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RECONSTRUCTION ALGEBRAS OF TYPE A

MICHAEL WEMYSS

Abstract. We introduce a new class of algebras, called reconstruction algebras, and present some of their basic properties. These non-commutative rings dictate in every way the process of resolving the Cohen-Macaulay singularities $\mathbb{C}^2/G$ where $G = \frac{1}{r}(1,a) \leq GL(2,\mathbb{C})$.

1. Introduction

It is not a new idea that non-commutative algebra in many ways dictates the process of desingularisation in algebraic geometry; this has been a theme in many recent papers (eg [Van04a], [BKR01], [Bri06]) however almost all research in this direction has taken place inside the relatively small sphere of Gorenstein singularities. For example when considering rings of invariants by small finite subgroups of $GL(n,\mathbb{C})$, the Gorenstein hypothesis forces the subgroups inside $SL(n,\mathbb{C})$.

For G a finite subgroup of $SL(2,\mathbb{C})$ it is well known that the preprojective algebra of the corresponding extended Dynkin diagram encodes the process of resolving the Gorenstein Kleinian singularity $\mathbb{C}[x,y]^G$. From the viewpoint of this paper, the preprojective algebra should be treated as an algebra that can be naturally associated to the dual graph of the minimal commutative resolution, from which we can gain all information about the process of desingularisation. Thus the preprojective algebra is defined with prior knowledge of the dual graph of the minimal resolution, but since it is Morita equivalent to the skew group ring we could alternatively use this purely algebraic ring. The question arises whether there are similar non-commutative algebras for finite subgroups of $GL(2,\mathbb{C})$.

The answer is yes [Wem08], and in this paper we prove it for the case of finite cyclic subgroups $G = \frac{1}{r}(1,a) \leq GL(2,\mathbb{C})$ (for notation see Section 2).

For such a group G, we associate to the dual graph of the minimal commutative resolution (complete with self-intersection numbers) a non-commutative ring $A_{r,a}$ which we call the *reconstruction algebra* and prove that $A_{r,a}$ is isomorphic to the endomorphism ring of the special Cohen-Macaulay modules in the sense of Wunram [Wun88]. This is important since it shows that for cyclic groups there is a structural correspondence (via the underlying quiver) between the special CM modules and the dual graph complete with self-intersection numbers, thus generalizing McKay's observation for finite subgroups of $SL(2,\mathbb{C})$ to finite cyclic subgroups of $GL(2,\mathbb{C})$.

The above is a correspondence purely on the level of the underlying quiver. However if we also add in the information of the relations we get more: in this paper we prove that the reconstruction algebra $A_{r,a}$

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- has centre $\mathbb{C}[x,y]^{\frac{1}{r}(1,a)}$ and so contains all the information regarding the singularity. Furthermore it is finitely generated over its centre, so is 'tractably' non-commutative (Corollary 3.26).
- contains enough information to construct the minimal resolution via a moduli space of finite dimensional representations (Theorem 4.5).
- contains exactly the same homological information as the minimal resolution through a derived equivalence (Theorem 5.8).

Although this paper studies cyclic subgroups of $GL(2,\mathbb{C})$ and therefore both the singularities $\mathbb{C}^2/G$ and their minimal resolutions are toric, the main ideas in this paper (for example the correspondence between the quiver and the dual graph) are independent of toric geometry and as such provide the correct framework for generalisation.

We also remark that in general the reconstruction algebra is *not* homologically homogenous in the sense of Brown-Hajarnavis [BH84]. This should not be surprising, as there are many other examples of non-commutative resolutions of sensible non-Gorenstein Cohen-Macaulay singularities which are not homologically homogeneous ([QS06] and [SdB06, 5.1(2)]). Non-commutative crepant resolutions have yet to be defined for Cohen-Macaulay singularities, however when $G \not\leq SL(2,\mathbb{C})$ the minimal resolution of $\mathbb{C}^2/G$ is not crepant yet is still important. Hence the rings we produce should certainly be examples of (non-crepant) non-commutative resolutions, whenever such a definition is conceived. The failure of the homologically homogeneous property suggests we ought to again think hard about the non-commutative analogue of smoothness.

In fact the reconstruction algebra $A_{r,a}$ should be the *minimal* non-commutative resolution in some rough sense; certainly there is the following picture of derived categories:

$$\begin{tikzcd} D^b(\mathfrak{coh}\widetilde{X}) & & D^b(\mathfrak{mod}\mathbb{C}[x,y]\#G) \\ & D^b(\mathfrak{mod}A_{r,a}) \arrow[ul, "\cong"'] \arrow[ur, hook] & \end{tikzcd}$$

so we should still perhaps view the skew group ring as a non-commutative resolution, just not the smallest one.

This paper is organized as follows - in Section 2 we define the reconstruction algebra associated to a labelled Dynkin diagram of type A and describe some of its basic structure. In Section 3 we prove that it is isomorphic to the endomorphism ring of some Cohen-Macaulay modules. In Section 4 the minimal resolution of the singularity $\mathbb{C}^2/\frac{1}{r}(1,a)$ is obtained via a certain moduli space of representations of the associated reconstruction algebra $A_{r,a}$, and in Section 5 we produce a tilting bundle which gives us our derived equivalence. In Section 6 we prove that $A_{r,a}$ is a prime ring and use this to show that the Azumaya locus of $A_{r,a}$ coincides with the smooth locus of its centre $\mathbb{C}[x,y]^{\frac{1}{r}(1,a)}$. This then gives a precise value for the global dimension of $A_{r,a}$, which shows that the reconstruction algebra need not be homologically homogeneous.

In this paper we work mostly in the unbounded derived categories where arbitrary coproducts exist. This allows us to use techniques such as Bousfield localisation and compactly generated categories to simplify some of the work needed to obtain bounded derived equivalences, which in turn saves us from having to prove at the beginning that the reconstruction algebra has finite global dimension.

Throughout we shall use $\mathrm{D}(\mathscr{A})$ to denote the unbounded derived category and $\mathrm{D}^b(\mathscr{A})$ to denote the bounded derived category. When working with quivers, we shall write xy to mean x **followed by** y**.** We work over the ground field $\mathbb{C}$ but any algebraically closed field of characteristic zero will suffice.

The moduli results in this paper have been independently discovered by Alastair Craw [Cra07]. The benefits of his approach is that the minimal resolution is produced by using global arguments (as opposed to my local arguments), however the technique here generalizes to the non-toric case [Wem08]. Also, here the non-commutative ring can be explicitly written down. Both approaches have their merits.

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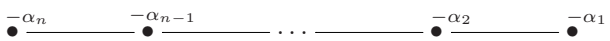
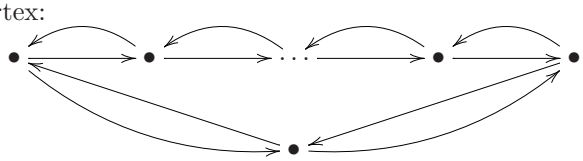
2. The Reconstruction Algebra of Type A

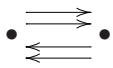
Consider, for positive integers $\alpha_i \geq 2$, the labelled Dynkin diagram of type A_n:



We call the vertex corresponding to α_i the ith vertex. To this picture we associate the double quiver of the extended Dynkin quiver, with the extended vertex called the 0th vertex:



Name this quiver Q'. For the sake of completeness note that for $n = 1$ by Q' we mean



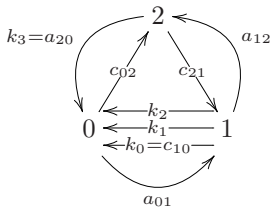
Now if any $\alpha_i > 2$, add an extra $\alpha_i - 2$ arrows from the ith vertex to the 0th vertex. Name this new quiver Q. Notice that when every $\alpha_i = 2$, $Q = Q'$ is exactly the underlying quiver of the preprojective algebra of type $\widetilde{A}_n$.

We label the arrows in Q as follows:

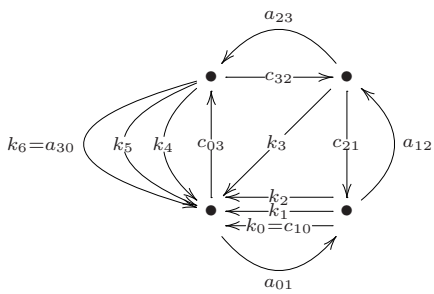
if $n = 1$	label the 2 arrows from 0 to 1 in Q' by a_1, a_2
	label the 2 arrows from 1 to 0 in Q' by c_1, c_2
	label the extra arrows due to α_1 by $k_1, \ldots k_{\alpha_1 - 2}$
if $n \geq 2$	label the clockwise arrows in Q' from i to $i-1$ by c_{ii-1} (and c_{0n})
	label the anticlockwise arrows in Q' from i to $i+1$ by a_{ii+1} (and a_{n0})
	label the extra arrows by $k_1, \ldots, k_{\sum(\alpha_i - 2)}$ anticlockwise

Note for example that c_{12} should be read 'clockwise from 1 to 2'. It is also convenient to write A_{ij} for the composition of anticlockwise paths a from vertex i to vertex j, and similarly C_{ij} as the composition of clockwise paths, where by C_{ii} (resp. A_{ii}) we mean not the empty path at vertex i but the path from i to i round each of the clockwise (resp. anticlockwise) arrows precisely once. For convenience we also denote $c_{10} := k_0$ and $a_{n0} := k_{1+\sum(\alpha_i - 2)}$.

Example 2.1. For $[\alpha_1, \alpha_2] = [4, 2]$ the quiver Q is



Example 2.2. For $[\alpha_1, \alpha_2, \alpha_3] = [4, 3, 4]$ the quiver Q is

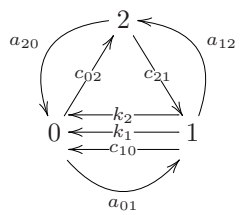


Denote by l_r the number of the vertex associated to the tail of the arrow k_r and denote $u_i := \max\{j : l_j = i\}$ and $v_i := \min\{j : l_j = i\}$. Because we have defined $k_0 := c_{10}$ and $k_{1+\sum(\alpha_i-2)} := a_{n0}$ it is always true that $v_1 = 0$ and $u_n = 1 + \sum(\alpha_i - 2)$. For $2 \leq i \leq n$ write $V_i := \max\{j : l_j < i\}$ and set $V_1 := 0$. In Example 2.2 above $u_1 = 2$, $v_3 = 4$, $V_{l_5} = V_3 = 3$ and $V_{l_2} = V_1 = 0$.

Definition 2.3. *For labels $[\alpha_1, \ldots, \alpha_n]$ with each $\alpha_i \geq 2$, define the reconstruction algebra of type A as the path algebra of the quiver Q subject to the following relations:*

$$
\begin{array}{llll}
\textit{if } n = 1 & c_2a_1 = c_1a_2 \textit{ and } a_1c_2 = a_2c_1 & & \\
 & k_1a_1 = c_2a_2 \textit{ and } a_1k_1 = a_2c_2 & & \\
 & k_ta_1 = k_{t-1}a_2 \textit{ and } a_1k_t = a_2k_{t-1} \ \forall\ 2 \leq t \leq \alpha_1 - 2. & & \\
\textit{if } n \geq 2 & \textit{Step 1:} & \textit{If } \alpha_1 = 2 & c_{10}a_{01} = a_{12}c_{21} \\
 & & \textit{If } \alpha_1 > 2 & k_sa_{01} = k_{s+1}C_{01}, a_{01}k_s = C_{01}k_{s+1} \ \forall\ 0 \leq s < u_1 \\
 & & & k_{u_1}a_{01} = a_{12}c_{21}. \\
 & & & \vdots \\
 & \textit{Step t:} & \textit{If } \alpha_t = 2 & c_{tt-1}a_{t-1t} = a_{tt+1}c_{t+1t} \\
 & & \textit{If } \alpha_t > 2 & c_{tt-1}a_{t-1t} = k_{v_t}C_{0t}, C_{0t}k_{v_t} = A_{0l_{V_t}}k_{V_t} \\
 & & & k_sA_{0t} = k_{s+1}C_{0t}, A_{0t}k_s = C_{0t}k_{s+1} \ \forall\ v_t \leq s < u_t \\
 & & & k_{u_t}A_{0t} = a_{tt+1}c_{t+1t} \\
 & & & \vdots \\
 & \textit{Step n:} & \textit{If } \alpha_n = 2 & c_{nn-1}a_{n-1n} = a_{n0}c_{0n}, c_{0n}a_{n0} = A_{0l_{V_n}}k_{V_n} \\
 & & \textit{If } \alpha_n > 2 & c_{nn-1}a_{n-1n} = k_{v_n}c_{0n}, c_{0n}k_{v_n} = A_{0l_{V_n}}k_{V_n} \\
 & & & k_sA_{0n} = k_{s+1}c_{0n}, A_{0n}k_s = c_{0n}k_{s+1} \ \forall\ v_n \leq s < u_n
\end{array}
$$

Example 2.4. The reconstruction algebra of type A associated to $[4,2]$ is



$$k_2a_{01} = a_{12}c_{21} \quad c_{10}a_{01} = k_1c_{02}c_{21} \quad k_1a_{01} = k_2c_{02}c_{21}$$
$$c_{21}a_{12} = a_{20}c_{02} \quad a_{01}c_{10} = c_{02}c_{21}k_1 \quad a_{01}k_1 = c_{02}c_{21}k_2$$
$$c_{02}a_{20} = a_{01}k_2$$

Example 2.5. The reconstruction algebra of type A associated to $[4,3,4]$ is the path algebra of the quiver in Example 2.2 subject to the relations

$$\begin{aligned}
c_{10}a_{01} &= k_1c_{03}c_{32}c_{21} & a_{01}c_{10} &= c_{03}c_{32}c_{21}k_1 & & \\
k_1a_{01} &= k_2c_{03}c_{32}c_{21} & a_{01}k_1 &= c_{03}c_{32}c_{21}k_2 & & \\
k_2a_{01} &= a_{12}c_{21} & c_{21}a_{12} &= k_3c_{03}c_{32} & c_{03}c_{32}k_3 &= a_{01}k_2 \\
k_3a_{01}a_{12} &= a_{23}c_{32} & c_{32}a_{23} &= k_4c_{03} & c_{03}k_4 &= a_{01}a_{12}k_3 \\
k_4a_{01}a_{12}a_{23} &= k_5c_{03} & a_{01}a_{12}a_{23}k_4 &= c_{03}k_5 & & \\
k_5a_{01}a_{12}a_{23} &= a_{30}c_{03} & a_{01}a_{12}a_{23}k_5 &= c_{03}a_{30} & &
\end{aligned}$$

Definition 2.6. *For $r, a \in \mathbb{N}$ with $\mathrm{hcf}(r,a) = 1$ and $r > a$ define the group $G = \frac{1}{r}(1,a)$ by*

$$G = \left\langle \zeta := \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^a \end{pmatrix} \right\rangle \leq GL(2,\mathbb{C}),$$

where ε is a primitive r^{th} root of unity.

Now consider the Jung-Hirzebruch continued fraction expansion of $\frac{r}{a}$, namely

$$\frac{r}{a} = \alpha_1 - \frac{1}{\alpha_2 - \frac{1}{\alpha_3 - \frac{1}{(\dots)}}} := [\alpha_1, \dots, \alpha_n]$$

with each $\alpha_i \geq 2$. The labelled Dynkin diagram of type A associated to this data is precisely the dual graph of the minimal resolution of $\mathbb{C}^2/\frac{1}{r}(1,a)$ [Rie77, Satz8].

Definition 2.7. *Define the reconstruction algebra $A_{r,a}$ associated to the group $G = \frac{1}{r}(1,a)$ to be the reconstruction algebra of type A associated to the data of the Jung-Hirzebruch continued fraction expansion of $\frac{r}{a}$.*

Note for the group $\frac{1}{r}(1, r-1)$, the reconstruction algebra $A_{r,r-1}$ is the reconstruction algebra of type A for the data $[\underbrace{2,\dots,2}_{r-1}]$. Since $V_n = 0$, $k_{V_n} = c_{10}$ and $l_{V_n} = 1$ this is precisely the preprojective algebra of type $\widetilde{A}_{r-1}$.

Example 2.8. Since $\frac{7}{2} = [4,2]$ the reconstruction algebra $A_{7,2}$ associated to the group $\frac{1}{7}(1,2)$ is precisely the algebra in Example 2.4.

Example 2.9. After noticing that $\frac{40}{11} = [4,3,4]$ we see that the reconstruction algebra $A_{40,11}$ associated to the group $\frac{1}{40}(1,11)$ is precisely the algebra in Example 2.5.

The following lemma is important later for certain duality arguments; geometrically it says that the reconstruction algebra is independent of the direction we view the dual graph of the minimal resolution:

Lemma 2.10. *The reconstruction algebra of type A associated to the data* $[\alpha_1, \dots, \alpha_n]$ *is the same as that associated to the data* $[\alpha_n, \dots, \alpha_1]$.

Proof. If $n = 1$ there is nothing to prove so assume $n \geq 2$. To avoid confusion write everything to do with the reconstruction algebra associated to $[\alpha_n, \dots, \alpha_1]$ in typeface fonts eg $\mathtt{a}_{0n}$, $\mathtt{A}_{03}$, $\mathtt{c}_{12}$ $\mathtt{C}_{0n-1}$, $\mathtt{k}_1$, $\mathtt{u}_n$ etc. Flip the quiver vertex numbers by the operation $'$ which takes 0 to itself (ie $0' = 0$), and reflects the other vertices in the natural line of symmetry (ie $1' = n$, $n' = 1$ etc), then the explicit isomorphism between the reconstruction algebras is given by

$$\begin{aligned} c_{ij} &\mapsto \mathtt{a}_{i'j'} \\ a_{ij} &\mapsto \mathtt{c}_{i'j'} \\ k_i &\mapsto \mathtt{k}_{n-i} \end{aligned}$$

Under this map $A_{ij} \mapsto \mathtt{C}_{i'j'}$ and $C_{ij} \mapsto \mathtt{A}_{i'j'}$, and furthermore the relations for the reconstruction algebra associated to $[\alpha_1, \dots, \alpha_n]$ read backwards are precisely the relations for the reconstruction algebra associated to $[\alpha_n, \dots, \alpha_1]$ read forwards. □

Now $A_{r,a}$ is supposed to encode all information about the singularity $\mathbb{C}[x,y]^{\frac{1}{r}(1,a)}$ as well as the resolution, so since $\mathbb{C}[x,y]^{\frac{1}{r}(1,a)}$ is determined by the continued fraction expansion of $\frac{r}{r-a}$ (by [Rie77, Satz1], see Lemma 3.5 below) it should be possible to read this directly from the quiver. Indeed this is true and to do it we must introduce some more notation.

Define $\sigma_1 = 1$ and inductively σ_s $(s \geq 1)$ to be the smallest vertex t with $t > \sigma_{s-1}$ and $\alpha_t > 2$ (if it exists), else $\sigma_s = n$. Stop this process when we reach n. Thus we have

$$1 = \sigma_1 < \ldots < \sigma_z = n.$$

Note if all $\alpha_t = 2$ this degenerates into $1 = \sigma_1 < \sigma_2 = n$.

Lemma 2.11. *For the group* $\frac{1}{r}(1,a)$ *with notation as above*

$$\frac{r}{r-a} = [\underbrace{2,\dots,2}_{u_{\sigma_1}-v_{\sigma_1}}, (\sigma_2-\sigma_1+2), \underbrace{2,\dots,2}_{u_{\sigma_2}-v_{\sigma_2}}, (\sigma_3-\sigma_2+2), \underbrace{2,\dots,2}_{u_{\sigma_3}-v_{\sigma_3}}, \dots, (\sigma_z-\sigma_{z-1}+2), \underbrace{2,\dots,2}_{u_{\sigma_z}-v_{\sigma_z}}]$$

Proof. This is just a reformulation of Riemenschneider duality, using the reconstruction algebra to give a slightly different interpretation of the Riemenschneider point diagram (see [Rie74, p223]). See also [Kid01, 1.2]. □

Example 2.12. By merely looking at the shape of $A_{40,11}$ in Example 2.2, by the above Lemma we can read off

$$\frac{40}{40-11} = [2,2,3,3,2,2]$$

Thus the shape of the reconstruction algebra $A_{r,a}$ determines the continued fraction expansion of $\frac{r}{r-a}$ which in turn determines the singularity $\mathbb{C}[x,y]^{\frac{1}{r}(1,a)}$ (see Lemma 3.5). We will prove in Corollary 3.26 that $Z(A_{r,a}) = \mathbb{C}[x,y]^{\frac{1}{r}(1,a)}$.

3. Special Cohen-Macaulay Modules

The reconstruction algebra is, by definition, constructed with prior knowledge of the minimal resolution. The aim of this section is to show that we could have defined it in a purely algebraic way by summing certain CM modules and looking at their endomorphism ring. More precisely in this section we shall show that the

reconstruction algebra is isomorphic as a ring to the endomorphism ring of the sum of the special CM modules. In the process, we shall see that the relations for the reconstruction algebra arise naturally through a notion which we call a web of paths.

Keeping the notation from the last section, consider the group $G = \frac{1}{r}(1, a) = \langle \zeta \rangle$.

Definition 3.1. *For $0 \leq t \leq r-1$ define*

$$S_t = \{f \in \mathbb{C}[x, y] : \zeta f = \varepsilon^t f\}.$$

These are precisely the non-isomorphic indecomposable maximal CM modules [Yos90, 10.10] over the CM singularity $X = \operatorname{Spec} \mathbb{C}[x, y]^G$. Of these, only some are important:

Definition 3.2. [Wun88] *The module S_t is said to be special if $S_t \otimes \omega_X/\text{torsion}$ is CM, where ω_X is the canonical module of $X = \operatorname{Spec} \mathbb{C}[x, y]^G$.*

Note that the ring S_0 is always special. There are in fact many equivalent characterisations of the special CM modules (see for example [Rie03, Thm 5]), some which refer to the minimal resolution and some that do not. For cyclic groups the combinatorics governing which CM modules are special is well understood.

Definition 3.3. *For $\frac{r}{a} = [\alpha_1, \ldots, \alpha_n]$ define the i-series and j-series as follows:*

$$\begin{array}{llll} i_0 = r & i_1 = a & i_t = \alpha_{t-1} i_{t-1} - i_{t-2} & \text{for } 2 \leq t \leq n+1 \\ j_0 = 0 & j_1 = 1 & j_t = \alpha_{t-1} j_{t-1} - j_{t-2} & \text{for } 2 \leq t \leq n+1 \end{array}$$

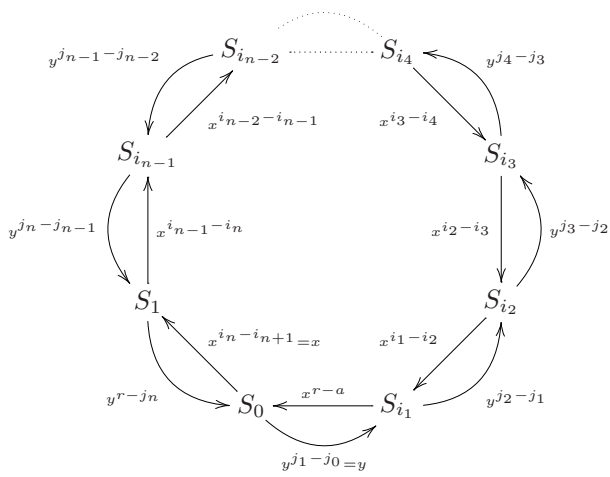
It's easy to see that

$$\begin{array}{ccccccccccc} i_0 = r & > & i_1 = a & > & i_2 & > & \ldots & > & i_n = 1 & > & i_{n+1} = 0 \\ j_0 = 0 & < & j_1 = 1 & < & j_2 = \alpha_1 & < & \ldots & < & j_n & < & j_{n+1} = r. \end{array}$$

It is the i-series which gives an easy combinatorial characterisation of the specials:

Theorem 3.4. [Wun87] *For $G = \frac{1}{r}(1, a)$ with $\frac{r}{a} = [\alpha_1, \ldots, \alpha_n]$, the special CM modules are precisely those S_{i_p} for $0 \leq p \leq n$. Furthermore if $1 \leq p \leq n$ then S_{i_p} is minimally generated by x^{i_p} and y^{j_p}.*

For $\frac{r}{a} = [\alpha_1, \ldots, \alpha_n]$ we now sum the specials and look at the endomorphism ring. Since $\operatorname{Hom}(S_{i_q}, S_{i_p}) \cong S_{i_p - i_q}$ certainly there are the following maps between the specials:



In general there will be more. If for any $1 \leq p \leq n$ it is true that $\alpha_p > 2$, then for each t such that $1 \leq t \leq \alpha_p - 2$, add an extra map from $S_{i_p} \to S_0$ labelled $x^{i_{p-1}-(t+1)i_p} y^{tj_p - j_{p-1}}$. Call the diagram complete with these extra arrows D. Define the natural map $\phi : \mathbb{C}Q \to \mathrm{End}(\oplus_{p=1}^{n+1} S_{i_p})$ by

$$\begin{array}{rcl} c_{0n} & \mapsto & x^{i_n - i_{n+1}} = x \\ c_{tt-1} & \mapsto & x^{i_{t-1}-i_t} \\ a_{n0} & \mapsto & y^{j_{n+1}-j_n} = y^{r-j_n} \\ a_{tt+1} & \mapsto & y^{j_{t+1}-j_t} \\ k_s & \mapsto & x^{i_{l_s-1}-((s-V_{l_s})+1)i_{l_s}} y^{(s-V_{l_s})j_{l_s} - j_{l_s-1}} \end{array}$$

where recall $V_1 := 0$. The remainder of this section is devoted to proving that ϕ is surjective, with kernel generated by the reconstruction algebra relations.

We firstly show surjectivity by proving that D contains all the generators of the maps between the specials, in that every other map between the specials is a finite sum of compositions of those in D.

To do this we argue that for any $0 \leq p \leq n$ we can see every $f \in \mathrm{Hom}(S_{i_p}, S_{i_p}) \cong \mathbb{C}[x,y]^G$ as a finite sum of compositions of arrows in D forming a cycle at vertex p. We then argue that given any two specials S_{i_p}, S_{i_q} we can see every $f \in \mathrm{Hom}(S_{i_p}, S_{i_q}) \cong S_{i_q - i_p}$ as a finite sum of compositions of arrows in D from vertex p to vertex q.

We begin by putting the generators of the ring $\mathbb{C}[x,y]^{\frac{1}{r}(1,a)}$ into a form suitable for our needs:

Lemma 3.5. *$\mathbb{C}[x,y]^{\frac{1}{r}(1,a)}$ is generated as a ring by the following invariants:*

$$\begin{array}{cc} & x^r \\ & x^{r-a}y \\ \text{if } \alpha_1 > 2 & \left\{ \begin{array}{c} x^{i_0 - 2i_1} y^{2j_1 - j_0} \\ \vdots \\ x^{i_0-(\alpha_1-1)i_1} y^{(\alpha_1-1)j_1 - j_0} \end{array} \right. \\ & \vdots \\ \text{if } \alpha_n > 2 & \left\{ \begin{array}{c} x^{i_{n-1}-2i_n} y^{2j_n - j_{n-1}} \\ \vdots \\ x^{i_{n-1}-(\alpha_n-1)i_n} y^{(\alpha_n-1)j_n - j_{n-1}} \end{array} \right. \\ & y^r \end{array}$$

where i and j are the i and j-series for the continued fraction expansion of $\frac{r}{a}$.

Proof. Denote by $\mathring{\imath}$ and $\mathring{\jmath}$ the i and j-series for the continued fraction expansion of $\frac{r}{r-a}$ then it is well known that the collection $x^{\mathring{\imath}_t} y^{\mathring{\jmath}_t}$ $(0 \leq t \leq 2 + \sum_{p=1}^{n}(\alpha_p - 2))$ generate the invariant ring [Rie77, Satz1]. We must put $\mathring{\imath}$ and $\mathring{\jmath}$ in terms of the i and j. By definition $x^{\mathring{\imath}_0} y^{\mathring{\jmath}_0} = x^r$ and $x^{\mathring{\imath}_1} y^{\mathring{\jmath}_1} = x^{r-a}y = x^{i_0 - i_1} y^{j_1 - j_0}$ and so the first two terms in the above list are correct.

Case 1: $\alpha_1 > 2$. Then $u_{\sigma_1} - v_{\sigma_1} = \alpha_1 - 2 > 0$ and so by Lemma 2.11 $\mathring{\imath}_2 = 2\mathring{\imath}_1 - \mathring{\imath}_0 = i_0 - 2i_1$ and $\mathring{\jmath}_2 = 2\mathring{\jmath}_1 - \mathring{\jmath}_0 = 2j_1 - j_0$, verifying the next in the list. Since the first $\alpha_1 - 2$ entries in the continued fraction expansion of $\frac{r}{r-a}$ are 2's, the first '$\alpha_1 > 2$ block' now follows easily.

Case 2: $\alpha_1 = 2$. Then the '$\alpha_2 > 2$ block' is empty.

In either case the last term we know is correct is $x^{i_0-(\alpha_1-1)i_1}y^{(\alpha_1-1)j_1-j_0}$ and the next block to check is the '$\alpha_{\sigma_2} > 2$ block'. To show that in this block the first term is correct we must prove (by definition of $\mathring{\imath}$ and $\mathring{\jmath}$, using Lemma 2.11 to tell us that the next term in the continued fraction expansion of $\frac{r}{r-a}$ is $\sigma_2 - \sigma_1 + 2$) that

$$(\sigma_2 - \sigma_1 + 2)(i_0 - (\alpha_1 - 1)i_1) - (i_0 - (\alpha_1 - 2)i_1) = i_{\sigma_2 - 1} - 2i_{\sigma_2}$$

and

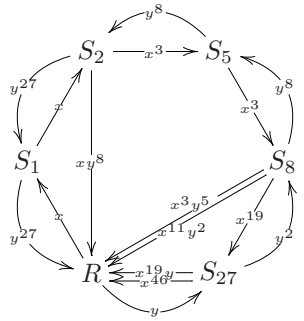
$$(\sigma_2 - \sigma_1 + 2)((\alpha_1 - 1)j_1 - j_0) - ((\alpha_1 - 2)j_1 - j_0) = 2j_{\sigma_2} - j_{\sigma_2 - 1}.$$

We show the first; the second is very similar. By grouping terms, the left hand side (LHS) equals $\sigma_2 i_0 - (\sigma_2\alpha_1 - \sigma_2 + 1)i_1$. But by definition of σ_2 it is easy to show that $i_{\sigma_2} = (\sigma_2 - 1)i_2 - (\sigma_2 - 2)i_1$ and so on substituting in $i_2 = \alpha_1 i_1 - i_0$ and grouping terms we obtain $-i_{\sigma_2} = LHS + (\alpha_1 - 1)i_1 - i_0$. But $(\alpha_1 - 1)i_1 - i_0 = i_2 - i_1 = \ldots = i_{\sigma_2} - i_{\sigma_2 - 1}$ gives $i_{\sigma_2 - 1} - 2i_{\sigma_2} = LHS$, as required.

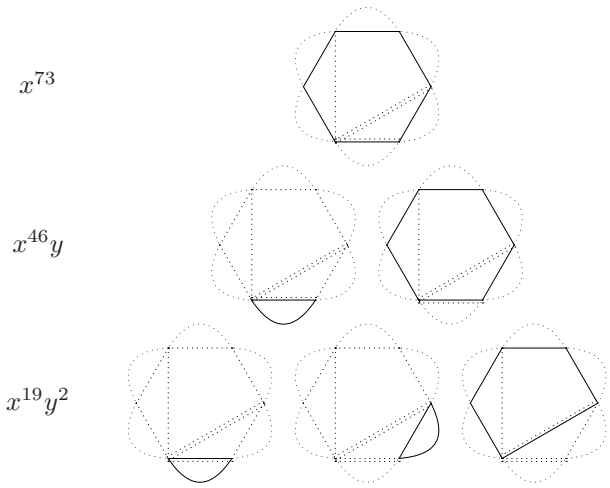
From the above we know that the first term in the '$\alpha_{\sigma_2} > 2$ block' is correct, and by Lemma 2.11 that the next $u_{\sigma_2} - v_{\sigma_2}$ terms in the continued fraction expansion of $\frac{r}{r-a}$ are all 2's. Thus the proof continues exactly as in Case 1 above, from which the induction step is clear. □

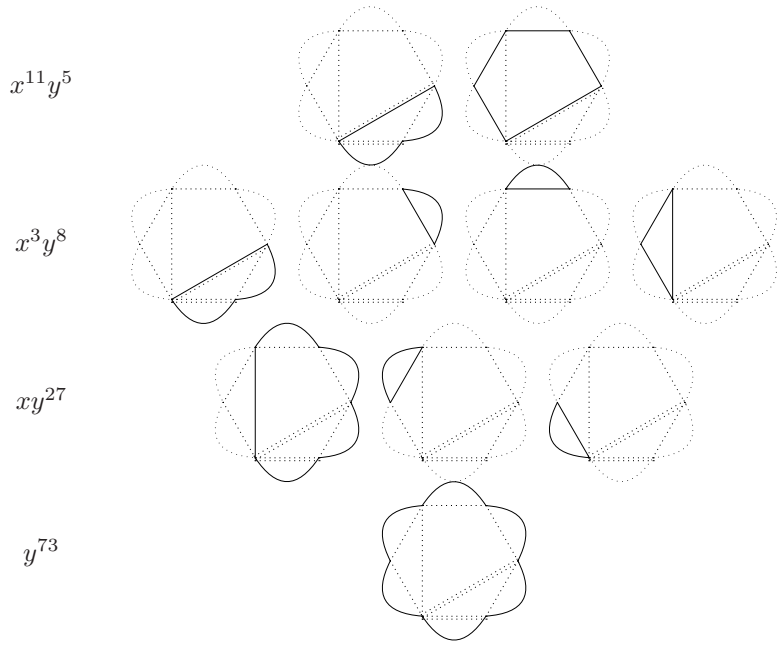
We now illustrate the correspondence between the generators of the invariant ring $\mathbb{C}[x,y]^{\frac{1}{r}(1,a)}$ and the cycles in D in an example.

Example 3.6. Consider the group $\frac{1}{73}(1,27)$. In this case the diagram D is



and we view the generators of the invariant ring $\mathbb{C}[x,y]^{\frac{1}{73}(1,27)}$ as follows:





The proof of the following lemma follows the pattern in the above example:

Lemma 3.7. *For any $0 \leq p \leq n$ we can view every $f \in \mathrm{Hom}(S_{i_p}, S_{i_p}) \cong \mathbb{C}[x,y]^G$ as a finite sum of compositions of arrows in D forming a cycle at vertex p.*

Proof. For every vertex we just need to justify that we can see the generators appearing in Lemma 3.5 as a cycle. To do this, we must introduce notation for the arrows in D and in light of the map ϕ above we just label the arrows in D by their notational counterpart in the reconstruction algebra. To simplify the exposition we also consider cycles up to rotation - for example by '$c_{10}a_{01}$ at vertices 0 and 1' we actually mean '$c_{10}a_{01}$ at vertex 1 and $a_{01}c_{10}$ at vertex 0'; similarly by 'C_{00} at vertex t' we mean C_{tt} (i.e. we rotate suitably so that the cycle starts and finishes at the vertex we want).

Now we can always see x^r at every vertex as C_{00}, and y^r at every vertex as A_{00}. We can always see $x^{r-a}y$ at vertices 0 and 1 as $c_{10}a_{01}$; for the remaining vertices how to view $x^{r-a}y$ depends on parameters. If $\alpha_1 > 2$ then we can see $x^{r-a}y$ everywhere as $C_{01}k_1$. If $\alpha_1 = 2$ then at all vertices t with $1 \leq t \leq \sigma_2$ we can see $x^{r-a}y$ as $c_{tt-1}a_{t-1t}$, and for all $\sigma_2 \leq t \leq n$ as $C_{0\sigma_2}k_1$. This takes care of $x^{r-a}y$, so we now move on to the remaining generators.

Case $\alpha_1 > 2$. If $\alpha_1 = 3$ then $x^{i_0-2i_1}y^{2j_1-j_0}$ can be seen at vertices 0 and 1 as k_1a_{01}, at all vertices t with $2 \leq t \leq \sigma_2$ as $c_{tt-1}a_{t-1t}$ and at all $\sigma_2 \leq t \leq n$ as $C_{0\sigma_2}k_{v_{\sigma_2}}$. If $\alpha_1 > 3$ then for all s with $2 \leq s \leq \alpha_1 - 2$, $x^{i_0-si_1}y^{sj_1-j_0}$ can be seen at all vertices as $C_{01}k_s$. Furthermore $x^{i_0-(\alpha_1-1)i_1}y^{(\alpha_1-1)j_1-j_0}$ can be seen at vertices 0 and 1 as $k_{u_1}a_{01}$, at all vertices t with $2 \leq t \leq \sigma_2$ as $c_{tt-1}a_{t-1t}$ and at all $\sigma_2 \leq t \leq n$ as $C_{0\sigma_2}k_{v_{\sigma_2}}$.

Case $\alpha_q > 2$ *for* $1 < q < n$. Denote $\gamma_q = \min\{T : q < T \leq n \text{ with } \alpha_T > 2\}$, or take $\gamma_q = n$ is this set is empty. Now if $\alpha_q = 3$ then $x^{i_{q-1}-2i_q}y^{2j_q-j_{q-1}}$ can be seen at all vertices t with $0 \leq t \leq q$ as $A_{0q}k_{u_q}$, at all vertices t with $q+1 \leq t \leq \gamma_2$ as $c_{tt-1}a_{t-1t}$ and at all $\gamma_2 \leq t \leq n$ as $C_{0\gamma_q}k_{v_{\gamma_q}}$. If $\alpha_q > 3$ then for all s with

$2 \leq s \leq \alpha_1 - 2$, $x^{i_{q-1}-si_q}y^{sj_q-j_{q-1}}$ can be seen at all vertices t with $0 \leq t \leq q$ as $A_{0q}k_{v_q+s-2}$ and at all vertices t with $q \leq t \leq n$ as $C_{0q}k_{v_q+s-1}$. Furthermore $x^{i_{q-1}-(\alpha_1-1)i_q}y^{(\alpha_1-1)j_q-j_{q-1}}$ can be seen at all vertices t with $0 \leq t \leq q$ as $A_{0q}k_{u_q}$, at all vertices t with $q+1 \leq t \leq \gamma_q$ as $c_{tt-1}a_{t-1t}$ and at all $\gamma_2 \leq t \leq n$ as $C_{0\gamma_2}k_{v_{\gamma_2}}$.

Case $\alpha_n > 2$. For all s with $2 \leq s \leq \alpha_n - 1$, $x^{i_{n-1}-si_n}y^{sj_n-j_{n-1}}$ can be seen at all vertices t as $A_{0n}k_{v_n+(s-2)}$. □

Lemma 3.8. *For any $0 \leq p \leq n$, every map $S_0 \to S_{i_p}$ can be seen in D.*

Proof. The case $p = 0$ is Lemma 3.7 so assume $p > 0$. Then $\mathrm{Hom}(S_0, S_{i_p}) \cong S_{i_p}$ which by Theorem 3.4 is generated as a module by x^{i_p} and y^{j_p}. Clearly both of these generators are paths in D and so since cycles at vertex p are all of $\mathbb{C}[x,y]^G$ we are done. □

Thus it remains to prove the following 2 statements:

(i) for any $0 \leq q < p \leq n$, every map $S_{i_p} \to S_{i_q}$ can be seen in D.
(ii) for any $0 < p < q \leq n$, every map $S_{i_p} \to S_{i_q}$ can be seen in D.

We shall see that we need only prove (i), then appealing to duality gives (ii) for free. In what follows we refer to the vertex S_{i_t} as the tth vertex.

Lemma 3.9. *For $0 \leq q < p \leq n$, if $x^{z_1}y^{z_2} \in S_{i_q-i_p}$ with $0 \leq z_1, z_2 \leq r-1$, then $x^{z_1}y^{z_2}$ factors as either*
(i) $(x^{i_{p-1}-i_p})A$ *for some* $A \in S_{i_q-i_{p-1}}$
(ii) $(x^{i_{p-1}-(t+1)i_p}y^{tj_p-j_{p-1}})B$ *for some* $B \in S_{i_q}$ *and some* $1 \leq t \leq \alpha_p - 2$
(iii) $(y^{j_{p+1}-j_p})C$ *for some* $C \in S_{i_q-i_{p+1}}$.

Proof. The case $p = n$ is trivial, so assume $p < n$. Clearly if $z_1 \geq i_{p-1} - i_p$ then we're in (i) so we can assume that $0 \leq z_1 < i_{p-1} - i_p$. Consider the invariant $x^{z_1}y^{z_2+(j_p-j_q)}$. Since we can see all invariants at every vertex, consider this as a path in D at the pth vertex. It must leave the vertex, and the hypothesis on z_1 means that it can't leave through the $x^{i_{p-1}-i_p}$ map.

Case 1: $\alpha_p = 2$ Then it must leave through the $y^{j_{p+1}-j_p}$ map to vertex $p+1$, i.e.

$$x^{z_1}y^{z_2+(j_p-j_q)} = y^{j_{p+1}-j_p}M$$

for some path M from vertex $p+1$ to q. Now from vertex $p+1$ the path M has to reach vertex p again. But this can only be reached in two ways, via the map $x^{i_{p-1}-i_p} = x^{i_p-i_{p+1}}$ from vertex $p+1$ to p, or via the map $y^{j_p-j_{p-1}}$ from vertex $p-1$ to p. The hypothesis on the x forces the later, so in particular the path factors through the 0 vertex. It may be true that there are cycles in the path that occur after the 0th vertex however since we have all cycles at all vertices we may move these cycles to the 0th vertex and hence assume that the path M finishes as the composition

$$y^{j_1-j_0}y^{j_2-j_1}\dots y^{j_p-j_{p-1}} = y^{j_p}$$

Hence we may write

$$x^{z_1}y^{z_2+(j_p-j_q)} = y^{j_{p+1}-j_p}Ay^{j_p}$$

for some path $A : S_{i_{p+1}} \to S_0$. But since $j_p \geq j_p - j_q$ we can cancel $y^{j_p-j_q}$ from both sides and write

$$x^{z_1}y^{z_2} = y^{j_{p+1}-j_p}A'$$

for some monomial A'. Necessarily $A' \in S_{i_q - i_{p+1}}$ and so we have the desired factorisation as in (iii).

Case 2: $\alpha_p > 2$. For notational ease denote the extra arrows leaving vertex p by $k_t = x^{i_{p-1}-(t+1)i_p} y^{tj_p - j_{p-1}}$. Now $x^{z_1} y^{z_2+(j_p-j_q)}$ must leave vertex p through the $y^{j_{p+1}-j_p}$ map to vertex $p+1$, or through one of the extra k_t. We argue case by case:

Suppose first that $x^{z_1} y^{z_2+(j_p-j_q)}$ leaves through the $y^{j_{p+1}-j_p}$ map to vertex $p+1$, i.e.

$$x^{z_1} y^{z_2+(j_p-j_q)} = y^{j_{p+1}-j_p} M$$

for some path M from vertex $p+1$ to p. If M leaves vertex $p+1$ through the $x^{i_p - i_{p+1}}$ map we are done since then

$$x^{z_1} y^{z_2+(j_p-j_q)} = x^{i_p - i_{p+1}} y^{j_{p+1}-j_p} M_1 = k_{\alpha_p - 2} y^{j_p} M_1$$

for some monomial M_1 and so since $j_p \geq j_p - j_q$ we may cancel and write $x^{z_1} y^{z_2} = k_{\alpha_p-2} M_1'$ for some monomial M_1' which necessarily belongs to S_{i_q}; this is a factorisation as in (ii). Hence we can assume that M leaves vertex $p+1$ via another path. Since $p+1 \leq n$ each of these paths has y component greater than or equal to $y^{j_{p+1}-j_p}$ and so we may write

$$x^{z_1} y^{z_2+(j_p-j_q)} = y^{2(j_{p+1}-j_p)} M_2$$

for some monomial M_2. But now $j_{p+1} - j_p > j_p - j_q$ so we may cancel and write $x^{z_1} y^{z_2} = y^{j_{p+1}-j_p} M_2'$ for some monomial M_2' which necessarily belongs to $S_{i_q - i_{p+1}}$; this is a factorisation as in (iii).

Now suppose that $x^{z_1} y^{z_2+(j_p-j_q)}$ factors through one of the extra arrows k_t out of vertex p. Thus $x^{z_1} y^{z_2+(j_p-j_q)} = k_t B$ for some $1 \leq t \leq \alpha_p - 2$ and some path B from 0 to p. By Lemma 3.8 there are 2 possibilities for B: either $B = x^{i_p} B_1$ or $B = y^{j_p} B_2$ for some invariants B_1 and B_2. We split the remainder of the proof into cases depending on the value of t:

If $t = 1$ then $x^{z_1} y^{z_2+(j_p-j_q)}$ is either $k_1 x^{i_p} B_1 = x^{i_{p-1}-i_p} y^{j_p - j_{p-1}} B_1$ which is impossible by the assumption on z_1, or it's equal to $k_1 y^{j_p} B_2$. But now $j_p - j_q \leq j_p$ and so after cancelling we may write $x^{z_1} y^{z_2} = k_1 B_2'$ for some monomial B_2' which necessarily belongs to S_{i_q}; this gives a factorization as in (ii).

The above argument takes care of $t = 1$ and so we are done if $\alpha_p = 3$. Hence the final case to consider is when $\alpha_p > 3$ and t is such that $1 < t \leq \alpha_p - 2$. Here $x^{z_1} y^{z_2+(j_p-j_q)}$ is either

$$k_t x^{i_p} B_1 = k_{t-1} y^{j_p} B_1 \quad \text{or} \quad k_t y^{j_p} B_2.$$

Again $j_p - j_q \leq j_p$ and so after cancelling we may write $x^{z_1} y^{z_2}$ as either

$$k_{t-1} B_1' \quad \text{or} \quad k_t B_2'$$

for some monomials B_2', B_2' which necessarily belong to S_{i_q}. This gives the required factorizations as in (ii), and completes the proof. □

The next two results are simple inductive arguments based on the previous lemma.

Corollary 3.10. *For any $0 \leq q < n$, every map $S_{i_n} \to S_{i_q}$ can be seen in D.*

Proof. Let $x^{z_1}y^{z_2} \in S_{i_q - i_n} = S_{i_q - 1}$ then by Lemma 3.7 we can remove cycles and so assume $0 \le z_1, z_2 \le r-1$. By Lemma 3.9 we know $x^{z_1}y^{z_2}$ either factors
(i) through vertex $n-1$ as $(x^{i_{n-1}-i_n})A$ for some map $A : S_{i_{n-1}} \to S_{i_q}$
(ii) through vertex 0 as $(x^{i_{n-1}-(t+1)i_n}y^{tj_n - j_{n-1}})B$ for some $B : S_0 \to S_{i_q}$ and some $1 \le t \le \alpha_n - 2$
(iii) through vertex 0 as $(y^{j_{n+1}-j_n})C$ for some $C \in S_{i_q - i_{p+1}} = S_{i_q} = \mathrm{Hom}(S_0, S_{i_q})$.
If we are in cases (ii) or (iii) then we're done by Lemma 3.8 since we can see both B and C in the diagram D. Hence assume case (i). If $q = n-1$ then we are done since by Lemma 3.7 we can see A in the diagram D. Hence we can assume that we are in case (i) with $q < n-1$. But now by Lemma 3.9 A either factors
(i) through vertex $n-2$ as $(x^{i_{n-2}-i_{n-1}})A'$ for some map $A' : S_{i_{n-2}} \to S_{i_q}$
(ii) through vertex 0 as $(x^{i_{n-2}-(t+1)i_{n-1}}y^{tj_{n-1}-j_{n-2}})B'$ for some $B' : S_0 \to S_{i_q}$ and some $1 \le t \le \alpha_{n-1} - 2$
(iii) through vertex n as $(y^{j_n - j_{n-1}})C'$ for some $C' : S_n \to S_q$.
Since we removed cycles from $x^{z_1}y^{z_2}$ at the beginning we can't be in case (iii). If in case (ii) then we're again done by Lemma 3.8, so can again assume in case (i). If $q = n-2$ then we again we are done, so we can suppose we are in case (i) with $q < n-2$. Proceed inductively; since $0 \le q$ this process must stop. □

Corollary 3.11. *For any $0 \le q < p \le n$, every map $S_{i_p} \to S_{i_q}$ can be seen in D.*

Proof. Fix q. This is now just a simple induction argument: if $p = n$ then the result is true by Corollary 3.10, so let $p < n$ and assume the result is true for larger p.

Let $x^{z_1}y^{z_2} \in S_{i_q - i_p}$ then by Lemma 3.7 we can remove cycles and so assume $0 \le z_1, z_2 \le r-1$. By Lemma 3.9 we know $x^{z_1}y^{z_2}$ either factors
(i) through vertex $p-1$ as $(x^{i_{p-1}-i_p})A$ for some map $A : S_{i_{p-1}} \to S_{i_q}$
(ii) through vertex 0 as $(x^{i_{p-1}-(t+1)i_p}y^{tj_p - j_{p-1}})B$ for some $B : S_0 \to S_{i_q}$ and some $1 \le t \le \alpha_p - 2$
(iii) through vertex $p+1$ as $(y^{j_{p+1}-j_p})C$ for some $C : S_{i_{p+1}} \to S_{i_q}$.
If we are in case (iii) then by inductive hypothesis we can see C in the diagram D and so we are done; if in case (ii) then by Lemma 3.8 we are also finished. Hence we can assume we are in case (i). If $q = p-1$ then we are done by Lemma 3.7 as we can see A in the diagram D, hence we can assume $q < p-1$. Thus the result follows by an identical argument as in Corollary 3.10 above - we've removed cycles and so A can't factor through S_{i_p}. □

To prove the corresponding statement in the opposite direction we appeal to duality. More precisely the singularity defined by $\frac{1}{r}(1,a)$ with $\frac{r}{a} = [\alpha_1, \ldots, \alpha_n]$ is isomorphic to the singularity $\frac{1}{r}(1,b)$ with $\frac{r}{b} = [\alpha_n, \ldots, \alpha_1]$ (note $b = j_n$); the isomorphism is given by swapping the x and y's. To avoid confusion refer to everything for the singularity $\frac{1}{r}(1,b)$ in typeface font, eg CM modules $\mathtt{S}_t$, i-series by $\mathtt{i}$, diagram $\mathtt{D}$ etc. The explicit isomorphism is given by

$$\begin{array}{ccc} S_0 & \to & \mathtt{S}_0 \\ x & \mapsto & \mathtt{y} \\ y & \mapsto & \mathtt{x} \end{array}$$

As in Lemma 2.10 flip the quiver vertex numbers by the operation $'$ which takes 0 to itself (ie $0' = 0$), and reflects the other vertices in the natural line of symmetry (ie $1' = n$, $n' = 1$ etc). Now for all $1 \le p \le n$ we have $i_p = \mathtt{j}_{p'}$ and $j_p = \mathtt{i}_{p'}$ thus

$$S_{i_p} = (x^{i_p}, y^{j_p})S_0 \cong (\mathtt{y}^{\mathtt{j}_{p'}}, \mathtt{x}^{\mathtt{i}_{p'}})\mathtt{S}_0 = \mathtt{S}_{\mathtt{i}_{p'}}.$$

Corollary 3.12. *For the singularity $\frac{1}{r}(1,a)$, for any $0 \leq p < q \leq n$, every map $S_{i_p} \to S_{i_q}$ can be seen in D.*

Proof. Under the duality above $x^{z_1}y^{z_2} : S_{i_p} \to S_{i_q}$ corresponds to $\mathtt{y}^{z_1}\mathtt{x}^{z_2} : \mathtt{S}_{\mathtt{i}_{p'}} \to \mathtt{S}_{\mathtt{i}_{q'}}$. But since $q' < p'$ we can by Corollary 3.11 view this in the diagram $\mathtt{D}$ as the composition of monomials whose powers are in terms of $\mathtt{i}$'s, $\mathtt{j}$'s and α_t's. Under the duality isomorphisms we can view this as a path in the diagram D. □

Summarizing Lemma 3.7, Corollary 3.11 and Corollary 3.12 we have:

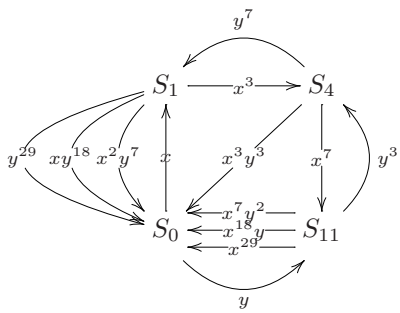
Proposition 3.13. *For any $\frac{1}{r}(1,a)$, for any $0 \leq p, q \leq n$, we can see every map $S_{i_p} \to S_{i_q}$ in the diagram D. Consequently the natural map $\mathbb{C}Q \to \mathrm{End}_R(\oplus_{p=1}^{n+1} S_{i_p})$ is surjective.*

This may seem abstract but in reality it is extremely useful if we actually want to compute some examples. One way to compute the endomorphism ring of the specials is to take the McKay quiver and corner (ie ignore a vertex and compose maps that pass through that vertex) the non-special vertices. Of course the larger the group the longer this computation; for the example $\frac{1}{40}(1,11)$ there are forty vertices in the McKay quiver and we need to corner thirty-six of them. Given any example $\frac{1}{r}(1,a)$, Proposition 3.13 saves us this long computation since the algorithm to produce the necessary diagram involves only the continued fraction expansion of $\frac{r}{a}$ and the associated i and j series, all of which are extremely quick to calculate.

Example 3.14. For $\frac{1}{40}(1,11)$, $\frac{40}{11} = [4,3,4]$ so the i and j-series are

$$\begin{array}{ccccccccc} i_0 = 40 & > & i_1 = 11 & > & i_2 = 4 & > & i_3 = 1 & > & i_4 = 0 \\ j_0 = 0 & < & j_1 = 1 & < & j_2 = 4 & < & j_3 = 11 & < & j_4 = 40. \end{array}$$

By Proposition 3.13 the endomorphism ring of the specials is

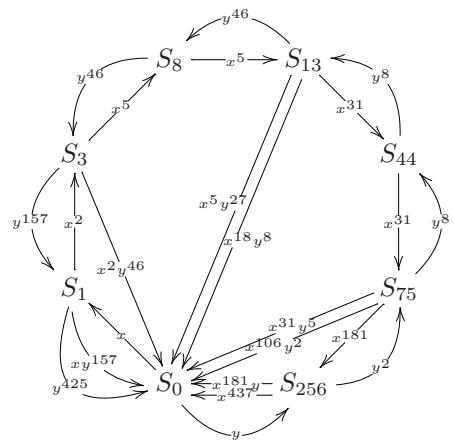


Notice the correspondence with Example 2.2.

Example 3.15. For the group $\frac{1}{693}(1,256)$, $\frac{693}{256} = [3,4,2,4,2,3,3]$ so the i and j series are

	0	1	2	3	4	5	6	7	8
i	693	256	75	44	13	8	3	1	0
j	0	1	3	11	19	65	111	268	693.

and further the endomorphism ring of the specials is



In Proposition 3.13 above we have shown that the natural map $\phi : \mathbb{C}Q \to \mathrm{End}_R(\oplus_{p=1}^{n+1} S_{i_p})$ is surjective and so we now show that the kernel is generated by the reconstruction algebra relations. To achieve this double index the arrows in $A_{r,a}$ as follows:

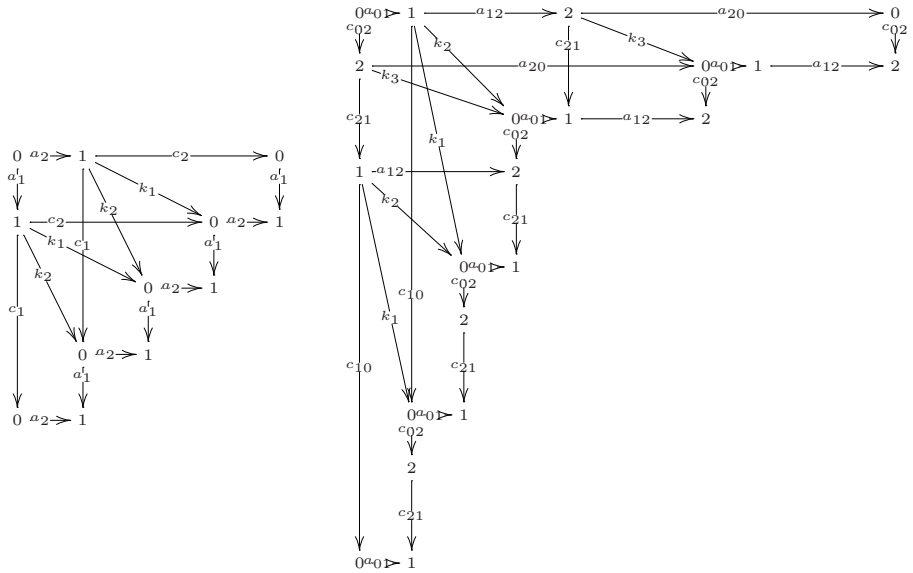
arrow	double index
c_{0n}	$(1,0)$
c_{tt-1}	$(i_{t-1}-i_t,0)$
a_{n0}	$(0,r-j_n)$
a_{tt+1}	$(0,j_{t+1}-j_t)$
k_s	$(i_{l_s-1}-((s-V_{l_s})+1)i_{l_s},(s-V_{l_s})j_{l_s}-j_{l_s-1})$

It is easy to see that the two terms in any relation for $A_{r,a}$ have the same double index and so the double index can be extended to all paths in $A_{r,a}$. We shall now show that if there exists a path of double index (z_1,z_2) leaving a vertex t in $A_{r,a}$ then the path is necessarily unique.

Definition 3.16. *For a given vertex t in $A_{r,a}$ define the web of paths leaving t as follows: place t in the $(0,0)$ position of a 2-dimensional grid, and for each arrow leaving t draw a line from $(0,0)$ to the double index of that arrow. Mark the end of this line by the head of the arrow. Continue in this way for all the heads of the arrows.*

This is best understood after consulting some examples. In the following two examples the web should be extended forever in the obvious direction; for practical purposes we draw only the start of the picture.

Example 3.17. The web of paths from vertex 0 in $A_{4,1}$ and $A_{11,3}$ begins respectively:

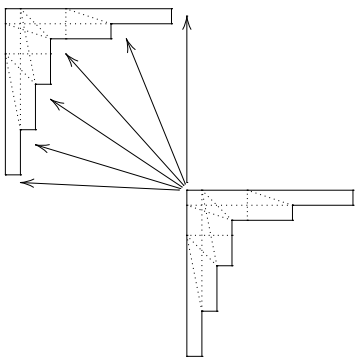


We call the points in the web of paths that lie in the set $\{(w,0) : 0 \leq w < n\}$ the left rail and similarly those that lie in the set $\{(0,w) : 0 \leq w < n\}$ are called the top rail.

Definition 3.18. *Draw the left rail and the top rail in the web of paths leaving vertex* 0*, and draw in every arrow leaving these vertices. Join the ends of these by using only vertical and horizontal paths, and call the resulting diagram* F*.*

The fact that this can always be done is due to the grading we put on $A_{r,a}$, together with simple combinatorics with continued fractions. The examples above show F for $A_{4,1}$ and $A_{11,3}$.

Now F generates the web of paths leaving 0 in the sense that all paths can be obtained by gluing on extra copies of F to the existing copy, as in the following picture



The copies of F glue together seamlessly due to the symmetry and repetition inside F - the boundary of F consists entirely of straight lines and crucially F contains (by definition) all paths from the rails so there can be no paths that leap over the boundary to create new paths.

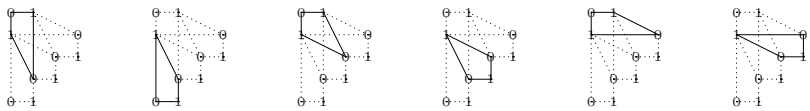
Since F generates the web of paths leaving 0 it is clear (since F can) that the web of paths can be subdivided into small 'squares'; we call these *basic squares*.

Definition 3.19. *A square is a pair of paths* $(p_1 \dots p_s, q_1 \dots q_t)$ *with* $tail(p_1) = tail(q_1)$ *and* $head(p_s) = head(q_t)$. *A square in* F *is called basic if* $p_i \notin \{q_1, \dots, q_t\}$ *for all* $1 \leq i \leq s$ *and* $q_j \notin \{p_1, \dots, p_s\}$ *for all* $1 \leq j \leq t$.

By the definition and structure of F it is clear that if all basic squares in F commute then all squares in F commute. Since F generates the web of paths this means all squares commute (since they are made from squares in F), giving us the required uniqueness of path.

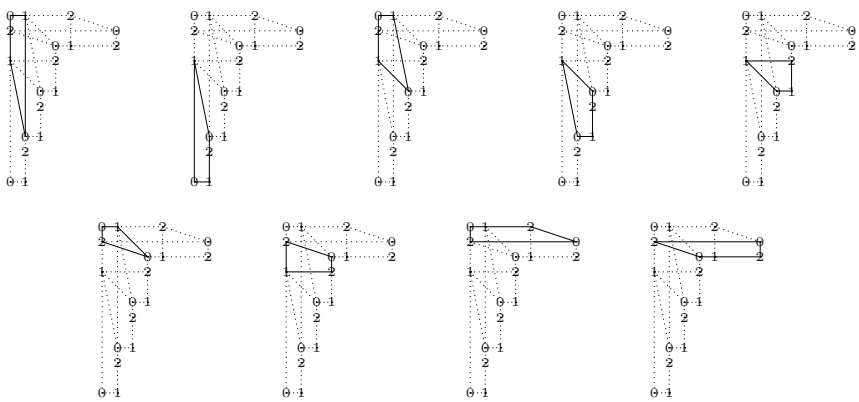
Because of the symmetry in F there are in fact repetitions of the basic squares inside F - more precisely the basic squares starting at the 1 on the top rail are the same as those starting at 1 on the left rail, etc. Thus by the symmetry in F it is clear that all the basic squares leaving the left rail are all the basic squares in F.

Example 3.20. The 6 basic squares in the example $A_{4,1}$ above are indicated by the following solid lines:



These are precisely the relations. Note that these prove the paths $a_2k_1a_1$ and $a_1c_1k_2$ coincide since that square can be subdivided into 2 basic squares, both of which commute.

Example 3.21. The 9 basic squares in the example $A_{11,3}$ above are



The first five diagrams are the five Step 1 relations, the last four the Step 2 relations.

Lemma 3.22. *For any double index* (z_1, z_2) *either there is precisely one path out of vertex* 0 *with that double index, or there are none.*

Proof. By the above we just need to prove that all the basic squares in F out of the left rail commute. This is just a bookkeeping exercise:
Case 1: $n = 1$. This is an easy extension of the $A_{4,1}$ example above.
Case 2: $n > 1$. We work through the basic squares leaving 1 (which we'll see, together with some basic squares leaving 0, correspond to the Step 1 relations) and then work upwards: if $\alpha_1 = 2$ then the only basic square leaving 1 is $c_{10}a_{01} = a_{12}c_{21}$,

so we may assume $\alpha_1 > 2$. Then, as in Example 3.21 we get $k_s a_{01} = k_{s+1} C_{01}$ and above it $a_{01} k_s = C_{01} k_{s+1}$ for all $0 \leq s < u_1$, then end with $k_{u_1} a_{01} = a_{12} c_{21}$. Thus all basic squares leaving 1 (and the corresponding ones leaving 0) on the left rail commute.
Now for basic squares leaving t on the left rail with $1 < t < n$ (if such t exist): if $\alpha_t = 2$ then the only basic square is $c_{tt-1} a_{t-1t} = a_{tt+1} c_{t+1t}$ so we may assume that $\alpha_t > 2$. Certainly we have the basic square $c_{tt-1} a_{t-1t} = k_{v_t} C_{0t}$ and above it $C_{0t} k_{v_t} = A_{0 l_{V_t}} k_{V_t}$. If $\alpha_t > 3$ we also have the basic squares $k_s A_{0t} = k_{s+1} C_{0t}$ and above it $A_{0t} k_s = C_{0t} k_{s+1}$ for all $v_t \leq s < u_t$. The final basic square out of t is $k_{u_t} A_{0t} = a_{tt+1} c_{t+1t}$.
For the basic squares leaving n on the left rail: if $\alpha_n = 2$ then the only basic square is $c_{nn-1} a_{n-1n} = a_{n0} c_{0n}$ and above it $c_{0n} a_{n0} = A_{0 l_{V_n}} k_{V_n}$. Hence assume $\alpha_n > 2$. Then $c_{nn-1} a_{n-1n} = k_{v_n} c_{0n}$ and above it $c_{0n} k_{v_n} = A_{0 l_{V_n}} k_{V_n}$. The only basic squares remaining are $k_s A_{0n} = k_{s+1} c_{0n}$ and above it $A_{0n} k_s = c_{0n} k_{s+1}$ for all $v_n \leq s < u_n$ (recall $k_{u_n} = a_{n0}$). □

Corollary 3.23. *For any double index (z_1, z_2) and any vertex t, either there is precisely one path out of vertex t with that double index, or there are none.*

Proof. To obtain the web of paths of vertex n delete the top row in the web of paths of vertex 0 and decrease the first index in every double index by 1. All squares in this web of paths commute because they do in the web of paths for vertex 0. For vertex $n-1$ delete all the rows above the $n-1$ on the left rail, and decrease the double indices accordingly. Again all squares in this web of paths commute since they do in the web of paths for vertex 0. Continue in this fashion. □

We now reach the main theorem which shows that the algebraically-constructed ring (the endomorphism ring of the specials) is isomorphic to the geometrically-constructed ring (the reconstruction algebra). For a third construction of the same non-commutative ring, see Section 5.

Theorem 3.24. *For $G = \frac{1}{r}(1, a)$, let $T_{r,a} = \oplus_{p=1}^{n+1} S_{i_p}$. Then*

$$A_{r,a} \cong \mathrm{End}_{\mathbb{C}[x,y]^G}(T_{r,a}).$$

Proof. We already have a map $\phi : \mathbb{C}Q \to \mathrm{End}(\oplus_{p=1}^{n+1} S_{i_p})$ which by Proposition 3.13 is surjective. It is straightforward to see that all the reconstruction relations are sent to zero and so belong to the kernel, since the double index of any relation corresponds to the double index (z_1, z_2) of the cycle $x^{z_1} y^{z_2}$ in the endomorphism ring that it represents. The content in the theorem is that there are no more relations. But this is just Corollary 3.23. □

Remark 3.25. If $a = r - 1$ then the group $\frac{1}{r}(1, r-1)$ is inside $SL(2, \mathbb{C})$, all CM modules are special and $T_{r,r-1} = \mathbb{C}[x, y]$ so this theorem reduces to the well known

$$A_{r,r-1} = \text{pre-projective algebra} \cong \mathrm{End}_{\mathbb{C}[x,y]^G}(\mathbb{C}[x, y]) \cong \mathbb{C}[x, y]\#G.$$

Corollary 3.26. *The centre of $A_{r,a}$ is $\mathbb{C}[x, y]^{\frac{1}{r}(1,a)}$ and furthermore $A_{r,a}$ is a finitely generated module over its centre. Consequently $A_{r,a}$ is a noetherian PI ring.*

Proof. Denote $R = \mathbb{C}[x, y]^{\frac{1}{r}(1,a)}$ then since CM modules are torsion-free there is an embedding of R into $\mathrm{End}_R(T_{r,a})$ which clearly has image inside the centre. It is the

whole of the centre since $\mathrm{Hom}_R(S_{i_p}, S_{i_p}) \cong R$ for all p. The fact that $A_{r,a}$ is finitely generated as an R-module is immediate from its description as an endomorphism ring. The rest is standard. □

4. Moduli Space of Representations and the Minimal Resolution

The minimal resolutions of cyclic quotient singularities are well understood by a construction of Fujiki (see for example [Wun87, 2.7]); more recently there is an easier algorithm using toric geometry techniques [Rei97] which coincides with the G-Hilb description ([Kid01], [Ish02]).

In this section we prove that for any group $G = \frac{1}{r}(1, a)$ we can obtain the minimal resolution of the singularity $\mathbb{C}^2/G$ from the moduli space of the reconstruction algebra $A_{r,a}$, thus giving yet another description of the minimal resolution. This may not be entirely surprising (by construction!), but it is important since by Theorem 3.24 we could have defined $A_{r,a}$ without prior knowledge of the minimal resolution.

For a summary of moduli space techniques we refer the reader to [Kin94], [Kin97]. For $G = \frac{1}{r}(1, 1)$ (ie the reconstruction algebra with the $n = 1$ relations) everything is trivial so we assume $n \geq 2$. With respect to the ordering of the vertices as in Section 2, fix for the rest of this paper the dimension vector $\alpha = (1, 1, \ldots, 1)$ and fix the generic stability condition $\theta = (-n, 1, \ldots, 1)$. The point is that when considering representations of this dimension vector the maps are just scalars so the relations reduce in complexity. As we shall see the stability condition is chosen to be 'blind' to the arrows $k_1, \ldots k_{\sum(\alpha_1 - 2)}$ and so we have a open covering of the moduli space by the same number of open sets as in the preprojective case (ie $n+1$ open sets). It is the relations that ensure each of the opens is still $\mathbb{C}^2$, and standard geometric arguments give minimality.

Definition 4.1. *For $0 \leq t \leq n$ define the open set W_t in $\mathrm{Rep}(A_{r,a}, \alpha)/\!\!/_\theta \mathrm{GL}$ as follows: W_0 is defined by the condition $C_{01} \neq 0$, W_n by the condition $A_{0n} \neq 0$ and for $1 \leq t \leq n-1$ W_t is defined by the conditions $C_{0t+1} \neq 0$ and $A_{0t} \neq 0$.*

Remark 4.2. For toric geometers, below is the dictionary between the above open sets and the standard toric charts on the minimal resolution. We also state for reference the result of Lemma 4.4, which gives the position of where (if we change basis so that the specified non-zero arrows in the definition of the open sets are actually the identity) the co-ordinates can be read off the quiver.

$$\begin{array}{ccccc}
W_0 & \longleftrightarrow & \mathrm{Spec}\ \mathbb{C}\left[x^r, \frac{y}{x^a}\right] & (c_{10}, a_{01}) \\
 & \vdots & & \\
\begin{array}{c} W_t \\ \textit{for } 1 \leq t \leq n-1 \end{array} & \longleftrightarrow & \mathrm{Spec}\ \mathbb{C}\left[\frac{x^{i_t}}{y^{j_t}}, \frac{y^{j_{t+1}}}{x^{i_{t+1}}}\right] & (c_{t+1t}, a_{tt+1}) \\
 & \vdots & & \\
W_n & \longleftrightarrow & \mathrm{Spec}\ \mathbb{C}\left[\frac{x}{y^{j_n}}, y^r\right] & (c_{0n}, a_{n0})
\end{array}$$

Lemma 4.3. *For the group $G = \frac{1}{r}(1, a)$, with notation as above $\{W_t : 0 \leq t \leq n\}$ forms an open cover of the moduli space $\mathrm{Rep}(A_{r,a}, \alpha)/\!\!/_\theta \mathrm{GL}$.*

Proof. Suppose $M \in \mathrm{Rep}(A_{r,a}, \alpha)$ is θ-stable. It is clear from the stability condition that we need, for every vertex $i \neq 0$, a non-zero path from vertex 0 to vertex i.

Now if $a_{01} = 0$ then to get to vertex 1 clearly we need $C_{01} \neq 0$ and so M is in W_0. Otherwise we can assume $a_{01} \neq 0$. If $a_{12} = 0$ then to get to vertex 2 we need $C_{02} \neq 0$ and so M is in W_1. Continuing in this fashion it is obvious that $\{W_t : 0 \leq t \leq n\}$ forms an open cover of the moduli space. □

Lemma 4.4. (i) *Each representation in W_0 is determined by $(c_{10}, a_{01}) \in \mathbb{C}^2$.*
(ii) *For every $1 \leq t \leq n-1$, each representation in W_t is determined by $(c_{t+1t}, a_{tt+1}) \in \mathbb{C}^2$.*
(iii) *Each representation in W_n is determined by $(a_{n0}, c_{0n}) \in \mathbb{C}^2$.*
Thus every open set in the cover is just $\mathbb{C}^2$.

Proof. (i) We can set $c_{0n} = c_{nn-1} = \ldots = c_{21} = 1$. We proceed anticlockwise round the vertices of the quiver (starting at the 0th vertex), showing at each stage that all arrows leaving the vertex are determined by c_{10} and a_{01}.
Vertex 0: trivial as the only arrows leaving are $c_{0n} = 1$ and a_{01}.
Vertex 1: If $\alpha_1 = 2$ then the only two arrows leaving are a_{12} and c_{10}. The Step 1 relations give $a_{12} = c_{10}a_{01}$. Thus we may assume that $\alpha_1 > 2$ so we have $c_{10} = k_0, k_1, \ldots, k_{u_1}, a_{12}$ leaving the vertex. But now the Step 1 relations give

$$\begin{aligned} c_{10}a_{01} &= k_1 \\ k_1 a_{01} &= k_2 \\ &\vdots \\ k_{u_1-1}a_{01} &= k_{u_1} \\ k_{u_i}a_{01} &= a_{12} \end{aligned}$$

so it is clear that $k_1, \ldots, k_{u_1}, a_{12}$ can be expressed in terms of c_{10} and a_{01}.
Vertex s for $1 < s < n$: If $\alpha_s = 2$ then only arrows leaving are $c_{ss-1} = 1$ and a_{ss+1}. The Step s relations give $a_{ss+1} = a_{s-1s}$ and by work at previous vertices we know that a_{s-1s} is determined by c_{10} and a_{01}; hence so is a_{s+1s}. Thus we may assume $\alpha_s > 2$ and so the arrows leaving vertex s are $k_{v_s}, \ldots, k_{u_s}, c_{ss-1} = 1, a_{ss+1}$. But by the Step s relations we know

$$\begin{aligned} k_{v_s} &= a_{s-1s} \\ k_{v_s+1} &= k_{v_s}A_{0s} \\ &\vdots \\ k_{u_s} &= k_{u_s-1}A_{0s} \\ a_{ss+1} &= k_{u_s}A_{0s} \end{aligned}$$

which, by work at the previous vertices, can all be expressed in terms of c_{10} and a_{01}.
Vertex n: If $\alpha_n = 2$ then again everything is trivial and so we may assume $\alpha_n > 2$ in which case the arrows $k_{v_n}, \ldots, k_{u_n} = a_{n0}, c_{nn-1} = 1$ leave vertex n. The step n relations give

$$\begin{aligned} k_{v_n} &= a_{n-1n} \\ k_{v_n+1} &= k_{v_n}A_{0n} \\ &\vdots \\ k_{u_n} &= k_{u_s-1}A_{0s} \end{aligned}$$

which again by work at the other vertices can be expressed in terms of c_{10} and a_{01}.
(iii) Follows immediately by Lemma 2.10.
(ii) We can set $c_{0n} = \ldots = c_{t+2\,t+1} = 1 = a_{01} = \ldots = a_{t-1t}$. As above we show that the arrows leaving every vertex are determined by c_{t+1t} and a_{tt+1}, but we now work anticlockwise from vertex $t+1$ to vertex 0, then work clockwise from vertex t to vertex 1:
Vertex $t+1$: If $\alpha_{t+1} = 2$ then the only arrows leaving are c_{t+1t} and a_{t+1t+2}. The relations give $a_{t+1t+2} = c_{t+1t}a_{tt+1}$. Hence we may assume $\alpha_{t+1} > 2$ and so the arrows leaving vertex $t+1$ are $k_{v_{t+1}}, \ldots, k_{u_{t+1}}, c_{t+1t}, a_{t+1t+2}$. The Step $t+1$ relations give

$$\begin{aligned} k_{v_{t+1}} &= c_{t+1t}a_{tt+1} \\ k_{v_{t+1}+1} &= k_{v_{t+1}}A_{0t+1} = k_{v_{t+1}}a_{tt+1} \\ &\vdots \\ k_{u_{t+1}} &= k_{u_{t+1}-1}A_{0t+1} = k_{u_{t+1}-1}a_{tt+1} \\ a_{t+1t+2} &= k_{u_{t+1}}A_{0t+1} = k_{u_{t+1}}a_{tt+1} \end{aligned}$$

which therefore can all be expressed in terms of c_{t+1t} and a_{tt+1}.
Vertex s for $n < s < t+1$: If $\alpha_s = 2$ then the only arrows leaving are $c_{ss-1} = 1$ and a_{ss+1}. The relation gives $a_{ss+1} = a_{s-1s}$ and by work at previous vertices we know that a_{s-1s} is determined by c_{t+1t} and a_{tt+1}; hence so is a_{ss+1}. Hence assume $\alpha_s > 2$ and so the arrows leaving are $k_{v_s}, \ldots, k_{u_s}, c_{ss-1}, a_{ss+1}$. The Step s relations give

$$\begin{aligned} k_{v_s} &= a_{s-1s} \\ k_{v_s+1} &= k_{v_s}A_{0s} = k_{v_s}A_{ts} \\ &\vdots \\ k_{u_s} &= k_{u_s-1}A_{0s} = k_{u_s-1}A_{ts} \\ a_{ss+1} &= k_{u_s}A_{0s} = k_{u_s}A_{ts} \end{aligned}$$

which by work at the previous vertices can all be expressed in terms of c_{t+1t} and a_{tt+1}.
Vertex n: Similar to the above case.
Vertex 0: Only arrows leaving are c_{0n} and a_{01}, both of which are 1.
We now start at vertex t and work clockwise:
Vertex t: If $\alpha_t = 2$ then the only arrows leaving are c_{tt-1} and a_{tt+1}; the relations give $c_{tt-1} = a_{tt+1}c_{t+1t}$. Hence assume $\alpha_t > 2$ and so the arrows leaving are $k_{v_t}, \ldots, k_{u_t}, c_{tt-1}, a_{tt+1}$. The Step t relations (read backwards) give

$$\begin{aligned} k_{u_t} &= a_{tt+1}c_{t+1t} \\ k_{u_t-1} &= k_{u_t}C_{0t} = k_{u_t}c_{t+1t} \\ &\vdots \\ k_{v_t} &= k_{v_t+1}C_{0t} = k_{v_t+1}c_{t+1t} \\ c_{tt-1} &= k_{v_t}C_{0t} = k_{v_t}c_{tt-1} \end{aligned}$$

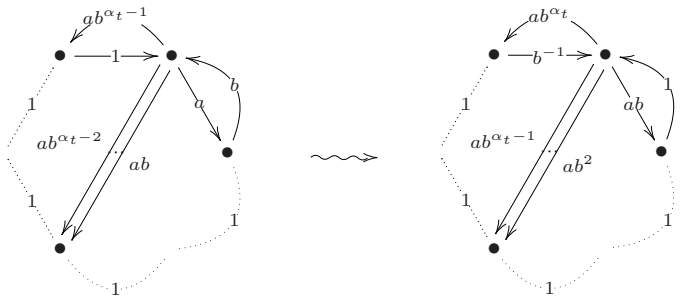
which therefore can all be expressed in terms of c_{t+1t} and a_{tt+1}.
Vertex s for $1 \leq s < t$: Similar to the above; read the Step s relations backwards and use work at the previous vertices. □

Theorem 4.5. *Keeping α and θ fixed from before,*

$$\mathrm{Rep}(A_{r,a},\alpha)/\!\!/_{\theta}\mathrm{GL} \longrightarrow \mathbb{C}^2/\tfrac{1}{r}(1,a)$$

is the minimal resolution of singularities.

Proof. Firstly note that W_{t-1} and W_t glue together to give $\mathscr{O}_{\mathbb{P}^1}(-\alpha_t)$ for each $1 \leq t \leq n$. To see this let $(a,b) \in W_{t-1}$ with $b \neq 0$, then simply changing basis at vertex t by dividing all arrows into vertex t by b and multiplying all arrows out by b gives us



$(a,b) \in W_{t-1}$ $\qquad$ $(b^{-1}, ab^{\alpha_t}) \in W_t$

Consequently above the singularity there is a string of $\mathbb{P}^1$'s each with self-intersection number ≤ -2. None of these can be contracted to give a smaller resolution. $\square$

Remark 4.6. For finite subgroups of $SL(2,\mathbb{C})$ the above theorem remains true if we replace the fixed θ by an arbitrary generic stability condition [CS98]. However in the $GL(2,\mathbb{C})$ case if we choose a different stability condition it is *not* true in the general that the above theorem holds, since the moduli may have components. A concrete example is $\frac{1}{3}(1,1)$. Thus in the non-Gorenstein case the question of stability is much more subtle.

5. Tilting Bundles

We want to show that the minimal resolution $\widetilde{X}$ of the singularity $\mathbb{C}^2/\frac{1}{r}(1,a)$ is derived equivalent to the reconstruction algebra $A_{r,a}$. To do this, we search for a tilting bundle. During the production of this paper this result has been independently proved by Craw [Cra07], who points out that it actually follows immediately from a result of Van den Bergh [Van04b, Thm B]. The proof here uses a simple trick involving an ample line bundle.

Definition 5.1. *Suppose $\mathfrak{T}$ is a triangulated category with small direct sums. An object $C \in \mathfrak{T}$ is called compact if for any set of objects X_i, the natural map*

$$\coprod \mathrm{Hom}(C, X_i) \to \mathrm{Hom}\left(C, \coprod X_i\right)$$

is an isomorphism.

Denote by $\langle \mathfrak{X} \rangle_{\oplus}$ the smallest full triangulated subcategory of $\mathfrak{T}$ closed under infinite sums containing $\mathfrak{X}$. Note this is necessarily closed under direct summands.

Definition 5.2. *Let $\mathfrak{T}$ be a triangulated category with small direct sums. We say $\mathfrak{T}$ is compactly generated if there exists a set of compact objects $\mathfrak{X}$ such that $\langle \mathfrak{X} \rangle_{\oplus} = \mathfrak{T}$.*

Definition 5.3. *A vector bundle $\mathscr{E}$ of finite rank is called a tilting bundle if*
(1) $\mathrm{Ext}^i(\mathscr{E},\mathscr{E}) = 0$ *for all* $i > 0$
(2) $\langle \mathscr{E} \rangle_{\oplus} = D(\mathfrak{Qcoh} X)$.

It is standard that if $\widetilde{X}$ has a tilting bundle $\mathscr{E}$ then $\widetilde{X}$ and $\mathrm{End}_{\widetilde{X}}(\mathscr{E})$ are derived equivalent (see Theorem 5.8 below). In our situation the bundle to consider comes from Wunram's geometric description of the special CM modules as full sheaves:

Definition 5.4. [Esn85] *A sheaf $\mathscr{F}$ on $\widetilde{X}$ is called full if*
(1) *$\mathscr{F}$ is locally free*
(2) *$\mathscr{F}$ is generated by global sections*
(3) $H^1(\widetilde{X}, \mathscr{F}^\vee \otimes \omega_{\widetilde{X}}) = 0$ *where $\omega_{\widetilde{X}}$ is the canonical module.*

Denoting the minimal resolution by $\widetilde{X} \xrightarrow{\pi} \mathbb{C}^2/G$ then given any CM module M it is true that

$$\widetilde{M} := \pi^* M/\text{torsion}$$

is a full sheaf. In fact full sheaves are in 1-1 correspondence with indecomposable CM modules by work of Esnault [Esn85]; the inverse map is global sections. Denote the functor $\mathrm{Hom}(-, R)$ by * and note that if M is any CM module then $M^* = \pi_*((\widetilde{M})^\vee)$.

The definition of special CM module was originally stated in terms of the corresponding full sheaf:

Lemma 5.5. [Wun88] *S_t is a special CM module* $\iff$ $H^1(\widetilde{S_t}^\vee) = 0$.

To obtain a derived equivalence between the minimal resolution $\widetilde{X}$ and the reconstruction algebra $A_{r,a}$ we shall show that $\mathscr{E} = \oplus_{p=1}^{n+1} \widetilde{S_{i_p}}$ is a tilting bundle, with endomorphism ring $A_{r,a}$. Firstly we prove the statement on the endomorphism ring: note that the following lemma shows that the three ways to produce a non-commutative ring all give the same answer.

Lemma 5.6. $\mathrm{End}_{\widetilde{X}}(\mathscr{E}) = \mathrm{End}_{\widetilde{X}}\left(\oplus_{p=1}^{n+1} \widetilde{S_{i_p}}\right) \cong \mathrm{End}_{\mathbb{C}[x,y]^G}\left(\oplus_{p=1}^{n+1} S_{i_p}\right) \cong A_{r,a}$.

Proof. The last isomorphism is Theorem 3.24. The first isomorphism follows from the statement

$$\mathrm{Hom}_{\widetilde{X}}\left(\widetilde{S_{i_p}}, \widetilde{S_{i_q}}\right) \cong \mathrm{Hom}_{\mathbb{C}[x,y]^G}(S_{i_p}, S_{i_q})$$

which is true since both are reflexive and isomorphic away from the singular point (see e.g. [Wem08, 3.1]). □

Now for every pair p, q with $1 \le p, q \le n$ by choosing a generic section of $\widetilde{S_{i_q}} \oplus \widetilde{S_{i_p}}$ we have a short exact sequence

$$0 \longrightarrow \mathscr{O} \longrightarrow \widetilde{S_{i_q}} \oplus \widetilde{S_{i_p}} \longrightarrow \widetilde{S_{i_q}} \otimes \widetilde{S_{i_p}} \longrightarrow 0\ .$$

(This can also be constructed locally, using the open cover in Theorem 4.5). Tensoring the above by $\widetilde{S_{i_p}}^\vee$ gives the exact sequence

$$(B_{p,q}) \qquad 0 \longrightarrow \widetilde{S_{i_p}}^\vee \longrightarrow \left(\widetilde{S_{i_p}}^\vee \otimes \widetilde{S_{i_q}}\right) \oplus \mathscr{O} \longrightarrow \widetilde{S_{i_q}} \longrightarrow 0\ .$$

Lemma 5.7. $\mathrm{Ext}^r(\mathscr{E},\mathscr{E}) = 0$ *for all* $r > 0$.

Proof. Since the singularity is rational $H^r(\mathscr{O}) = 0$ for all $r > 0$. Further $\widetilde{S_{i_p}}$ is generated by global sections so $H^1(\widetilde{S_{i_p}}) = 0$, thus using the short exact sequence $B_{p,p}$

$$0 \longrightarrow \widetilde{S_{i_p}}^{\vee} \longrightarrow \mathscr{O}^2 \longrightarrow \widetilde{S_{i_p}} \longrightarrow 0$$

and Lemma 5.5 it is true that $H^r(\widetilde{S_{i_p}}) = H^r(\widetilde{S_{i_p}}^{\vee}) = 0$ for all $r > 0$.

But using the $B_{p,q}$ together with these facts shows that $H^r(\widetilde{S_{i_p}}^{\vee} \otimes \widetilde{S_{i_q}}) = 0$ for all $r > 0$ and $1 \leq p, q \leq n$. Hence

$$\mathrm{Ext}^r(\mathscr{E}, \mathscr{E}) = \bigoplus_{p,q=1}^{n+1} H^r(\widetilde{S_{i_p}}^{\vee} \otimes \widetilde{S_{i_q}}) = 0 \text{ for all } r > 0.$$

□

Theorem 5.8. *Let $\widetilde{X}$ be the minimal resolution of the singularity $\mathbb{C}^2/\frac{1}{r}(1,a)$, let $A_{r,a}$ be the corresponding reconstruction algebra and let $\mathscr{E} = \oplus_{p=1}^{n+1} \widetilde{S_{i_p}}$. Then*
(1) $\mathbf{R}\mathrm{Hom}(\mathscr{E}, -)$ *induces an equivalence between $D(\mathfrak{Qcoh}\widetilde{X})$ and $D(\mathfrak{Mod}A_{r,a})$*
(2) *This equivalence restricts to an equivalence between $D^b(\mathfrak{Qcoh}\widetilde{X})$ and $D^b(\mathfrak{Mod}A_{r,a})$*
(3) *This equivalence restricts to an equivalence between $D^b(\mathfrak{coh}\widetilde{X})$ and $D^b(\mathfrak{mod}A_{r,a})$*
(4) *Since $\widetilde{X}$ is smooth, $A_{r,a}$ has finite global dimension.*

Proof. By Lemma 5.6 and Lemma 5.7 we need only prove that $\langle \mathscr{E} \rangle_{\oplus} = D(\mathfrak{Qcoh}\widetilde{X})$. Since the first chern class of $\mathscr{L} := \widetilde{S_{i_1}} \otimes \ldots \otimes \widetilde{S_{i_n}}$ consists entirely of ones, $\mathscr{L}$ is ample and so it is true by [Nee96, 1.10] that $D(\mathfrak{Qcoh}X) = \langle \mathscr{L}^{-\otimes n} : n \in \mathbb{N} \rangle_{\oplus}$. Hence it suffices to prove that $\langle \mathscr{E} \rangle_{\oplus}$ contains all negative tensors of the ample line bundle $\mathscr{L}$. But using the short exact sequences $B_{p,q}$ together with suitable tensors of them, (which give triangles) this is indeed true: by the sequence $B_{p,p}$ it follows that $\langle \mathscr{E} \rangle_{\oplus}$ contains $\widetilde{S_{i_p}}^{\vee}$. Now after tensoring $B_{p,p}$ by $\widetilde{S_{i_p}}^{\vee}$ it follows that $\langle \mathscr{E} \rangle_{\oplus}$ contains $\widetilde{S_{i_p}}^{\otimes -2}$. Continuing in this fashion $\langle \mathscr{E} \rangle_{\oplus}$ contains $\widetilde{S_{i_p}}^{\otimes -t}$ for all $t \geq 0$ and all i_p. Now by considering the sequence $B_{p,q}$ tensored by $\widetilde{S_{i_q}}^{\vee}$ it follows that $\langle \mathscr{E} \rangle_{\oplus}$ contains $\widetilde{S_{i_p}}^{\vee} \otimes \widetilde{S_{i_q}}^{\vee}$. Continuing in this manner a simple inductive argument shows that $\langle \mathscr{E} \rangle_{\oplus}$ contains $(\widetilde{S_{i_1}} \otimes \ldots \otimes \widetilde{S_{i_n}})^{\otimes -t}$ for all $t \geq 0$. The result is now standard (see e.g. [HdB, 7.6]). □

Denoting by $\langle \mathscr{E} \rangle$ the smallest thick full triangulated subcategory containing $\mathscr{E}$, it is true by Neeman-Ravenel [Nee92] that $\langle \mathscr{E} \rangle$ coincides with the compact objects of $D(\mathfrak{Qcoh}\widetilde{X})$. By [Nee96, 2.3] these are precisely the perfect complexes, which since $\widetilde{X}$ is smooth are the whole of $D^b(\mathfrak{coh}\widetilde{X})$. Thus it is also true that $\langle \mathscr{E} \rangle = D^b(\mathfrak{coh}\widetilde{X})$.

6. Homological Considerations

It is well known that the preprojective algebra of an extended Dynkin diagram is a homologically homogeneous ring of global dimension 2. We observed in Theorem 5.8 that for general labels $[\alpha_1, \ldots, \alpha_n]$ the reconstruction algebra of type A also has finite global dimension, thus it is natural to ask its value and whether it is homologically homogeneous (i.e. all its simple modules have the same projective dimension).

We shall prove in this section that

$$\mathrm{gldim} A_{r,a} = \begin{cases} 2 \text{ if } a = r-1 \\ 3 \text{ else} \end{cases}$$

and so $A_{r,a}$ is homologically homogeneous only when $r = a-1$, ie when $G = \frac{1}{r}(1,a) \leq SL(2,\mathbb{C})$. Furthermore we show that the projective resolutions of the simples in the non-Azumaya locus are determined by the intersection theory.

We firstly show that the Azumaya locus of $A_{r,a}$ coincides with the smooth locus of its centre $R = \mathbb{C}[x,y]^G$. Such a problem has been considered in [Mac09] in a slightly more general setting, although here we give a direct argument. The reason we desire such a result is that we can then 'ignore' the simples in the Azumaya locus (i.e. those simple A-modules whose R-annihilator lies in the Azumaya locus defined below) as they correspond to smooth points and so their projective dimensions are easily controlled.

Definition 6.1. *$A = A_{r,a}$ is a noetherian ring module finite over its centre $R = \mathbb{C}[x,y]^{\frac{1}{r}(1,a)}$. Define*

$$\mathscr{A}_A \quad = \quad \{\mathfrak{m} \in \mathrm{max} R : A_{\mathfrak{m}} \text{ is Azumaya over } R_{\mathfrak{m}}\}$$

where $\mathrm{max}R$ *is the set of maximal ideals of R. The set $\mathscr{A}_A$ is called the Azumaya locus of A.*

Lemma 6.2. *$A = A_{r,a}$ is a prime ring of PI degree $n+1$.*

Proof. Since R is a domain $\mathbf{0}$ is a prime ideal, so denote $F = R_{\mathbf{0}}$ (R localized at the zero ideal) to be the quotient field of R. Since CM modules are torsion-free and $A \cong \mathrm{End}_R(\oplus_{p=1}^{n+1} S_{i_p})$, non-zero elements in R are not zero-divisors in A and so

$$A \subseteq A \otimes F = A_{\mathbf{0}} \cong \mathrm{End}_{R_{\mathbf{0}}}(\oplus_{p=1}^{n+1} S_{i_p \mathbf{0}}) = \mathrm{End}_{R_{\mathbf{0}}}(\oplus_{p=1}^{n+1} R_{\mathbf{0}}) = \mathrm{End}_F(F^{n+1}),$$

since the CM modules S_{i_p} are free of rank 1 away from the singular locus of R. Thus $A \subseteq A \otimes F = A_{\mathbf{0}}$ with $A_{\mathbf{0}}$ a classical right quotient ring of A. Since $A_{\mathbf{0}}$ is simple, necessarily A is prime [GW04, 6.17]. Now A is a prime PI ring, so by definition its PI-degree is equal to $\dim_F(F \otimes (\oplus_{p=1}^{n+1} S_{i_p})) = \dim_F(F^{n+1}) = n+1$. □

Throughout this section we denote by $\mathfrak{m}_0$ the unique singular point of R.

Lemma 6.3. $\mathscr{A}_A = \mathrm{max}R \backslash \{\mathfrak{m}_0\}$.

Proof. By Theorem 3.24, $A_{\mathfrak{m}} \cong \mathrm{End}_{R_{\mathfrak{m}}}(\oplus_{p=1}^{n+1} S_{i_p \mathfrak{m}})$ for all $\mathfrak{m} \in \mathrm{max}R$. But CM modules are free on the smooth locus, so if $\mathfrak{m} \neq \mathfrak{m}_0$ then $S_{i_p \mathfrak{m}} \cong R_{\mathfrak{m}}$ for all p. Consequently $A_{\mathfrak{m}} \cong M_{n+1}(R_{\mathfrak{m}})$ for any $\mathfrak{m} \neq \mathfrak{m}_0$, where $M_{n+1}(R_{\mathfrak{m}})$ is the ring of $(n+1)$-square matrices over $R_{\mathfrak{m}}$ and thus $A_{\mathfrak{m}}$ is Azumaya over $R_{\mathfrak{m}}$. This proves that $\mathrm{max}R \backslash \{\mathfrak{m}_0\} \subseteq \mathscr{A}_A$. Equality holds since A is a prime affine $\mathbb{C}$-algebra, finitely generated over its centre with finite global dimension. For such rings it is well-known that the Azumaya locus and the singular locus are disjoint (see e.g. [BG02, III.1.8]). Alternatively just observe that the one-dimensional simple corresponding to vertex 0 is a simple A-module annihilated by $\mathfrak{m}_0$ and this does not have maximal dimension $n+1$ (equal to the PI degree). □

Corollary 6.4. *For all $\mathfrak{m} \in \mathscr{A}_A$,* $\mathrm{gldim} A_{\mathfrak{m}} = 2$.

Proof. By the above lemma $\mathfrak{m} \neq \mathfrak{m}_0$ with $A_{\mathfrak{m}} \cong M_{n+1}(R_{\mathfrak{m}})$. Since global dimension passes over morita equivalence we have $\mathrm{gldim} A_{\mathfrak{m}} = \mathrm{gldim} R_{\mathfrak{m}} = 2$ where the last equality holds since R is equi-codimensional [Eis95, 13.4]. □

The hard work in the global dimension proof comes in computing the projective resolutions of the 1-dimensional simples corresponding to the vertices of $A_{r,a}$.

Let Q be the quiver of the reconstruction algebra, denote its vertices by Q_0 and its arrows by Q_1. Denote the relations by $\mathcal{R} = \{R_t\}$ and note that they are all admissible (i.e. contain no path of length ≤ 1) and basic (i.e. each R_t is a linear combination of paths, each with common head and tail). Denote by D_j the one-dimensional simple corresponding to vertex $j \in Q_0$. As is standard, for any paths p we denote $t(p)$ to be the tail of p, $h(p)$ to be the head. Consider the following complex

$$\bigoplus_{t(R_i)=j} e_{h(R_i)}A \xrightarrow{d_2} \bigoplus_{t(a)=j} e_{h(a)}A \xrightarrow{d_1} e_jA \to D_j \to 0.$$

where the left hand sum is over all relations with tail j and the right hand sum is over all arrows with tail j. The maps d_1 and d_2 are given as

$$d_2 : (g_i) \mapsto (\textstyle\sum_i \partial_a R_i g_i)_a \qquad d_1 : (f_a) \mapsto \textstyle\sum_a af_a$$

respectively, where for any arrow a and any path p we define

$$\partial_a p = \begin{cases} q & \text{if } p = aq \\ 0 & \text{else} \end{cases}$$

and extend by linearity. It is easy to see that the above is a complex which is exact at e_jA. Moreover it is also exact at $\oplus_{t(a)=j} e_{h(a)}A$: to see this denote by I the ideal of relations and note first that we may write $I = \sum R_i\mathbb{C}Q + Q_1I$. Now if $(f_a) \in \ker d_1$ then $\sum_{a:t(a)=j} af_a \in I$ and so we may find g_i such that $\sum_{a:t(a)=j} af_a - \sum_{R_i\in\mathcal{R}} R_ig_i \in Q_1I$. For any $a \in Q_1$ such that $t(a) = j$ we apply ∂_a to this expression to obtain $f_a \equiv \sum_{R_i\in\mathcal{R}} \partial_a R_i g_i \bmod I$. Thus (f_a) is the image of (g_i) under d_2, as required.

Lemma 6.5. *If $\alpha_t = 2$ for some $1 \leq t \leq n$ then the simple D_t at vertex t has projective resolution*

$$0 \longrightarrow e_tA \longrightarrow e_{t-1}A \oplus e_{t+1}A \longrightarrow e_tA \longrightarrow D_t \longrightarrow 0$$

where if $t = n$ take $t + 1 = 0$. Hence $\mathrm{pd}(D_t) = 2$.

Proof. We just need to show that d_2 is injective. But here d_2 is the map

$$\begin{array}{rcl} e_tA & \to & e_{t-1}A \oplus e_{t+1}A \\ g & \mapsto & (a_{t-1t}g, -c_{t+1t}g) \end{array}$$

and if $g \in \ker d_2$ then $a_{t-1t}g = 0$ from which (viewing as polynomials via Theorem 3.24) we deduce that $g = 0$. Since the first syzygy is not projective (else on localizing it would contradict the depth lemma), it follows that $\mathrm{pd}(D_t) = 2$. □

Corollary 6.6. *If $\alpha_1 = \ldots = \alpha_n = 2$ then the simple D_0 at vertex 0 has projective resolution*

$$0 \longrightarrow e_0A \longrightarrow e_nA \oplus e_1A \longrightarrow e_0A \longrightarrow D_0 \longrightarrow 0$$

and so $\mathrm{pd}(D_0) = 2$.

Proof. By hypothesis the quiver is symmetric and so the 0th vertex is indistinguishable from the other vertices. The result now follows from Corollary 6.5 above. □

Lemma 6.7. *If $\alpha_t > 2$ then the simple D_t at vertex t has projective resolution*

$$0 \longrightarrow (e_tA)^{\alpha_t-1} \longrightarrow (e_{t-1}A) \oplus (e_0A)^{\alpha_t-2} \oplus (e_{t+1}A) \longrightarrow e_tA \longrightarrow D_t \longrightarrow 0$$

and so $\mathrm{pd}(D_t) = 2$.

Proof. Again we just need to show that d_2 is injective. Here d_2 is the map sending $(g_i) \in (e_tA)^{\alpha_t-1}$ to

$$(a_{t-1t}, -C_{0t}, 0, \ldots, 0)g_1 + \sum_{i=1}^{\alpha_t-3} (\underbrace{0, \ldots, 0}_{i}, A_{0t}, -C_{0t}, \underbrace{0, \ldots, 0}_{\alpha_t-i-2})g_{i+1} + (0, \ldots, 0, A_{0t}, -c_{t+1t})g_{\alpha_t-1}$$

where the convention is that the sum is empty if $\alpha_t = 3$. Now if $(g_i) \in \ker d_2$ then by inspecting the first summand we see that $a_{t-1t}g_1 = 0$ and so $g_1 = 0$. Now by inspecting the second summand (and using the fact $g_1 = 0$) we see that $A_{0t}g_2 = 0$ and so $g_2 = 0$. Proceeding inductively gives $g_1 = \ldots = g_{\alpha_t-1} = 0$ and so d_2 is injective, as required. □

Lemma 6.8. *If some $\alpha_t > 2$ then the simple D_0 at vertex 0 has projective resolution*

$$0 \longrightarrow \bigoplus_{i=1}^{n} (e_iA)^{\alpha_i-2} \longrightarrow (e_0A)^{1+\sum(\alpha_t-2)} \longrightarrow e_nA \oplus e_1A \longrightarrow e_0A \longrightarrow D_0 \longrightarrow 0$$

and so $\mathrm{pd}(D_0) = 3$.

Proof. Denote $\gamma = \sum_{t=1}^{n}(\alpha_t - 2)$ then by assumption $\gamma \geq 1$. Here d_2 is the map

$$\begin{array}{ccc} (e_0A)^{1+\gamma} & \to & e_nA \oplus e_1A \\ (g_i) & \mapsto & \sum_{t=1}^{1+\gamma}(\hat{C}_{nl_t}k_t, -\hat{A}_{1l_{t-1}}k_{t-1})g_i \end{array}$$

where

$$\hat{C}_{ij} = \begin{cases} C_{ij} & i \neq j \\ e_i & i = j \end{cases} \quad \text{and} \quad \hat{A}_{ij} = \begin{cases} A_{ij} & i \neq j \\ e_i & i = j \end{cases}$$

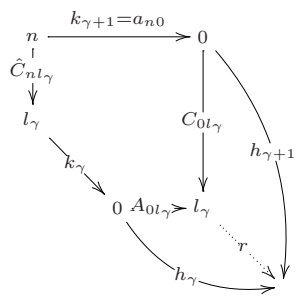
and recall $k_0 = c_{10}$ and $k_{\gamma+1} = a_{n0}$. We firstly claim that the kernel of d_2 is

$$K_3 := \sum_{i=1}^{\gamma} (\underbrace{0, \ldots, 0}_{i-1}, A_{0l_i}, -C_{0l_i}, \underbrace{0, \ldots, 0}_{\gamma-i})e_{l_i}A.$$

To prove this claim we proceed by induction. Take $(h_i) = (h_1, \ldots, h_{\gamma+1})$ belonging to the kernel of d_2. If $h_1 = \ldots = h_{\gamma-1} = 0$ then

$$\hat{C}_{nl_\gamma}k_\gamma h_\gamma = -\hat{C}_{nl_{\gamma+1}}k_{\gamma+1}h_{\gamma+1} = -a_{n0}h_{\gamma+1}$$

and so viewing everything as polynomials in the web of paths we have



We deduce that $h_\gamma = A_{0l_\gamma} r$ and $h_{\gamma+1} = -C_{0l_\gamma} r$ for some $r \in e_{l_\gamma} A$ by viewing everything as polynomials and examining the x and y components. Thus

$$(h_i) = (0, \dots, 0, h_\gamma, h_{\gamma+1}) = (0, \dots, 0, A_{0l_\gamma}, -C_{0l_\gamma})r \in K_3$$

and so the claim is true when $h_1 = \dots = h_{\gamma-1} = 0$. Thus assume that the claim is true for any $(\underbrace{0, \dots, 0}_{i}, h_{i+1}, \dots, h_{\gamma+1})$ belonging to the kernel with $1 \leq i \leq \gamma - 1$; we shall now show that the claim is true for any $(\underbrace{0, \dots, 0}_{i-1}, h_i, \dots, h_{\gamma+1})$ belonging to the kernel: certainly

$$\hat{C}_{nl_i} k_i h_i = -\sum_{t=i+1}^{\gamma+1} \hat{C}_{nl_t} k_t h_t \tag{1}$$

and so by viewing everything as polynomials and examining the y components we see that

$$y \text{ component of } h_i \geq (y \text{ component of } k_{i+1}) - (y \text{ component of } k_i) = j_{l_i}$$

Thus

$$\begin{tikzcd} 0 \arrow[r, "A_{0l_i}"] \arrow[dr, "h_i"'] & l_i \arrow[d, dotted] \\ & \end{tikzcd}$$

and so by viewing everything as polynomials we get that $h_i = A_{0l_i} r$ for some $r \in e_{l_i} A$. Hence

$$\hat{C}_{nl_i} k_i h_i = \hat{C}_{nl_i} k_i A_{0l_i} r = \hat{C}_{nl_{i+1}} k_{i+1} C_{0l_i} r$$

and so (1) becomes

$$\hat{C}_{nl_{i+1}} k_{i+1} (C_{0l_i} r + h_{i+1}) + \sum_{t=i+2}^{\gamma+1} \hat{C}_{nl_t} k_t h_t = 0.$$

But also

$$-\hat{A}_{1l_{i-1}} k_{i-1} h_i = -\hat{A}_{1l_{i-1}} k_{i-1} A_{0l_i} r = -\hat{A}_{1l_i} k_i C_{0l_i} r$$

and so by the inductive hypothesis

$$(\underbrace{0, \dots, 0}_{i}, C_{0l_i} r + h_{i+1}, h_{i+2}, \dots, h_{\gamma+1}) \in K_3.$$

But now

$$\begin{aligned}(\underbrace{0, \dots, 0}_{i-1}, h_i, \dots, h_{\gamma+1}) = (\underbrace{0, \dots, 0}_{i-1}, A_{0l_i}, -C_{0l_i}, 0, \dots, 0)r \\ + (\underbrace{0, \dots, 0}_{i}, C_{0l_i} r + h_{i+1}, h_{i+2}, \dots, h_{\gamma+1})\end{aligned}$$

and so $(\underbrace{0, \dots, 0}_{i-1}, h_i, \dots, h_{\gamma+1}) \in K_3$. Thus by induction the claim is established, so the kernel is K_3.

We have an obvious surjection

$$\bigoplus_{i=1}^{\gamma} e_{l_i} A = \bigoplus_{i=1}^{n} (e_i A)^{\alpha_i - 2} \to \Omega^3 K_3 = \sum_{i=1}^{\gamma} (\underbrace{0, \dots, 0}_{i-1}, A_{0l_i}, -C_{0l_i}, \underbrace{0, \dots, 0}_{\gamma-i}) e_{l_i} A$$

which, by using a similar argument as in Lemma 6.7 is also injective. This proves that $\mathrm{pd}(D_0) \leq 3$. Since the first and second syzygies are not projective equality holds. □

Summarizing what we have proved

Theorem 6.9. *Consider $G = \frac{1}{r}(1,a)$ and $A = A_{r,a}$. Then for $1 \leq t \leq n$ the simple D_t at vertex t has projective resolution*

$$0 \longrightarrow (e_tA)^{\alpha_t - 1} \longrightarrow (e_{t-1}A) \oplus (e_0A)^{\alpha_t - 2} \oplus (e_{t+1}A) \longrightarrow e_tA \longrightarrow D_t \longrightarrow 0$$

(where if $t = n$ take $t+1 = 0$) and so $\mathrm{pd}(D_t) = 2$. Further the simple D_0 at vertex 0 has projective resolution

$$0 \longrightarrow \bigoplus_{i=1}^{n} (e_iA)^{\alpha_i - 2} \longrightarrow (e_0A)^{1+\sum(\alpha_t - 2)} \longrightarrow e_nA \oplus e_1A \longrightarrow e_0A \longrightarrow D_0 \longrightarrow 0$$

and so
(i) *If $G \leq SL(2,\mathbb{C})$ (i.e. all $\alpha_t = 2$) then $\mathrm{pd}(D_0) = 2$.*
(ii) *If $G \not\leq SL(2,\mathbb{C})$ (i.e. some $\alpha_t > 2$) then $\mathrm{pd}(D_0) = 3$.*

Proof. For $1 \leq t \leq n$ if $\alpha_t = 2$ use Lemma 6.5; if $\alpha_t > 2$ then use Lemma 6.7. For the 0th vertex use either Corollary 6.6 or Lemma 6.8. □

All the hard work in the global dimension statement has now been done - to finish the proof we use standard ring theoretic methods involving the Azumaya locus.

Theorem 6.10.

$$\mathrm{gldim} A_{r,a} = \begin{cases} 2 & \textit{if } a = r-1 \\ 3 & \textit{else} \end{cases}$$

Proof. It is well-known by [Rai87] that

$$\mathrm{gldim} A = \sup\{\mathrm{pd}_A S : S \text{ simple right } R \text{ module}\}.$$

Let S be such a simple and consider $\mathrm{ann}_R S$; it is a maximal ideal of R (see e.g. [BG02, III.1.1(3)]). There are two possibilities
(i) $\mathrm{ann}_R S$ lies in the Azumaya locus. Then

$$\mathrm{pd}_A S = \sup\{\mathrm{pd}_{A_{\mathfrak{m}}} S_{\mathfrak{m}} : \mathfrak{m} \in \max R\} = \mathrm{pd}_{A_{\mathrm{ann}_R S}} S_{\mathrm{ann}_R S} \leq \mathrm{gldim} A_{\mathrm{ann}_R S} = 2.$$

by Corollary 6.4.
(ii) $\mathrm{ann}_R S$ does not lie in the Azumaya locus, so by Lemma 6.3 $\mathrm{ann}_R S = \mathfrak{m}_0$. Now the maximal number of non-isomorphic simple A-modules V with $\mathrm{ann}_R V = \mathfrak{m}_0$ is equal to the PI degree of A ([BG02, III.1.1(3)]) which we already know is $n+1$. But it is clear that $D_0, \ldots, D_n$ are all annihilated by $\mathfrak{m}_0$ and so consequently these must be all the simple A-modules annihilated by $\mathfrak{m}_0$. Thus S must be one of $D_0, \ldots, D_n$, and so by Theorem 6.9 we know that the projective dimension is either 2 or 3. Combining (i) and (ii) gives the desired result. □

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