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THE EXTREMAL ALGEBRA ON TWO HERMITIANS WITH SQUARE 1

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Abstract. Let \( Ea(u, v) \) be the extremal algebra determined by two hermitians \( u \) and \( v \) with \( u^2 = v^2 = 1 \). We show that: \( Ea(u, v) = \{ f + gu : f, g \in C(\mathbb{T}) \} \), where \( \mathbb{T} \) is the unit circle; \( Ea(u, v) \) is \( C^* \)-equivalent to \( C^*(G) \), where \( G \) is the infinite dihedral group; most of the hermitian elements \( k \) of \( Ea(u, v) \) have the property that \( k^n \) is hermitian for all odd \( n \) but for no even \( n \); any two hermitian words in \( G \) generate an isometric copy of \( Ea(u, v) \) in \( Ea(u, v) \).

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1. Introduction. This is a continuation of [2], except that we are concerned here only with the extremal Banach algebra \( Ea(u, v) \) determined by two hermitian involutions \( u \) and \( v \) (we use involution here in the group sense, namely that \( u^2 = v^2 = 1 \)). In [2] we presented \( Ea(u, v) \) as an abstract completion of a group algebra. Here we present it as a specific algebra of pairs of continuous functions on the unit circle and we prove that it is even \( C^* \)-equivalent for the natural star operation on \( Ea(u, v) \) which makes the generators \( u \) and \( v \) unitary elements. The hermitian element defined by \( h = (i/2)(uv - vu) \) has the remarkable property that \( h^n \) is hermitian for every odd \( n \) but for no even \( n \); and yet the subalgebra generated by \( h \) is \( C^* \)-equivalent to \( C[1, 1] \). The algebra \( Ea(u, v) \) is equivalent to the \( C^* \)-algebra of the infinite dihedral group \( G \). We give a simple explicit description of the space of hermitian elements in \( Ea(u, v) \); we also show that most of the hermitian elements \( k \) of \( Ea(u, v) \) have the property that \( k^n \) is hermitian for all odd \( n \) but for no even \( n \). Permutations of \( G \) induce (isometric) automorphisms of \( C^*(G) \). We show that there are also many (isometric) isomorphisms onto subalgebras of \( Ea(u, v) \).

We use without comment some elementary properties of hermitians which may be found in [1].

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2. $E_a(u,v)$ is $C^*$-equivalent. We repeat here some essential notation from [2]. We write $G$ for the infinite dihedral group generated by $x$ and $u$, where $u^2 = 1$ and $ux = x^{-1} u$. In relation to the algebra $E_a(u,v)$ we have $x = uv$. Put $H = \{x^n : n \in \mathbb{Z}\}$ so that $G = H \cup H u$. Put $A_0 = \mathbb{C}[G]$ and recall the (algebra) involutions $^*$ and $^\dagger$ given by

$$
\left( \sum \alpha_g g \right)^* = \sum \alpha_g g^{-1}, \quad \left( \sum \alpha_g g \right)^\dagger = \sum \alpha_g g^{-1}.
$$

For $a \in A_0$ we note that $a^* = a^\dagger \iff a \in \mathbb{R}[G]$. Let $J \subseteq \mathbb{C}[G]$ be the set of all finite products of elements of the form $p = \cos \theta + i \sin \theta x^n u$, where $\theta \in \mathbb{R}, n \in \mathbb{Z}$. Since $(x^n u)^{-1} = x^n u$, we have $p^* = \cos \theta - i \sin \theta x^n u \in J$, and $p^* p = 1 = pp^*$. It follows that, for all $a \in J$, $a^* a = 1$ and $a^* \in J$. Hence $J$ is a group in $A_0$. Identities such as $\cos \theta + i \sin \theta x u = (iu)(\cos \theta + i \sin \theta v)(-iu)$ show that this is the $J$ of [2]. Since $x^n = (-ix^n u)(iu)$, we have $x^n \in J$ and so $H \subseteq J$.

Since $G = H \cup H u$, each $a \in A_0$ can be written as $a = b + cu$ with $b, c \in \mathbb{C}[H]$. For $c \in \mathbb{C}[H]$, we have $uc = c^\dagger u$. For $b, c \in \mathbb{C}[H]$, this gives $(b + cu)^\dagger = b^\dagger + c^\dagger u$; define an involution $^\dagger$ on $A_0$ by $(b + cu)^\dagger = b^\dagger - cu$. By contrast, note that $(b + cu)^\dagger = b^\dagger + cu$. Then $a^* = a^\dagger$ for $a \in A_0$, and $a^* = a^\dagger$ if $a \in J$. For $a \in A_0$, $a^* = a^\dagger$ if and only if $a = b + icu$ for some $b, c \in \mathbb{R}[H]$; then $a^* a = b^* b + c^* c = aa^*$. 

**Lemma 2.1.** Let $K = \{a \in A_0 : a^* = a^\dagger, \ a^* a = 1\}$. Then $J = K$.

**Proof.** From the above, $J \subseteq K$. Clearly $K$ is also a group. Any $a \in K$, $a \not\in (H)u(Hu)$, may be written

$$
a = a_p x^p + \cdots + a_m x^m + i\beta_1 x^1 u + \cdots + i\beta_n x^n u
$$

where $p, q, m, n \in \mathbb{Z}, p \leq m, q \leq n, a_p \alpha_p \beta_q \beta^2_1 \neq 0$ and $\alpha_k, \beta_k \in \mathbb{R}$ for all $k$. Suppose that $m - p > n - q$. Then the coefficient of $\sum_{k=-p}^{m-p}$ in $a^* a$ is $\alpha_p \alpha_m \neq 0$. Since $a^* a = 1$, this coefficient is 0. Similarly we rule out $n - q > m - p$. Therefore $m - p = n - q$; call this common value the length of $a$. We show that $a \in K$ implies $a \in J$ by induction on the length of $a$. If $a$ has length 0 then $a = a_p x^p + i\beta_n x^1 u = x^p (a_p + i\beta_n x^{q-p} u) \in J$, since $1 = a^* a = a_p^2 + \beta^2_n$. Suppose that our claim holds for elements of length less than $N$, and consider $a$ as above of length $N$. For $\theta \in \mathbb{R}$, $\cos \theta + i \sin \theta x^{q-p} u \in J \subseteq K$ and so $a' = a(\cos \theta + i \sin \theta x^{q-p} u) \in K$. Here $a'$ has the form of (1) with $a_p$ replaced by $a_p' = a_p \cos \theta - \beta_1 \sin \theta$. We choose $\theta$ so that $a_p' = 0$. Then $a'$ has length less than $N$ and, by hypothesis, $a' \in J$. Therefore $a \in A_0$, as required. 

As in [2], we now define a norm on $A_0$ by

$$
\|a\| = \inf \left\{ \sum_{k=1}^N |a_k| : a = \sum_{k=1}^N a_k a_k, N \in \mathbb{N}, a_k \in \mathbb{C}, a_k \in J \right\}.
$$

Let $\mathbb{T}$ denote the unit circle in $\mathbb{C}$. With each element $b = \sum a_n x^n$ of $\mathbb{C}[H]$ we associate the function on $\mathbb{T}$ given by $b(\xi) = \sum a_n \xi^n$. We can now regard $A_0$ as the set of all elements $f + gu$ where $f, g$ are polynomials in $\xi$ and $\xi^{-1} = \overline{\xi}$ on $\mathbb{T}$. We also have a representation as $2 \times 2$ matrices of functions of $\xi \in \mathbb{T}$ by

$$
\pi(f + gu) = \begin{pmatrix} f(\xi) & g(\xi) \\ g(\overline{\xi}) & f(\overline{\xi}) \end{pmatrix}.
$$
The involutions on $\mathbb{C}[\mathcal{H}]$ correspond to
\[ f^\ast(\zeta) = \overline{f(\zeta)}, \quad f^\dagger(\zeta) = f(\zeta). \]

We have $\mathcal{J} = \mathcal{J}^\ast = \mathcal{J}^\dagger$, and so $\ast$ and $\dagger$ are isometric for $\| \cdot \|$. We write $| \cdot |_\infty$ for the supremum norm over $\mathbb{T}$. Of course the element $x$ corresponds to the function $x(\zeta) = \zeta$.

**Lemma 2.2.** Let $f \in \mathbb{R}[\mathcal{H}]$ with $|f|_\infty < 1$. Then there exists $g \in \mathbb{R}[\mathcal{H}]$ such that $f^*f + g^*g = 1$.

*Proof.* Put $F = 1 - f^*f$, so that $F$ is a positive trigonometric polynomial with real coefficients. By [3, pp 117–8], $F$ can be written as $g^*g$, and the proof in [3] shows that the trigonometric polynomial $g$ also has real coefficients. □

**Corollary 2.3.** For $f \in \mathbb{R}[\mathcal{H}]$, we have $\|f\| = |f|_\infty$. For $f \in \mathbb{C}[\mathcal{H}]$, we have
\[ |f|_\infty \leq \|f\| \leq 2|f|_\infty. \]
The completion of $(\mathbb{C}[\mathcal{H}], \| \cdot \|)$ is $C(\mathbb{T})$, with $|f|_\infty \leq \|f\| \leq 2|f|_\infty$ for all $f \in C(\mathbb{T})$.

*Proof.* Let $f \in \mathbb{R}[\mathcal{H}]$ with $|f|_\infty < 1$. By Lemma 2.2, there exists $g \in \mathbb{R}[\mathcal{H}]$ such that $f^*f + g^*g = 1$. Then $a = f \pm ig$ satisfies $a^* = a^\dagger$ and $a^\dagger a = 1$. By Lemma 2.1, $f \pm ig \in \mathcal{J}$. Therefore $\|f \pm ig\| = 1$, and $\|f\| \leq 1$. By linearity, $\|f\| \leq |f|_\infty$ for $f \in \mathbb{R}[\mathcal{H}]$. For $b + icu \in \mathcal{J}$, we have
\[ |b(\zeta)|^2 + |c(\zeta)|^2 = (b^*b + c^*c)(\zeta) = 1 \]
and so $|b(\zeta)| \leq 1$ for $\zeta \in \mathbb{T}$. Hence, for $f \in \mathbb{C}[\mathcal{H}]$, $\|f\| \geq |f(\zeta)|$, and so $\|f\| \geq |f|_\infty$, which gives $\|f\| = |f|_\infty$ for $f \in \mathbb{R}[\mathcal{H}]$.

Let $f = \sum a_n x^n \in \mathbb{C}[\mathcal{H}]$. Note that $f^\dagger = \sum \overline{a_n} x^n$, and $|f^\dagger|_\infty = |f|_\infty$. Thus $f + f^\dagger \in \mathbb{R}[\mathcal{H}]$ and $|f + f^\dagger|_\infty \leq 2|f|_\infty$. This gives $\|f + f^\dagger\| \leq 2|f|_\infty$.

Also, $i(f - f^\dagger) \in \mathbb{R}[\mathcal{H}]$, which gives $\|f - f^\dagger\| \leq 2|f|_\infty$ and hence $\|f\| \leq 2|f|_\infty$. The final part follows by the Stone-Weierstrass theorem. □

The involutions $\ast$ and $\dagger$ extend in the natural way to $C(\mathbb{T})$, and a routine approximation argument gives the next corollary. Define $C_S(\mathbb{T}) = \{ f \in C(\mathbb{T}) : f^\ast = f^\dagger \}$.

**Corollary 2.4.** Let $f \in C_S(\mathbb{T})$. Then $\|f\| = |f|_\infty$.

We define a norm $| \cdot |$ on $\mathbb{C}[G]$ by $|a| = \sup\{|ab|_2 : b \in \ell^2(G), |b|_2 = 1\}$, where $\|\sum \beta_g x_g\|_2 = (\sum |\beta_g|^2)^{1/2}$. The completion of $(\mathbb{C}[G], | \cdot |)$ is the $C^*$-algebra $C^*(G)$.

**Lemma 2.5.** Let $\mathcal{L}$ be a subgroup of $G$. Let $a = \sum_{g \in \mathcal{L}} \alpha_g g \in \mathbb{C}[G]$, and $d = \sum_{g \in \mathcal{L}} \alpha_g g$ its projection in $\mathbb{C}[\mathcal{L}]$. Then $|d| \leq |a|$.

*Proof.* Write $a = d + f$ where $f \in \text{lin}(G \setminus \mathcal{L})$. If $b \in \ell^2(\mathcal{L})$ then $db \in \ell^2(\mathcal{L})$ and $fb \in \ell^2(G \setminus \mathcal{L})$. Therefore $|ab|_2 = |db + fb|_2 \geq |db|_2$. Taking the supremum over $|b|_2 = 1$, we have $|a| \geq |d|$. □
Note that, with the notation of Lemma 2.5, \(|d|\) is the same whether taken over \(\mathcal{L}\) or \(\mathcal{G}\).

**Theorem 2.6.** As algebras,

\[
Ea(u, v) = C^*(\mathcal{G}) = \{f + gu : f, g \in C(T)\},
\]

with \(uf = f^*u\) and \(|f| = |f|_\infty\) \((f \in CS(T))\). For \(a \in Ea(u, v)\), \(|a| \leq \|a\| \leq 4|a|\).

**Proof.** Let \(a = f + gu\) with \(f, g \in \mathbb{C}[\mathcal{H}]\). Lemma 2.5 gives \(|f| \leq |a|\). Since \(au = g + fu\), also \(|g| \leq |au| = |a|\). We have \(|f| = |f|_\infty\). From the Stone-Weierstrass theorem we deduce that \(C^*(\mathcal{G}) = \{f + gu : f, g \in C(T)\}\).

It is now enough to prove that \(|a| \leq \|a\| \leq 4|a|\) for \(a = f + gu, f, g \in \mathbb{C}[\mathcal{H}]\). We have that \(|a| \leq \|a\|\) by the extremal nature of \(\|\cdot\|\). Also, \(\|a\| \leq \|f\| + \|g\| \leq 2|f| + 2|g| \leq 4|a|\) by Corollary 2.3.

\[\Box\]

**Corollary 2.7.** The extremal Banach algebra on one generator with all odd powers hermitian is \(C^*\)-equivalent with \(|\cdot| \leq \|\cdot\| \leq 2|\cdot|\) where \(\|\cdot\|\) is the extremal norm and \(|\cdot|\) the \(C^*\)-norm.

We extend * and \(\dagger\) to \(Ea(u, v)\) by the earlier formulæ. For the above matrix representation, \(\dagger\) gives the adjugate matrix.

3. Properties of \(Ea(u,v)\). We begin by identifying the space of hermitian elements in \(Ea(u,v)\). In [2] we noted the obvious hermitian elements (in \(A_0\)) given by \(x^n u\) \((n \in \mathbb{Z})\), 1 and \(i(x^n - x^{-n})\) \((n \in \mathbb{N})\). As expected, the space \(H\) of hermitian elements of \(Ea(u,v)\) is the closed real linear span of these elements. In fact, we can give a more elegant, and useful, description in terms of the involutions * and \(\dagger\).

**Theorem 3.1.** We have \(H = \{h \in Ea(u,v) : h^* = h, \ h + h^\dagger \in \mathbb{R}\}\).

**Proof.** Suppose that \(h \in Ea(u,v)\) with \(h^* = h\) and \(h + h^\dagger = \alpha \in \mathbb{R}\). Replacing \(h\) by \(h - \alpha/2\), we assume that \(\alpha = 0\). We approximate \(h\) by elements \(k\) in \(A_0\) satisfying \(k = k^* = -k^\dagger\). We verify that \(k\) is a real linear combination of elements \(x^n u\) and \(i(x^n - x^{-n})\) for \(n \in \mathbb{Z}\). Hence \(k\), and so its limit \(h\), is hermitian.

Now suppose that \(h \in H\). By extremality, \(h\) is also hermitian in \(C^*(\mathcal{G})\), and so \(h^* = h\). Let \(\zeta \in T\) and \(\beta \in \mathbb{C}\). Define a linear functional \(\phi\) on \(A\) by

\[
\phi(b + cu) = (1 - 2\beta)b(1) + \beta b(\zeta) + \beta b(\overline{\zeta}) \quad (b, c \in C(T)).
\]

Then \(\phi(1) = 1\). If \(b + cu \in \mathcal{J}\) then, as in Corollary 2.3, \(-1 \leq b(1) \leq 1, |b(\zeta)| \leq 1\) and \(b(\overline{\zeta}) = \overline{b(\zeta)}\). These give \(|\phi(b + cu)| \leq \max\{1, |1 - 4\beta|\}\). If \(|1 - 4\beta| \leq 1\) then \(|\phi(\mathcal{J})| \leq 1\) and so \(\|\phi\| \leq 1\). For these \(\beta, \phi\) is a support functional of 1. Hence \(\phi(h) \in \mathbb{R}\). Write \(h = f + gu\) with \(f, g \in C(T)\). We deduce that \(f(1) \in \mathbb{R}\) and \(f(\zeta) + f(\overline{\zeta}) = 2f(1)\). Therefore \(h + h^\dagger = f + f^\dagger = 2f(1)\), as required.

\[\Box\]

The proof of the next result is routine.
Proposition 3.2. The centre $Z$ of $Ea(u, v)$ is given by $Z = \{ f \in C(\mathbb{T}) : f = f^\dagger \}$ and $Z \cap H = \mathbb{R}$.

We show that most hermitian elements $h$ of $Ea(u, v)$ have the property that $h^n$ is hermitian for all odd $n$ but for no even $n$. On the other hand, when $h$ contains a non-zero multiple of the identity, we usually have no other power hermitian. We remark that these latter hermitians cannot generate the extremal algebra on one hermitian generator because they generate $C^*$-equivalent subalgebras.

Let $H_0 = \{ h \in A : h^* = -h \}$, so that $H_0 \subset H$.

Theorem 3.3 Let $n \in \mathbb{N}$.

1. If $h \in H_0$ and $n$ is odd then $h^n \in H$.
2. If either $h \in H_0$ and $h^n \in H$ with $n$ even, or $h \in H \setminus H_0$ and $h^n \in H$ with $n > 1$, then $P(h) = 0$ for some quadratic polynomial $P$.

Proof. (1) Since $h = h^* = -h^\dagger$, we have $h^n = h^{*n} = -h^{n^*}$ for $n$ odd, and so $h^n \in H_0$.

(2) For some $\lambda, \mu \in \mathbb{R}$, $h + h^\dagger = \lambda$ and $h^n + h^{\dagger n} = \mu$, where $\lambda = 0$ and $n$ is even, or $\lambda \neq 0$ and $n > 1$. Consider the even polynomial $Q(\zeta) = \zeta^n + (\lambda - \zeta)^n - \mu$, which has at most two real zeros. Then $Q(h) = 0$, and each factor $h - \zeta$ of $Q(h)$ with $\zeta \notin \mathbb{R}$ may be cancelled since $h$ has real spectrum. This leaves a real quadratic $P$ with $P(h) = 0$. \qed

An example of the situation in Theorem 3.3 (2) is $h = i(x - x^{-1}) + (x + x^{-1})u$. Here $h \in H_0$ and $h^2 = 4$. In these cases, $h^n \in H$ ($n \in \mathbb{N}$).

The infinite dihedral group $G$ has many subgroups which are isomorphic to $G$ and hence the $C^*$-algebra generated by such is isometrically isomorphic to $C^*(G)$. There are natural related questions to ask for $Ea(u, v)$. Since $\|u\| = \|u^{-1}\| = 1$, the mapping $a \to uau$ is an isometric monomorphism of $Ea(u, v)$. Thus the closed subalgebra generated by $u, uuv$ is a copy of $Ea(u, v)$. Equally for the closed subalgebra generated by $vuv, v$. By applying these two mappings repeatedly we easily see that the closed subalgebra generated by $x^n u, x^{n+1} u$ is a copy of $Ea(u, v)$ for any $n \in \mathbb{Z}$. On the other hand, this simple method will not identify for us the closed subalgebra generated by $x^n u, x^n v$ (i.e. $xu, x^{-2} u$). We show in fact that any two hermitian elements $x^m u, x^n u$ with $m, n \in \mathbb{Z}, m \neq n$ generate a copy of $Ea(u, v)$.

Let $A_S = \{ a \in Ea(u, v) : a^\dagger = a \}$. We easily verify that $A_S = \{ f + igu : f, g \in C_G(\mathbb{T}) \}$. Also, $A_S$ is a real $C^*$-algebra with the involution $*$ and norm $\| \cdot \|$. 

Proposition 3.4. We have $\| a \| = | a |$ for $a \in A_S$.

Proof. Let $a \in A_S$ with $| a | < 1$. By [4], $a$ is a convex combination of elements of the form $\cos b e^c$, where $b, c \in A_S$, $b^* = b$, $e^* = -c$. Then $b \in C_G(\mathbb{T})$, $b$ is real valued, $\cos b \in C_G(\mathbb{T})$ and so $\| \cos b \| = | \cos b |_\infty \leq 1$. Also, $(ic)^* = -ic^* = ic = -(ic)^\dagger$ and so $ic \in H_0, \| ec \| = 1$. Therefore $\| a \| \leq 1$. It follows that $\| a \| \leq | a |$ for all $a \in A_S$. But $| a | \leq \| a \|$ by Theorem 2.6. Hence $\| a \| = | a |$. \qed

Theorem 3.5. Let $x^m u, x^n u$ be any two hermitian words in $G$ (where $m, n \in \mathbb{Z}$). Then they generate an isometric copy of $Ea(u, v)$ in $Ea(u, v)$.
Proof. In $G$, $x^n u$ and $x^n u$ generate an isomorphic subgroup $G_1 = H_1 \cup K_1$, where $H_1 \subseteq H$ and $K_1 \subseteq Hu$. Define a norm $\| \cdot \|_1$ on $\mathbb{C}[G_1]$ via $\mathcal{J}_1 = \{ a \in \mathbb{C}[G_1] : a^* = a^\dagger, \ a^* a = 1 \}$. The involutions $\ast, \dagger \ast \cap \mathbb{C}[G_1]$ agree with those of $\mathbb{C}[G]$. Then $(\mathbb{C}[G_1], \| \cdot \|_1)$ is an isometric copy of $A_0$. It is enough to show that $\| \cdot \|_1$ is just the restriction of the norm $\| \cdot \|$ of $A_0$.

Let $a \in \mathcal{J}$, and $d$ its projection in $\mathbb{C}[G_1]$. By Lemma 2.5, $|d| \leq |a| = 1$. From $a^* = a^\dagger$ we deduce that $d^* = d^\dagger$. By Proposition 3.4, $\|d\|_1 = |d|$. Hence $\|d\|_1 \leq 1$.

Now consider $a \in \mathbb{C}[G_1]$. Suppose $a = \sum \alpha_k a_k$ with $a_k \in \mathbb{C}$ and $a_k \in \mathcal{J}$. Let $d_k$ be the projection of $a_k$ in $\mathbb{C}[G_1]$. Then $\|a\|_1 = \|\sum \alpha_k a_k\|_1 = \|\sum \alpha_k d_k\|_1 \leq \sum |\alpha_k|$. It follows that $\|a\|_1 \leq \|a\|$. But since $\mathcal{J}_1 \subseteq \mathcal{J}$, $\|a\|_1 \geq \|a\|$. Therefore $\|a\|_1 = \|a\|$. 

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