}^{l} v_\eta^h w_{\mu,\eta}^{j,k} = 0$ for all $\mu = 1, \cdots, n$ whenever $h \neq k$. This together with the property shown above, i.e.,

$$qa_j = 0 \ \forall j = 1, \cdots, n \implies t_i^{***}(q, a) = 0$$

for all $(q, a) \in \mathbb{R}_+^l \times \mathbb{R}_+^{nl}$, implies $T_{i,h,(j,k)} = t_i^{***}\left(v^h, w^{j,k}\right) = 0$ for all $h \neq k$.

Note that for every $k, h = 1, \cdots, l$ with $h \neq k$, we have $\sum_{\eta=1}^{l}(v_\eta^k + v_\eta^h) w_{j,\eta}^{j,k} = 1$ and $\sum_{\eta=1}^{l}(v_\eta^k + v_\eta^h) w_{\mu,\eta}^{j,k} = 0$ for all $\mu \neq j$, and likewise $\sum_{\eta=1}^{l}(v_\eta^k + v_\eta^h) w_{j,\eta}^{j,h} = 1$ and $\sum_{\eta=1}^{l}(v_\eta^k + v_\eta^h) w_{\mu,\eta}^{j,h} = 0$ for all $\mu \neq j$. Therefore we have

$$t_i^{***}(v^k + v^h, w^{j,k}) = t_i^{***}(v^k + v^h, w^{j,h}).$$

Since $t_i^{***}(v^h, w^{j,k}) = t_i^{***}(v^k, w^{j,h}) = 0$, we have

$$t_i^{***}(v^k, w^{j,k}) = t_i^{***}(v^h, w^{j,h}),$$

which implies

$$T_{i,k,(j,k)} = T_{i,h,(j,h)}$$

for all $k, h = 1, \cdots, l$.

Hence, we can let $\widehat{T}_{i,j} = T_{i,k,(j,k)}$ for arbitrary $k = 1, \cdots, l$, and obtain the form

$$t_i^{***}(p, e) = \sum_{j=1}^{n} \widehat{T}_{i,j} \sum_{k=1}^{l} p_k e_{j,k}.$$

From the properties already obtained it holds $\widehat{T}_{i,j} \geq 0$ for all $j = 1, \cdots, n$ and

$$\sum_{j=1}^{n} \widehat{T}_{i,j} = 1.$$

By the property following from Restricted Anonymity, we have $\widehat{T}_{i,i} = \widehat{T}_{j,j}$, $\widehat{T}_{i,j} = \widehat{T}_{j,i}$ and $\widehat{T}_{i,\mu} = \widehat{T}_{i,\nu}$ for all i, j and $\mu, \nu \neq i$. That is, $\widehat{T}$ is a symmetric $n \times n$ bi-stochastic matrix such that all the diagonal entries are equal to each other and all the off-diagonal entries are equal to each other.

Hence $\widehat{T}$ takes the form

$$\widehat{T} = \lambda I + (1 - \lambda) E,$$

where I is the $n \times n$ identity matrix and E is the $n \times n$ matrix with all diagonal entries being 0 and all off-diagonal entries being $\frac{1}{n-1}$. ■

Note that in the above class: (i) when $\lambda = 1$ it is the case of no redistribution, where $t_i(p, e) = pe_i$ for each i; (ii) when $\lambda = \frac{1}{n}$ it is the case of equal division of social income, where $t_i(p, e) = p\bar{e}$ for each i with $\bar{e} = \frac{1}{n}\sum_{j=1}^{n} e_j$, and (iii) when $\lambda = 0$ it is the case of receiving the others' average income, where $t_i(p, e) = p\bar{e}_{-i}$.

The class of income redistribution codes we have characterized *violates* so-called recursive invariance (except when $\lambda = \frac{1}{n}$), which states in the context of quasi-linear bargaining that once the prescribed allocation is taken to be the initial endowment reapplying the rule does not change the allocation (see Chun (1989), Green (2005)). This shows that where the original endowments are coming from does really matter in our argument. In other words, there is a distinction between endowments before redistribution, for which individuals are only partially responsible in general, and endowments after redistribution for which the individuals are now supposed to be responsible.

5 Conclusion

In this paper we have axiomatically studied the problem of resource allocation in which each individual is taken to be responsible but in general only partially for his initial endowment. In our argument how much individuals are responsible is rather calibrated as a part of pinning down the solution.

We have characterized a class of allocation rules which lie between the Walras rule taking initial endowments as they are and the Walras rule starting from equal division, which are parametrized by income redistribution codes.

Also, we have characterized a one-parameter family of income redistribution codes, in which one extreme corresponds to the case that each individual is 100% responsible for his initial endowment and the other extreme corresponds to the case that everybody is not responsible for this endowment at all.

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