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A REMARK ON RATIONAL CHEREDNIK ALGEBRAS AND DIFFERENTIAL OPERATORS ON THE CYCLIC QUIVER

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Abstract. We show that the spherical subalgebra $U_{k,\ell}$ of the rational Cherednik algebra associated to $S_n \rtimes C_\ell$, the wreath product of the symmetric group and the cyclic group of order $\ell$, is isomorphic to a quotient of the ring of invariant differential operators on a space of representations of the cyclic quiver of size $\ell$. This confirms a version of [5, Conjecture 11.22] in the case of cyclic groups. The proof is a straightforward application of work of Oblomkov [12] on the deformed Harish-Chandra homomorphism, and of Crawley-Boevey, [3] and [4], and Gan and Ginzburg [7] on preprojective algebras.

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1. Introduction.

1.1. The representation theory of symplectic reflection algebras has links with a number of subjects including algebraic combinatorics, resolutions of singularities, Lie theory and integrable systems. There is a family of symplectic reflection algebras associated to any symplectic vector space $V$ and finite subgroup $\Gamma \leq \text{Sp}(V)$, but a simple reduction allows one to study those subgroups $\Gamma$ which are generated by symplectic reflections (i.e. by elements whose set of fixed points is of codimension two in $V$). This essentially focuses attention on two cases:

(1) $\Gamma = W$, a finite complex reflection group, acting on $V = \mathfrak{h} \oplus \mathfrak{h}^*$ where $\mathfrak{h}$ is a reflection representation of $W$;

(2) $\Gamma = S_n \rtimes K$, where $K$ is a finite subgroup of $\text{SL}_2(\mathbb{C})$, acting naturally on $(\mathbb{C}^2)^n$.

The representation theory in the first case is mysterious at the moment: several important results are known but there is no general theory yet. On the other hand a geometric point of view on the representation theory in the second case is beginning to emerge. A key fact is that in this case the singular space $V/\Gamma$ admits a crepant resolution of singularities: the representation theory of the symplectic reflection algebra is then expected to be closely related to the resolution. In the case $\Gamma = S_n$ (i.e. $K$ is trivial) there are two approaches to this: the first is via noncommutative algebraic geometry [8] the second via sheaves of differential operators [7]. In this paper we extend the second approach to the groups $\Gamma = \Gamma_n = S_n \rtimes C_\ell$.

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1.2. To state our result we need to introduce a little notation here. Let $Q$ be the cyclic quiver with $\ell$ vertices and cyclic orientation. Choose an extending vertex (in this case any vertex) 0. Then let $Q_{\infty}$ be the quiver obtained by adding one vertex named $\infty$ to $Q$ that is joined to 0 by a single arrow.

We shall consider representation spaces of these quivers. Let $\delta = (1, 1, \ldots, 1)$ be the affine dimension vector of $Q$, and set $\epsilon = e_\infty + n\delta$, a dimension vector for $Q_{\infty}$. Let $\text{Rep}(Q, n\delta)$ and $\text{Rep}(Q_{\infty}, \epsilon)$ be the representation spaces of these quivers with the given dimension vectors. There is an action of $G = \prod_{r=0}^{\ell-1} \mathbb{G}L_n(\mathbb{C})$ on both these spaces.

In fact, the action of the scalar matrices in $G$ is trivial on $\text{Rep}(Q, n\delta)$ (but not on $\text{Rep}(Q_{\infty}, \epsilon)$) and so in this case the action descends to an action of $PG = G/\mathbb{C}^*$. Let $X = \text{Rep}(Q, n\delta) \times \mathbb{P}^{n-1}$. There is an action of $PG$ on $X$.

1.3. Let $D(\text{Rep}(Q_{\infty}, \epsilon))$ denote the ring of differential operators on the affine space $\text{Rep}(Q_{\infty}, \epsilon)$, $D_X(nk)$ the sheaf of twisted differential operators on $X$ and its algebra of global sections. The group action of $G$ (respectively $PG$) on $\text{Rep}(Q_{\infty}, \epsilon)$ (respectively $X$) differentiates to an action of $g = \text{Lie}(G)$ (respectively $pg = \text{Lie}(PG)$) by differential operators. This gives mappings

$$\hat{\tau} : g \rightarrow D(\text{Rep}(Q_{\infty}, \epsilon)), \quad \tau : pg \rightarrow D_X(nk).$$

1.4. Let $U_{k, c}$ be the spherical subalgebra of type $S_n \wr C_\ell$. (This is defined in Section 3.4.)

**Theorem.** For all $(k, c)$ there are isomorphisms of algebras

$$\left( \frac{D(\text{Rep}(Q_{\infty}, \epsilon))}{I_{k, c}} \right)^G \cong \left( \frac{D_X(nk)}{I_c} \right)^{PG} \cong U_{k, c},$$

where $I_{k, c}$ is the left ideal of $D(\text{Rep}(Q_{\infty}, \epsilon))$ generated by $(\hat{\tau} - \chi_{k, c})(g)$ and $I_c$ is the left ideal of $D_X(nk)$ generated by $(\tau - \chi_c)(pg)$ for suitable characters $\chi_{k, c} \in g^*$ and $\chi_c \in pg^*$.

(These are defined in Section 4.)

Note that it is a standard fact that the left hand side is an algebra. The proof of the theorem has two parts. One part constructs a filtered homomorphism from the left hand side to the right hand side using as its main input the work of Oblomokov [12]. The other part proves that the associated graded homomorphism is an isomorphism and is a simple application of results of Crawley–Boevey [3] and [4], and of Gan and Ginzburg [7].

1.5. We give an application of this result in Section 4. For related pairs $(k, c)$ and $(k', c')$ we construct “shift functors”

$$U_{k, c}\text{-mod} \rightarrow U_{k', c'}\text{-mod}$$

using differential operators. We expect these to be a useful tool in the representation theory of Cherednik algebras, deserving of careful study.

1.6. While writing this down, we were informed that the general version of [5, Conjecture 11.22] has been proved in [6]. This result is more general than the work...
presented here and requires a new approach and ideas to overcome problems that simply do not arise for the case $\Gamma = S_n \wr C_\ell$.

2. Quivers.

2.1. Once and for all fix integers $\ell$ and $n$. We assume that both are greater than 1. Set $\eta = e^{2\pi i/\ell}$.

2.2. Let $Q$ be the cyclic quiver with $\ell$ vertices and cyclic orientation. Choose an extending vertex (in this case any vertex) 0. Then let $Q_\infty$ be the quiver obtained by adding one vertex named $\infty$ to $Q$ that is joined to 0 by a single arrow. Let $\overline{Q}$ and $\overline{Q}_\infty$ denote the double quivers of $Q$ and $Q_\infty$ respectively, obtained by inserting an arrow $a^*$ in the opposite direction to every arrow $a$ in the quiver.

We shall consider representation spaces of these quivers. Let $\delta = (1, 1, \ldots, 1)$ be the affine dimension vector of $Q$, and set $\epsilon = e_\infty + n \delta$, a dimension vector for $Q_\infty$.

Recall that $\text{Rep}(Q, n\delta) = \oplus_{r=0}^{\ell-1} \text{Mat}_n(\mathbb{C}) = \{(X_0, X_1, \ldots, X_{\ell-1}) : X_r \in \text{Mat}_n(\mathbb{C}) \}$

and

$\text{Rep}(Q_\infty, \epsilon) = \oplus_{r=0}^{\ell-1} \text{Mat}_n(\mathbb{C}) \oplus \mathbb{C}^n = \{(X_0, X_1, \ldots, X_{\ell-1}, i) : X_r \in \text{Mat}_n(\mathbb{C}), i \in \mathbb{C}^n \}$

Let $G = \prod_{r=0}^{\ell-1} GL_n(\mathbb{C})$ be the base change group. If $g = (g_0, \ldots, g_{\ell-1})$, then $g$ acts on $\text{Rep}(Q, n\delta)$ by

$g \cdot (X_0, X_1, \ldots, X_{\ell-1}) = (g_0 X_0 g_1^{-1}, g_1 X_1 g_2^{-1}, \ldots, g_{\ell-1} X_{\ell-1} g_0^{-1})$

and on $\text{Rep}(Q_\infty, \epsilon)$ by

$g \cdot (X_0, X_1, \ldots, X_{\ell-1}, i) = (g_0 X_0 g_1^{-1}, g_1 X_1 g_2^{-1}, \ldots, g_{\ell-1} X_{\ell-1} g_0^{-1}, g_0 i)$.

The action of the scalar subgroup $\mathbb{C}^*$ is trivial in the first action (but not the second) and so we can consider the first action, as a $PG$-action where $PG = G/\mathbb{C}^*$. Let $g$ and $pg$ be the Lie algebras of $G$ and $PG$, respectively.

2.3. Let $\eta^{\text{reg}} \subset \mathbb{C}^n$ be the affine open subvariety consisting of points $x = (x_1, \ldots, x_n)$ such that

(i) if $i \neq j$ then $x_i \neq \eta^m x_j$ for all $m \in \mathbb{Z}$,
(ii) for each $1 \leq i \leq n \ x_i \neq 0$.

This is the subset of $\mathbb{C}^n$ on which $\Gamma_n = S_n \wr C_\ell$ acts freely.
2.4. We can embed $h_{\text{reg}}$ into $\text{Rep}(Q, n\delta)$ by first considering a point $x = (x_1, \ldots, x_n) \in h_{\text{reg}}$ as a diagonal matrix $X = \text{diag}(x_1, \ldots, x_n)$ and then sending this to $\tilde{X} = (X, X, \ldots, X)$. We denote the image of $h_{\text{reg}}$ in $\text{Rep}(Q, n\delta)$ by $S$.

Let $T_\Delta$ be the subgroup of $G$ with elements $(T, T, \ldots, T)$ where $T$ is a diagonal matrix in $GL_n(\mathbb{C})$. Then $T_\Delta$ is the stabiliser of $S$. Now consider the mapping

$$\pi : G/T_\Delta \times h_{\text{reg}} \rightarrow \text{Rep}(Q, n\delta)$$

given by $\pi(gT_\Delta, x) = g \cdot X$. If we let $G$ act on $G/T_\Delta \times h_{\text{reg}}$ by left multiplication, then $\pi$ is a $G$-equivariant mapping.

**Lemma.** $\pi$ is an étale mapping with covering group $\Gamma_n$. In fact, its image $\text{Rep}(Q, n\delta)_{\text{reg}}$ is open in $\text{Rep}(Q, n\delta)$ and we have an isomorphism

$$\omega : G/T_\Delta \times \Gamma_n h_{\text{reg}} \rightarrow \text{Rep}(Q, n\delta)_{\text{reg}}.$$

**Proof.** Let $S = \{X : x \in h_{\text{reg}}\}$. Set $N_G(S) = \{g \in G : g \cdot S = S\}$ and $Z_G(S) = \{g \in G : g \cdot X = X \text{ for all } X \in S\}$.

Suppose that $g \cdot X = Y$ for some $X, Y \in S$. This implies that for each $0 \leq i \leq \ell - 1$

$$g_i \text{diag}(x)^i g_i^{-1} = \text{diag}(y)^i.$$

The hypotheses on $h_{\text{reg}}$ imply that both $\text{diag}(x)^i$ and $\text{diag}(y)^i$ are regular semisimple in $\text{Mat}_n(\mathbb{C})$. Two such elements are conjugate if and only if $g_i \in N_{GL_n(\mathbb{C})}(T) = T \cdot S_n$, where $T$ is the diagonal subgroup of $GL_n(\mathbb{C})$. Hence there exists $\sigma \in S_n$ such that for all $i$ we have $g_i = t_i \sigma$ for some $t_i \in T$, and for all $1 \leq r \leq n$ we have that $x_{\sigma(r)}^i = y_r^i$. Hence $x_{\sigma(r)} = \eta^m y_r$ for some $m_r \in \mathbb{Z}$. Now we find that $Y = g \cdot X$ implies that $\text{diag}(y)^i = t_1 t_{i+1}^{-1} \text{diag}(\eta^m y_r)$. Since $y_r \neq 0$ this shows that $t_{i+1} = \text{diag}(\eta^m_t) t_i$ for each $i$. Hence we find that $gT_\Delta = (\sigma, \text{diag}(\eta^m) \sigma, \ldots, \text{diag}(\eta^m)^{\ell - 1} \sigma)T_\Delta$.

In particular, if $X = Y$ we see from above that each $m_r = 0$, so that $Z_G(S) = T_\Delta$. Thus the group $\Gamma_n$ is isomorphic to $N_G(S)/Z_G(S)$ via the homomorphism that sends $(\eta^m, \ldots, \eta^m) \sigma$ to $(\sigma, \text{diag}(\eta^m) \sigma, \ldots, \text{diag}(\eta^m)^{\ell - 1} \sigma)T_\Delta$.

Now suppose that $\pi(gT_\Delta, x) = \pi(hT_\Delta, y)$. Then $(h^{-1} g) \cdot X = Y$ and so we see that $h^{-1} g \in N_G(S)$. This shows that $\pi$ is the composition

$$G/T_\Delta \times h_{\text{reg}} \longrightarrow G/T_\Delta \times \Gamma_n h_{\text{reg}} \longrightarrow \text{Rep}(Q, n\delta)_{\text{reg}}.$$

The first mapping factors out the action of $\Gamma_n$, and since $\Gamma_n$ acts freely on $h_{\text{reg}}$ this is an étale mapping. Hence, to prove the lemma, it suffices to show that $\text{Rep}(Q, n\delta)_{\text{reg}}$ is open in $\text{Rep}(Q, n\delta)$.

We claim first that $\text{Rep}(Q, n\delta)_{\text{reg}}$ is the set $O$ of representations of $Q$ that decompose into $n$ simple modules of dimension $\delta$ and whose endomorphism ring is $n$-dimensional. To prove this, observe that any element of $\text{Rep}(Q, n\delta)_{\text{reg}}$ is isomorphic to a representation of the form $X$ and so it decomposes into the $n$ indecomposable modules $X_1, \ldots, X_n$ of dimension $\delta$, where $X_i = (x_i, x_i, \ldots, x_i)$. (The condition $x_i \neq 0$ implies simplicity.) Now the representation $X_i$ is isomorphic to the representation $(1, 1, \ldots, 1, x_i^f)$. By hypothesis $x_i^f \neq x_j^f$ and so we deduce that the representations $x_i$ are pairwise non-isomorphic which ensures that the endomorphism ring of $X$ is $n$-dimensional. This proves the inclusion $\text{Rep}(Q, n\delta)_{\text{reg}} \subseteq O$. On the other hand, if $V$ belongs to $O$ then $\bar{V} = V_1 \oplus \ldots \oplus V_n$, where each $V_i$ is isomorphic to a representation
(1, 1, . . . , 1, v_i), for some non-zero scalars v_i. Moreover, since dim End(V) = n the v_i must be pairwise distinct. Now, let η_i be an ℓ-th root of v_i. Then V_i is isomorphic to (η_i, . . . , η_i). Therefore V is isomorphic to the representation X, where x = (η_1 , . . . , η_n).

Now we must show that O is open in Rep(Q, nδ). We use first the fact that the canonical decomposition of the vector nδ is δ + δ + · · · + δ, [13, Theorem 3.6]. This means that the representations of Rep(Q, nδ) whose indecomposable components all have dimension δ form an open set. Now, consider the morphism f from Rep(Q, δ) to C that sends the representation (λ_1, . . . , λ_ℓ) to the product λ_1 . . . λ_ℓ. The open set f^{−1}((C^*) consists of the simple representations of dimension vector δ. Therefore the subset of Rep(Q, nδ) consisting of representations which decompose as the sum of n simple representations of dimension vector δ is open. On the other hand, the function from Rep(Q, nδ) to N that sends a representation V to dim End(V) is upper semi-continuous. Thus \{ V : dim End(v) ≤ n \} is an open set in Rep(Q, nδ). Intersecting these two sets shows that O is open, as required. □

2.5. Now we are going to move from Q to Q_∞ and so we start with the following inclusion:

\{ ([gT_δ, x], i) : g_0^{−1}i is a cyclic vector for diag(x) \} ⊂ (G/ T_δ × Γ_n, \mathfrak{h}_{reg}) × C^n.

By applying ω^{−1} × id_C, the left-hand side corresponds to an open subset of Rep(Q, nδ) × C^n = Rep(Q_∞, ε). Call that set U_∞. This is a G-invariant open set since the G-action on triples is given by

h · ([gT_δ, x], i) = ([hgT_δ, x], h_0i)

so that g_0^{−1}i is cyclic for diag(x) if and only if (h_0g_0)^{−1}h_0i is cyclic for diag(x). Observe too that U_∞ is an affine variety. Indeed it is defined by the non-vanishing of the morphism

s : (G/ T_δ × Γ_n, \mathfrak{h}_{reg}) × C^n → C

which sends ([gT_δ, x], i) to (g_0^{−1}i) ∧ diag(x) · (g_0^{−1}i) ∧ · · · ∧ diag(x)^{n−1} · (g_0^{−1}i).

**Lemma.** The G-action on U_∞ is free and projection onto the second component

π_2 : U_∞ → \mathfrak{h}_{reg} / Γ_n

is a principal G-bundle.

**Proof.** Suppose that h · ([gT_δ, x], i) = ([gT_δ, x], i). Then [g^{−1}hgT_δ, x] = [T_δ, x] and so, by Lemma 2.4, g^{−1}hg ∈ T_δ.

We have that h_0i = i. Setting \tilde{i} = g_0^{−1}i implies that g_0^{−1}h_0g_0\tilde{i} = i'. By hypothesis \tilde{i}' is a cyclic vector for diag(x). Hence with respect to the standard basis \{e_j\}, \tilde{i}' decomposes as \sum λ_je_j, where each λ_j is non-zero. Therefore the only diagonal matrix that fixes \tilde{i}' is the identity element. In other words g_0^{−1}h_0g_0 = I_n. Since g^{−1}hg ∈ T_δ this implies that g^{−1}hg = id. Thus h = id and this proves that the action is free.

It remains to prove that each fibre of π_2 is a G-orbit. We take ([gT_δ, x], i) ∈ π_2^{−1}([x]). This equals g · ([T_δ, x], g_0^{−1}i). Now g_0^{−1}i is a cyclic vector for diag(x) and so it has the form \sum λ_je_j with each λ_j non-zero. Let \tilde{i} = diag(λ_1, . . . , λ_n) and consider
\[ \mathcal{L} = (t, \ldots , t) \in T_{\Delta}. \] We have
\[
[gT_{\Delta}, x], i = g\mu^{-1}([T_{\Delta}, x], g_0^{-1} i) = g\left([T_{\Delta}, x], \sum_{j=1}^{n} e_j \right).
\]
This proves that each fibre of \( \pi_2 \) is indeed a \( G \)-orbit.

\section{2.6}

Consider the representation space for the doubled quiver \( \overline{Q}_\infty \):
\[
\text{Rep}(\overline{Q}_\infty, \epsilon) = \{(X_0, \ldots , X_{\ell-1}, Y_0, \ldots , Y_{\ell-1}, i, j) : X_r, Y_r \in \text{Mat}_n(\mathbb{C}), i \in \mathbb{C}^n, j \in (\mathbb{C}^*)^n \} = \{(X, Y, i, j)\}.
\]
We can naturally identify it with \( T^* \text{Rep}(Q_\infty, \epsilon) \). The group \( G \) acts on the base and hence on the total space of the cotangent bundle. The resulting moment map
\[
\mu : \text{Rep}(\overline{Q}_\infty, \epsilon) \to g^* \cong g
\]
is given by
\[
\mu(X, Y, i, j) = [X, Y] + ij.
\]

**Theorem** (Gan–Ginzburg, Crawley–Boevey). Let \( \mu^{-1}(0) \) denote the scheme-theoretic fibre of \( \mu \).

1. \( \mu^{-1}(0) \) is reduced, equidimensional and a complete intersection.
2. The moment map \( \mu \) is flat.
3. \( \mathbb{C}[\mu^{-1}(0)]^G \cong \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\Gamma_n} \).

**Proof.** (i) This is proved in [7, Theorem 3.2.3].
(ii) This follows from [3, Theorem 1.1] and the dimension formula in [7, Theorem 3.2.3(iii)].
(iii) This is [4, Theorem 1.1]. \( \square \)

\section{2.7}

Let \( \mathcal{X} = \{(X, i) \in \text{Rep}(Q, n\delta) \times \mathbb{P}^{n-1} \} \). This space is the quotient of the (quasi-affine) open subvariety
\[
U = \{(X, i) : i \neq 0\} \subset \text{Rep}(Q_\infty, \epsilon)
\]
by the scalar group \( \mathbb{C}^* \). Thus there is an action of \( \text{PG} \) on \( \mathcal{X} \).

Since
\[
T^* \mathbb{P}^{n-1} = \{(i, j) : i \neq 0, ji = 0\}/\mathbb{C}^*
\]
we have
\[
T^* \mathcal{X} = \{(X, Y, i, j) \in \text{Rep}(\overline{Q}_\infty, \epsilon) : i \neq 0, ji = 0\}/\mathbb{C}^*.
\]
The \( \text{PG} \) action on \( \mathcal{X} \) gives rise to a moment map
\[
\mu_\mathcal{X} : T^* \mathcal{X} \to \mathfrak{p}^* \cong \mathfrak{p}^*.
\]
Let
\[ \mu_{\mathcal{X}}^{-1}(0) = \{(X, Y, i, j) \in \text{Rep}(\mathcal{O}_\infty, \epsilon) : i \neq 0, ji = 0, \ [X, Y] + ij = 0\}/\mathbb{C}^* \]
denote the scheme theoretic fibre of 0.

**Proposition.** There is an isomorphism \( \mathbb{C}[\mu_{\mathcal{X}}^{-1}(0)]^{PG} \cong \mathbb{C}[^* \oplus h^*]^{T_{h}}. \)

**Proof.** Consider the \( G \)-equivariant open subvariety of \( \mu^{-1}(0) \) given by the non-vanishing of \( i \). The variety \( \mu^{-1}(0) \) is determined by the conditions \( [X, Y] + ij = 0 \), and so if we take the trace of this equation then we see that \( 0 = T(ij) = Tr(ij) = ji \). Thus \( \{(X, Y, i, j) \in \text{Rep}(\mathcal{O}_\infty, \epsilon) : i \neq 0, ji = 0\} \cap \mu^{-1}(0) \) is an open subvariety of \( \mu^{-1}(0) \) and so, in particular, is reduced by Theorem 2.6(1). Hence factoring out by the action of \( \mathbb{C}^* \leq G \) shows that \( \mu_{\mathcal{X}}^{-1}(0) \) is reduced and that there is a \( PG \)-equivariant morphism
\[ \mu_{\mathcal{X}}^{-1}(0) \rightarrow \mu^{-1}(0)/\mathbb{C}^*. \]

This induces an algebra map
\[ \alpha : \mathbb{C}[\mu^{-1}(0)]^G \rightarrow \mathbb{C}[\mu_{\mathcal{X}}^{-1}(0)]^{PG}. \]

We now follow some of the proof of [7, Lemma 6.3.2]. Write \( O_1 \) for the conjugacy class of rank one nilpotent matrices in \( gl(n) \), and let \( \overline{O}_1 \) denote the closure of \( O_1 \) in \( gl(n) \). The moment map \( \nu : T^* [\mathfrak{gl}^{n-1}] \rightarrow \mathfrak{gl}(n)^* \cong \mathfrak{gl}(n) \) that sends \( (i, j) \) to \( ij \) gives a birational isomorphism \( T^* [\mathfrak{gl}^{n-1}] \rightarrow \overline{O}_1 \). Let \( J \subset \mathbb{C}[\mathfrak{gl}(n)] = \mathbb{C}[Z] \) be the ideal generated by all \( 2 \times 2 \) minors of the matrix \( Z \) and also by the trace function. Then \( J \) is a prime ideal whose zero scheme is \( \overline{O}_1 \) and the pullback morphism \( \nu^* : \mathbb{C}[\mathfrak{gl}(n)]/J \rightarrow \mathbb{C}[T^* [\mathfrak{gl}^{n-1}]] \)
is a graded isomorphism.

Now the moment map \( \mu_{\mathcal{X}} : T^* \mathcal{X} \rightarrow \mathfrak{g}^* \) factors as the composite
\[ T^* \mathcal{X} = T^* \text{Rep}(Q, n\delta) \times T^* [\mathfrak{gl}^{n-1}] \rightarrow T^* \text{Rep}(Q, n\delta) \times \overline{O}_1 \overset{\theta}{\rightarrow} \mathfrak{pg}^*, \]
where the first mapping is \( id \times \nu \) and the second mapping \( \theta \) sends \( (X, Y, Z) \) to \( [X, Y] + Z_0 \), where \( Z_0 \) indicates that we place the matrix \( Z \) on the copy of \( \mathfrak{gl}(n) \) associated to the vertex 0. We have a graded algebra isomorphism
\[ \mathbb{C}[T^* \text{Rep}(Q, n\delta)] \otimes \mathbb{C}[\mathfrak{gl}(n)]/J \rightarrow \mathbb{C}[T^* \mathcal{X}]. \]

Now write \( \mathbb{C}[X, Y, Z] = \mathbb{C}[T^* \text{Rep}(Q, n\delta) \times \mathfrak{gl}(n)] \), and let \( \mathbb{C}[X, Y, Z][X, Y] + Z_0 \) denote the ideal in \( \mathbb{C}[X, Y, Z] \) generated by all \( \ell \) matrix entries of the \( \ell \) matrices \( [X, Y] + Z_0 \). Let \( I \) denote the ideal \( \mathbb{C}[X, Y, Z][X, Y] + Z_0 \subset \mathbb{C}[X, Y] \otimes J \subset \mathbb{C}[X, Y, Z] \). From the above we have
\[ \mathbb{C}[\mu_{\mathcal{X}}^{-1}(0)] \cong \mathbb{C}[T^* \text{Rep}(Q, n\delta) \times \overline{O}_1]/\mathbb{C}[T^* \text{Rep}(Q, n\delta) \times \overline{O}_1]\theta^*(\mathfrak{gl}(n)) = \mathbb{C}[X, Y, Z]/I. \]

Define an algebra homomorphism \( r : \mathbb{C}[X, Y, Z] \rightarrow \mathbb{C}[X, Y] \) by sending \( P \in \mathbb{C}[X, Y, Z] \) to the function \( (X, Y) \mapsto P(X, Y, [X, Y]_0) \). Obviously \( r \) induces an isomorphism \( \mathbb{C}[X, Y, Z]/\mathbb{C}[X, Y, Z][X, Y] + Z_0 \cong \mathbb{C}[X, Y]/I_1 \), where \( I_1 \) is the ideal of \( \mathbb{C}[\text{Rep}(Q, n\delta)] = \mathbb{C}[X, Y] \) generated by the elements
\[ \sum_{h(a)=i} X_a X_{a'} - \sum_{t(a)=i} X_{a'} X_a \]
for all \( i \) not equal to zero. Observe that the linear function \( P : (X, Y, Z) \mapsto \text{Tr}Z = \text{Tr}([X, Y] + Z_0) \) belongs to the ideal \( \mathbb{C}[X, Y, Z][X, Y] + Z_0 \). We deduce that the mapping \( r \) sends \( \mathbb{C}[X, Y] \otimes J \) to the ideal generated by

\[
\text{rank} \left( \sum_{h(a) = 0} X_a X_{a'} - \sum_{t(a) = 0} X_{a'} X_a \right) \leq 1.
\]

Thus we obtain algebra isomorphisms

\[
\mathbb{C}[\mu_X^{-1}(0)] \cong \mathbb{C}[X, Y, Z]/I \cong \mathbb{C}[T^* \text{Rep}(Q, n\delta)]/I_2,
\]

where \( I_2 \) is the ideal generated by the elements

\[
\sum_{h(a) = i} X_a X_{a'} - \sum_{t(a) = i} X_{a'} X_a,
\]

for all \( 1 \leq i \leq \ell - 1 \), and

\[
\text{rank} \left( \sum_{h(a) = 0} X_a X_{a'} - \sum_{t(a) = 0} X_{a'} X_a \right) \leq 1.
\]

By [10, Theorem 1] the \( G \)-invariant (respectively \( PG \)-invariant) elements of \( \mathbb{C}[\text{Rep}(\overline{Q}_\infty, \epsilon)] \) (respectively \( \mathbb{C}[\text{Rep}(\overline{Q}, n\delta)] \)) are generated by traces along oriented cycles. Since all oriented cycles in \( \overline{Q} \) are oriented cycles in \( \overline{Q}_\infty \) we have a surjective composition of algebra homomorphisms

\[
\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\Gamma_n} \cong \mathbb{C}[\mu_X^{-1}(0)]^G \longrightarrow \mathbb{C}[\mu_X^{-1}(0)]^{PG} \longrightarrow \left( \frac{\mathbb{C}[\text{Rep}(\overline{Q}, n\delta)]}{I_2} \right)^{PG}, \quad (2.7.1)
\]

where the first isomorphism is Theorem 2.6(3). The left hand side is a domain of dimension \( 2 \dim \mathfrak{h} \) and so, to see that the mapping is an isomorphism, it suffices to prove that the right hand side also has dimension \( 2 \dim \mathfrak{h} \).

Let \( I_3 \) be the ideal of \( \mathbb{C}[\text{Rep}(\overline{Q}, n\delta)] \) generated by the elements

\[
\sum_{h(a) = i} X_a X_{a'} - \sum_{t(a) = i} X_{a'} X_a
\]

for all \( i \). This is the ideal of the zero fibre of the moment map for the \( PG \)-action on \( \text{Rep}(\overline{Q}, n\delta) \). This ideal contains \( I_2 \) since the rank condition on the matrices is implied by the commutator condition. Hence there is a surjective mapping

\[
\frac{\mathbb{C}[\text{Rep}(\overline{Q}, n\delta)]^{PG}}{I_2^{PG}} \longrightarrow \frac{\mathbb{C}[\text{Rep}(\overline{Q}, n\delta)]^{PG}}{I_3^{PG}}.
\]

We do not know yet whether the right hand side is reduced or not, but by [4, Theorem 1.1] the reduced quotient of the right hand side is the ring of functions of the variety \( (\mathfrak{h} \oplus \mathfrak{h}^*)/\Gamma_n \). As this variety has dimension \( 2 \dim \mathfrak{h} \) we deduce that the composition in (2.7.1) is an isomorphism, and hence that

\[
\mathbb{C}[\mu_X^{-1}(0)]^{PG} \cong \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\Gamma_n}.
\]

\[\square\]
Remark. In passing let us note that the commutativity of the following diagram

\[
\begin{array}{ccc}
\mathbb{C}[T^* \text{Rep}(Q, n\delta)] & \overset{\iota}{\longrightarrow} & \mathbb{C}[T^* \text{Rep}(Q, n\delta)] \otimes \mathbb{C}[T^* \mathbb{P}^{n-1}] \\
\downarrow \varphi^* & & \downarrow \iota \\
\mathbb{C}[T^* \text{Rep}(Q, n\delta)] \otimes \mathbb{C}[\mathcal{O}_1] & \longrightarrow & \mathbb{C}[T^* \text{Rep}(Q, n\delta)]/I_2
\end{array}
\]

where \(\iota(f) = f \otimes 1\), shows that \(\text{im} \, \iota\) maps surjectively onto \(\mathbb{C}[\mu^{-1}_X(0)]\).

3. Differential operators.

3.1. Symplectic reflection algebras. Let \(C_\ell\) be the cyclic subgroup of \(SL_2(\mathbb{C})\) generated by \(\sigma = \text{diag}(\eta, \eta^{-1})\). The vector space \(V = (\mathbb{C}^2)^n\) admits an action of \(S_n \ltimes C_\ell = S_n \ltimes (C_\ell)^n\). Here \((C_\ell)^n\) acts by extending the natural action of \(C_\ell\) on \(\mathbb{C}^2\), whilst \(S_n\) acts by permuting the \(n\) copies of \(\mathbb{C}^2\). For an element \(\gamma \in C_\ell\) and an integer \(1 \leq i \leq n\) we write \(\gamma_i\) to indicate the element \((1, \ldots, \gamma, \ldots, 1) \in (C_\ell)^n\) which is non-trivial in the \(i\)-th factor.

3.2. The elements \(S_n \ltimes C_\ell\) whose fixed points are a subspace of codimension two in \(V\) are called symplectic reflections. In this case their conjugacy classes are of two types.

\(S\) The elements \(s_{ij}\gamma_j \gamma_i^{-1}\) where \(1 \leq i, j \leq n\), \(s_{ij} \in S_n\) is the transposition that swaps \(i\) and \(j\), and \(\gamma \in C_\ell\).

\(C_\ell\) The elements \(\gamma_i\) for \(1 \leq i \leq n\) and \(\gamma \in C_\ell \setminus \{1\}\).

There is a unique conjugacy class of type \((S)\) and \(\ell - 1\) of type \((C_\ell)\) (depending on the non-trivial element we choose from \(C_\ell\)). We shall consider a conjugation invariant function from the set of symplectic reflections to \(\mathbb{C}\). We can identify it with a pair \((k, c)\) where \(k \in \mathbb{C}\) and \(c\) is an \((\ell - 1)\)-tuple of complex numbers: the function sends elements from \((S)\) to \(k\) and the elements \((\sigma^{m})_j\) to \(c_m\).

3.3. There is a symplectic form on \(V\) that is induced from \(n\) copies of the standard symplectic form \(\omega\) on \(\mathbb{C}^2\). If we pick a basis \(\{x_i, y_i\}\) for \(\mathbb{C}^2\) such that \(\omega(x_i, y_j) = 1\), then we can extend this naturally to a basis \(\{x_i, y_i : 1 \leq i \leq n\}\) of \(V\) such that the \(x_i\)'s and the \(y_i\)'s form Lagrangian subspaces and \(\omega(x_i, y_j) = \delta_{ij}\). We let \(TV\) denote the tensor algebra on \(V\): with our choice of basis this is just the free algebra on generators \(x_i, y_j\) for \(1 \leq i \leq n\). The symplectic reflection algebra \(H_{k,c}\) associated to \(S_n \ltimes C_\ell\) is the quotient of \(TV \ast (S_n \ltimes C_\ell)\) by the following relations:

\[
\begin{align*}
x_i x_j &= x_j x_i, & y_i y_j &= y_j y_i & \text{(for all } 1 \leq i, j \leq n), \\
y_i x_j - x_i y_j &= 1 + k \sum_{j \neq i} \sum_{\gamma \in C_\ell} s_{ij} \gamma_j \gamma_i^{-1} + \sum_{\gamma \in C_\ell \setminus \{1\}} c_{\gamma} \gamma_i & \text{(for } 1 \leq i \leq n), \\
y_i x_j - x_j y_i &= -k \sum_{m=0}^{\ell - 1} \eta^m s_{ij} \sigma^m_j (\sigma^m_j)^{-1} & \text{(for } i \neq j).\end{align*}
\]

(NB: my \(k\) is \(-k\) for Oblomkov.)
3.4. The spherical algebra. The symmetrising idempotent of the group algebra \( C(S_n \wr C_\ell) \) is given by
\[
e = \frac{1}{|S_n \wr C_\ell|} \sum_{w \in S_n \wr C_\ell} w.\]
The subalgebra \( eH_{k,c}e \) is denoted by \( U_{k,c} \) and called the spherical algebra. It will be our main object of study.

3.5. Rings of differential operators. Recall the definition of \( \mathcal{X} \) from 2.7. Let \( D_\mathcal{X}(nk) \) denote the sheaf of twisted differential operators on \( \mathcal{X} \) and let \( D(\mathcal{X}, nk) \) be its algebra of global sections. This is simply the tensor product \( D(\text{Rep}(Q, n\delta)) \otimes D_{\mathbb{P}^{n-1}}(nk). \) (The twisted differential operators on \( \mathbb{P}^{n-1} \) can be defined as follows. Let \( A_n = \mathbb{C}[x_1, \ldots, x_n, \partial_1, \ldots, \partial_n] \) be the \( n \)-th Weyl algebra. This is a graded algebra with \( \text{deg}(x_i) = 1 \) and \( \text{deg}(\partial_j) = -1. \) The degree zero component is the subring generated by the operators \( x_i \partial_j \) which, under the commutator, generate the Lie algebra \( \mathfrak{gl}(n). \) Call this subring \( R. \) Let \( E = \sum_{i=1}^n x_i \partial_i \in R \) be the Euler operator. Then \( D(\mathbb{P}^{n-1}, nk) \) is the quotient of \( R \) by the two-sided ideal generated by \( E - nk. \))

The group action of \( PG \) on \( \mathcal{X} \) differentiates to an action of \( \mathfrak{pg} \) on \( \mathcal{X} \) by differential operators. This gives a mapping
\[
\tau: \mathfrak{pg} \longrightarrow D_\mathcal{X}(nk). \tag{3.5.1}
\]
(One way to understand this is to start back with \( U \subset \text{Rep}(Q, \delta) \) and look at the \( G \) action on \( U. \) Differentiating the \( G \)-action gives an action of \( \mathfrak{g} \) by differential operators on \( U, \) \( \hat{\tau}: \mathfrak{g} \longrightarrow D_U. \) Since \( \mathbb{C}^* \) acts trivially on \( \text{Rep}(Q, n\delta) \) and by scaling on \( i \in \text{Rep}(Q, \delta), \) we find that \( \hat{\tau}(\text{id}) = 1 \otimes \mathbf{1}, \) where \( \text{id} = (I_n, I_n, \ldots, I_n) \in \mathbb{C} \subset \mathfrak{g}. \) Thus we get an action of \( \mathfrak{pg} \) on \( (D_U/D_U(1 \otimes \mathbb{C} - nk))^\mathbb{C}^* = D_\mathcal{X}(nk)). \)

3.6. Recall the Lie algebra \( \mathfrak{g} = \text{Lie}(G) \) and its quotient \( \mathfrak{pg} = \text{Lie}(PG) \) which is simply \( \mathfrak{g}/\mathbb{C} \cdot \text{id}, \) where \( \text{id} = (I_n, \ldots, I_n) \in \mathfrak{g}. \) Let \( \chi_\mathcal{X}: \mathfrak{g} \longrightarrow \mathbb{C} \) send an element \( (X) = (X_0, \ldots, X_{\ell-1}) \in \mathfrak{g} \) to
\[
\chi_\mathcal{X}(X) = \sum_{r=0}^{\ell-1} C_r \text{Tr}(X_r),
\]
where \( C_r = \ell^{-1}(1 - \sum_{m=1}^{\ell-1} \eta^{mr}c_m) \) for \( 1 \leq r \leq \ell - 1 \) and \( C_0 = \ell^{-1}(1 - \ell - \sum_{m=1}^{\ell-1} c_m). \) Observe that
\[
\chi_\mathcal{X}(\text{id}) = \text{Tr}(I_n) \sum_{r=0}^{\ell-1} C_r = n \sum_{r=0}^{\ell-1} \sum_{m=0}^{\ell-1} -\eta^{rm}c_m = 0.
\]
In particular \( \chi_\mathcal{X} \) is actually a character of \( \mathfrak{pg}. \)

Let \( \chi_k: \mathfrak{g} \longrightarrow \mathbb{C} \) send an element \( (X) = (X_0, \ldots, X_{\ell-1}) \) to \( \chi_k(c) = k \text{ Tr}(X_0). \) We shall be regularly using the character \( \chi_{k,c} \in \mathfrak{g}^* \) defined by \( \chi_{k,c} = \chi_c + \chi_k. \)

3.7. Let us recall Oblomkov’s deformed Harish–Chandra homomorphism [12]. By Lemma 2.4, \( S = \omega(\tau^{reg}/\Gamma_n) \) is a subset of \( \text{Rep}(Q, n\delta)^{reg} \) which is a slice for the
operator $E_i$ that sends $U$ onto its 0-th summand $[X_{i}]$. The Lie algebra $\mathfrak{g}$ acts on $W_k'$ by projection onto its 0-th summand $\mathfrak{g}(n)$, and then by the natural action of $\mathfrak{g}(n)$ on polynomials (so that $E_{ij}$ acts as $y_i \partial / \partial y_j$). With this action the identity matrix in $\mathfrak{g}(n)$ becomes the Euler operator $E$ which acts by multiplication by $-nk$. Thus we can make $W_k'$ a $\mathfrak{g}$-module by twisting $W_k'$ by the character $\chi_k$ since then id acts trivially. If we call this module $\hat{W}_k$, then $\hat{W}_k = W_k' \otimes \chi_k$. Now define $\text{Fun}'$ to be the space of functions on $\text{Rep}(Q, n\delta)$ of the form

$$f = \tilde{f} \prod_{i=0}^{\ell-1} \det(X_i)^{r_i},$$

where $\tilde{f}$ is a rational function on $\text{Rep}(Q, n\delta)^{\text{reg}}$ regular on $S$, $r_i = \sum_{j=0}^{i} C_j + \sigma$ and $\sigma = \ell^{-1} \sum_{j=0}^{\ell-1} s C_x$. Then $(\text{Fun}' \otimes W_k)^{\mathfrak{g}}$ is a space of $(\mathfrak{g}, \chi_i)$-semi-invariant functions defined on a neighbourhood of $S$ that take values in $W_k$. This space is a free $\mathbb{C}[\mathfrak{h}^{\text{reg}}]^{\mathfrak{g}}$-module of rank 1, the isomorphism being given by restriction to $S$. (Note that the determinant of an element of the form $(X_1, \ldots, X_\ell)$ is $\det(X)^{\sum r_i} = 1$ as $\sum r_i = 0$.) Any $\mathfrak{g}$-invariant differential operator $D$ acts on such a function $f$. Oblomkov defines his homomorphism to be the restriction of $D(f)$ to $S$.

### 3.8

We can view the procedure above in terms of $\text{Rep}(Q, \epsilon)$. By Lemma 2.5 we use $S_\infty = S \times (1, \ldots, 1) \in U_\infty$ as a slice for the $G$-action. The space $S \times (\mathbb{C}^*)^n$ is a closed subset of $U_\infty$ since the condition that $i$ be cyclic for $\text{diag}(x_1, \ldots, x_n)$ is equivalent to $i \in (\mathbb{C}^*)^n$. Thus functions on a neighbourhood of $S_\infty$ in $U_\infty$ can be identified with functions from a neighbourhood of $S$ taking values in $W_k$. In particular, we can consider elements on $(\text{Fun}' \otimes W_k)^{\mathfrak{g}}$ first as $(\mathfrak{g}, \chi_{k,c})$-semi-invariant functions from a neighbourhood of $S$ taking values in $W_k'$ and hence as $(\mathfrak{g}, \chi_{k,c})$-semiinvariant functions on an open set in a neighbourhood of $S_\infty$. We can apply any element of $D \in D(U_\infty)^{\mathfrak{g}}$ to these $(\mathfrak{g}, \chi_{k,c})$-semiinvariant functions and then restrict to $S_\infty$ to get a homomorphism

$$\bar{\mathfrak{g}}_{k,c} : D(U_\infty)^{\mathfrak{g}} \rightarrow D(\mathfrak{h}^{\text{reg}} / \Gamma_n).$$

### 3.9

Since $\text{Rep}(Q, \epsilon) = \text{Rep}(Q, n\delta) \times \mathbb{C}^n$ there is a mapping

$$\mathfrak{G} : D(\text{Rep}(Q, n\delta))^{\mathfrak{g}} \rightarrow D(U_\infty)^{\mathfrak{g}}$$

that sends $D \in D(\text{Rep}(Q, n\delta))^{\mathfrak{g}}$ to $(D \otimes 1)$. Oblomkov's homomorphism is $\bar{\mathfrak{g}}_{k,c} \circ \mathfrak{G}$.

### 3.10

Differentiating the $G$-action on $U_\infty$ gives a Lie algebra homomorphism $\tilde{\tau} : \mathfrak{g} \rightarrow \text{Vect}(U_\infty)$ which we extend to an algebra map

$$\tilde{\tau} : U(\mathfrak{g}) \rightarrow D(U_\infty).$$

By Lemma 2.5, $U_\infty$ is a principal $G$-bundle over $\mathfrak{h}^{\text{reg}} / \Gamma_n$, and so a generalisation of [14, Corollary 4.5] shows that the kernel of $\bar{\mathfrak{g}}_{k,c}$ is $(D(U_\infty)(\tilde{\tau} - \chi_{k,c})(\mathfrak{g}))^g$. Moreover, since the finite group $\Gamma_n$ acts freely on $\mathfrak{h}^{\text{reg}}$ we can identify $D(\mathfrak{h}^{\text{reg}} / \Gamma_n)$ with $D(\mathfrak{h}^{\text{reg}})^{\Gamma_n}$. 


3.11. Recall that
\[ D_X(nk) \cong \left( \frac{D_U}{D_U(\bar{\tau} - \chi_k)(C \cdot \text{id})} \right)^C. \]
Hence we have
\[ \left( \frac{D_U}{D_U(\bar{\tau} - \chi_k, c)(\mathfrak{g})} \right)^G \cong \left( \frac{D_X(nk)}{D_X(nk)(\tau - \chi_c)(\mathfrak{g})} \right)^{\text{PG}}, \tag{3.11.1} \]
where \( U = \{(X, i) : i \neq 0\} \subset \text{Rep}(Q_{\infty}, n\delta) \) as in 2.7. We consider the restriction mapping \( D_U \rightarrow D(U_{\infty}) \). Composing the global sections of the isomorphism above with this restriction and the homomorphism \( \mathcal{F}_{k, c} \) gives
\[ R_k'_{k, c} : \left( \frac{D(\mathfrak{X}, nk)}{D(\mathfrak{X}, nk)(\tau - \chi_c)(\mathfrak{g})} \right)^{\text{PG}} \rightarrow D(\mathfrak{h}_{\text{reg}})^{\Gamma_{\nu}}. \]

3.12. Let
\[ \delta_{k, c}(x) = \delta^{-k-1}\delta_{\Gamma}, \]
where \( \delta = \prod_{1 \leq i < j \leq n}(x_i^f - x_j^f) \) and \( \delta_{\Gamma} = \prod_{i=1}^{n} x_i \). Define a twisted version of \( R_{k, c}' \) above by
\[ R_k_{k, c}(D) = \delta_{k, c}^{-1} \circ R_{k, c}'(D) \circ \delta_{k, c} \]
for any differential operator \( D \).

3.13. Our main result is as follows.

**Theorem.** For all values of \( k \) and \( c \), the homomorphism \( R_{k, c} \) has image \( \text{im} \theta_{k, c} \). In particular we have an isomorphism
\[ \theta_{k, c}^{-1} \circ R_{k, c} : \left( \frac{D(\mathfrak{X}, nk)}{D(\mathfrak{X}, nk)(\tau - \chi_c)(\mathfrak{g})} \right)^{\text{PG}} \rightarrow U_{k, c}. \]

**Proof.** Let us abuse notation by writing \( U_{k, c} \) for the image of \( U_{k, c} \) in \( D(\mathfrak{h}_{\text{reg}})^{\Gamma_{\nu}} \) under \( \theta_{k, c} \). Since \( \mathfrak{X} = \text{Rep}(Q, n\delta) \times \mathbb{P}^{n-1} \), there is a mapping given by
\[ D(\text{Rep}(Q, n\delta))^{\text{PG}} \rightarrow D(\mathfrak{X}, nk)^{\text{PG}} \rightarrow D(\mathfrak{h}_{\text{reg}})^{\Gamma_{\nu}}, \]
that sends \( D \in D(\text{Rep}(Q, n\delta))^{\text{PG}} \) to \( R_{k, c}(D \otimes 1) \). Recall \( \tau \) from (3.5.1). Since \( \text{gr} \tau = \mu_{\mathfrak{X}}^* \), we have an inclusion \( \text{gr}(D(\mathfrak{X}, nk))\mu_{\mathfrak{X}}^*(\mathfrak{g}) \subseteq \text{gr}(D(\mathfrak{X}, nk)(\tau - \chi_c)(\mathfrak{g})) \). This gives a graded surjection
\[ p : \left( \frac{\text{gr} D(\mathfrak{X}, nk)}{\text{gr}(D(\mathfrak{X}, nk))\mu_{\mathfrak{X}}^*(\mathfrak{g})} \right)^{\text{PG}} \rightarrow \text{gr} \left( \frac{D(\mathfrak{X}, nk)}{D(\mathfrak{X}, nk)(\tau - \chi_c)(\mathfrak{g})} \right)^{\text{PG}}. \]
By Remark 2.7 the composition
\[
\text{gr } D(\text{Rep}(Q, n\delta))^{PG} \to \text{gr } D(X, nk)^{PG} \to \left( \frac{\text{gr } D(X, nk)}{\text{gr } (D(X, nk))^{\mu_X^*(g)}} \right)^{PG}
\]
is surjective. Thus the homomorphism
\[
D(\text{Rep}(Q, n\delta))^{PG} \to \left( \frac{D(X, nk)}{D(X, nk)(\tau - \chi_c)(g)} \right)^{PG}
\]
is also surjective. In particular, by 3.9 this implies that the image of \( \mathcal{R}_{k,c} \) equals the image of Oblomkov’s Harish–Chandra homomorphism, which, by [12, Theorem 2.5], is \( U_{k,c} \).

Thus we have a filtered surjective homomorphism
\[
\mathcal{R}_{k,c} : \left( \frac{D(X, nk)}{D(X, nk)(\tau - \chi_c)(g)} \right)^{PG} \to U_{k,c}
\]
Thus the dimension of the left hand side is at least \( 2 \dim \mathfrak{h} = \dim U_{k,c} \). By Proposition 2.7
\[
\left( \frac{\text{gr } D(X, nk)}{\text{gr } (D(X, nk))^{\mu_X^*(g)}} \right)^{PG} \cong \mathbb{C}[\mu_X^{-1}(0)]^{PG} \cong \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^\ast]^{T^\ast}.
\]
Hence \( p \) is a surjection from a domain of dimension \( 2 \dim \mathfrak{h} \) onto an algebra of dimension at least \( 2 \dim \mathfrak{h} \) and so is an isomorphism. It follows that \( (D(X, nk)/D(X, nk)(\tau - \chi_c)(g))^{pg} \) is a domain of dimension \( 2 \dim \mathfrak{h} \). This implies that \( \mathcal{R}_{k,c} \) is an isomorphism.  

\[ \square \]


4.1. The Holland-Schwarz Lemma. We wish to understand the space
\[
\frac{D(\text{Rep}(Q_\infty, \epsilon))}{D(\text{Rep}(Q_\infty, \epsilon))(\hat{\tau} - \chi_{k,c})(g)}.
\]
As we observed in the proof of Theorem 3.13 there is a natural surjective homomorphism
\[
\frac{\text{gr } D(\text{Rep}(Q_\infty, \epsilon))}{\text{gr } (D(\text{Rep}(Q_\infty, \epsilon))^{\mu^*(g)})} \to \text{gr } \left( \frac{D(\text{Rep}(Q_\infty, \epsilon))}{D(\text{Rep}(Q_\infty, \epsilon))(\hat{\tau} - \chi_{k,c})(g)} \right).
\]
It turns out that this is an isomorphism.

LEMA (Schwarz, Holland). The homomorphism (4.1.1) is an isomorphism of \( \mathbb{C}[T^* \text{Rep}(Q_\infty, \epsilon)] \)-modules.

Proof. This is [9, Lemma 2.2] since, by Theorem 2.6 (2), the moment map \( \mu \) is flat.  

\[ \square \]
4.2. This lets us prove the first part of the isomorphism in the statement of Theorem 1.4.

**Lemma.** There is an algebra isomorphism

\[
\left( \frac{D(\text{Rep}(Q, \epsilon))}{D(\text{Rep}(Q, \epsilon))(\hat{\tau} - \chi_{k,e})(g)} \right)^G \cong \left( \frac{D(\mathcal{X}, nk)}{D(\mathcal{X}, nk)(\tau - \chi_{e})(pg)} \right)^{PG}.
\]

**Proof.** We have a natural $pg$-equivariant mapping

\[
D(\text{Rep}(Q, \epsilon))^C \rightarrow D^C_U \rightarrow D(\mathcal{X}(nk))
\]

which induces a homomorphism

\[
D(\text{Rep}(Q, \epsilon))^G \rightarrow \left( \frac{D(\mathcal{X}, nk)}{D(\mathcal{X}, nk)(\tau - \chi_{e})(pg)} \right)^{PG}.
\]

This is surjective since, as we observed in the proof of Theorem 3.13, the image of $D(\text{Rep}(Q, \epsilon))^G$ spans the right hand side. By (3.11.1) the kernel of this homomorphism includes the ideal $(D(\text{Rep}(Q, \epsilon)), \epsilon)(\hat{\tau} - \chi_{k,e})(g))^G$. Hence we have a surjective homomorphism

\[
\left( \frac{D(\text{Rep}(Q, \epsilon))}{D(\text{Rep}(Q, \epsilon))(\hat{\tau} - \chi_{k,e})(g)} \right)^G \rightarrow \left( \frac{D(\mathcal{X}, nk)}{D(\mathcal{X}, nk)(\tau - \chi_{e})(pg)} \right)^{PG}.
\]

(4.2.1)

By Lemma 4.1 and Proposition 2.7, there is an isomorphism

\[
\left( \frac{\text{gr} D(\text{Rep}(Q, \epsilon))}{\text{gr} D(\text{Rep}(Q, \epsilon))(\hat{\tau} - \chi_{k,e})(g)} \right)^G \cong \left( \frac{\text{gr} D(\text{Rep}(Q, \epsilon))}{\text{gr} D(\text{Rep}(Q, \epsilon))\mu^*(\mathfrak{g})} \right)^G = \mathbb{C}[\mu^{-1}(0)]^G = \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^G.
\]

This shows that the algebra on the left is a domain of dimension $2\dim \mathfrak{h}$ and so (4.2.1) is also injective, as required. $\square$

4.3. Shifting. The previous two lemmas provide us with an interesting series of bimodules. Given a character $\Lambda$ of $G$ we define

\[
B_{k,e}^\Lambda = \left( \frac{D(\text{Rep}(Q, \epsilon))}{D(\text{Rep}(Q, \epsilon))(\hat{\tau} - \chi_{k,e})(g)} \right)^\Lambda
\]

to be the set of $(G, \Lambda)$-semiinvariants. Thanks to Lemma 4.2 and Theorem 3.13 this is a right $U_{k,e}$-module. Now observe that if $\chi \in \mathfrak{g}$ and $D \in D(\text{Rep}(Q, \epsilon))^\Lambda$ then

\[
[\tau(x), D] = \lambda(x)D,
\]
where $\lambda = d \Lambda$. Hence $B_{k,c}^\Lambda$ is a left \((D(\text{Rep}(Q_{\infty}, \epsilon))/D(\text{Rep}(Q_{\infty}, \epsilon))(\hat{\tau} - \chi_{k,c} - \lambda)(g))^G\)-module and so tensoring sets up a \textit{shift functor}

$$S_{k,c}^\Lambda : \left( \frac{D(\text{Rep}(Q_{\infty}, \epsilon))}{D(\text{Rep}(Q_{\infty}, \epsilon))(\hat{\tau} - \chi_{k,c} - \lambda)(g)} \right)^G \text{-mod} \rightarrow \left( \frac{D(\text{Rep}(Q_{\infty}, \epsilon))}{D(\text{Rep}(Q_{\infty}, \epsilon))(\hat{\tau} - \chi_{k,c} - \lambda)(g)} \right)^G \text{-mod}. $$

4.4. The character group of $G$ is isomorphic to $\mathbb{Z}^\ell$ via

$$(i_0, \ldots, i_{\ell-1}) \mapsto \left( (g_0, \ldots, g_{\ell-1}) \mapsto \prod_{r=0}^{\ell-1} \det(g_r)^{i_r} \right).$$

Corresponding to the standard basis element $\epsilon_i$ is the character $\chi_i$ of $g$ that sends $X \in g$ to $\text{Tr}(X_i)$.

**Lemma.** The bimodule $B_{k,c}^{\epsilon_i}$ above is a $(U_{k',c'}, U_{k,c})$-bimodule, where $k' = k + 1$ and $c' = c + (1 - \eta^{-i}, 1 - \eta^{-2i}, \ldots, 1 - \eta^{-(\ell-1)i})$.

**Proof.** Recall that $(k, c)$ corresponds to the character of $g$ we called $\chi_{k,c}$ which is defined as

$$\chi_{k,c}(X) = (C_0 + k) \text{Tr}(X_0) + \sum_{j=1}^{\ell-1} C_j \text{Tr}(X_j),$$

where $C_r = \ell^{-1}(1 - \sum_{m=1}^{r-1} \eta^{mr} c_m)$ for $1 \leq r \leq \ell - 1$ and $C_0 = \ell^{-1}(1 - \ell - \sum_{m=1}^{\ell-1} c_m)$. We need to calculate $(k', c')$ so that $\chi_{k,c} + \chi_i = \chi_{k',c'}$. We have

$$\chi_{nk,c} + \chi_i(X) = (C_0 + k) \text{Tr}(X_0) + \text{Tr}(X_j) + \sum_{j=1}^{\ell-1} C_j \text{Tr}(X_j)$$

$$= (C_0' + k') \text{Tr}(X_0) + \sum_{j=1}^{\ell-1} C_j' \text{Tr}(X_j).$$

Calculation shows that $k' = k + 1$ and that if $i = 0$ then $C_j' = C_j$ and otherwise

$$C_j' = C_j + \begin{cases} -1 & \text{if } j = 0, \\ 1 & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

These unpack to give $c_m' = c_m + 1 - \eta^{-mi}$.

4.5. **Question.** Thus for each $0 \leq i \leq \ell - 1$ we have a \textit{shift functor}

$$S_i : U_{k,c} \text{-mod} \rightarrow U_{k+1,c'} \text{-mod}$$

where $c'$ is as above. When is this an equivalence of categories?
REMARK. Shift functors are also constructed in [1] and [15]. Hopefully they agree with the functors here.

REFERENCES