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THE EXTREMAL ALGEBRA ON TWO HERMITIANS WITH SQUARE 1

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Abstract. Let $Ea(u, v)$ be the extremal algebra determined by two hermitians u and v with $u^2 = v^2 = 1$. We show that: $Ea(u, v) = \{f + gu : f, g \in C(\mathbb{T})\}$, where $\mathbb{T}$ is the unit circle; $Ea(u, v)$ is C^*-equivalent to $C^*(\mathcal{G})$, where $\mathcal{G}$ is the infinite dihedral group; most of the hermitian elements k of $Ea(u, v)$ have the property that k^n is hermitian for all odd n but for no even n; any two hermitian words in $\mathcal{G}$ generate an isometric copy of $Ea(u, v)$ in $Ea(u, v)$.

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1. Introduction. This is a continuation of [**2**], except that we are concerned here only with the extremal Banach algebra $Ea(u, v)$ determined by two hermitian involutions u and v (we use *involution* here in the group sense, namely that $u^2 = v^2 = 1$). In [**2**] we presented $Ea(u, v)$ as an abstract completion of a group algebra. Here we present it as a specific algebra of pairs of continuous functions on the unit circle and we prove that it is even C^*-equivalent for the natural star operation on $Ea(u, v)$ which makes the generators u and v unitary elements. The hermitian element defined by $h = (i/2)(uv - vu)$ has the remarkable property that h^n is hermitian for every odd n but for no even n; and yet the subalgebra generated by h is C^*-equivalent to $C[-1, 1]$. The algebra $Ea(u, v)$ is equivalent to the C^*-algebra of the infinite dihedral group $\mathcal{G}$. We give a simple explicit description of the space of hermitian elements in $Ea(u, v)$; we also show that most of the hermitian elements k of $Ea(u, v)$ have the property that k^n is hermitian for all odd n but for no even n. Permutations of $\mathcal{G}$ induce (isometric) automorphisms of $C^*(\mathcal{G})$. We show that there are also many (isometric) isomorphisms onto subalgebras of $Ea(u, v)$.

We use without comment some elementary properties of hermitians which may be found in [**1**].

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2. *Ea*(*u*,*v*) is *C-equivalent.** We repeat here some essential notation from [**2**]. We write $\mathcal{G}$ for the infinite dihedral group generated by x and u, where $u^2 = 1$ and $ux = x^{-1}u$. In relation to the algebra $Ea(u, v)$ we have $x = uv$. Put $\mathcal{H} = \{x^n : n \in \mathbb{Z}\}$ so that $\mathcal{G} = \mathcal{H} \cup \mathcal{H}u$. Put $A_0 = \mathbb{C}[\mathcal{G}]$ and recall the (algebra) involutions * and † given by

$$\left(\sum \alpha_g g\right)^* = \sum \overline{\alpha}_g g^{-1}, \quad \left(\sum \alpha_g g\right)^\dagger = \sum \alpha_g g^{-1}.$$

For $a \in A_0$ we note that $a^* = a^\dagger \Longleftrightarrow a \in \mathbb{R}[\mathcal{G}]$. Let $\mathcal{J} \subseteq \mathbb{C}[\mathcal{G}]$ be the set of all finite products of elements of the form $p = \cos\theta + i\sin\theta x^n u$, where $\theta \in \mathbb{R}, n \in \mathbb{Z}$. Since $(x^n u)^{-1} = x^n u$, we have $p^* = \cos\theta - i\sin\theta x^n u \in \mathcal{J}$, and $p^*p = 1 = pp^*$. It follows that, for all $a \in \mathcal{J}$, $a^*a = 1 = aa^*$ and $a^* \in \mathcal{J}$. Hence $\mathcal{J}$ is a group in A_0. Identities such as $\cos\theta + i\sin\theta\, xu = (iu)(\cos\theta + i\sin\theta\, v)(-iu)$ show that this is the $\mathcal{J}$ of [**2**]. Since $x^n = (-ix^n u)(iu)$, we have $x^n \in \mathcal{J}$ and so $\mathcal{H} \subseteq \mathcal{J}$.

Since $\mathcal{G} = \mathcal{H} \cup \mathcal{H}u$, each $a \in A_0$ can be written as $a = b + cu$ with $b, c \in \mathbb{C}[\mathcal{H}]$. For $c \in \mathbb{C}[\mathcal{H}]$, we have $uc = c^\dagger u$. For $b, c \in \mathbb{C}[\mathcal{H}]$, this gives $(b + cu)^* = b^* + c^{*\dagger}u$; define an involution ‡ on A_0 by $(b + cu)^\ddagger = b^\dagger - cu$. By contrast, note that $(b + cu)^\dagger = b^\dagger + cu$. Then $a^{*\ddagger} = a^{\ddagger *}$ for $a \in A_0$, and $a^* = a^\ddagger$ if $a \in \mathcal{J}$. For $a \in A_0$, $a^* = a^\ddagger$ if and only if $a = b + icu$ for some $b, c \in \mathbb{R}[\mathcal{H}]$; then $a^*a = b^*b + c^*c = aa^*$.

LEMMA 2.1. *Let* $\mathcal{K} = \{a \in A_0 : a^* = a^\ddagger,\ a^*a = 1\}$. *Then* $\mathcal{J} = \mathcal{K}$.

Proof. From the above, $\mathcal{J} \subseteq \mathcal{K}$. Clearly $\mathcal{K}$ is also a group. Any $a \in \mathcal{K}$, $a \notin \pm(\mathcal{H} U i\mathcal{H}u)$, may be written

$$a = \alpha_p x^p + \cdots + \alpha_m x^m + i\beta_q x^q u + \cdots + i\beta_n x^n u \tag{1}$$

where $p, q, m, n \in \mathbb{Z}$, $p \leq m$, $q \leq n$, $\alpha_p\alpha_m\beta_q\beta_n \neq 0$ and $\alpha_k, \beta_k \in \mathbb{R}$ for all k. Suppose that $m - p > n - q$. Then the coefficient of x^{m-p} in a^*a is $\alpha_p\alpha_m \neq 0$. Since $a^*a = 1$, this coefficient is 0. Similarly we rule out $n - q > m - p$. Therefore $m - p = n - q$; call this common value the length of a. We show that $a \in \mathcal{K}$ implies $a \in \mathcal{J}$ by induction on the length of a. If a has length 0 then $a = \alpha_p x^p + i\beta_q x^q u = x^p(\alpha_p + i\beta_q x^{q-p}u) \in \mathcal{J}$, since $1 = a^*a = \alpha_p^2 + \beta_q^2$. Suppose that our claim holds for elements of length less than N, and consider a as above of length N. For $\theta \in \mathbb{R}$, $\cos\theta + i\sin\theta\, x^{q-p}u \in \mathcal{J} \subseteq \mathcal{K}$ and so $a' = a(\cos\theta + i\sin\theta\, x^{q-p}u) \in \mathcal{K}$. Here a' has the form of (1) with α_p replaced by $\alpha'_p = \alpha_p\cos\theta - \beta_q\sin\theta$. We choose θ so that $\alpha'_p = 0$. Then a' has length less than N and, by hypothesis, $a' \in \mathcal{J}$. Therefore $a \in \mathcal{J}$, as required. □

As in [**2**], we now define a norm on A_0 by

$$\|a\| = \inf\left\{\sum_1^N |\alpha_k| : a = \sum_1^N \alpha_k a_k, N \in \mathbb{N}, \alpha_k \in \mathbb{C}, a_k \in \mathcal{J}\right\}.$$

Let $\mathbb{T}$ denote the unit circle in $\mathbb{C}$. With each element $b = \sum \alpha_n x^n$ of $\mathbb{C}[\mathcal{H}]$ we associate the function on $\mathbb{T}$ given by $b(\zeta) = \sum \alpha_n \zeta^n$. We can now regard A_0 as the set of all elements $f + gu$ where f, g are polynomials in ζ and $\zeta^{-1} = \overline{\zeta}$ on $\mathbb{T}$. We also have a representation as 2×2 matrices of functions of $\zeta \in \mathbb{T}$ by

$$\pi(f + gu) = \begin{pmatrix} f(\zeta) & g(\zeta) \\ g(\overline{\zeta}) & f(\overline{\zeta}) \end{pmatrix}.$$

The involutions on $\mathbb{C}[\mathcal{H}]$ correspond to

$$f^*(\zeta) = \overline{f(\zeta)}, \qquad f^\dagger(\zeta) = f(\overline{\zeta}).$$

We have $\mathcal{J} = \mathcal{J}* = \mathcal{J}\ddagger$, and so $*$ and $\ddagger$ are isometric for $\|\cdot\|$. We write $|\cdot|_\infty$ for the supremum norm over $\mathbb{T}$. Of course the element x corresponds to the function $x(\zeta) = \zeta$.

LEMMA 2.2. *Let $f \in \mathbb{R}[\mathcal{H}]$ with $|f|_\infty < 1$. Then there exists $g \in \mathbb{R}[\mathcal{H}]$ such that $f^*f + g^*g = 1$.*

Proof. Put $F = 1 - f^*f$, so that F is a positive trigonometric polynomial with real coefficients. By [**3**, pp 117–8], F can be written as g^*g, and the proof in [**3**] shows that the trigonometric polynomial g also has real coefficients. □

COROLLARY 2.3. *For $f \in \mathbb{R}[\mathcal{H}]$, we have $\|f\| = |f|_\infty$. For $f \in \mathbb{C}[\mathcal{H}]$, we have $|f|_\infty \le \|f\| \le 2|f|_\infty$. The completion of $(\mathbb{C}[\mathcal{H}], \|\cdot\|)$ is $C(\mathbb{T})$, with $|f|_\infty \le \|f\| \le 2|f|_\infty$ for all $f \in C(\mathbb{T})$.*

Proof. Let $f \in \mathbb{R}[\mathcal{H}]$ with $|f|_\infty < 1$. By Lemma 2.2, there exists $g \in \mathbb{R}[\mathcal{H}]$ such that $f^*f + g^*g = 1$. Then $a = f \pm igu$ satisfy $a^* = a^\ddagger$ and $a^*a = 1$. By Lemma 2.1, $f \pm igu \in \mathcal{J}$. Therefore $\|f \pm igu\| = 1$, and $\|f\| \le 1$. By linearity, $\|f\| \le |f|_\infty$ for $f \in \mathbb{R}[\mathcal{H}]$. For $b + icu \in \mathcal{J}$, we have

$$|b(\zeta)|^2 + |c(\zeta)|^2 = (b^*b + c^*c)(\zeta) = 1$$

and so $|b(\zeta)| \le 1$ for $\zeta \in \mathbb{T}$. Hence, for $f \in \mathbb{C}[\mathcal{H}]$, $\|f\| \ge |f(\zeta)|$, and so $\|f\| \ge |f|_\infty$, which gives $\|f\| = |f|_\infty$ for $f \in \mathbb{R}[\mathcal{H}]$.

Let $f = \sum \alpha_n x^n \in \mathbb{C}[\mathcal{H}]$. Note that $f^{*\dagger} = \sum \overline{\alpha_n} x^n$, and $|f^{*\dagger}|_\infty = |f|_\infty$. Thus $f + f^{*\dagger} \in \mathbb{R}[\mathcal{H}]$ and $|f + f^{*\dagger}|_\infty \le 2|f|_\infty$. This gives $\|f + f^{*\dagger}\| \le 2|f|_\infty$. Also, $i(f - f^{*\dagger}) \in \mathbb{R}[\mathcal{H}]$, which gives $\|f - f^{*\dagger}\| \le 2|f|_\infty$ and hence $\|f\| \le 2|f|_\infty$. The final part follows by the Stone-Weierstrass theorem. □

The involutions $*$ and $\dagger$ extend in the natural way to $C(\mathbb{T})$, and a routine approximation argument gives the next corollary. Define $C_S(\mathbb{T}) = \{f \in C(\mathbb{T}) : f^* = f^\dagger\}$.

COROLLARY 2.4. *Let $f \in C_S(\mathbb{T})$. Then $\|f\| = |f|_\infty$.*

We define a norm $|\cdot|$ on $\mathbb{C}[\mathcal{G}]$ by $|a| = \sup\{|ab|_2 : b \in \ell^2(\mathcal{G}),\ |b|_2 = 1\}$, where $|\sum \beta_g g|_2 = (\sum |\beta_g|^2)^{1/2}$. The completion of $(\mathbb{C}[\mathcal{G}], |\cdot|)$ is the C^*-algebra $C^*(\mathcal{G})$.

LEMMA 2.5. *Let $\mathcal{L}$ be a subgroup of $\mathcal{G}$. Let $a = \sum_{g \in \mathcal{G}} \alpha_g g \in \mathbb{C}[\mathcal{G}]$, and $d = \sum_{g \in \mathcal{L}} \alpha_g g$ its projection in $\mathbb{C}[\mathcal{L}]$. Then $|d| \le |a|$.*

Proof. Write $a = d + f$ where $f \in \mathrm{lin}(\mathcal{G} \setminus \mathcal{L})$. If $b \in \ell^2(\mathcal{L})$ then $db \in \ell^2(\mathcal{L})$ and $fb \in \ell^2(\mathcal{G} \setminus \mathcal{L})$. Therefore $|ab|_2 = |db + fb|_2 \ge |db|_2$. Taking the supremum over $|b|_2 = 1$, we have $|a| \ge |d|$. □

Note that, with the notation of Lemma 2.5, $|d|$ is the same whether taken over $\mathcal{L}$ or $\mathcal{G}$.

THEOREM 2.6. *As algebras,*

$$Ea(u, v) = C^*(\mathcal{G}) = \{f + gu : f, g \in C(\mathbb{T})\},$$

with $uf = f^\dagger u$ *and* $|f| = |f|_\infty$ $(f \in C_S(\mathbb{T}))$. *For* $a \in Ea(u, v)$, $|a| \le \|a\| \le 4|a|$.

Proof. Let $a = f + gu$ with $f, g \in \mathbb{C}[\mathcal{H}]$. Lemma 2.5 gives $|f| \le |a|$. Since $au = g + fu$, also $|g| \le |au| = |a|$. We have $|f| = |f|_\infty$. From the Stone-Weierstrass theorem we deduce that $C^*(\mathcal{G}) = \{f + gu : f, g \in C(\mathbb{T})\}$.

It is now enough to prove that $|a| \le \|a\| \le 4|a|$ for $a = f + gu$, $f, g \in \mathbb{C}[\mathcal{H}]$. We have that $|a| \le \|a\|$ by the extremal nature of $\|\cdot\|$. Also, $\|a\| \le \|f\| + \|g\| \le 2|f| + 2|g| \le 4|a|$ by Corollary 2.3. □

COROLLARY 2.7. *The extremal Banach algebra on one generator with all odd powers hermitian is* C^**-equivalent with* $|\cdot| \le \|\cdot\| \le 2|\cdot|$ *where* $\|\cdot\|$ *is the extremal norm and* $|\cdot|$ *the* C^**-norm.*

We extend $*$ and $\ddagger$ to $Ea(u, v)$ by the earlier formulæ. For the above matrix representation, $\ddagger$ gives the adjugate matrix.

3. Properties of *Ea(u,v)*. We begin by identifying the space of hermitian elements in $Ea(u, v)$. In [**2**] we noted the obvious hermitian elements (in A_0) given by $x^n u$ $(n \in \mathbb{Z})$, 1 and $i(x^n - x^{-n})$ $(n \in \mathbb{N})$. As expected, the space H of hermitian elements of $Ea(u, v)$ is the closed real linear span of these elements. In fact, we can give a more elegant, and useful, description in terms of the involutions $*$ and $\ddagger$.

THEOREM 3.1. *We have* $H = \{h \in Ea(u, v) : h^* = h,\ h + h^\ddagger \in \mathbb{R}\}$.

Proof. Suppose that $h \in Ea(u, v)$ with $h^* = h$ and $h + h^\ddagger = \alpha \in \mathbb{R}$. Replacing h by $h - \alpha/2$, we assume that $\alpha = 0$. We approximate h by elements k in A_0 satisfying $k = k^* = -k^\ddagger$. We verify that k is a real linear combination of elements $x^n u$ and $i(x^n - x^{-n})$ for $n \in \mathbb{Z}$. Hence k, and so its limit h, is hermitian.

Now suppose that $h \in H$. By extremality, h is also hermitian in $C^*(\mathcal{G})$, and so $h^* = h$. Let $\zeta \in \mathbb{T}$ and $\beta \in \mathbb{C}$. Define a linear functional ϕ on A by

$$\phi(b + cu) = (1 - 2\beta)b(1) + \beta b(\zeta) + \beta b(\overline{\zeta}) \qquad (b, c \in C(\mathbb{T})).$$

Then $\phi(1) = 1$. If $b + cu \in \mathcal{J}$ then, as in Corollary 2.3, $-1 \le b(1) \le 1$, $|b(\zeta)| \le 1$ and $b(\overline{\zeta}) = \overline{b(\zeta)}$. These give $|\phi(b + cu)| \le \max\{1, |1 - 4\beta|\}$. If $|1 - 4\beta| \le 1$ then $|\phi(\mathcal{J})| \le 1$ and so $\|\phi\| \le 1$. For these β, ϕ is a support functional of 1. Hence $\phi(h) \in \mathbb{R}$. Write $h = f + gu$ with $f, g \in C(\mathbb{T})$. We deduce that $f(1) \in \mathbb{R}$ and $f(\zeta) + f(\overline{\zeta}) = 2f(1)$. Therefore $h + h^\ddagger = f + f^\ddagger = 2f(1)$, as required. □

The proof of the next result is routine.

PROPOSITION 3.2. *The centre Z of $Ea(u, v)$ is given by $Z = \{f \in C(\mathbb{T}) : f = f^{\dagger}\}$ and $Z \cap H = \mathbb{R}$.*

We show that most hermitian elements h of $Ea(u, v)$ have the property that h^n is hermitian for all odd n but for no even n. On the other hand, when h contains a non-zero multiple of the identity, we usually have no other power hermitian. We remark that these latter hermitians cannot generate the extremal algebra on one hermitian generator because they generate C^*-equivalent subalgebras.

Let $H_0 = \{h \in A : h^* = h = -h^{\ddagger}\}$, so that $H_0 \subset H$.

THEOREM 3.3 *Let $n \in \mathbb{N}$.*

(1) *If $h \in H_0$ and n is odd then $h^n \in H$.*

(2) *If either $h \in H_0$ and $h^n \in H$ with n even, or $h \in H \setminus H_0$ and $h^n \in H$ with $n > 1$, then $P(h) = 0$ for some quadratic polynomial P.*

Proof. (1) Since $h = h^* = -h^{\ddagger}$, we have $h^n = h^{n*} = -h^{n\ddagger}$ for n odd, and so $h^n \in H_0$.

(2) For some $\lambda, \mu \in \mathbb{R}$, $h + h^{\ddagger} = \lambda$ and $h^n + h^{\ddagger n} = \mu$, where $\lambda = 0$ and n is even, or $\lambda \neq 0$ and $n > 1$. Consider the even polynomial $Q(\zeta) = \zeta^n + (\lambda - \zeta)^n - \mu$, which has at most two real zeros. Then $Q(h) = 0$, and each factor $h - \zeta$ of $Q(h)$ with $\zeta \notin \mathbb{R}$ may be cancelled since h has real spectrum. This leaves a real quadratic P with $P(h) = 0$. □

An example of the situation in Theorem 3.3 (2) is $h = i(x - x^{-1}) + (x + x^{-1})u$. Here $h \in H_0$ and $h^2 = 4$. In these cases, $h^n \in H$ $(n \in \mathbb{N})$.

The infinite dihedral group $\mathcal{G}$ has many subgroups which are isomorphic to $\mathcal{G}$ and hence the C^*-algebra generated by such is isometrically isomorphic to $C^*(\mathcal{G})$. There are natural related questions to ask for $Ea(u, v)$. Since $\|u\| = \|u^{-1}\| = 1$, the mapping $a \to uau$ is an isometric monomorphism of $Ea(u, v)$. Thus the closed subalgebra generated by u, uvu is a copy of $Ea(u, v)$. Equally for the closed subalgebra generated by vuv, v. By applying these two mappings repeatedly we easily see that the closed subalgebra generated by $x^n u, x^{n+1}u$ is a copy of $Ea(u, v)$ for any $n \in \mathbb{Z}$. On the other hand, this simple method will not identify for us the closed subalgebra generated by uvu, vuv (i.e. $xu, x^{-2}u$). We show in fact that any two hermitian elements $x^m u, x^n u$ with $m, n \in \mathbb{Z}, m \neq n$ generate a copy of $Ea(u, v)$.

Let $A_S = \{a \in Ea(u, v) : a^* = a^{\ddagger}\}$. We easily verify that $A_S = \{f + igu : f, g \in C_S(\mathbb{T})\}$. Also, A_S is a real C^*-algebra with the involution $*$ and norm $|\cdot|$.

PROPOSITION 3.4. *We have $\|a\| = |a|$ for $a \in A_S$.*

Proof. Let $a \in A_S$ with $|a| < 1$. By **[4]**, a is a convex combination of elements of the form $\cos b\, e^c$, where $b, c \in A_S$, $b^* = b$, $c^* = -c$. Then $b \in C_S(\mathbb{T})$, b is real valued, $\cos b \in C_S(\mathbb{T})$ and so $\|\cos b\| = |\cos b|_\infty \leq 1$. Also, $(ic)^* = -ic^* = ic = -(ic)^{\ddagger}$ and so $ic \in H_0$, $\|e^c\| = 1$. Therefore $\|a\| \leq 1$. It follows that $\|a\| \leq |a|$ for all $a \in A_S$. But $|a| \leq \|a\|$ by Theorem 2.6. Hence $\|a\| = |a|$. □

THEOREM 3.5. *Let $x^m u$, $x^n u$ be any two hermitian words in $\mathcal{G}$ (where $m, n \in \mathbb{Z}$). Then they generate an isometric copy of $Ea(u, v)$ in $Ea(u, v)$.*

Proof. In $\mathcal{G}$, $x^m u$ and $x^n u$ generate an isomorphic subgroup $\mathcal{G}_1 = H_1 \cup K_1$, where $H_1 \subseteq H$ and $K_1 \subseteq Hu$. Define a norm $\| \cdot \|_1$ on $\mathbb{C}[\mathcal{G}_1]$ via $\mathcal{J}_1 = \{a \in \mathbb{C}\ [\mathcal{G}_1] : a^* = a^\ddagger,\ a^* a = 1\}$. The involutions $*$, $\ddagger$ of $\mathbb{C}[\mathcal{G}_1]$ agree with those of $\mathbb{C}[\mathcal{G}]$. Then $(\mathbb{C}[\mathcal{G}_1], \| \cdot \|_1)$ is an isometric copy of A_0. It is enough to show that $\| \cdot \|_1$ is just the restriction of the norm $\| \cdot \|$ of A_0.

Let $a \in \mathcal{J}$, and d its projection in $\mathbb{C}[\mathcal{G}_1]$. By Lemma 2.5, $|d| \leq |a| = 1$. From $a^* = a^\ddagger$ we deduce that $d^* = d^\ddagger$. By Proposition 3.4, $\|d\|_1 = |d|$. Hence $\|d\|_1 \leq 1$.

Now consider $a \in \mathbb{C}[\mathcal{G}_1]$. Suppose $a = \sum \alpha_k a_k$ with $\alpha_k \in \mathbb{C}$ and $a_k \in \mathcal{J}$. Let d_k be the projection of a_k in $\mathbb{C}[\mathcal{G}_1]$. Then $\|a\|_1 = \|\sum \alpha_k a_k\|_1 = \|\sum \alpha_k d_k\|_1 \leq \sum |\alpha_k|$. It follows that $\|a\|_1 \leq \|a\|$. But since $\mathcal{J}_1 \subseteq \mathcal{J}$, $\|a\|_1 \geq \|a\|$. Therefore $\|a\|_1 = \|a\|$. □

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