# Consistent bilateral assignment

**Herve Moulin**
*University of Glasgow and Higher School of Economics, St Petersburg*

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**Abstract**

In the bilateral assignment problem, source $a$ holds the amount $r_a$ of resource of type $a$, while sink $i$ must receive the total amount $x_i$ of the various resources. We look for assignment rules meeting the powerful separability property known as Consistency: "every subassignment of a fair assignment is fair". They are essentially those rules selecting the feasible flow minimizing the sum $\sum_{i,a} W(y_{ia})$, where $W$ is smooth and strictly convex.

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## 1 The problem and the punchline

A preference relation over profiles of variables $(w_1, w_2, \cdots)$ is *separable* in those variables, if a change affecting only a subset of the variables can be evaluated in complete ignorance of the values of the other unchanged variables, as long as we know they do not change. This has obvious and well known advantages in terms of computational simplicity and the ability to decentralize choice. For tractability, most utility functions of empirical economic models are separable, and the same applies to standard definitions of social welfare as a preference relation over individual welfares (e.g., [18]).

Fair division problems such as the one we study here call for a *division rule*, solving each problem in a given family of problems. Such rules are not necessarily derived from maximizing a social preference over the feasible allocations so the above notion of separability does not apply directly; however the widely studied properties called *Consistency* and *Inverse Consistency* (explained in the next paragraph, and defined in Section 3) convey a very similar idea without invoking a preference relation, social or otherwise. As we recall below, in the simple *rationing problem* any reasonable consistent rule actually maximizes a separable preference relation. Our results extend this connection between the two concepts to the bilateral assignment problem.

Bilateral assignment is one of the oldest models in Operations Research. In the shipping interpretation favoured in the flow-graph literature ([2]), there is a set $N$ of retailers (aka *sinks*) indexed by $i$, and a set $A$ of warehouses (aka *sources*) indexed by $a$; warehouse $a$ stores the amount $r_a$ of some commodity, while retailer $i$ can absorb the total amount $x_i$ of commodity (irrespective of the warehouse they come from). Critically, the assignment problem is *balanced*: $\sum_N x_i = \sum_A r_a$. In the simple version we discuss here, there are no constraints on the flow $y_{ia}$ from warehouse $a$ to retailer $i$. Common instances of the assignment problem include dividing jobs (measured in hours) between workers, allocating students of different types (ethnic, academic) to schools, etc.. *Consistency* of a division rule means that, if it selects a flow matrix $[y_{ia}]_{N\times A}$ to ship goods from warehouses in $A$ to retailers in $N$, it will also select the submatrix $[y_{ia}]_{N'\times A'}$ in order to ship the corresponding goods from the subset $A'$ of warehouses to the subset $N'$ of retailers. In the words of Balinski and Young ([5]), "every part of a fair allocation must be fair". *Converse Consistency* says that in order to check that the initial (large) flow matrix is fair, it is enough to check that every $2\times 2$ submatrix is fair. Taken together, Consistency and its inverse make *local* fairness equivalent to *global* fairness, with all the decentralization advantages of separable preferences.

Which assignment we deem the "fairest" depends of course upon the context. The most common principle is *proportionality.* For instance the numerous measures of segregation discussed in the literature (e.g., [10]) view the proportional assignment as the ideal of zero segregation: each school should have the same mix of the various types of students, each type of job should have the same gender balance, and so on. But other fairness principles are out there as well: if the entry $y_{ia}$ is the load of a truck from the warehouse $a$ to the retailer $i$, we may want to equalize the loads across all trucks, perhaps for efficiency reasons; full equality is not feasible as soon as the $x_i$ or the $r_a$ are not identical, so we must pick the "most egalitarian" assignment according to some approximation criterion: for instance we can minimize the variance of the entries $y_{ia}$, as in the Minimal Trade model of the transportation literature (see [11]).

To fix ideas, and stress the connections of our model with the extensive literature on dividing a single commodity according to individual claims, the *rationing* or *bankruptcy* models (on which more below), we speak of the set of *resources* $A$ and *agents* $N$. In this interpretation the agents observe passively the assignment selected by the benevolent manager, and care about its fairness; but unlike in the rationing problem we take as given the total resource allocated to each agent, so the agents care only about the distribution of the given total between the different resources. An alternative interpretation, in the spirit of the Operations Research literature in the two previous paragraphs, is the one-person story of the manager who is bound by certain standards of rationality like Consistency and looks for a systematic principle to solve any assignment problem; in that view the individual retailers of the shipping story do not care anymore about the origin of their deliveries.

Our punchline is that if the manager accepts the Consistency axiom, and two additional fairness requirements[1], he will end up using one of a fairly simple family of division rules.

The first fairness axiom is the standard "horizontal equity", requiring a symmetric treatment of the agents $i$, and the same for the resources $a$. The second is a natural yet powerful *Monotonicity* property: transferring resources from source $a$ to source $b$ results in a (weak) increase of the flow $y_{ib}$ to each agent $i$ (and a similar statement for transferring capacity from agent $i$ to agent $j$). Together with Consistency, these properties essentially characterizes the class of assignment rules minimizing a separable sum $\sum_{N\times A} W(y_{ia})$, where $W$ is a smooth and strictly convex (anti)welfare function. We call these assignment rules *welfarist*. This result is strikingly similar to the famous Debreu-Gorman theorem ([7]) showing that, essentially, a separable preference relation as in the first paragraph above is represented by an additively separable utility $U_1(w_1) + U_2(w_2) + \cdots$. The "essential" qualification, in our model as in Debreu-Gorman, refers to additional boundary and continuity assumptions that eliminate borderline solutions.

From a technical standpoint Young's Theorem ([25]) on the parametric representation of rationing rules is the most relevant to our result, because its statement is similar; moreover it is a key component of our proof. More on this in the next section, reviewing the relevant literature.

Section 3 defines the model and the Consistency axioms. Welfarist assignment rules are introduced in Section 4. They are consistent (CSY) as well as inverse consistent (CSY$^{-1}$). A natural one-dimensional subset of such rules obtains by imposing *Scale Invariance*: then $W$ is mostly a power function $W^q(z) = \pm z^q$ (Lemma 2). This family includes the minimal variance rule ($W^2(z) = z^2$), and the proportional rule ($W^1(z) = z\ln(z)$). Section 5 introduces the monotonicity and boundary conditions used in our main result: the characterization theorem is stated in Section 6, where we also give a sketch of its proof. Section 7 gathers concluding comments and some open questions. All substantial proofs are in Section 8.

## 2 Related literature

The Consistency property appears first (under the name Uniformity) in the work of Balinski and Young on apportionment ([5]), then in a paper by Balinski and Demange on the proportional approximation of a given assignment ([4]). It has been applied to a variety of models, from solutions of cooperative games to competitive equilibrium, the allocation of indivisible goods, matching, and more. See [22], [23] for an extensive discussion of the CSY and CSY$^{-1}$ concepts and their versatile applications.

One instance where Consistency has proven particularly useful is the familiar *rationing* model: the amount $r$ of a *single* resource must be divided between agents labeled $i$ who each have a legitimate claim $x_i$, but there is not enough

[1] Consistency has no bearing on fairness: for instance it is compatible with rules giving priorities to certain agents.

resource to meet all claims: $r < \sum_N x_i$. Aumann and Maschler ([3]) use CSY to justify the rationing rule known as "Talmudic", then Young ([25]) shows that a symmetric, continuous, and consistent rationing rule solves a separably additive program of the form $\arg\min_y \sum_i V(x_i, y_i)$. This resembles our own welfarist rules solving $\arg\min_y \sum_{ia} W(y_{ia})$, though in our model the Consistency assumption applies in two dimensions hence has more bite. Two surveys on the rationing model are [14] and [21].

The companion paper [15] develops a capacited version of the assignment problem, where a different exogenous constraint (lower and upper bounds) applies to each entry $y_{ia}$. Taking for granted that the proportional assignment should be selected if it is compatible with the constraints, it shows that minimizing the total entropy $\sum_{ia} En(y_{ia})$ is the uniquely consistent way to accomodate the constraints, if we do not endow them (the constraints) with any normative content.

The model of *bipartite rationing*, recently introduced by this author and Sethuraman ([16], [17]) generalizes both the standard one-resource rationing model two paragraphs above and the assignment model here. Each retailer $i$ in $N$ has a total demand $x_i$, each warehouse $a$ in $A$ has capacity $r_a$, and we assume $\sum_N x_i \geq \sum_A r_a$; moreover transfers are confined to a given bipartite graph. The *bipartite proportional method* in [16] generalizes the proportional rule (2) here: assuming as in the previous paragraph that the rule must be proportional in any one-resource problem, it turns out that, once again, the only consistent way to extend it to multiple resources is by minimizing total entropy of the flow.

In the current paper we are entirely agnostic about the assignment rule in the one-resource case, and our welfarist assignment rules include much more than the proportional rule. As explained in Section 7.2, it is a reasonable conjecture that our main characterization result can be extended to the bipartite rationing model.

Erlanson and Szwagrzak ([8]) discuss a generalization of standard rationing with multiples resources, where each agent has a claim on each type of resource (unlike here). They characterize a family of welfarist rationing rules similar to ours by requiring both Consistency and Independence of Irrelevant Alternatives. IIA is the familiar route to choice rules maximizing some preference relation, so the combination of CSY and IIA is very powerful. It does not apply to our model because the sets of feasible assignments in two distinct problems are never nested.

# 3 Assignment problems and rules

The sets $N$ of agents (or sinks) and $A$ of resources(or sources) are both finite with generic elements $i$ and $a$ respectively. We use the notation $z_D = \sum_{d \in D} z_d$, and write $iA$, $Na$ instead of $\{i\} \times A$ and $N \times \{a\}$ when this causes no confusion.

An *assignment problem* $P = (N, A, x, r)$ *specifies*
*the total allocation* $x_i$ *of each agent, so* $x \in \mathbb{R}_+^N$

*the endowment $r_a$ of the resource of type $a$, so $r \in \mathbb{R}_+^A$ meeting the budget balance equation $x_N = r_A = s$.*

A *feasible assignment* is a matrix $y \in \mathbb{R}_+^{N \times A}$ such that $y_{iA} = x_i$ for all $i \in N$, and $y_{Na} = r_a$ for all $a \in A$. We write $\mathcal{P}$ for the set of assignment problems, and $\Phi(P)$ for the set of feasible assignments of problem $P$. Note that $\mathcal{P}$ contains problems of arbitrary dimensions $|N|$ and $|A|$.

An *assignment rule* selects $y \in \Phi(P)$ for every $P \in \mathcal{P}$. We restrict attention to rules treating all agents, and all resources, symmetrically. We also require that a small change in the demands $x_i, r_a$ should have only a small influence on the solution.

If $\sigma$ is a bijection of $N$, from the new name $i$ to the old name $\sigma(i)$, and $y \in \Phi(P)$ is an assignment with the old names, the same assignment with the new names is $y^\sigma$: $y^\sigma_{ia} = y_{\sigma(i)a}$; define similarly $x^\sigma$, and write $P^\sigma = (N, A, x^\sigma, r)$.

**Definition 1** *An assignment rule $F$ chooses for every $P = (N, A, x, r) \in \mathcal{P}$ an assignment $F(P) = y \in \Phi(P)$, and meets the following properties:*

- **Symmetry in $N$ ($N$-SYM)**: *for any $P \in \mathcal{P}$ and bijection $\sigma$ of $N$, $F(P)^\sigma = F(P^\sigma)$*
- **Symmetry in $A$ ($A$-SYM)**: *same property after exchanging the roles of $N$ and $A$*
- **Continuity (CONT)** *of the mapping $\mathcal{P} \ni (x, r) \to F(P)$, for any fixed $N, A$*

*We write $\mathcal{F}$ for the set of assignment rules.*

The next property is critical. Given a rule $F$, a pair $N, A$, and a matrix $y \in \mathbb{R}_+^{N \times A}$, we say that $y$ is *$F$-fair* if the rule $F$ chooses $y$ in the problem $(N, A, x, r)$ where $x_i = y_{iA}$ for all $i$, and $r_a = y_{Na}$ for all $a$.

- **Consistency (CSY):** *every submatrix of an $F$-fair matrix is $F$-fair*

If $x_i = 0$ for some $i$, then $y_{ia} = 0$ for all $a$, so in the submatrix obtained after deleting row $i$ all sums in rows and columns are the same. Consistency allows us to simply delete $i$ altogether; similarly if $r_a = 0$. Thus we can always assume for convenience $x, r \gg 0$.

Finally, the property of Converse Consistency ensures that global fairness can be verified locally. It is an appealing property of the rules we discuss, but it will not be used in the characterization result.

- **Inverse Consistency (CSY$^{-1}$):** *if every $2 \times 2$ submatrix of the $N \times A$ matrix $y$ is $F$-fair, then $y$ is $F$-fair as well*

# 4 Welfarist assignment rules

We introduce a rich family of consistent assignment rules. Fix a strictly convex and smooth function $W$ on $]0, \infty[$, of which the derivative $W'$ is continuous and strictly increasing on $]0,\infty[$. Define $W(0)=\lim_{z\to+0}W(z)$, which could be $+\infty$, and $W'(0)=\lim_{z\to 0}W'(z)$, which could be $-\infty$.

**Definition 2** *The $W$-welfarist assignment rule $F^W$ ($W$-rule for short) selects at any $P\in\mathcal{P}$ the assignment*

$$F(P)=\arg\min_{y\in\Phi(P)}\sum_{N\times A}W(y_{ia}) \tag{1}$$

It is well defined because $W$ is strictly convex and $\Phi(P)$ is convex and compact.

**Lemma 1** *The assignment rule $F^W$ is symmetric, continuous, consistent, and inverse consistent.*

**Proof** Continuity follows from Berge Theorem and Symmetry is clear. For CSY fix an $F^W$-fair matrix $y$ and $P=(N,A,x,r)$ the corresponding assignment problem. If the submatrix $y_{[M\times B]}\in\mathbb{R}_+^{M\times B}$ where $M\subseteq N,B\subseteq A$ is not $F^W$-fair, there is some $y'_{[M\times B]}$ with identical sums in all rows and columns such that $\sum_{M\times B}W(y'_{ia})<\sum_{M\times B}W(y_{ia})$, so that $(y'_{[M\times B]},y_{[N\times A\setminus M\times B]})$ is feasible in $P$ and improves the objective. Finally Inverse Consistency is a consequence of statement *iii)* in Lemma 4 below, as explained in Section 6.$\blacksquare$

We give some examples. As discussed in the introduction, the most natural assignment rule is *proportional*: it guarantees that in any two rows (resp. columns) the proportions of any two resources (resp. agents' shares) are identical. Formally we have (recall $s=x_N=r_A$):

$$F(P)=y \text{ where } y_{ia}=\frac{x_i\times r_a}{s} \text{ for all } i,a \tag{2}$$

$$\Longleftrightarrow y\in\Phi(P) \text{ and } y_{ia}\times y_{jb}=y_{ib}\times y_{ja} \text{ for all } i,j,a,b$$

This is $F^W$ in Definition 2 if we choose for $W$ the *entropy* function $En(z)=z\ln(z)$. The claim follows easily from the KKT optimality conditions applied to the convex and smooth program $\arg\min_{\Phi(P)}\sum_{N\times A}En(y_{ia})$ (or from statement *iii)* in Lemma 4).

If we choose now $W(z)=-\ln(z)$, the $W$-welfarist rule picks $y$ maximizing the Nash product of all the entries $y_{ia}$.

For $W(z)=z^2$ the $W$-welfarist rule chooses $y$ minimizing the variance $\frac{1}{|N|\times|A|}\sum_{N\times A}(y_{ia}-\frac{s}{|N|\times|A|})^2$.

All three rules just mentioned meet a familiar property:

- **Scale Invariance (SI)**: for any $P\in\mathcal{P}$ and $\delta>0$, $F(\delta\times P)=\delta\times F(P)$, where $P=(x,r)$ and $\delta\times P=(\delta x,\delta r)$

This axiom captures a one-dimensional family of welfarist assignment rules.

**Lemma 2:** *The $W$-welfarist assignment rule is scale invariant if and only if $W$ is one of the following functions $W^q,q\in\mathbb{R}$:*

$W^1(z)=z\ln(z)$ *(the proportional rule)*

$W^q(z)=z^q$ *for $q>1$ and for $q<0$*

$W^0(z)=-\ln(z)$ *(maximizing the Nash product)*

$W^q(z) = -z^q$ *for* $0 < q < 1$
Proof in Section 8.

When $q \to \pm\infty$ the pointwise limits of the $W^q$-rules are interesting in their own right, in particular because they are not welfarist in the sense of Definition 2 but meet the same basic properties. If $q \to -\infty$ the limit of $W^q$ is the *maxmin rule* $F^{-\infty}$, choosing first the subset of $\Phi(P)$ where the smallest entry of $y$ is as large as possible, then the sub-subset of those maximizing the next smallest entry, and so on.[2] Similarly when $q \to +\infty$ the $W^q$-rule converges pointwise to the *minmax rule* $F^{+\infty}$, minimizing first the largest entry of $y$, then the next largest entry and so on.

The Symmetry and Consistency properties of $F^q$ are preserved by pointwise convergence, and a direct argument shows that both rules $F^{\pm\infty}$ are continuous as well. They are also scale invariant.

# 5 Monotonicity and limit properties

We need three new axioms for our characterization result: (two variants of) a monotonicity property and two boundary conditions on individual shares.

The monotonicity property considers shifts of endowments toward a certain resource, that keeps total endowment constant; or shifts toward the allocation of a certain agent, leaving total allocation constant. In the former case we require that all agents get a weakly larger share of the resource in question, in the latter case that the agent in question gets a weakly larger share of every resource. As is often the case in separability results, we actually need a slightly stronger property guaranteeing strict increases in some cases.

To define formally the two variants of the property we say that $r'$ *is an a-shift of* $r$ (where $r, r' \in \mathbb{R}_+^A$ and $a \in A$) if $r'_a > r_a, r'_b \leq r_b$ for all $b \neq a$ and $r'_A = r_A$. We say similarly that $x'$ *is an i-shift of* $x$ if $x'_i > x_i, x'_j \leq x_j$ for all $j \neq i$ and $x'_N = x_N$. In the two following definitions, $y$ and $y'$ refer to the assignments selected before and after the shift:

- **Monotonicity (MON):** *if* $r'$ *is an a-shift of* $r$ *then* $y_{ia} \leq y'_{ia}$ *for all* $i$*; if* $x'$ *is an i-shift of* $x$*, then* $y_{ib} \leq y'_{ib}$ *for all* $b$
- **Monotonicity$^*$ (MON$^*$)**: *if* $r'$ *is an a-shift of* $r$ *and* $0 < y_{ia} < x_i$ *for some* $i$*, then* $y_{ia} < y'_{ia}$*; if* $x'$ *is an i-shift of* $x$ *and* $0 < y_{ia} < r_a$ *for some* $a$*, then* $y_{ia} < y'_{ia}$

The main result uses two additional properties to pin down two sublasses of welfarist assignment rules. The first one rules out zero entries whenever possible:

[2] The first step of this algorithm picks the smallest row-sum $x_i$ and the smallest column-sum $r_a$, then finds the smallest of $\frac{x_i}{|A|}$ and $\frac{r_a}{|N|}$: if the former, we fill the $i$-row with equal entries $\frac{x_i}{|A|}$, if the latter we fill the $a$-column with equal entries $\frac{r_a}{|N|}$; after substracting the corresponding entries from the column sums in the former case, and from the row sums in the latter case, we repeat.

- **Positive Entries (PE)**: *fix any $i, a$ and $P = (N, A, x, r)$ with $y = F(P)$*

$$\{x_i > 0 \text{ and } r_a > 0\} \Longrightarrow y_{ia} > 0$$

The next property puts restrictions on problems where one entry is much larger than the others. Given a problem $P = (N, A, x, r)$ the range of entry $y_{ia}$ for $y \in \Phi(P)$ is the following interval $[LB_{ia}, UB_{ia}]$:

$$(x_i - r_{A \diagdown a})_+ = (r_a - x_{N \diagdown i})_+ = LB_{ia} \leq y_{ia} \leq UB_{ia} = \min\{x_i, r_a\}$$

(with the notation $(z)_+ = \max\{z, 0\}$). In the statement of the next axiom, we fix $N, A$.

- **Bounded Entries (BE)**: *for any $i, a$ and any sequence $P^\delta = (N, A, x^\delta, r^\delta)$, $\delta = 1, 2, \cdots$*

$$\{\lim_{\delta \to \infty} LB^\delta_{ia} = \infty \text{ and } \sup_{\delta \to \infty} \{UB^\delta_{jb}\} < \infty \text{ for all } (j, b) \neq (i, a)\}$$

$$\Longrightarrow \lim_{\delta \to \infty} \{y^\delta_{ia} - LB^\delta_{ia}\} = 0$$

This says that an entry of the assignment that has to be (by feasibility) arbitrarily larger than any other entry, should be kept as small as permitted by feasibility.

**Lemma 3** *The $W$-welfarist assignment rule*
*i) is Monotonic$^*$,*
*ii) meets Positive Entries if and only if $W'(0) = -\infty$,*
*iii) meets Bounded Entries if and only if $W'(\infty) = +\infty$.*
Proof in Section 8.

It is easy to check that MON$^*$ and Continuity together imply MON. Therefore all welfarist rules meet MON.

The scale invariant welfarist rules $F^{W^q}$ (Lemma 2) meet PE if and only if $q \leq 1$; they meet BE if and only if $q \geq 1$. The proportional rule $F^{W^1}$ is the only one in this family meeting BE and PE.

The maxmin rule $F^{-\infty}$ and the minmax rules $F^{+\infty}$ meet MON but fail MON$^*$; $F^{-\infty}$ is PE but no BE and $F^{+\infty}$ is BE but no PE.

*Remark 1 Another standard axiom in one-dimensional rationing is Ranking: a larger claim gets a weakly larger share of the resources. We define two versions of the axiom, both of them useful in the proof of our main result (Section 8):*
*Ranking (RKG): fix any $i, j$ and $P = (N, A, x, r)$ with $y = F(P)$*

$$x_i > x_j \Longrightarrow y_{ia} \geq y_{ja} \text{ for all } a$$

*Ranking$^*$ (RKG$^*$): fix any $i, j$ and $P = (N, A, x, r)$ with $y = F(P)$*

$$\{x_i > x_j \text{ and } y_{ia} > 0\} \Longrightarrow y_{ia} > y_{ja} \text{ for all } a$$

*All welfarist rules meet both Ranking axioms.*[3]

[3] Check that MON$^*$ and $N$-SYM imply RKG$^*$. This is clear if $y_{ja} = 0$. If $y_{ja} > 0$ we

# 6 Characterizing welfarist assignment rules

**Theorem**

*i) An assignment rule $F$ is Consistent, Monotonic*, and satisfies Positive Entries if and only if it is a $W$-welfarist rule where* $W'(0) = -\infty$;

*ii) An assignment rule $F$ is Consistent, Monotonic*, and satisfies Bounded Entries if and only if it is a $W$-welfarist rule where* $W'(\infty) = +\infty$.

The "if" statements follow from Lemma 1 and Lemma 3. We give a rough sketch of the long proof of "only if", detailed in the Appendix.

First we apply the KKT conditions to program (1), and derive the critical additive structure structure of the assignment $F^W(P)$ for any $P$. To state this result we use the extension $\Gamma$ of the inverse of $W'$ to the entire real line. The domain of $W'$ is $\mathbb{R}_+ \cup \{+\infty\}$ and its range is the interval $[W'(0), W'(\infty)[$ such that $W'(0) \geq -\infty$ and $W'(\infty) \leq \infty$ . We set: $\Gamma(\alpha) = 0$ if $\alpha \leq W'(0)$ ; $\Gamma(\alpha) = (W')^{-1}(\alpha)$ if $W'(0) \leq \alpha < W'(\infty)$ ; $\Gamma(\alpha) = \infty$ if $\alpha \geq W'(\infty)$. Thus $\Gamma$ is continuous, weakly increasing, and strictly so in the interval $]W'(0), W'(\infty)[$.

**Lemma 4** *Fix $P \in \mathcal{P}$ such that $x, r \gg 0$, and an assignment $y \in \Phi(P)$. The three following statements are equivalent:*
*i)* $y = F^W(P)$
*ii) there exist $\alpha \in \mathbb{R}^N, \beta \in \mathbb{R}^A$ such that* $y_{ia} = \Gamma(\alpha_i + \beta_a)$ for all $i, a$
*iii) for all $i, j, a, b$ :* $W'(y_{ia}) + W'(y_{jb}) < W'(y_{ib}) + W'(y_{ja}) \Longrightarrow y_{ib} \times y_{ja} = 0$
Proof in Section 8.

Note that the equivalence holds also when some $x_i, r_a$ are zeros, provided we allow then $\alpha_i = -\infty$, $\beta_a = -\infty$, and extend the domain of $\Gamma$ to include $-\infty$.

Lemma 4 implies that the matrix $y$ is $F^W$-fair if and only if it can be written in the additive form *ii*), and if and only if every $2 \times 2$ submatrix meets property *iii*). In particular this implies that $F^W$ is inverse consistent.

Statement *iii*) implies $W'(y_{ia})+W'(y_{jb}) = W'(y_{ib})+W'(y_{ja})$ whenever these four entries of $y$ are strictly positive. In some cases this gives a closed form of the assignment rule: for instance the proportional rule $F^{W^1}$, $\Gamma$ is the exponential function so that $y \gg 0$ and the above equality amounts to $y_{ia} \times y_{jb} = y_{ib} \times y_{ja}$ hence $y_{ia} = \frac{x_i \times r_a}{s}$.

Another example is the rule $F^{W^0}$ maximizing the Nash product $\Pi_{N \times A} y_{ia}$: $\Gamma$ is the inverse function so we still get $y \gg 0$ and statements *ii*), *iii*) write:

$$\{y_{ia} = \frac{1}{\alpha_i + \beta_a} \text{ for all } i, a\} \Longleftrightarrow \{\frac{1}{y_{ia}} + \frac{1}{y_{jb}} = \frac{1}{y_{ib}} + \frac{1}{y_{ja}} \text{ for all } i, j, a, b\}$$

but no closed form representation of $y$ follows.

Consider finally $F^{W^2}$ choosing the assignment with the smallest variance. We have $\Gamma(z) = (z)_+$, and *ii*) gives $y_{ia} = (\alpha_i + \beta_a)_+$. Therefore $y_{ia} + y_{jb} =$

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have $y_{ia} < x_i$. Change $P$ to $P'$ where $x'_i = x'_j = \frac{x_i + x_j}{2}$ and the rest is as in $P$: by MON* $y'_{ia} < y_{ia}, y'_{ja} > y_{ja}$, and by $N$-SYM $y'_{ia} = y'_{ja}$. Finally CONT plus RKG* together imply RKG.

$y_{ib} + y_{ja}$ holds only when those entries are strictly positive. If all entries are, we get

$$\{y_{ia} = \alpha_i + \beta_a \text{ for all } i, a\} \Longleftrightarrow \{y_{ia} + y_{jb} = y_{ib} + y_{ja} \text{ for all } i, j, a, b\}$$

$$\Longleftrightarrow y_{ia} = \frac{1}{|A|} x_i + \frac{1}{|N|} r_a - \frac{1}{|A| \times |N|} s$$

and this closed form is correct if and only if $\frac{1}{|A|} x_i + \frac{1}{|N|} r_a \geq \frac{1}{|A| \times |N|} s$ for all $i, a$.

*Remark 2 The maxmin rule $F^{-\infty}$ and the minmax rule $F^{+\infty}$ are not adressed by Lemma 4 because they are not welfarist. But they can be parametrized in very similar way: the interesting details are in Subsection 8.2.*

We sketch now the proof of the Theorem. The first step associates a *standard rationing rule* to any rule $F \in \mathcal{F}$ meeting CSY.

A rationing problem is $(N, x, t)$, where $x \in \mathbb{R}_+^N$ is the profile of demands, $t \geq 0$ is the amount to be divided, and $t \leq x_N$. A rationing rule $h$ finds for every problem a division $y = h(N, x, t)$ of $t$ among $N$ such that $0 \leq y \leq x$ and $y_N = t$. Although a rationing problem is not a special case of an assignment problem (there is a single resource but budget balance does not hold), it is straightforward to adapt the definition of the properties $N$-SYM, CONT, and CSY.

For any $F \in \mathcal{F}$ we define $h^F$ as follows. Given a rationing problem $(N, x, t)$ we construct the assignment problem $P = (N, \{a, b\}, x, r)$ where $r_a = t$ and $r_b = x_N - t$, and set $h^F(N, x, t)$ to be the $a$-column of $F(P)$. The rule $h^F$ meets SYM, CONT, and CSY so by Young's Theorem ([25]) $h^F$ is parametrized by a continuous function $\theta(x_i, \lambda)$, weakly increasing in $\lambda \in \mathbb{R} \cup \{-\infty, +\infty\}$ and such that $\theta(x_i, -\infty) = 0$ and $\theta(x_i, \infty) = x_i$:

$$h^F(N, x, t) = y \Longleftrightarrow \{\exists \lambda : y_i = \theta(x_i, \lambda) \text{ for all } i, \text{ and } \sum_N \theta(x_i, \lambda) = t\} \quad (3)$$

Moreover by $A$-SYM of $F$ the rationing rule $h^F$ is *self dual*: in any problem $(N, x, t)$ it distributes losses and gains in the same way

$$h^F(N, x, t) + h^F(N, x, x_N - t) = x \quad (4)$$

We use an alternative parametrization $\pi(z, \lambda)$ of $h^F$. For $z \geq 0$ and $\lambda \in \mathbb{R}$, $\pi(z, \lambda)$ solves the following equation in $y_i$:

$$y_i = \pi(z, \lambda) \Leftrightarrow y_i = \theta(z + y_i, \lambda)$$

(where the index $i$ is just to remind us that $y_i$ is one-dimensional). Thus $z$ is the loss of an agent with demand $z + y_i$. In fact existence of a solution to this equation is not guaranteed for all $z, \lambda$, so the actual proof has to proceed more carefully and use the additional assumptions MON*, PE, and BE.

The key consequence of CSY is that if the matrix $y$ is $F$-fair, for any two columns $a,b$ there is a unique parameter $\lambda_b^a \in \mathbb{R}$ such that $y_{ib} = \pi(y_{ia}, \lambda_b^a)$ for all $i$. Indeed the reduced $N \times \{a,b\}$ matrix $[y_{[a]}, y_{[b]}]$ is $F$-fair, therefore $y_{[b]}$ is just $h^F(N, x^{ab}, r_b)$ for the profile of demands $x_i^{ab} = y_{ia} + y_{ib}$. Conversely if we fix a column $y_{[a]} \in \mathbb{R}_+^N$ and arbitrary parameters $\lambda_b^a$ for each $b$, we can show that the matrix with $b$-column $y_{[b]} = \pi(y_{[a]}, \lambda_b^a)$ is $F$-fair.

This allows us to define an inner product $\lambda * \mu$ by the equality $\pi(\pi(z,\lambda),\mu) = \pi(z, \lambda * \mu)$, for all $z$. Then CSY implies that the operation $\lambda * \mu$ is associative, as well as continuous and strictly increasing, so by the well known Associativity Theorem ([1], [12]) we get an additive representation of the form $\lambda * \mu = g^{-1}(g(\lambda) + g(\mu))$. We can rescale the parameters $\lambda$ so that $\lambda * \mu = \lambda + \mu$, and check that the range of $\alpha \to \pi(1,\alpha)$ is $\mathbb{R}_+$.

Finally we take an arbitrary $F$-fair matrix $y$ and fix a resource $a$. For every $i$ there is $\alpha_i$ such that $y_{ia} = \pi(1,\alpha_i)$. For every $b \neq a$ there is a $\lambda_b$ such that $y_{[b]} = \pi(y_{[a]}, \lambda_b)$ thus $y_{ib} = \pi(\pi(1,\alpha_i),\lambda_b) = \pi(1, \alpha_i + \lambda_b)$ for all $i,b$; setting $\lambda_a = 0$ we now have the parametric representation in statement *ii*) of Lemma 3 for the function $\Gamma(z) = \pi(1,z)$, therefore $F = F^W$ where $W$ is a primitive of $\Gamma^{-1}$.■

*Remark 3 Tightness of the characterization result. It is easy to see that new assignment rules emerge if we drop one of the following axioms:*
*Consistency: take* $F^{W_1}$ *for some cardinalities of* $N$ *and* $A$*, and a different* $F^{W_2}$ *otherwise;*
*Monotonicity*: take the maxmin rule* $F^{-\infty}$ *for statement i), and the minmax rule* $F^{+\infty}$ *for statement ii);*
*Symmetry in* $N$*: take the rule minimizing* $\sum_{N\times A} W_i(y_{ia})$ *over* $\Phi(P)$*, where* $W_i$ *is strictly convex and varies with i;*
*Symmetry in A: take similarly* $\arg\min_{y\in\Phi(P)} \sum_{N\times A} W_a(y_{ia})$
*Positive Entries: take* $F^{W^2}$ *in Lemma 2, for which* $(W^2)'(0) = 0$*;*
*Bounded Entries: take* $F^{W^{\frac{1}{2}}}$ *in Lemma 2, for which* $(W^2)'(\infty) = 0$;

*However we are unable to decide if dropping either one of Continuity or Monotonicity introduces additional rules.*

# 7 Concluding comments

## 7.1 A new family of rationing rules

As explained in the previous section each welfarist rule $F^W$ defines a standard rationing rule $h^W$: these rules form an interesting new family defined as follows:

$$h^W(N,x,t) = \arg\min_{0 \le y \le x\ ;\ y_N = t} \left(\sum_N W(y_i) + W(x_i - y_i)\right) \tag{5}$$

*Case 1* $W'(0) = -\infty$ The solution $y$ of (5) is strictly positive (provided $x \gg 0, t > 0$) and is characterized by the existence of some $\lambda \in \mathbb{R}$ such that

$W'(y_i) - W'(x_i - y_i) = \lambda$ for all $i$. Thus a parametric representation of $h$ as in (3) is

$$y_i = \theta(x_i, \lambda) \Longleftrightarrow W'(y_i) - W'(x_i - y_i) = \lambda$$

and $\theta$ is continuous and strictly increasing in both variables.

For instance consider the Nash scale invariant rule $F^{W^0}$ in Lemma 2: it defines the rationing rule $h^{W^0}$ maximizing the product $\Pi_i y_i(x_i - y_i)$, of which a parametrization is

$$\frac{1}{y_i} - \frac{1}{x_i - y_i} = \lambda \Longleftrightarrow y_i = \frac{x_i}{\sqrt{1+\lambda^2 x_i^2} + 1 - \lambda x_i} \text{ for } \lambda \in \mathbb{R}$$

Figure 1 depicts the graphs of $x_i \to \theta(x_i, \lambda)$. Similar shapes obtain for $h^{W^{\frac{1}{2}}}$ and $h^{W^{-1}}$ see Figures 2,3.

When $q \to -\infty$ the rule $h^{W^q}$ converges pointwise to the celebrated Talmudic rule $h^{W^{-\infty}}$ ([3]) parametrized by

$$\theta(x_i, \lambda) = \min\{\frac{x_i}{2}, -\frac{1}{\lambda}\} \text{ for } \lambda \leq 0 \text{ ; } \theta(x_i, \lambda) = \max_i\{\frac{x_i}{2}, x_i - \frac{1}{\lambda}\} \text{ for } \lambda \geq 0$$

*Case 2 $W'(0)$ is finite* The solution of (5) may have some null coordinates, and the parametrization takes the following form:

$$\theta(x_i, \lambda) = 0 \text{ if } \lambda \leq W'(0) - W'(x_i)$$

$$y_i = \theta(x_i, \lambda) \text{ solves } W'(y_i) - W'(x_i - y_i) = \lambda \text{ if } W'(0) - W'(x_i) \leq \lambda \leq W'(x_i) - W'(0)$$

$$\theta(x_i, \lambda) = x_i \text{ if } W'(x_i) - W'(0) \leq \lambda$$

For instance $h^{W^2}$ minimizes the variance of the joint profile of gains $y_i^2$ and losses $(x_i - y_i)^2$, and is parametrized as follows

$$\theta(x_i, \lambda) = \frac{x_i}{2} + \frac{\lambda}{2} \text{ if } -x_i \leq \lambda \leq x_i$$

(and 0 or $x_i$ elsewhere) See Figure 4. Similarly $h^{W^3}$ is parametrized as

$$\theta(x_i, \lambda) = \frac{x_i^2 + \lambda}{2x_i} \text{ if } -x_i^2 \leq \lambda \leq x_i^2$$

See Figure 5.

### 7.2 An open question

Recall from Section 2 that a bilateral rationing problem $P = (N, A, x, r)$ differs from an assignment problem only in that the balancedness condition is replaced by the deficit inequality $\sum_N x_i \geq \sum_A r_a$. Then a feasible assignment $y$ is a matrix in $\mathbb{R}_+^{N \times A}$ such that $y_{Na} = r_a$ for all $a$, and $y_{iA} \leq x_i$ for all $i$. Consistency of the rule $F$ becomes: if $F(P) = y$ and we drop resource $a$, the reduced problem $P' = (N, A \diagdown a, x', r_{-a})$ with the reduced claims $x'_i = x_i - y_{ia}$, has the same solution $F(P') = y_{-a}$; and a similar statement holds if we drop an agent $i$ and reduce the amount of resource $a$ to $r_a - y_{ia}$. As noted in [16], [17], for any strictly convex function $W$ and convex function $V$, the program $\arg\min_y \sum_{N\times A} W(y_{ia}) + \sum_N V(x_i - y_{iA})$ delivers a consistent (as well as symmetric and continuous) bilateral rationing rule. Extending the characterization result of our Theorem to bilateral rationing, with the help of appropriate monotonicity and boundary assumptions, is worthy of future research.

### 7.3 Asymmetric assignment rules

Symmetry is a critical assumption in Young's Theorem for standard rationing problems. It is easy to define asymmetric consistent rationing rules by minimizing $\sum_i V_i(x_i, y_i)$, but it has proven difficult to capture the entire class of Consistent and Continuous rules. A flurry of recent literature characterizes a variety of interesting subclasses: [13], [6], [24], [19], [20], [9]. In our assignment model, an obvious family of asymmetric consistent assignment rules minimizes a welfarist objective of the form $\sum_{N\times A} W_{ia}(y_{ia})$. A challenging open question is to capture all or part of this family under additional axiomatic requirements.

# 8 Appendix: proofs

### 8.1 Lemma 2

The result plays no essential role so we only sketch its proof.That all rules $F^{W^p}$ are SI is clear. Conversely it follows from statement *iii*) in Lemma 3 that for any $F^W$ the assignment matrix $\begin{matrix} a & b \\ c & d \end{matrix}$ where all entries are strictly positive and $W'(a) + W'(d) = W'(b) + W'(c)$, is $F^W$-fair. Therefore SI implies, for all positive $a, b, c, d, \delta$:

$$W'(a) + W'(d) = W'(b) + W'(c) \Longleftrightarrow W'(\delta a) + W'(\delta d) = W'(\delta b) + W'(\delta c)$$

Assume for simplicty that $W'$ is differentiable (it must be so almost everywhere; we omit the details of the general argument). Fix $a, c$, and $\delta$; for any small enough $\varepsilon$ there exists $\lambda > 0$ such that $W'(a+\varepsilon) - W'(a) = W'(c+\lambda\varepsilon) - W'(c)$. By the equivalence above, this amounts to $W'(\delta a + \delta\varepsilon) - W'(\delta a) = W'(\delta c + \lambda\delta\varepsilon) - W'(\delta c)$. Therefore $W''(a) - \lambda W''(c) = o(\varepsilon)$ and $W''(\delta a) - \lambda W''(\delta c) = o(\varepsilon)$, where $\lambda$ depends on $\varepsilon$ but must converge by differentiability of $W'$. This gives $W''(a) \cdot W''(\delta c) = W''(c) \cdot W''(\delta a)$, from which follows, after rescaling $W''$, $W''(ab) = W''(a) \cdot W''(b)$ for all $a, b > 0$. Thus $W''$ is a power function by standard arguments.$\blacksquare$

### 8.2 Lemma 4

We prove Lemma 4 before Lemma 3 because it is used in the proof of the latter.

We must show that the three following statements are equivalent:
*i)* $y = F^W(P)$
*ii) there exist two vectors* $\alpha \in \mathbb{R}^N, \beta \in \mathbb{R}^A$ *such that* $y_{ia} = \Gamma(\alpha_i + \beta_a)$ for all $i, a$
*iii)* for all $i, j, a, b$ : $\{W'(y_{ia}) + W'(y_{jb}) < W'(y_{ib}) + W'(y_{ja})\} \Longrightarrow y_{ib} \times y_{ja} = 0$
**Statement** *i)* $\Longleftrightarrow$ *ii)*: The assignment $y$ minimizes $\sum_{N\times A} W(y_{ia})$ under the following equality and inequality constraints:

$$y_{iA} = x_i \ , \ y_{Na} = r_a \ ; \ y_{ia} \geq 0 \text{ for all } i, a$$

The Lagrangien of the problem is $\mathcal{L}(y, \alpha, \beta, \gamma)$, where $\alpha \in \mathbb{R}^N, \beta \in \mathbb{R}^A$ and $\gamma \in \mathbb{R}_+^{N\times A}$:

$$\mathcal{L}(y, \alpha, \beta, \gamma) = \sum_{(N\times A)} W(y_{ia}) -$$

$$-\sum_N \alpha_i(y_{iA} - x_i) - \sum_A \beta_a(y_{Na} - r_a) - \sum_{N\times A\setminus\tau(P)} \gamma_{ia} y_{ia}$$

From $|N\times A| \geq 2\max\{|N|, |A|\}$ the linear mapping $\mathbb{R}^{N\times A} \ni y \to (y_{iA}, y_{Na}) \in \mathbb{R}^{N\cup A}$ is of maximal rank. Also there is some feasible and strictly positive $y$ because $x, r \gg 0$, hence the qualification constraints hold. Therefore there exist some KKT multipliers $\alpha, \beta, \gamma$, such that

$$\min_{y\in\Phi(P)} \sum_{N\times A} W(y_{ia}) = \min_{y\in\mathbb{R}^{N\times A}} \mathcal{L}(y, \alpha, \beta, \gamma)$$

Moreover $y$ solves the Left Hand program above if and only if 1) $y$ minimizes $\mathcal{L}$ over the entire space; 2) $y \in \Phi(P)$; 3) and $\gamma_{ia} y_{ia} = 0$ for all $ia$.

If $y = F^W(P)$ it cancels the derivative of $\mathcal{L}(y, \alpha, \beta, \gamma)$:

$$W'(y_{ia}) = \alpha_i + \beta_a + \gamma_{ia} \text{ for all } i, a$$

If $y_{ia} > 0$ this implies $W'(y_{ia}) = \alpha_i + \beta_a \Longleftrightarrow y_{ia} = \Gamma(\alpha_i + \beta_a)$. If $y_{ia} = 0$ we get $W'(y_{ia}) = W'(0) \geq \alpha_i + \beta_a \Longrightarrow \Gamma(\alpha_i + \beta_a) = 0 = y_{ia}$.

Conversely pick a feasible $y$ as in statement *ii)*: $y_{ia} = \Gamma(\alpha_i + \beta_a)$ implies $W'(y_{ia}) \geq \alpha_i + \beta_a$, therefore $y$ cancels the derivative of $\mathcal{L}(y, \alpha, \beta, \gamma)$ for some $\gamma$ meeting the complementarity conditions 3) above. As $\mathcal{L}$ is strictly convex and smooth this means that $y$ is the absolute minimum of $\mathcal{L}(y, \alpha, \beta, \gamma)$ and we conclude $y = F^W(P)$.

**Statement** *ii)* $\Longrightarrow$ *iii)* Pick $y \in \Phi(P)$ and $\alpha \in \mathbb{R}^N, \beta \in \mathbb{R}^A$ as in *ii)*. Fix any $i, j, a, b$ such that $y_{ib}, y_{ja}$ are both strictly positive: we must show $W'(y_{ia}) + W'(y_{jb}) \geq W'(y_{ib}) + W'(y_{ja})$. By construction of $\Gamma$ we have $y_{ia} = \Gamma(\alpha_i + \beta_a) \Longrightarrow W'(y_{ia}) \geq \alpha_i + \beta_a$ and similarly $W'(y_{jb}) \geq \alpha_j + \beta_b$. On the other hand $W'(y_{ib}) = \alpha_i + \beta_b$ and $W'(y_{ib}) = \alpha_i + \beta_b$, and the claim follows.

**Statement** *iii)* $\Longrightarrow$ *ii)* Fixing a matrix $y$ meeting *iii)*, we must show that $y = F^W(P)$. We proceed by induction on the number of zeros in $y$. If $y \gg 0$ property *iii)* gives the diagonal equality $W'(y_{ia}) + W'(y_{jb}) = W'(y_{ib}) + W'(y_{ja})$

for all $i, j, a, b$. Fixing agent 1 and resource $a^*$ we write this as $y_{jb} = \Gamma(W'(y_{1b}) + W'(y_{ja^*}) - W'(y_{1a^*}))$ for all $j, b$. Setting $\alpha_j = W'(y_{ja^*}) - W'(y_{1a^*})$ and $\beta_b = W'(y_{1b})$ gives the parametrization of $y$ in *ii*), so we are done.

Next we assume the claim proven for up to $K$ zeros in $y$, and assume $y$ has $K+1$ zeros. Fix $1, a^*$ such that $y_{1a^*} = 0$ and define $A_+ = \{a | y_{1a} > 0\}$ and $A_0 = \{a \neq a^* | y_{1a} = 0\}$, and $N_+ = \{i | y_{ia^*} > 0\}, N_0 = \{i \neq 1 | y_{ia^*} = 0\}$, where both $A_+$ and $N_+$ are non empty. Note that for all $i \in N_+, a \in A_+$ assumption *iii*) implies $W'(y_{ia^*}) + W'(y_{1a}) \leq W'(y_{1a^*}) + W'(y_{ia})$; as $W'(y_{1a^*}) = W'(0) < W'(y_{ia^*}), W'(y_{1a})$ we deduce $W'(y_{ia}) > W'(y_{ia^*}), W'(y_{1a}) > W'(0)$. Then $y_{ia} > 0$ and the following diagonal equality holds:

$$W'(y_{ia}) = W'(y_{ia^*}) + W'(y_{1a}) - W'(y_{1a^*}) \text{ for all } i \in N_+, a \in A_+ \tag{6}$$

The inductive assumption implies that the $(N \diagdown 1) \times (A \diagdown a^*)$-submatrix of $y$ is $F^W$-fair, so by property *ii*) there exist $\alpha \in \mathbb{R}^{N \diagdown 1}, \beta \in \mathbb{R}^{A \diagdown a^*}$ such that $y_{ia} = \Gamma(\alpha_i + \beta_a)$ for all $i \neq 1, a \neq a^*$. We extend this definition to 1 and $a^*$ as follows

$$\alpha_1 = \alpha_i + W'(y_{1a}) - W'(y_{ia}) \text{ for any } i \in N_+, a \in A_+ \tag{7}$$

$$\beta_{a^*} = \beta_a + W'(y_{ia^*}) - W'(y_{ia}) \text{ for any } i \in N_+, a \in A_+ \tag{8}$$

and we check first that these definitions makes sense. The sum in (7) does not depend on $i, j \in N_+$ if $\alpha_i - W'(y_{ia}) = \alpha_j - W'(y_{ja})$ which follows from (6) and $W'(y_{ia}) = \alpha_i + \beta_a$ in $N_+ \times A_+$. The same sum does not depend on $a, b \in A_+$ if $W'(y_{1a}) - W'(y_{ia}) = W'(y_{1b}) - W'(y_{ib})$ which is the diagonal equality in $\{1, i\} \times \{a, b\}$ where all entries of $y$ are strictly positive. A similar argument shows that (8) defines $\beta_{a^*}$ unambiguously.

It remains to check the equality $y_{ia} = \Gamma(\alpha_i + \beta_a)$ in $(\{1\} \times A) \cup (N \times \{a^*\})$. We prove this in $\{1\} \times A$ and omit the similar argument in $N \times \{a^*\}$. Together (7), (8) and (6) give

$$\alpha_1 + \beta_{a^*} = W'(y_{1a}) + W'(y_{ia^*}) - W'(y_{ia}) = W'(y_{1a^*}) = W'(0) \text{ for any } i \in N_+, a \in A_+$$

so that $\Gamma(\alpha_1 + \beta_{a^*}) = 0 = y_{1a^*}$.

Next for $a \neq a^*$ the definition (7) gives $\alpha_1 + \beta_a = \alpha_i + \beta_a + W'(y_{1a}) - W'(y_{ia})$ for any $i \in N_+$. Distinguish two cases. If $a \in A_+$ we showed above $W'(y_{ia}) = \alpha_i + \beta_a$ therefore $\alpha_1 + \beta_a = W'(y_{1a})$ so we are done. If $a \in A_0$ we have $y_{1a} = 0$ and $y_{ia} = \Gamma(\alpha_i + \beta_a)$. By construction of $\Gamma$ we have $W'(\Gamma(\alpha)) \geq \alpha$ for all $\alpha$, hence

$$\alpha_1 + \beta_a = W'(0) + (\alpha_i + \beta_a - W'(y_{ia})) \leq W'(0)$$

ad we conclude $\Gamma(\alpha_1 + \beta_a) = 0 = y_{1a}$ as desired. This completes the induction step.$\blacksquare$

*Remark 4 The maxmin rule $F^{-\infty}$ and the minmax rule $F^{+\infty}$ are not adressed by Lemma 4 because they are not welfarist. But they can be parametrized in very similar way.*

**Lemma 4$^{\infty}$** *Fix $P \in \mathcal{P}$ and $y \in \Phi(P)$. The three following statements are equivalent*
*i)* $y = F^{-\infty}(P)$ (resp. $y = F^{+\infty}(P)$)
*ii) there exist two vectors $\alpha \in \mathbb{R}_+^N, \beta \in \mathbb{R}_+^A$ such that*

$$y_{ia} = \min\{\alpha_i, \beta_a\} \text{ for all } i, a \quad (\text{resp. } y_{ia} = \max\{\alpha_i, \beta_a\} \text{ for all } i, a)$$

*iii)* $\min\{y_{ia}, y_{jb}\} = \min\{y_{ib}, y_{ja}\}$ *for all* $i, j, a, b$
(resp. $\max\{y_{ia}, y_{jb}\} = \max\{y_{ib}, y_{ja}\}$ for all $i, j, a, b$)

*In particular the two rules $F^{\pm\infty}$ are inverse consistent.*

*We give the proof for the maxmin rule, the proof for the minmax rule is similar. We show first that statements ii) and iii) are equivalent. Clearly ii) $\Longrightarrow$ iii). Now assume iii) and set $\gamma = \min_{ia} y_{ia}$: for any $i \neq j$ and $a \neq b$ $\{y_{ia} > \gamma$ and $y_{jb} > \gamma\} \Longrightarrow \{y_{ib} > \gamma$ and $y_{ja} > \gamma\}$. Therefore $\{ia | y_{ia} > \gamma\}$ is a rectangle, and its complement $C$ is a union of rows and columns. We set $\alpha_i = \beta_a = \gamma$ for all such columns and rows (typically a single row or column). Then we repeat this construction inside $(N \times A) \diagdown C$, find the union $C'$ or columns and/or rows where $y_{ia}$ is minimal, and set $\alpha_i, \beta_a$ as this smallest value $\gamma'$. Now for every entry $ia$ where each coordinate is in $C \cup C'$ the desired equality $y_{ia} = \min\{\alpha_i, \beta_a\}$ holds. Moreover $C \cup C'$ is also a union of rows and columns, so the iteration is now clear.*

*We check next i) $\Longrightarrow$ iii). Fix $P \in P$ and $y = F^{-\infty}(P)$ and call $k_1, k_2, \cdots, k_p, \cdots,$ the sequence of columns or rows successively filled in the algorithm defining $y$. Each entry $y_{ia}$ appears in exactly one such column or row denoted $k_{p(ia)}$. Moreover all entries appearing in $k_p$ are equal to some $z_p$ and the sequence $z_p$ is weakly increasing. Note that if $z_p = z_{p+1}$ the order in which they appear is arbitrary. For any $i, j, a, b$ if $p = p(ia)$ is the earliest appearance among the four entries of $\{, i, j\} \times \{a, b\}$, then $y_{ib} = y_{ia} = z_p$ and/or $y_{ja} = y_{ia} = z_p$; if $y_{ib} = z_p$ then $y_{jb}, y_{ja} \geq z_p$ as they appear later, so $\min\{y_{ia}, y_{jb}\} = \min\{y_{ib}, y_{ja}\}$; the argument is similar if $y_{ja} = z_p$, completing the proof.*

*We show finally ii) $\Longrightarrow$ i). Assume $y$ is feasible and $y_{ia} = \min\{\alpha_i, \beta_a\}$ for all $i, a$. Suppose $\alpha_i = \min_{j,a}\{\alpha_j, \beta_a\}$ so that $y_{ia} = \alpha_i = \frac{x_i}{|A|}$ for all $a$: clearly $x_i = \min_j x_j$ and $r_a = y_{Na} \geq |N|\alpha_i$ implying $\frac{r_a}{|N|} \geq \frac{x_i}{|A|}$ for all $a$. Therefore row $i$ is the first step in the algorithm defining $F^{-\infty}(P)$; then we iterate by choosing the smallest among the remaining $\alpha_j, \beta_a$, and so on.*■

## 8.3 Lemma 3

We fix a welfarist rule $F^W$ throughout the proof.
**Statement** *i)* For any two problems $P, P'$, with $y = F^W(P), y' = F^W(P')$ we set $d_{ia} = y'_{ia} - y_{ia}$ and $\partial_{ia} = (\alpha'_i + \beta'_a) - (\alpha_i + \beta_a)$ where the latter comes from the parametric representation in statement *ii)* of Lemma 4. The following facts are easily checked, and require no assumption on the shift from $P$ to $P'$. First because $\Gamma$ is weakly increasing

$$d_{ia} > 0 \Longrightarrow \partial_{ia} > 0 \text{ and } d_{ia} < 0 \Longrightarrow \partial_{ia} < 0 \text{ for all } i, a$$

For all $i,j,a,b$, the identity $\partial_{ia}+\partial_{jb}=\partial_{ib}+\partial_{ja}$ gives

$$\{\partial_{ia},\partial_{jb}\geq 0,\partial_{ib}\leq 0,\text{ with at least one strict inequality}\}\Longrightarrow \partial_{ja}>0\} \quad (9)$$

$$d_{ia},d_{jb}>0 \text{ and } d_{ib},d_{ja}<0 \text{ is impossible}$$

Now we fix $P,P'$ as in the premises of MON* with $a_1$ the resource increasing strictly in size.

*Step 1* We show first $y_{ia}\leq y'_{ia}$ for all $i$, i.e., property MON. We assume $y_{1a_1}>y'_{1a_1}\iff d_{1a_1}<0$ and derive a contradiction. Note that $\partial_{1a_1}<0$ as well. Because $x_1$ is the same in $P$ and $P'$ there is some $a_2$ such that $d_{1a_2}>0$, hence $\partial_{1a_2}>0$ as well. As $r_{a_2}$ decreases weakly there is some agent 2 such that $d_{2a_2},\partial_{2a_2}<0$. Applying (9) to $1,2,a_1,a_2$ (with a change of sign) we get $\partial_{2a_1}<0$ and $d_{2a_1}\leq 0$ (but we cannot claim $d_{2a_1}<0$ as $y_{2a_1},y'_{2a_1}$ could both be zero). We have shown

$$\partial_{1a_1},\partial_{2a_1},\partial_{2a_2}<0<\partial_{1a_2}$$

We cannot have $A=\{a_1,a_2\}$ because $d_{2a_1}\leq 0,d_{2a_2}<0$ but $x_2$ does not change. Thus we can pick $a_3$ such that $d_{2a_3},\partial_{2a_3}>0$. Now we apply (9) to $1,2,a_2,a_3$ to get $\partial_{1a_3}>0$ and $d_{1a_3}\geq 0$; as $r_{a_3}$ decreases weakly there some agent 3 such that $d_{3a_3},\partial_{3a_3}<0$ (so $N\neq\{1,2\}$). Applying (9) successively to $1,3,a_1,a_3$ and to $2,3,a_2,a_3$ we get $\partial_{3a_1},\partial_{3a_2}<0$ hence $d_{3a_1},d_{3a_2}\leq 0$, so we have now

$$\{\partial_{ka_l}<0 \text{ for } 1\leq l\leq k\leq 3\} \text{ and } \{\partial_{ka_l}>0 \text{ for } 1\leq k<l\leq 3\} \quad (10)$$

As above we cannot have $A=\{a_1,a_2,a_3\}$ because $d_{3a_1}+d_{3a_3}+d_{3a_3}<0$ and $x_3$ stays put.

The induction pattern is now clear: we can choose $a_4$ such that $d_{3a_4}>0$; after using (9) to show $\partial_{ka_4}>0$, $d_{ka_4}\geq 0$ for $k=1,2$, we can choose $4\in N$ such that $d_{4a_4}<0$ and show $\partial_{4a_l}<0$ for $l=1,2,3$; then we reach the pattern (10) where 4 replaces 3, and again we cannot have exhausted $A$. The contradiction follows because $N$ and $A$ are finite.

*Step 2* Now we consider an entry $1a_1$ such that $0<y_{1a_1}<x_1$. By step 1 we have $y_{1a_1}\leq y'_{1a_1}$ and we must prove $y_{1a_1}=y'_{1a_1}$ is impossible. We assume this equality and derive a contradiction. Thus $d_{1a_1}=0$, and $\partial_{1a_1}=0$ as well because $y_{1a_1}>0$ and $\Gamma$ increases strictly when it takes positive values. We distinguish two cases.

*Case 1* We assume there is at least one resource $a_2$ such that $d_{1a_2}>0$, hence $\partial_{1a_2}>0$ as well. Then we proceed exactly as in Step 1 to construct agent 2, resource $a_3$, agent 4, etc..The only difference is that $d_{1a_1},\partial_{1a_1}=0$ instead of $d_{1a_1},\partial_{1a_1}<0$, but each time the application of (9) involves entry $1a_1$ the equality $\partial_{1a_1}=0$ is enough. And in the pattern (10) we make an exception for $\partial_{1a_1}$. Then we derive the same contradiction.

*Case 2* We are left with the case where $d_{1a}\leq 0$ for all $a\neq a_1$. Because $d_{1a_1}=0$ and $x_1$ is constant, this gives $d_{1a}=0$ for all $a$. This does not allow in general to sign $\partial_{1a}$ if $y_{1a}=0$. But the premises of MON* specify $y_{1a_1}>0$ and $\Gamma$ increases strictly when it takes positive values, therefore $\partial_{1a_1}=0$. Furthermore $y_{1a_1}<x_1$ implies there exists some $a_2$ such that $y_{1a_2}>0$, therefore $\partial_{1a_2}=0$ as well. As $r_{a_1}<r'_{a_1}$, there an agent 2 such that $d_{2a_1}>0$, and $\partial_{2a_1}>0$ as well. Apply now (9) to $1,2,a_1,a_2$, to get $\partial_{2a_2}>0$ hence $d_{2a_2}\geq 0$. As $x_2$ is constant and $d_{2a_1}+d_{2a_2}>0$ there is an $a_3$ such that $d_{2a_3},\partial_{2a_3}<0$. Again (9) at $1,2,a_1,a_3$ gives $\partial_{1a_3}<0$, but as $d_{1a_3}=0$ we get $y_{1a_3}=y'_{1a_3}=0$.

Check $y_{2a_2}>0$: indeed $d_{2a_3}<0$ gives $y_{2a_3}>0$ so $y_{2a_2}=0$ would give the $\{1,2\}\times\{a_2,a_3\}$-submatrix of $y$: $\begin{bmatrix} y_{1a_2} & 0 \\ 0 & y_{2a_3} \end{bmatrix}$, with the diagonal terms strictly positive, in contradiction of statement $iii)$ in Lemma 3. Now $y_{2a_2},\partial_{2a_2}>0$ imply $d_{2a_2}>0$, and in turn $d_{1a_2}+d_{2a_2}>0$ while $r_{a_2}$ decreases at least weakly: so there is an agent 3 such that $d_{3a_2},\partial_{3a_2}<0$. Apply (9) at $1,3,a_1,a_2$ and at $2,3,a_2,a_3$ to get respectively $\partial_{3a_1}<0\Rightarrow d_{3a_1}\leq 0$ and $\partial_{3a_3}<0\Rightarrow d_{3a_3}\leq 0$, and finally $d_{3a_1}+d_{3a_2}+d_{3a_3}<0$ while $x_3$ stays put. Then we can pick $a_4$ such that $d_{3a_4},\partial_{3a_4}>0$.

From then on we construct sequences $3,4,5,\cdots$, and $a_1,a_2,a_3,a_4,\cdots$, such that

$$\partial_{ka_l}<0 \text{ for } 3\leq l\leq k, \text{ and } \partial_{ka_l}>0 \text{ for } 1\leq k<l$$

For instance to pick agent 4 we check first $\partial_{1a_4},\partial_{2a_4}>0$ by (9) at $1,3,a_1,a_4$ then at $2,3,a_1,a_4$, so that $d_{1a_4},d_{2a_4}\geq 0$ and $d_{1a_4}+d_{2a_4}+d_{3a_4}>0$. Then as usual there is some 4 such that $d_{4a_4},\partial_{4a_4}<0$, and we check $\partial_{4a_l}<0$ for $l=1,2,3$, by (9) at $3,4,a_l,a_4$. And so on. This construction cannot stop, so we reach a contradiction.

**Statement** $ii)$ If for some problem $P$ and some entry of $y=F^W(P)$ we have $y_{ia}=0$, then by statement $ii)$ in Lemma 4 and the construction of $\Gamma$ we get $\alpha_i+\beta_a\leq W'(0)$ which requires $W'(0)>-\infty$. Conversely if $W'(0)$ is finite, by continuity of $W'$ there is some $z,0<z<1$, such that $W'(1)+W'(0)>2W'(z)$. Then in the $2\times 2$ problem $P=(\{1,2\},\{a,b\},x=(1+z,z),r=(1+z,z))$ the rule $F^W$ chooses $\begin{bmatrix} 1 & z \\ z & 0 \end{bmatrix}$.

**Statement** $iii)$ The function $\Gamma$ for large values of $z$ is as follows:
Case 1: $W'(\infty)=\infty$, then $\Gamma(\alpha)$ is always finite and $\lim_{\alpha\rightarrow\infty}\Gamma(\alpha)=\infty$
Case 2: $W'(\infty)=C<\infty$, then $\lim_{\alpha\rightarrow C}\Gamma(\alpha)=\infty$, and $\Gamma(\alpha)=\infty$ for $\alpha\geq C$
We must show that $F^W$ meets BE in case 1, but not in case 2.

In case 1 pick $i,a$ and a sequence of problems $P^t=(N,A,x^t,r^t)$ as in the premises of BE; by statement $ii)$ of Lemma 4 there are parameters $\alpha^t\in\mathbb{R}^N,\beta^t\in\mathbb{R}^A$ such that $y^t_{jb}=\Gamma(\alpha^t_j+\beta^t_b)$ for all $j,b$ and $t$. We can always choose $\alpha^t,\beta^t$ such that $\alpha^t_i=\beta^t_a$, by adding a constant to all $\alpha^t_i$ and taking it away from the $\beta^t_a$. Set $\lambda^t=\alpha^t_i=\beta^t_a$. By assumption $y^t_{ia}=\Gamma(2\lambda^t)\rightarrow\infty$ as $t\rightarrow\infty$, therefore $\lambda^t\rightarrow\infty$. For $j\neq i$ we know that $\Gamma(\alpha^t_j+\lambda^t)$ is uniformly bounded in $t$, hence $\alpha^t_j\rightarrow-\infty$; similarly $\beta^t_b\rightarrow-\infty$ for any $b\neq a$: we conclude $y^t_{jb}=\Gamma(\alpha^t_j+\beta^t_b)\rightarrow\Gamma(-\infty)=0$ for all $j\neq i$ and $b\neq a$. Feasibility implies $x^t_j-y^t_{ja}=\sum_{b\neq a}y^t_{jb}\rightarrow 0$ for all $j\neq i$; then from $y^t_{ia}+\sum_{j\neq i}y^t_{ja}=r_a$ we deduce $y^t_{ia}-(r_a-x^t_{N\diagdown i})\rightarrow 0$, as was to be proved.

In case 2 we invoke the continuity of $W'$ to choose $z$ such that

$$W'(z) = \frac{1}{3}W'(0) + \frac{2}{3}C \Longleftrightarrow z = \Gamma(\frac{2}{3}C + \frac{1}{3}W'(0))$$

Then we consider the sequence $P^t = (\{1,2\}, \{a,b\}, x^t = (1+t, z), r^t = (1+t, z))$ where clearly $LB^t_{1a} = (1+t-z)_+ \to \infty$ with $t$, while $UB^t_{1b}, UB^t_{2a}, UB^t_{2b}$ remain bounded by $z$. In the parametrization of $y^t$ in statement $ii)$ of Lemma 4, we can choose $\alpha^t_i, \beta^t_b$ respecting the symmetry of $P^t$, so that for some $\lambda^t, \mu^t \in \mathbb{R}$ we have

$$y^t_{1a} = \Gamma(2\lambda^t) \text{ ; } y^t_{1b} = y^t_{2a} = \Gamma(\lambda^t + \mu^t) \text{ ; } y^t_{2b} = \Gamma(2\mu^t)$$

From $\Gamma(2\lambda^t) \to \infty$ we get $\lambda^t \to \frac{1}{2}C$. If $(y^t_{1a} - (1+t-z)) \to 0$ ,as requested by BE, then $y^t_{2b} \to 0$ by feasibility. By construction of $\Gamma$ the statement $\Gamma(2\mu^t) \to 0$ means $\limsup_{t\to\infty} \mu^t \leq \frac{1}{2}W'(0)$ therefore

$$\limsup_{t\to\infty} \{\Gamma(\lambda^t + \mu^t) + \Gamma(2\mu^t)\} \leq \Gamma(\frac{1}{2}W'(0) + \frac{1}{2}C)$$

which contradicts $z = \Gamma(\lambda^t + \mu^t) + \Gamma(2\mu^t)$ for all $t$.■

## 8.4 Theorem

We only need to prove the "only if" statement. We fix an assignment rule $F$ meeting CSY and MON$^*$.

**Step 1** *We discuss properties of the rationing rule $h^F$ associated with $F$, defined after the statement of the Theorem.*

We write simply $h$ in lieu of $h^F$ from now on, and choose a parametrization $\theta(x_i, \lambda)$ of $h$ as in (3). Recall $\lambda \in \mathbb{R} \cup \{-\infty, +\infty\}$, $\theta$ is continuous and weakly increasing in $\lambda$, and $\theta(x_i, -\infty) = 0$, $\theta(x_i, \infty) = x_i$. In particular $\theta(0, \lambda) = 0$ for all $\lambda$. Without loss of generality we assume that $\theta$ is "proper" in the sense that the two functions $\theta(\cdot, \lambda)$ and $\theta(\cdot, \lambda')$ are different if $\lambda \neq \lambda'$.

Self-duality of $h$ ((4)) implies $h(N, x, \frac{x_N}{2}) = \frac{1}{2}x$, therefore there is a parameter $\lambda^*$ such that $\theta(x_i, \lambda^*) = \frac{x_i}{2}$ for all $x_i$. In all examples of welfarist rules in Figures 1 to 5, we have $\lambda^* = 0$. We will say that $\lambda$ is *low if* $\lambda < \lambda^*$ and *high* if $\lambda > \lambda^*$. For a low $\lambda$ we have $\theta(z, \lambda) \leq \frac{z}{2}$ therefore we can have $z > 0$ and $\theta(z, \lambda) = 0$, but not $z > 0$ and $\theta(z, \lambda) = z$; for a high $\lambda$ the latter is possible but not the former.

Ranking (Remark 1 Section 5) implies at once that $\theta(z, \lambda)$ increases weakly in $z$. By self-duality $x_i > x_j \Longrightarrow x_i - h_i(N, x, t) \geq x_j - h_j(N, x, t)$, which then means that $\theta(z, \lambda)$ is weakly 1-contracting in $z$: $z < z' \Longrightarrow \theta(z', \lambda) - \theta(z, \lambda) \leq z' - z$.

For a low $\lambda$ we set $\tau_\lambda = \sup\{z | \theta(z, \lambda) = 0\}$ so that $\theta(z, \lambda) = 0$ exactly on $[0, \tau_\lambda]$ and $\theta(z, \lambda) < z$ for all $z$.

For a high $\lambda$ we set $\tau_\lambda = \sup\{z | \theta(z, \lambda) = z\}$ so that $\theta(z, \lambda) = z$ exactly on $[0, \tau_\lambda]$ (by weak 1-contraction) and $\theta(z, \lambda) > 0$ for all $z > 0$.

Whether $\lambda$ is high or low, we say that $z$ is *$\lambda$-regular* if $z \in ]\tau_\lambda, \infty[$, i.e., if $0 < \theta(z,\lambda) < z$. Note that all values are $\lambda^*$-regular, while there is no $\pm\infty$-regular value.

For the scale invariant welfarist rules $F^{W^q}$ of Lemma 2, $\tau_\lambda$ is zero if $q \leq 1$ and positive otherwise: for instance $F^{W^2}$ (Figure 4) has $\tau_\lambda = \frac{|\lambda|}{2}$, and $F^{W^3}$ (Figure 5) has $\tau_\lambda = \sqrt{|\lambda|}$.

Property MON$^*$ applied to two-resource problems implies that $t \to h_i(N,x,t)$ is strictly increasing whenever $0 < h_i(N,x,t) < x_i$, for all $i$. This in turn means that $\theta(z,\lambda)$ increases strictly in $\lambda$ at $\lambda$-regular values. Suppose, on the contrary, $0 < \theta(z,\lambda) = \theta(z,\lambda') < z$ and $\lambda < \lambda'$. Because $\theta$ is proper there is some $z'$ such that $\theta(z',\lambda) < \theta(z',\lambda')$, but then the strict monotonicity of $t \to h_i(N,x,t)$ is violated going from $(\{1,2\},(z,z'),\theta(z,\lambda)+\theta(z',\lambda))$ to $(\{1,2\},(z,z'),\theta(z,\lambda')+\theta(z',\lambda'))$.

Ranking$^*$ implies similarly $\{x_i > x_j,\ t > 0, \text{ and } h_i(N,x,t) > 0\} \Longrightarrow h_i(N,x,t) > h_j(N,x,t)$. From this we see easily that $\theta(z,\lambda)$ increases strictly in $z$ whenever $\theta(z,\lambda) > 0$. Apply RKG$^*$ to the dual of $h$, which is $h$ itself, to get $\{x_i > x_j,\ t < x_N, \text{ and } h_i(N,x,t) < x_i\} \Longrightarrow x_i - h_i(N,x,t) > x_j - h_j(N,x,t)$. Therefore $\theta(z,\lambda)$ is strictly 1-contracting in $z$ whenever $\theta(z,\lambda) < z$. Gathering our results we have shown

*$\theta(z,\lambda)$ increases strictly in $z$ whenever $\theta(z,\lambda) > 0$, which is true for all $z > 0$ if $\lambda \geq \lambda^*$, and all $\lambda$-regular values if $\lambda < \lambda^*$*

*$\theta(z,\lambda)$ increases strictly in $\lambda$ at all $\lambda$-regular values*

*$\theta(z,\lambda)$ is strictly 1-contracting in $z$ at all $\lambda$-regular values: for all $z' \in \mathbb{R}$*

$$z < z' \Longrightarrow 0 < \theta(z',\lambda) - \theta(z,\lambda) < z' - z \ ; \ z' < z \Longrightarrow 0 < \theta(z,\lambda) - \theta(z',\lambda) < z - z' \tag{11}$$

Fix $\lambda$ and consider a pair $N,x$, such that $|N| \geq 2$ and each $x_i$ is $\lambda$-regular. Self duality applied to the problem $(N,x,t = \sum_N \theta(x_i,\lambda))$ implies there is some $\lambda'$ such that $\theta(x_i,\lambda) + \theta(x_i,\lambda') = x_i$ for all $i$. Fix $\lambda$ and take two pairs $N,x$, $N',x'$ such that $N,N'$ overlap but are not nested: because $\theta$ increases strictly in $\lambda$ for those values of $x_i, x'_i$, the parameter $\lambda'$ is the same for $N,x$ and $N',x'$; therefore it does not depend on $N,x$ at all. This defines the "inverse" $\lambda^{-1}$ of $\lambda$ by the identity

$$\theta(z,\lambda) + \theta(z,\lambda^{-1}) = z \tag{12}$$

for all $\lambda$-regular values $z$. Moreover $(\lambda^{-1})^{-1} = \lambda$. In particular $(\lambda^*)^{-1} = \lambda^*$ and $\infty^{-1} = -\infty$. Note that $\lambda^{-1}$ is high if and only if $\lambda$ is low and vice versa,because $\theta(z,\lambda^*) = \frac{z}{2}$.

We check now $\tau_\lambda = \tau_{\lambda^{-1}}$ and that (12) holds in fact for all $z$. Suppose $\lambda$ is low. Then (12) holds for $z = \tau_\lambda$ by CONT, and implies $\tau_{\lambda^{-1}} \geq \tau_\lambda$. The opposite inequality obtains by applying (12) at $\lambda^{-1}$. Now (12) holds as well on $[0,\tau_\lambda]$ by definition of $\tau_\lambda, \tau_{\lambda^{-1}}$.

**Step 2** *We define an alternative parametrization $\pi(z,\lambda)$ of the rationing rule $h$.*

Keep in mind that in this step all variables are of dimension 1.

For all $z \geq 0$ and $-\infty \leq \lambda \leq +\infty$, $\pi(z,\lambda)$ solves the following equation in $y$ :

$$y = \pi(z,\lambda) \Leftrightarrow y = \theta(z+y,\lambda) \tag{13}$$

Such a solution may not exist: for instance if $\lambda = +\infty$ it boils down to $y = z+y$. On the other hand for $\lambda = -\infty$ we have $\pi(z,-\infty) = 0$ for all $z_i$, and $\theta(z,\lambda^*) = \frac{z}{2}$ implies $\pi(z,\lambda^*) = z$ for all $z$, so $\pi$ is well defined for $\lambda = -\infty, \lambda^*$.

We claim that if $\lambda$ is low, $\lambda < \lambda^*$, $\pi(z,\lambda)$ is uniquely defined for all $z$. We have $0 \leq \theta(z+0,\lambda)$ and $\theta(z+y,\lambda) \leq \theta(z+y,\lambda^*) = \frac{z+y}{2} \leq y$ for $y \geq z$, so (13) has a solution by continuity of $\theta$ in $z$. Suppose $z$ is $\lambda$-regular and (13) has two solutions $y, y'$ and $y < y'$: then $\theta(z+y',\lambda) - \theta(z+y,\lambda) = y'-y$, a contradiction of the 1-contracting property (11) because $z+y$ is still $\lambda$-regular. Now if $z \leq \tau_\lambda$ one solution of (13) is $y = 0$, and if there was another solution $y'$, $z+y'$ would be $\lambda$-regular and the previous argument would apply. This proves the claim, and that $\pi(z,\lambda) = 0$ for $z \leq \tau_\lambda$.

We turn to the case of a high value $\lambda > \lambda^*$, where two difficulties arise. First equation (13) has multiple solutions at $z = 0$ if $\tau_\lambda > 0$ (recall $\theta(y,\lambda) = y$ on $[0,\tau_\lambda]$), and in this case we set $\pi(0,\lambda) = [0,\tau_\lambda]$. Thus $\pi$ can be multi-valued, but only for $z = 0$. Indeed pick $z > 0$ and assume $y, y'$ are two solutions of (13): both $z+y$ and $z+y'$ are above $\tau_\lambda$ by definition of the latter, and $\theta(z+y',\lambda) - \theta(z+y,\lambda) = y'-y$ contradicts (11).

The next problem is that the equation may have no solution at all. Define for all $\lambda > 0$

$$\kappa_\lambda = \lim_{x\to\infty} \{x - \theta(x,\lambda)\} \tag{14}$$

This number is positive (because $\theta(x,\lambda) \equiv x$ holds only if $\lambda = +\infty$) and possibly infinite. For instance $\kappa_\lambda = \infty$ for $F^{W^2}$ and $F^{W^3}$ on Figures 4,5 because the slope $\partial_x\theta(x,\lambda)$ goes to $\frac{1}{2}$; but for $\lambda$ positive $\kappa_\lambda = \frac{1}{2\lambda}$ for $F^{W^0}$ on Figure 1 because $\partial_x\theta(x,\lambda)$ goes to 1, and similarly $\kappa_\lambda = \frac{1}{\sqrt{\lambda}}$ for $F^{W^{-1}}$ on Figure 2, $\kappa_\lambda = \frac{1}{4\lambda^2}$ for $F^{W^{\frac{1}{2}}}$ on Figure 3.

We claim that for a high $\lambda$ equation (13) has a solution if and only if $z < \kappa_\lambda$. Fix such a pair $z, \lambda$, and choose $x$ such that $z < x - \theta(x,\lambda) \Longleftrightarrow \theta(z+y,\lambda) < y$ for $y = x-z$, so the "if" statement follows as above by continuity of $\theta$ in $z$ and $0 \leq \theta(z+0,\lambda)$. If $z \geq \kappa_\lambda$, then $x - \theta(x,\lambda) < z$ for all $x$ as $x - \theta(x,\lambda)$ increases strictly. Hence $y < \theta(z+y,\lambda)$ for $y = x-z$, where $y$ ranges over $\mathbb{R}_+$, and the claim is proven.

Now for a low value $\lambda \leq \lambda^*$ we have $\kappa_\lambda = \infty$, so the claim is true for all $\lambda$.

*To sum up, $\pi(z,\lambda)$ is defined for all $z < \kappa_\lambda$ and only there; it is then single-valued, except possibly at zero if $\lambda > \lambda^*$: $\pi(0,\lambda) = [0,\tau_\lambda]$.*

*Moreover we have the following self-duality properties, for all $\lambda, -\infty \leq \lambda \leq +\infty$:*

$$\lim_{z\to\kappa_\lambda} \pi(z,\lambda) = \kappa_{\lambda^{-1}} \tag{15}$$

$$y = \pi(z,\lambda) \Leftrightarrow z = \pi(y,\lambda^{-1}) \text{ for all } z < \kappa_\lambda \tag{16}$$

The first statement is clear for $\lambda = \pm\infty$. For $\lambda \in \mathbb{R}$ note that (12) and (14) together imply $\kappa_{\lambda^{-1}} = \lim_{x\to\infty} \theta(x,\lambda)$. If $\kappa_\lambda = \infty$ the equality $\lim_{z\to\infty} \pi(z,\lambda) = \lim_{x\to\infty} \theta(x,\lambda)$ is clear by definition of $\pi$. If $\kappa_\lambda < \infty$, then $\lambda \geq \lambda^*$ and $\kappa_{\lambda^{-1}} = \infty$, so we must check $\lim_{z\to\kappa_\lambda} \pi(z,\lambda) = \infty$. Fix an arbitrary large $w > 0$, $x$ such that $x > w + \kappa_\lambda$, and $z$ such that $x - \theta(x,\lambda) < z < \kappa_\lambda$. Then $\pi(z,\lambda)$ is well defined and $\theta(z + (x-z), \lambda) > (x-z)$ implying $\pi(z,\lambda) > (x-z)$ because $\theta$ is 1-contracting in $z$, and in turn $\pi(z,\lambda) > w$. The second statement follows from (12) once we see that $y < \kappa_{\lambda^{-1}}$ is a consequence of $\lim_{z\to\kappa_\lambda} \pi(z,\lambda) = \kappa_{\lambda^{-1}}$.

Clearly $\pi$ is continuous in $(z,\lambda)$ because $\theta$ is so the graph of $\pi$ is closed. In particular for $\lambda > \lambda^*$ the correspondence $z \to \pi(z,\lambda)$ is upper-heni continuous at 0.

We check finally the monotonicity properties of $\pi$:

$\pi(z,\lambda)$ *increases weakly in* $z$ *and* $\lambda$

$\pi(z,\lambda)$ *increases strictly in* $z$ *if* $\lambda \leq \lambda^*$ *and* $z \geq \tau_\lambda$*, and if* $\lambda > \lambda^*$

$\pi(z,\lambda)$ *increases strictly in* $\lambda$ *if* $\lambda \leq \lambda^*$ *and* $z \geq \tau_\lambda$*, and if* $\lambda > \lambda^*$ *and* $z > 0$

For the second claim fix $\lambda \leq \lambda^*$, $z, z'$ such that $\tau_\lambda < z < z'$, and set $y = \pi(z,\lambda), y' = \pi(z',\lambda)$. Then $\theta(\cdot,\lambda)$ increases strictly from $z+y$ to $z'+y$, therefore $y < \theta(z'+y,\lambda)$, while $y' = \theta(z'+y',\lambda)$; so $y' = y$ is impossible, and $y' < y$ implies that $t \to t - \theta(z'+t,\lambda)$ decreases from $y'$ to $y$, in contradiction of (11). Next take $\lambda > \lambda^*$, $z, z'$ such that $0 < z < z' < \kappa_\lambda$, and set $y = \pi(z,\lambda), y' = \pi(z',\lambda)$. We showed in step 1 that $\theta(\cdot,\lambda)$ increases strictly everywhere, moreover $y, y' > \tau_\lambda$ implies that $\theta(\cdot,\lambda)$ is strictly 1-contracting between $z+y$ and $z'+y'$, therefore the same argument applies. The case $z = 0 < z'$ is left to the reader for brevity.

For the third claim fix $\lambda, \lambda'$ such that $\lambda < \lambda' \leq \lambda^*$, $z \geq \tau_\lambda \geq \tau_{\lambda'}$ and $y = \pi(z,\lambda), y' = \pi(z,\lambda')$. As $\theta$ increases strictly in $\lambda$ (shown in step 1) we have $y < \theta(z+y,\lambda')$, while $y' = \theta(z+y',\lambda')$. Thus $y' \leq y$ contradicts (11) exactly as in the previous paragraph. For the case $\lambda^* \leq \lambda < \lambda'$, $0 < z < \kappa_{\lambda'} \leq \kappa_\lambda$, and $y = \pi(z,\lambda), y' = \pi(z,\lambda')$, we have $y > \tau_\lambda, y' > \tau_{\lambda'}$, so $z+y$ is $\lambda$-regular: therefore $\theta(z+y,\cdot)$ increases strictly from $\lambda$ to $\lambda'$, and the contracting property (11) holds, so the same argument applies.

Finally $\pi(z,\lambda) = 0$ whenever $z < \tau_\lambda$ implies the first claim.

**Step 3** *The role of assumptions PE and BE.*

Positive Entries implies $h_i(N,x,t) > 0$ if $x_i, t > 0$. This is clearly equivalent to $\theta(x_i,\lambda) > 0$ when $x_i > 0$ and $\lambda > -\infty$ (recall from Step 1 that $\theta$ is a proper parametrization). By definition of $\tau_\lambda$, we deduce $\tau_\lambda = 0$ for a low $\lambda$. Now $\lambda \to \lambda^{-1}$ exchanges low and high values, and we showed $\tau_\lambda = \tau_{\lambda^{-1}}$ at the end of step1. So for all $\lambda > -\infty$ we have $\tau_\lambda = 0$, i.e., all positive values are $\lambda$-regular. In particular

**PE implies** $\tau_\lambda \equiv 0$ *so* $\pi(z,\lambda) > 0$ *if and only if* $z > 0$*;* also $\pi(z,\lambda)$ *is single-valued , continuous, and strictly increasing in both variables everywhere on* $0 \leq z < \kappa_\lambda$: for low $\lambda$ we have $\kappa_\lambda = \infty$; for high $\lambda$, $\pi(z,\lambda)$ *is not defined for* $z \geq \kappa_\lambda$.

Applying Bounded Entries to the sequence of assignment problems $(N, (x_1 + \delta, x_{-1}), (t, x_N + \delta - t))$ we see that $h(N, (x_1+\delta, x_{-1}), t) = y \to (t, 0_{-1})$ as $\delta$ goes to $\infty$. We check that this implies $\lim_{x\to\infty}\theta(x,\lambda)=\infty$ for all $\lambda>-\infty$. Assume $\lambda$ is such that $\lim_{x\to\infty}\theta(x,\lambda)=\omega<\infty$ and derive a contradiction. Note first that $\lambda$ must be low. Choose $w$ such that $\theta(w,\lambda)=\frac{3}{4}\omega$, and $x_1$ large enough that it is $\lambda$-regular and $x_1>w$. Consider the solution $(\theta(x_1,\lambda'),\theta(w,\lambda'))$ of the two-person problem $(\{1,2\},(x_1,w),t=\frac{3}{2}\omega)$: from $x_1>w$, $\theta(w,\lambda)+\theta(w,\lambda)=t$, and $\theta(\cdot,\lambda)$ increases strictly by $\lambda$-regularity, we deduce $\lambda'<\lambda$. Thus

$$\theta(x_1,\lambda')<\theta(x_1,\lambda)<\omega=t-\frac{\omega}{2}$$

so that $\theta(w,\lambda')\geq\frac{\omega}{2}$ no matter how large $x_1$, a contradiction of BE.

Pick now a low $\lambda$, and recall from step 2 that $\pi(z,\lambda)$ is defined for all $z$. By the very definition (13) of $\pi$, $\lim_{z\to\infty}\theta(z,\lambda)=\infty$ implies $\lim_{z\to\infty}\pi(z,\lambda)=\infty$, and by (15) this gives $\kappa_{\lambda^{-1}}=\infty$, in other word $\pi(z,\lambda^{-1})$ is defined for all $z$ as well. We have shown

**BE implies $\kappa_\lambda\equiv\infty$:** *$\pi(z,\lambda)$ is defined and continuous everywhere; for low $\lambda$ it is zero iff $0\leq z\leq\tau_\lambda$ and for high $\lambda$ we have $\pi(0,\lambda)=[0,\tau_\lambda]$; elsewhere $\pi(z,\lambda)$ strictly increases in both variables.*

**Step 4** *We characterize the $F$-fairness of the matrix $y\in\mathbb{R}_+^{N\times A}$ in terms of $\pi$.*

Writing the $a$-column of $y$ as $y_{[a]}$, we prove two facts.

**Fact 1:** *If $y$ is $F$-fair and $y_{Nb}>0$ for all $b\in A$ ($y$ has no null column), then for any $a\in A$ there is a unique parameter $\lambda_b\in\mathbb{R}$, one for each $b\neq a$, such that*

$$y_{ib}=\pi(y_{ia},\lambda_b)\text{ for all }i\in N\text{ , all }b\in A\diagdown\{a\}\qquad(17)$$

Indeed for each $b\neq a$ the reduced $N\times\{a,b\}$ matrix $[y_{[a]},y_{[b]}]$ is $F$-fair by CSY, therefore $y_{[b]}$ is just $h(N,x^{ab},r_b)$ for the profile of demands $x_i^{ab}=y_{ia}+y_{ib}$. By definition of the parametrization $\theta$, there is some $\lambda_b$ such that $y_{ib}=\theta(x_i^{ab},\lambda_b)$. We cannot have $\lambda_b=\pm\infty$ because no column of $y$ is null. Thus $y_{ib}=\theta(x_i^{ab},\lambda_b)\Longleftrightarrow y_{ib}=\pi(y_{ia},\lambda_b)$ for all $i$, as claimed. Uniqueness follows if $\pi(z,\lambda)$ increases strictly in $\lambda$. By step 2 if this is not true we must have either $z=0$ (and $\lambda>\lambda^*$) or $z<\tau_\lambda$ and $\pi(z,\lambda)=0$: neither case can arise because in (17) $y_{ia}$ is positive for some $i$, and $y_{jb}$ for some $j$.

System (17) is written below as $y_{[b]}=\pi(y_{[a]},\lambda_b)$. Now we prove the converse of Fact 1:

**Fact 2:** *Fix $a\in A$ and for each $b\neq a$ some parameter $\lambda_b\in\mathbb{R}$. Choose a column $y_{[a]}$ such that $0\leq y_{ia}<\min_{b\in A\diagdown\{a\}}\kappa_{\lambda_b}$ for all $i$, and define $y_{[b]}=\pi(y_{[a]},\lambda_b)$. Then the matrix $y=[y_{[a]},y_{[b]}]_{b\in A\diagdown a}$ is $F$-fair.*

*Proof* If $y_{[a]}=0$ the statement is trivial, so we assume without loss $y_{[a]}\neq 0$. If $y_{[b]}=0$ for some $b$ as well we simply drop it and prove that the reduced matrix is $F$-fair: by CSY augmenting a $F$-fair matrix by a null column maintains its fairness. So we assume from now on $y_{[b]}\neq 0$ for all $b$. Let $x,r$ be the sums of rows and columns of $y$, and $\widetilde{y}=F(x,r)$. We show $y=\widetilde{y}$. By Fact 1 there are parameters $\widetilde{\lambda}_b$ such that $\widetilde{y}_{[b]}=\pi(\widetilde{y}_{[a]},\widetilde{\lambda}_b)$ for all $b\neq a$. Assume first $y_{[a]}=\widetilde{y}_{[a]}$. As $y$ and $\widetilde{y}$ have the same column sums, this implies $\sum_N\pi(y_{ia},\lambda_b)=\sum_N\pi(y_{ia},\widetilde{\lambda}_b)$ for all $b\neq a$. If this equality holds with $\lambda_b\neq\widetilde{\lambda}_b$ then $\pi(y_{ia},\lambda_b)=$

$\pi(y_{ia}, \widetilde{\lambda}_b)$ for all $i$ because $\pi$ increases weakly in $\lambda$ (step 2) implying $y_{[b]} = \widetilde{y}_{[b]}$; this equality is also true if $\lambda_b = \widetilde{\lambda}_b$ so we are done.

Assume next $y_{[a]} \neq \widetilde{y}_{[a]}$ and partition $N$ as $N^+ = \{i \in N | y_{ia} > \widetilde{y}_{ia}\}$ and $N^- = \{i \in N | y_{ia} \leq \widetilde{y}_{ia}\}$, both non empty. Set $\lambda_a = \widetilde{\lambda}_a = \lambda^*$ so that $y_{[b]} = \pi(y_{[a]}, \lambda_b)$ and $\widetilde{y}_{[b]} = \pi(\widetilde{y}_{[a]}, \widetilde{\lambda}_b)$ hold for all $b$. Define $A^+ = \{b \in A | \lambda_b \geq \widetilde{\lambda}_b\}$, containing $a$, and $A^- = \{b \in A | \lambda_b < \widetilde{\lambda}_b\}$, which could be empty. Write $d_{ib} = y_{ib} - \widetilde{y}_{ib}$. The monotonicity properties of $\pi$ (step 2) imply $d_{ib} \leq 0$ if $i \in N^-, b \in A^-$ and $d_{ib} \geq 0$ if $i \in N^+, b \in A^+$; moreover $d_{ia} > 0$ for $i \in N^+$ and $a$ is in $A^+$. Therefore

$$\sum_{N^+ \times A^+} d_{ib} > 0 \ ; \ \sum_{N^- \times A^-} d_{ib} \leq 0$$

Sum up all columns in $A^+$

$$\sum_{N^+ \times A^+} d_{ib} + \sum_{N^- \times A^+} d_{ib} = 0 \Longrightarrow \sum_{N^- \times A^+} d_{ib} < 0$$

then sum all rows in $N^-$

$$\sum_{N^- \times A^-} d_{ib} + \sum_{N^- \times A^+} d_{ib} = 0 \Longrightarrow \sum_{N^- \times A^+} d_{ib} \geq 0$$

a contradiction.

**Step 5** *We define an inner product $\lambda * \mu$ for the parameters $\lambda, \mu \in \mathbb{R}$, not necessarily distinct.*

We use the equality $\pi(\pi(z, \lambda), \mu) = \pi(z, \lambda * \mu)$, for all $z$ such that $\pi(\pi(z, \lambda), \mu)$ is well defined. Problems can arise if $z$ is too large, i.e., $z \geq \kappa_\lambda$ or $\pi(z, \lambda) \geq \kappa_\mu$, then the expression is not defined; or if $z$ is too small, i.e., $z < \tau_\lambda$ so $\pi(z, \lambda) = 0$ and $\pi(\pi(z, \lambda), \mu)$ is multi-valued. Our assumptions PE or BE allow us to focus on one type of problem only.

**Case 1 PE holds** *We claim that $\pi(\pi(z, \lambda), \mu)$ is well defined if and only if $z < \pi(\kappa_\mu, \lambda^{-1})$, with the convention $\pi(z, \lambda) = \kappa_{\lambda^{-1}}$ if $z \geq \kappa_\lambda$.*

Supose first $\lambda \leq \lambda^*$ so that $\pi(z, \lambda)$ is well defined for all $z$ and we only need $\pi(z, \lambda) < \kappa_\mu$. By (15) we have $\lim_{z \to \infty} \pi(z, \lambda) = \kappa_{\lambda^{-1}}$, so there are no restrictions on $z$ if $\kappa_\mu \geq \kappa_{\lambda^{-1}}$, just as we claim: $\pi(\kappa_\mu, \lambda^{-1}) = \kappa_\lambda = \infty$. If on the other hand $\kappa_\mu < \kappa_{\lambda^{-1}}$, then the only restriction is $\pi(z, \lambda) < \kappa_\mu \Longleftrightarrow z < \pi(\kappa_\mu, \lambda^{-1})$ (by (16)). Suppose next $\lambda > \lambda^*$, then we need $z < \kappa_\lambda$ and $\pi(z, \lambda) < \kappa_\mu$. Now $\lim_{z \to \kappa_\lambda} \pi(z, \lambda) = \kappa_{\lambda^{-1}} = \infty$. If $\kappa_\mu < \infty$ we have $\pi(z, \lambda) < \kappa_\mu \Longleftrightarrow z < \pi(\kappa_\mu, \lambda^{-1})$ and $\pi(\kappa_\mu, \lambda^{-1}) < \lim_{z \to \infty} \pi(z, \lambda^{-1}) = \kappa_\lambda$, proving the claim. If $\kappa_\mu = \infty$, we only need $z < \kappa_\lambda$, and with our convention $\pi(\kappa_\mu, \lambda^{-1}) = \kappa_\lambda$. The claim is proven.

**Case 2 BE holds** *We claim that $\pi(\pi(z, \lambda), \mu)$ is well defined and strictly positive if $z > \pi(\tau_\mu, \lambda^{-1})$.*

Recall from step 3 that $\pi(z, \lambda)$ is well defined everywhere, single valued if $z > 0$, and strictly positive if $z > \tau_\lambda$. To ensure $\pi(\pi(z, \lambda), \mu) > 0$ we need $\mu \geq \lambda^*$ and/or $\pi(z, \lambda) > \tau_\mu$; by (16) the latter inequality is $z > \pi(\tau_\mu, \lambda^{-1})$.

**Defining** $\lambda * \mu$. We fix $\lambda, \mu$ and set in case 1 $J(\lambda,\mu) = ]0, \pi(\kappa_\mu, \lambda^{-1})[$, while in case 2 $J(\lambda,\mu) = ]\pi(\tau_\mu, \lambda^{-1}), \infty[$. We also choose $A = \{a, b, c\}$ and some $N$ with $|N| \geq 3$. For an arbitrary profile $z = (z_i) \in J(\lambda,\mu)^N$ we construct the $N \times \{a, b, c\}$ assignment matrix $y$ with strictly positive entries:

$$y_{ia} = z_i \ , \ y_{ib} = \pi(z_i, \lambda) \ , \ y_{ic} = \pi(\pi(z_i, \lambda), \mu) \text{ for all } i$$

Setting $z_i' = y_{ib}$, and using (16), an equivalent description of the matrix is

$$y_{ia} = \pi(z_i', \lambda^{-1}) \ , \ y_{ib} = z_i' \ , \ y_{ic} = \pi(z_i', \mu) \text{ for all } i$$

The matrix $y$ is $F$-fair by Fact 2 in step 4 applied to the latter description of $y$. By Fact 1 applied to the former description of $y$, there exists a parameter $\rho$ such that $y_{ic} = \pi(y_{ia}, \rho)$. Thus we have $\pi(\pi(z_i, \lambda), \mu) = \pi(z_i, \rho)$ for all $z_i$. Clearly $\rho$ is unique because $\pi(z_i, \rho)$ increases strictly in $\rho$ (step 2), and in fact it does not depend at all on the choice of the $z_i$-s: if we take two profiles $N, z$ and $N', z'$ such that $N, N'$ overlap in at least one coordinate $z_1 = z_1'$, then $\pi(z_1, \rho)$ is the same for both profiles, implying that $\rho$ did not change. We define $\lambda * \mu = \rho$ and we have

$$\pi(\pi(z, \lambda), \mu) = \pi(z, \lambda * \mu) \text{ for all } z \in J(\lambda, \mu) \tag{18}$$

The identity $\pi(z, \lambda^*) = z$ means that $\lambda^*$ is the neutral element of this operation, and (16) implies $\lambda * \lambda^{-1} = \lambda^*$. Note that we do not define $\lambda * \mu$ when one of them is $\pm\infty$.

**Step 6** *We show that $\lambda * \mu$ is an associative product, and derive an additive representation of this operation from the Associativity Functional Equation ([1], [12]).*

For any three parameters $\lambda, \mu, v$, associativity follows the repeated application of (18):

$$\pi(z, (\lambda*\mu)*v) = \pi(\pi(z, (\lambda*\mu)), v) = \pi(\pi(\pi(z, \lambda), \mu), v) = \pi(\pi(z, \lambda), \mu*v) = \pi(z, \lambda*(\mu*v))$$

where those expressions are all well defined for $z$ in a positive interval $[0, K[$ if PE holds, or in an interval $]K, \infty[$ if BE is true. For instance under PE $\pi(\pi(\pi(z, \lambda), \mu), v)$ is well defined whenever $\pi(z, \lambda)$ is well defined, and $\pi(z, \lambda) < \pi(\kappa_v, \mu^{-1})$, which amounts to $z < \kappa_\lambda$ and $z < \pi(\pi(\kappa_v, \mu^{-1}), \lambda^{-1})$. We omit the similar arguments for the other four terms, and under assumption BE.

Associativity of $*$ implies the identity $(\lambda*\mu)*(\mu^{-1}*\lambda^{-1}) = \lambda^*$, i.e., $(\lambda*\mu)^{-1} = \mu^{-1} * \lambda^{-1}$.

We check now that $(\lambda, \mu) \to \lambda * \mu$ is continuous and strictly increasing in both variables. For continuity pick any two $\overline{\lambda}, \overline{\mu}$ and observe that in a small enough neighborhood of $(\overline{\lambda}, \overline{\mu})$, equation (18) in $z$ holds on a non empty open interval: under PE this is because $\pi(\kappa_\mu, \lambda^{-1})$ decreases weakly in both $\lambda$ and $\mu$; under BE the same is true of $\pi(\tau_\mu, \lambda^{-1})$. As $\pi$ is strictly monotonic in both variables by construction, the product $\lambda * \mu$ is defined in this neighborhood of $(\overline{\lambda}, \overline{\mu})$ by equation (18) at a single value $z$: therefore the graph of $(\lambda, \mu) \to \lambda * \mu$ is closed by continuity of $\pi$, so this mapping is continuous at $(\overline{\lambda}, \overline{\mu})$. By the same argument it increases strictly in $\lambda$ and $\mu$.

By the Associativity Theorem (section 6.2 in [1]) an associative, continuous, and strictly monotonic product $*$ in $\mathbb{R}$ is represented as follows by a continuous and strictly increasing function $g$ on $\mathbb{R}$:

$$\lambda * \mu = g^{-1}(g(\lambda) + g(\mu)) \text{ for all } \lambda, \mu \in \mathbb{R}$$

In particular $g(\lambda^*) = 0$ is the neutral element, and $\lambda * \lambda^{-1} = \lambda^*$ becomes $g(\lambda^{-1}) = -g(\lambda)$.

Thus $g$ is an homeomorphism of $\mathbb{R}$ into its range, and its range must be $\mathbb{R}$: it is an interval stable by addition and symmetry around 0. We use now the new variable $\beta = g(\lambda)$ to parametrize the rationing rule $h$: we set $\widetilde{\theta}(z, \beta) = \theta(z, g^{-1}(\beta))$ for $\beta \in \mathbb{R}$, and $\widetilde{\theta}(z, -\infty) = 0$, $\widetilde{\theta}(z, +\infty) = z$. The rule $h$ is still represented by $\widetilde{\theta}$ through property (3), and the entire discussion of steps 1 to 5, including the definition of $\widetilde{\pi}$ through (13), the domain restrictions captured by the critical parameters $\widetilde{\tau}_\beta = \tau_{g^{-1}(\beta)}$ , $\widetilde{\kappa}_\beta = \kappa_{g^{-1}(\beta)}$, and the regularity properties of $\widetilde{\theta}$ and $\widetilde{\pi}$, are preserved. The advantage of this parametrization is that equation (18) takes the form

$$\widetilde{\pi}(\widetilde{\pi}(z, \beta), \gamma) = \widetilde{\pi}(z, \beta + \gamma) \text{ for all } z \in J(\lambda, \mu) \tag{19}$$

Now the duality operation is just $\beta^{-1} = -\beta$ and $\beta^* = 0$.

**Step 7** *We show that $F$ is a $W$-welfarist assignment rule.*

The goal is to construct the function $\Gamma$ in statement *ii*) of Lemma 3, then derive $W$ from $\Gamma = W'^{-1}$. First we check that the range of $\beta \to \widetilde{\pi}(1, \beta)$ over all $\beta$ for which it is defined (i.e., such that $1 < \widetilde{\kappa}_\beta$), is $\mathbb{R}_+$. By continuity of $\widetilde{\theta}$, for any $\Delta > 0$ there exists $\beta$ such that $\Delta - \widetilde{\theta}(\Delta + 1, \beta) = 0$, because this expression is $\Delta > 0$ at $\beta = -\infty$ and $-1$ at $\beta = \infty$. This equality is just $\Delta = \widetilde{\pi}(1, \beta)$. Note that $\widetilde{\pi}(1, \beta)$ is defined for all $\beta \leq 0$ (by step 2, e.g., $\widetilde{\pi}(1, 0) = 1$) and for $\beta <$, where $C = \min\{\beta | \widetilde{\kappa}_\beta \leq 1\}$ (by step 2 $\pi(z, \lambda)$ is defined *iff* $z < \kappa_\lambda$). Under BE we have $C = +\infty$, but this is not necessarily the case under PE.

We pick now an arbitrary problem $P = (N, A, x, r)$ with all $x_i, r_a > 0$, and set $y = F(N, A, x, r)$. Fix an arbitrary $a \in A$ and use Fact 1 in step 4: for each $b \neq a$ there is a parameter $\beta_b$ such that $y_{[b]} = \widetilde{\pi}(y_{[a]}, \beta_b)$. On the other hand the argument in the previous paragraph shows that for each $i$ there is a parameter $\alpha_i$ such that $y_{ia} = \widetilde{\pi}(1, \alpha_i)$. Combining these equations and (19) gives $y_{ib} = \widetilde{\pi}(y_{ia}, \beta_b) = \widetilde{\pi}(\widetilde{\pi}(1, \alpha_i), \beta_b) = \widetilde{\pi}(1, \alpha_i + \beta_b)$ for all $i$, all $b \in A \diagdown a$. We set $\beta_a = 0$ so that we have

$$y_{ib} = \widetilde{\pi}(1, \alpha_i + \beta_b) \text{ for all } i \in N, b \in A \tag{20}$$

Now we define $\Gamma$ from $\mathbb{R} \cup \{\pm\infty\}$ onto $\mathbb{R}_+$:

$$\Gamma(-\infty) = 0 \text{ ; } \Gamma(\alpha) = \widetilde{\pi}(1, \alpha) \text{ if } -\infty < \alpha < C \text{ ; } \Gamma(\alpha) = \infty \text{ if } \alpha \geq C$$

From the properties of $\widetilde{\pi}$ in step 2 $\Gamma$ is continuous and weakly increasing, strictly so if $\Gamma(\alpha) > 0$; also $\Gamma(0) = 1$. From the more detailed properties in step 3 we have also

$\rightarrow$ if PE holds $C$ may be finite, and $\Gamma$ is an increasing homeomorphism from $[-\infty, C]$ into $[0, \infty]$;
$\rightarrow$ if BE holds $C = \infty$ and $\Gamma(\alpha) = 0$ for all $\alpha \leq D = \max\{\beta | \beta \leq 0 \text{ and } \widetilde{\tau}_\beta \geq 1\}$ (which could be $-\infty$); then $\Gamma$ is an increasing homeomorphism from $[D, \infty]$ into $[0, \infty]$.

We define $W' : [0, \infty] \rightarrow \mathbb{R}$ to be the inverse of $\Gamma$ in the following precise sense:

$$\text{for } 0 < t < \infty : W'(t) = \alpha \Longleftrightarrow \Gamma(\alpha) = t$$

$$\text{if PE holds: } W'(0) = -\infty \text{ and } W'(+\infty) = C$$

$$\text{if BE holds: } W'(0) = D \text{ and } W'(+\infty) = +\infty$$

so that $W'$ is continous and strictly increasing.

Then (20) says precisely $y_{ia} = \Gamma(\alpha_i + \beta_a)$ for all $i, a$, and $\Gamma$ is constructed from $W'$ exactly as in Lemma 3 section 5, therefore $y = F^W(P)$ where $W$ is any primitive of $W'$.■

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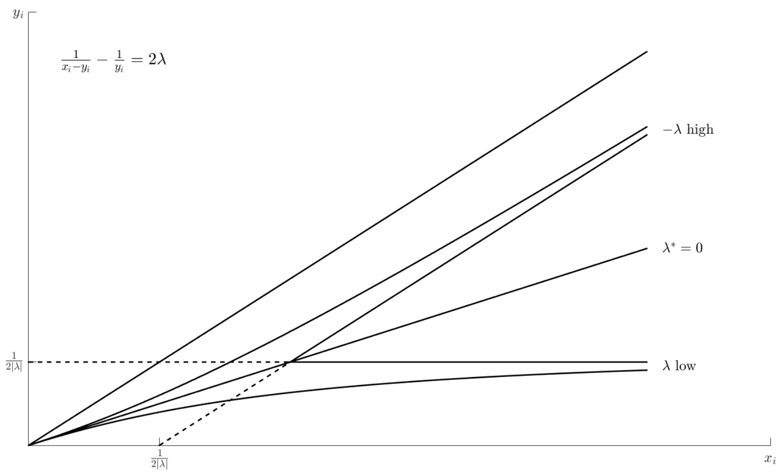


Figure 1: $W^0 : \frac{1}{x_i - y_i} - \frac{1}{y_i} = 2\lambda$

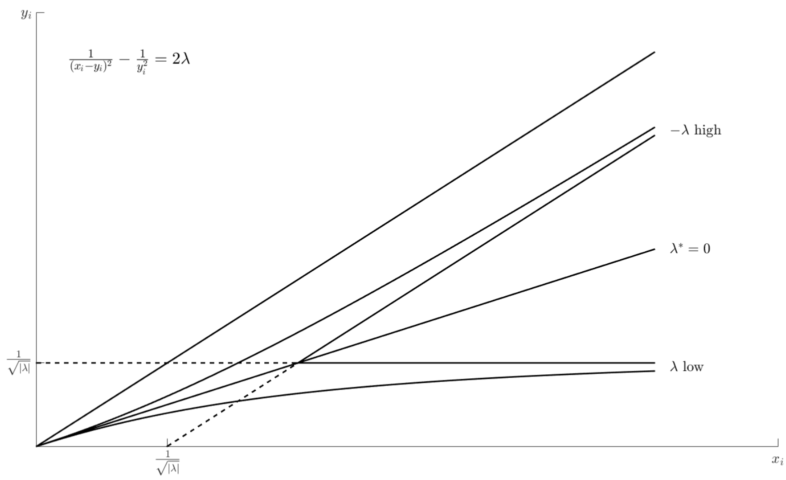


Figure 2: $\mathrm{W}^{-1} : \frac{1}{(x_i - y_i)^2} - \frac{1}{y_i^2} = \lambda$

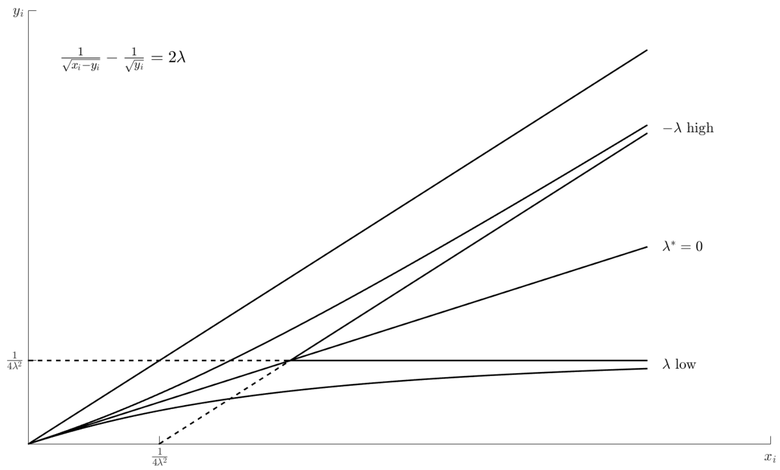


Figure 3: $W^{\frac{1}{2}} : \frac{1}{\sqrt{x_i - y_i}} - \frac{1}{\sqrt{y_i}} = \lambda$

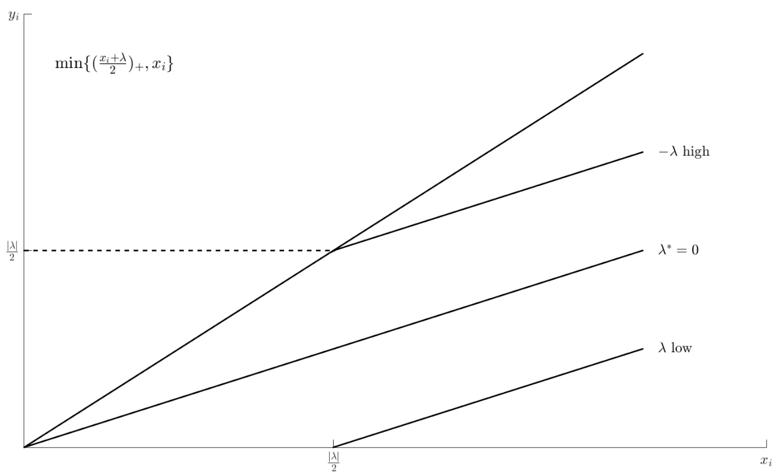


Figure 4: $\mathrm{W}^2 : y_i = \min\{(\frac{x_i+\lambda}{2})_+, x_i\}$

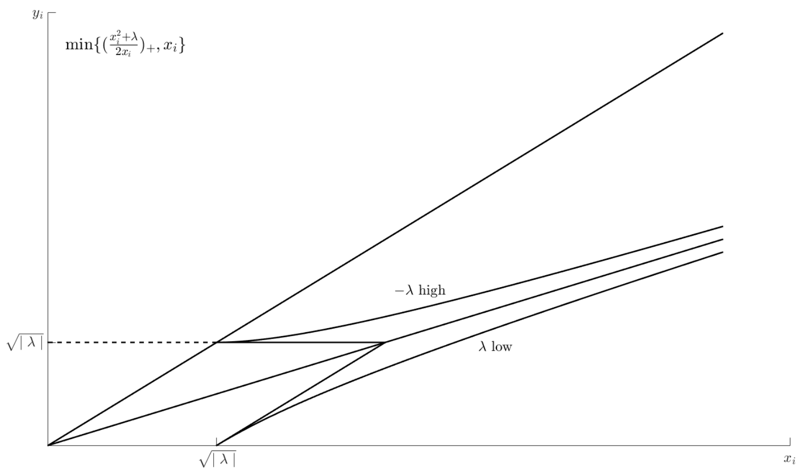


Figure 5: $\mathrm{W}^3 : y_i = \min\{(\frac{x_i^2+\lambda}{2x_i})_+, x_i\}$