LARGE TIME APPROXIMATION FOR SHEARING MOTIONS

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Abstract. Small- and large-amplitude oscillatory shear tests are widely used by experimentalists to measure, respectively, linear and nonlinear properties of visco-elastic materials. These tests are based on the quasi-static approximation according to which the strain varies sinusoidally with time after a number of loading cycles. Despite the extensive use of the quasi-static approximation in solid mechanics, few attempts have been made to justify rigorously such an approximation. The validity of the quasi-static approximation is studied here in the framework of the Mooney-Rivlin Kelvin-Voigt visco-elastic model by solving the equations of motion analytically. For a general nonlinear model, the quasi-static approximation is instead derived by means of a perturbation analysis.

Key words. Shearing motion, Mooney-Rivlin Kelvin-Voigt visco-elastic model, SAOS and LAOS tests.

AMS subject classifications. 74D05, 74D10, 74H10, 74H40

1. Introduction.

According to Truesdell [24], *the most illuminating homogeneous static deformation* is the simple shear deformation. Denoting (X, Y, Z) and (x, y, z) the Cartesian coordinates of a particle P of a given body $\mathcal{B}$ in the reference and current configurations, respectively, the simple shear deformation is given by the following equations

$$x = X + KY, \quad y = Y, \quad z = Z, \tag{1}$$

where the constant K is called the *amount of shear*. The simple shear deformation (1) is a homogeneous isochoric deformation and therefore it is a universal solution to all nonlinear incompressible isotropic materials (see for instance the textbook by Tadmor *et al.* [23]). In the linear theory of elasticity the infinitesimal deformation of the form (1) is associated with an infinitesimal shear stress $\boldsymbol{\sigma} = S(\boldsymbol{i} \otimes \boldsymbol{k} + \boldsymbol{k} \otimes \boldsymbol{i})$, S being a constant. This fact does not carry over to the framework of finite elasticity [7]. Indeed, the simple shear test in the framework of the theory of linear elasticity is a well defined experiment (see for example the BS ISO 8013 standard [3]), but in the theory of nonlinear elasticity it is not easy to model because of the unequal normal stresses needed to achieve the required simple shear deformation [18].

In his celebrated paper [16] Mooney notices that "*when a sample of soft rubber is stretched by an imposed tension, neither the force-elongation nor the stress-elongation relationship agrees with Hooke's law. On the other hand, if the sample is sheared by a shearing stress, or traction, Hooke's law is obeyed over a very wide range in deformation*". Mooney's statement is imprecise. In fact, as pointed out by Destrade *et al.* [7], for homogeneous, isotropic, non-linearly elastic materials the form of the homogeneous deformation consistent with the application of a Cauchy shear stress is not simple shear, in contrast to the situation in linear elasticity. Instead, it consists of a triaxial stretch superposed on a classical simple shear deformation, for which the amount of shear cannot be greater than 1. In other words, the faces of a cubic block cannot be slanted by an angle greater than 45° by the application of a pure

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shear stress alone. Mooney [16] ignored that in the framework of the nonlinear theory of elasticity the slanted surfaces of the sample are not stress-free. Both normal and shear traction must be applied on the inclined faces of the block to maintain the homogeneous deformation (1). Nevertheless, in his efforts at deriving the most general strain energy density function such that Hooke's law is obeyed in simple shear, Mooney [16] derived the celebrated *Mooney-Rivlin model*: the starting point of the modern theory of nonlinear elasticity. Very recently, Mangan *et al.* [13] showed that Mooney-Rivlin model is only a special case of the most general strain energy function such that Hooke's law is obeyed in simple shear.

In many experimental tests it is common practice to idealize the deformation that occurs in the real world as a simple shear deformation. For instance, the dynamic oscillatory shear tests that are used in rheometry to investigate a wide range of soft matter and complex fluids [8] are performed by subjecting a material to a sinusoidal deformation and measuring the resulting mechanical response as a function of time [13]. These oscillatory tests are usually divided into two regimes. In one regime a linear visco-elastic response is a suitable idealization of the experimental results found at small amplitude oscillatory shear (SAOS) deformations. In the other regime the material response is nonlinear as a consequence of large amplitude oscillatory shear (LAOS) deformations.

Clearly, LAOS tests present all the issues pointed out by Destrade *et al.* [7] for the classical static simple shear tests. In addition, in the dynamic context a new problem occurs for both the SAOS and LAOS tests. If the amount of shear in (1) is a function of time, say $K = K(t)$, the corresponding motion is neither a solution to the balance equation of linear momentum nor a self-equilibrated motion. The simple shear deformation (1) with $K = K(t)$ is an admissible motion only in the framework of a quasi-static approximation derived from the equations of motion by ignoring the inertia terms.

In solids mechanics there have been very few attempts to justify rigorously the quasi-static approximation. The quasi-static approximation is widely employed (see, for instance, [2] and [19]), but it is not completely clear when it represents a good approximation of the exact solutions to the equations of motion.

A general discussion of the quasi-static approximation in solid mechanics can be found in [11]. In the literature very few mathematical results to study this approximation can be reported. From a mathematical perspective the quasi-static approximation can be obtained by means of a singular perturbation analysis of the dynamic theory [20].

The aim of this paper is to investigate the validity of the quasi-static approximation in the framework of the Mooney-Rivlin Kelvin-Voigt viscoelastic model. Our results represent a first step toward a rigorous justification of the SAOS procedure. The advantage of considering the Mooney-Rivlin Kelvin-Voigt viscoelastic model is that the equation governing shear motions is linear and this allows a rigorous and detailed analysis of the problem. On the other hand, our asymptotic results for nonlinear models provide some insights into the LAOS procedure.

The plan of the paper is as follows. In Sections 2 and 3 we introduce the governing equations and the initial and boundary conditions. The basic properties of the solutions to the resulting initial-boundary value problem (IBVP) are established in Section 4. The exact solution to the IBVP governing shearing motions is derived in Section 5 and it is specialized to the case of oscillating boundaries in Section 6. Then, by considering the behaviour of the *exact* solution at large times we derive the quasi-static approximation. For large amplitude shear oscillations we instead derive the quasi-static approximation by means of a perturbation analysis (Section 7).

2. Constitutive equations. Let $\boldsymbol{X} = X\boldsymbol{i} + Y\boldsymbol{j} + Z\boldsymbol{k}$ be the position vector (relative to an origin O) of a particle P of a body $\mathcal{B}$ at the initial time $t = 0$, and $\boldsymbol{x} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ be the position vector (relative to the same origin O) of the same particle at time t. Choose the configuration occupied by $\mathcal{B}$ at the initial time as the reference configuration and denote it $\mathcal{B}_r$. A motion of the body $\mathcal{B}$ in the time interval $(0, T)$ is a mapping $\boldsymbol{\chi}$ defined in $\mathcal{B}_r \times (0, T)$ such that, for any $t \in (0, T)$, $\boldsymbol{\chi}_t \equiv \boldsymbol{\chi}(\cdot, t)$ is one-to-one, and $\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, t)$. The configuration of the solid at time t, $\mathcal{B}_t = \boldsymbol{\chi}_t(\mathcal{B}_r) = \boldsymbol{\chi}(\mathcal{B}_r, t)$, is called current configuration. The deformation gradient $\boldsymbol{F}$ and the left Cauchy-Green tensor $\boldsymbol{B}$ associated with the motion $\boldsymbol{\chi}$ are the second-order Cartesian tensors defined as

$$\boldsymbol{F} = \frac{\partial \boldsymbol{\chi}}{\partial \boldsymbol{X}}, \quad \boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T, \tag{2}$$

respectively, and the strain-rate tensor is instead given by

$$\boldsymbol{D} = \frac{1}{2}\left(\dot{\boldsymbol{F}}\boldsymbol{F}^{-1} + \boldsymbol{F}^{-T}\dot{\boldsymbol{F}}^T\right), \tag{3}$$

where the superimposed dot denotes the material time derivative. In the sequel we shall consider a solid made of an incompressible visco-elastic material. Such a solid can then undergo only isochoric motions, that is motions such that $\det \boldsymbol{F} = 1$ and, for smooth enough motions, $\mathrm{tr}\boldsymbol{D} = 0$.

The elastic part of the model is characterized by a strain-energy density (measured per unit volume in the undeformed state)

$$W = W(I_1, I_2), \tag{4}$$

where I_1 and I_2 are the first and second principal invariants of $\boldsymbol{B}$:

$$I_1 = \mathrm{tr}\boldsymbol{B}, \quad I_2 = \frac{1}{2}\left[(\mathrm{tr}\boldsymbol{B})^2 - \mathrm{tr}\boldsymbol{B}^2\right] = \mathrm{tr}\boldsymbol{B}^{-1}. \tag{5}$$

For consistency of the model (4) with linear elasticity in the limit of small strains, it is necessary that

$$W_1(3,3) + W_2(3,3) = \frac{\mu}{2}, \tag{6}$$

where the subscript i ($i = 1, 2$) denotes differentiation with respect to I_i and μ is the infinitesimal shear modulus. Since throughout this paper we shall assume that the strain energy function (4) satisfies the strong ellipticity condition, the infinitesimal shear stress is assumed to be positive [18].

The strong ellipticity condition is satisfied by many strain energy functions, including the Mooney-Riviln model

$$W = \frac{C}{2}(I_1 - 3) + \frac{D}{2}(I_2 - 3), \tag{7}$$

where, in virtue of (6), the non-negative constants C and D are such that $C + D = \mu$; the generalized Varga model [12, 25]

$$W_V = c(i_1 - 3) + d(i_2 - 3), \quad c > 0,\ d > 0,\ c + d = 2\mu, \tag{8}$$

where i_1 and i_2 are the first and second principal invariants of the left stretch tensor $\boldsymbol{V} = \boldsymbol{B}^{1/2}$; the Fung-Demiray model [6]

$$W_{FD} = \frac{\mu}{2\kappa}\left\{\exp\left[\kappa(I_1 - 3)\right] - 1\right\}, \tag{9}$$

where κ is a positive constant; and the Gent model [9]

$$W_G = -\frac{\mu J_m}{2} \ln\left(1 - \frac{I_1 - 3}{J_m}\right), \quad J_m > 0, \tag{10}$$

where J_m is a constant and the range of deformation is limited by the condition that $I_1 < J_m + 3$. Note that both the Fung-Demiray and Gent models tend to the neo-Hookean model

$$W_{nH} = \frac{\mu}{2}(I_1 - 3) \tag{11}$$

as $J_m \to +\infty$ and $\kappa \to 0$, respectively. Moreover, in plane strain deformations (and hence in shearing motions) Mooney-Rivlin model reduces to (11).

The elastic part $\boldsymbol{\sigma}^E$ of the Cauchy stress tensor $\boldsymbol{\sigma}$ can be derived from the strain-energy function (4) through the following equation

$$\boldsymbol{\sigma}^E = -p\boldsymbol{I} + 2W_1\boldsymbol{B} - 2W_2\boldsymbol{B}^{-1}, \tag{12}$$

where p is a Lagrange multiplier associated with the constraint of incompressibility. Regarding the dissipative part of the stress $\boldsymbol{\sigma}^D$, in a nonlinear setting the constitutive equation for $\boldsymbol{\sigma}^D$ may be very complex, but here, for the sake of illustration and simplicity, only materials whose Cauchy stress representation contains a term linear in the symmetric part of the velocity gradient $\boldsymbol{D}$, and no other dependence on $\boldsymbol{D}$, will be considered. We then assume that the viscous stress $\boldsymbol{\sigma}^D$ is of the form

$$\boldsymbol{\sigma}^D = 2\nu\boldsymbol{D}, \tag{13}$$

where the constant ν is the shear viscosity that, in virtue of the second law of thermodynamics, is positive. Consequently, the Cauchy stress tensor $\boldsymbol{\sigma} = \boldsymbol{\sigma}^E + \boldsymbol{\sigma}^D$ is given by the following constitutive equation

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + 2W_1\boldsymbol{B} - 2W_2\boldsymbol{B}^{-1} + 2\nu\boldsymbol{D}. \tag{14}$$

Finally, we recall that, in the absence of body forces, the equation of motion reads

$$\rho\boldsymbol{a} = \mathrm{div}\boldsymbol{\sigma} \tag{15}$$

where ρ is the (constant) mass density of the material and

$$\boldsymbol{a} = \left.\frac{\partial^2\boldsymbol{\chi}}{\partial t^2}\right|_{\boldsymbol{X}=\boldsymbol{\chi}_t^{-1}(\boldsymbol{x})} \tag{16}$$

is the spatial description of the acceleration.

3. Basic equations. Our aim is to investigate what happens in the shearing motion of a block made of a viscoelastic material of length L, width B and height H. Specifically, the motion is given by

$$x = X + u(Z,t), \quad y = Y, \quad z = Z, \tag{17}$$

where the function u is as yet unknown. Straightforward computations give

$$\boldsymbol{B} = \boldsymbol{I} + u_Z^2 \boldsymbol{i}\otimes\boldsymbol{i} + u_Z(\boldsymbol{i}\otimes\boldsymbol{k} + \boldsymbol{k}\otimes\boldsymbol{i}), \tag{18a}$$

$$\boldsymbol{B}^{-1} = \boldsymbol{I} + u_Z^2 \boldsymbol{k}\otimes\boldsymbol{k} - u_Z(\boldsymbol{i}\otimes\boldsymbol{k} + \boldsymbol{k}\otimes\boldsymbol{i}), \tag{18b}$$

$$\boldsymbol{D} = \frac{u_{Zt}}{2}(\boldsymbol{i}\otimes\boldsymbol{k} + \boldsymbol{k}\otimes\boldsymbol{i}), \tag{18c}$$

$$I_1 = I_2 = 3 + u_Z^2, \tag{18d}$$

where the subscript notation for differentiation is adopted. From (14) and (18) the shear stress σ_{13} is found to be

$$\sigma_{13} = \underbrace{2(W_1 + W_2)u_Z}_{\sigma_{13}^E} + \underbrace{\nu u_{Zt}}_{\sigma_{13}^D}. \tag{19}$$

Next, in view of (6), (14), (17) and (18), the equations of motion (15) read

$$\begin{cases} \rho u_{tt} = -p_x + [2(W_1 + W_2)u_Z]_Z + \nu u_{ZZt}, \\ 0 = -p_y, \\ 0 = [p - 2W_1 + 2W_2(1 + u_Z^2)]_Z. \end{cases} \tag{20}$$

We now assume that the normal stress vanishes on the boundary $Z = H$. Thus, with the aid of (14) and (18), we derive the boundary condition

$$0 = \boldsymbol{\sigma}(x, y, H, t)\boldsymbol{k}\cdot\boldsymbol{k} = [-p + 2W_1 - 2W_2(1 + u_Z^2)]_{|Z=H}. \tag{21}$$

Then, from (20) and (21) we deduce that the Lagrange multiplier p is given by

$$p = p(Z,t) = 2W_1 - 2W_2(1 + u_Z^2). \tag{22}$$

In this way, the equations of motion (20) reduce to the single partial differential equation

$$\rho u_{tt} = [2(W_1 + W_2)u_Z]_Z + \nu u_{ZZt}. \tag{23}$$

Since our main goal is to justify the SAOS procedure, for most part of this paper we shall be interested in a shearing regime such that, setting

$$U = \sup_{(Z,t)\in[0,H]\times[0,+\infty[} |u(Z,t)|, \tag{24}$$

$$U^2 \ll H^2. \tag{25}$$

As a consequence of this assumption and the consistency condition (6),

$$W_1(I_1, I_2) + W_2(I_1, I_2) = W_1(3,3) + W_2(3,3) + O\left(\frac{U^2}{H^2}\right) = \frac{\mu}{2} + O\left(\frac{U^2}{H^2}\right), \tag{26}$$

whence, to a first approximation, the elastic response of the material is linear and equation (23) reduces to the following linear partial differential equation

$$\rho u_{tt} = \mu u_{ZZ} + \nu u_{ZZt}. \tag{27}$$

Equation (27) represents the *exact* equation of balance of linear momentum when the strain-energy function W is given by the Mooney-Rivlin model (7).

Obviously, equation (27) can be solved provided that both initial and boundary conditions are prescribed. To this end, since the solid occupies the reference configuration $\mathcal{B}_r = [0, L] \times [0, B] \times [0, H]$ at the initial time $t = 0$ we require that

$$u(Z, 0) = 0 \quad \forall Z \in [0, H], \tag{28}$$

while we prescribe the initial velocity profile by

$$u_t(Z, 0) = f(Z) \quad \forall Z \in [0, H], \tag{29}$$

where f is a given function of the height Z. We further assume that the only nonzero component of the displacement field $\boldsymbol{x} - \boldsymbol{X}$ satisfies the boundary conditions

$$u(0, t) = g_0(t), \quad u(H, t) = g_H(t) \qquad \forall t \geq 0, \tag{30}$$

g_0 and g_H being given functions of time. The initial and boundary conditions are compatible providing that

$$g_0(0) = g_H(0) = 0, \quad f(0) = \dot{g}_0(0), \quad f(H) = \dot{g}_H(0). \tag{31}$$

In SAOS and LAOS tests between parallel plates $g_0(t) \equiv 0$ and $g_H(t) \equiv A \sin(\omega t)$, A and ω being constants (see Section 6).

We conclude this section by pointing out that very few analytical results for the IBVP (27)–(30) are reported in the literature. To the best of our knowledge, the only solution to (27)–(30) that has been studied in details is the one corresponding to the Stokes first problem [17, 21].

4. Basic properties of the solutions. We shall first establish some qualitative features of the solutions to the IBVP (27)–(30). We start with the uniqueness of the solution to the IBVP (27)–(30).

Proposition 1. *Let u_1 and u_2 be generalized solutions to the IBVP* (27)–(30)*. Then*

$$u_1(Z, t) = u_2(Z, t) \quad \textit{for a.e. } Z \in [0, H], \forall t \in [0, +\infty[. \tag{32}$$

Proof. The hypothesis implies that $w \equiv u_1 - u_2$ satisfies the following IBVP

$$\begin{cases} \rho w_{tt} = \mu w_{ZZ} + \nu w_{ZZt}, \\ w(Z, 0) = 0, \quad w_t(Z, 0) = 0, \\ w(0, t) = w(H, t) = 0. \end{cases} \tag{33}$$

Multiplying $(33)_1$ by w_t, integrating over $[0, H]$ and taking into account the boundary conditions $(33)_3$ yield

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^H \left(\rho w_t^2 + \mu w_Z^2\right) \mathrm{d}Z = -2\nu \int_0^H w_{Zt}^2 \mathrm{d}Z \leq 0. \tag{34}$$

Therefore, denoting $\|\cdot\|_2$ the $\mathrm{L}^2[0,H]$-norm, $\rho\|w_t(\cdot,t)\|_2^2+\mu\|w_Z(\cdot,t)\|_2^2$ is a non-negative non-increasing function of time that, by virtue of the initial conditions $(33)_2$, vanishes at $t=0$. Then, in virtue of the boundary conditions $(33)_3$, w vanishes for a.e. $Z\in[0,H]$ for all $t\in[0,+\infty[$. ∎

Proposition 2. *Assume that $f\equiv 0$, g_0 and g_H are bounded, and*

$$\Lambda_m=\min\left\{\inf_{t\geq 0} g_0(t),\inf_{t\geq 0} g_H(t)\right\}\leq 0 \tag{35}$$

and

$$\Lambda_M=\max\left\{\sup_{t\geq 0} g_0(t),\sup_{t\geq 0} g_H(t)\right\}\geq 0. \tag{36}$$

Let u be the generalized solution to (27)–(30). Then

$$u(Z,t)\in[\Lambda_m,\Lambda_M]\quad \textit{for a.e. } Z\in[0,H],\ \forall t\in[0,+\infty[. \tag{37}$$

Moreover, if g_0 and g_H are continuously differentiable with bounded first derivatives such that

$$\tilde\Lambda_m=\min\left\{\inf_{t\geq 0} \dot g_0(t),\inf_{t\geq 0} \dot g_H(t)\right\}\leq 0 \tag{38}$$

and

$$\tilde\Lambda_M=\max\left\{\sup_{t\geq 0} \dot g_0(t),\sup_{t\geq 0} \dot g_H(t)\right\}\geq 0, \tag{39}$$

then the only non-zero component of the velocity field $v=u_t$ satisisfies the inequalities

$$\tilde\Lambda_m\leq v(Z,t)\leq\tilde\Lambda_M\quad \textit{for a.e. } Z\in[0,H],\ \forall t\in[0,+\infty[. \tag{40}$$

Proof. Given $\phi:[0,H]\times[0,+\infty[\to\mathbb{R}$, we define

$$\phi_-(Z,t)\equiv\min\{\phi(Z,t),0\},\quad \phi_+(Z,t)\equiv\max\{\phi(Z,t),0\}. \tag{41}$$

From (35) and (36) it follows that both $(u-\Lambda_m)_-$ and $(u-\Lambda_M)_+$ satisfy the IBVP (33). Therefore, by virtue of Proposition 1 we deduce that

$$(u-\Lambda_m)_-=(u-\Lambda_M)_+=0\quad \text{for a.e. } Z\in[0,H],\ \forall t\in[0,+\infty[, \tag{42}$$

whence (37) is proved.

Next, the only nonzero component of the velocity $v=u_t$ satisfies the IBVP

$$\begin{cases}\rho v_{tt}=\mu v_{ZZ}+\nu v_{ZZt},\\ v(Z,0)=0,\quad v_t(Z,0)=0,\\ v(0,t)=\dot g_0(t),\quad v(H,t)=\dot g_H(t).\end{cases} \tag{43}$$

Then, by following the same arguments as in the proof of Proposition 1 one proves the uniqueness of the solution to the IBVP (43) and, by following similar arguments as in the proof of (37), one can prove inequalities (40). ∎

The next result shows that, on a long time scale, the solution to the IBVP (27)–(30) is not affected by the velocity field at the initial time.

PROPOSITION 3. *Let u and $\bar{u}$ be generalized solutions to the partial differential equation (27) satisfying the initial condition (28) and the boundary conditions (30). Assume that $\bar{u}_t(Z,0) = [(H-Z)\dot{g}_0(0) + Z\dot{g}_H(0)]/H$ for all $Z \in [0,H]$. Then, irrespective of the initial condition that u_t satisfies, $\|u-\bar{u}\|_2 \to 0$ as $t \to +\infty$.*

Proof. Assume that $u(Z,0) = f(Z)$, with $f \in \mathrm{L}^2[0,d]$. Then, $w \equiv u - \bar{u}$ is the solution to the following IBVP:

$$(44) \qquad \begin{cases} \rho w_{tt} = \mu w_{ZZ} + \nu w_{ZZt}, \\ w(Z,0) = 0, \quad w_t(Z,0) = f(Z) - \dfrac{(H-Z)\dot{g}_0(0) + Z\dot{g}_H(0)}{H}, \\ w(0,t) = w(H,t) = 0. \end{cases}$$

Solving the IBVP (44) by means of the method of separation of variables gives

$$(45) \qquad w(Z,t) = \sum_{n=1}^{+\infty} \left[a_n N_n(t) \sin\left(\frac{n\pi Z}{H}\right) \right],$$

where

$$(46) \qquad a_n = \sqrt{\frac{2}{H}} \int_0^H \left[f(Z) - \frac{(H-Z)\dot{g}_0(0) + Z\dot{g}_H(0)}{H} \right] \sin\left(\frac{n\pi Z}{H}\right) \mathrm{d}Z$$

are the Fourier coefficients of $f(Z) - [(H-Z)\dot{g}_0(0) + Z\dot{g}_H(0)]/H$ with respect to the Hilbert basis $\mathscr{B} = \left\{ \sqrt{\frac{2}{H}} \sin\left(\frac{n\pi Z}{H}\right) \right\}_{n\in\mathbb{N}}$ of the functional space $\mathcal{X} = \{h \in \mathrm{L}^2[0,H] : h(0) = h(H) = 0\}$,

$$(47) \qquad N_n(t) = \sqrt{\frac{2}{H}} \exp\left(-\frac{\nu n^2\pi^2}{2\rho H^2} t\right) \times \begin{cases} \dfrac{\sinh(\lambda_n t)}{\lambda_n} & \text{if } \mu < \dfrac{\nu^2 n^2 \pi^2}{4\rho H^2}, \\ t & \text{if } \mu = \dfrac{\nu^2 n^2 \pi^2}{4\rho H^2}, \\ \dfrac{\sin(\lambda_n t)}{\lambda_n} & \text{if } \mu > \dfrac{\nu^2 n^2 \pi^2}{4\rho H^2}, \end{cases}$$

and

$$(48) \qquad \lambda_n = \frac{n\pi}{2\rho H} \sqrt{\left| \frac{\nu^2 n^2 \pi^2}{H^2} - 4\rho\mu \right|}.$$

Next, from (45)–(48) we deduce that

$$(49) \qquad \|w(\cdot,t)\|_2^2 = \frac{H}{2} \sum_{n=1}^{+\infty} a_n^2 N_n^2(t) \to 0 \quad \text{as } t \to +\infty$$

which completes the proof. ∎

Let $\|\cdot\|$ be the $\mathrm{C}^0[0,H]$-norm. The following Proposition shows how the previous result can be improved by making assumptions on the initial velocity profile.

PROPOSITION 4. *Let u and $\bar{u}$ be generalized solutions to the partial differential equation (27) satisfying the initial condition (28) and the boundary conditions (30). Assume that $\bar{u}_t(Z,0) = [(H-Z)\dot{g}_0(0) + Z\dot{g}_H(0)]/H$ for all $Z \in [0,H]$ and $u_t(Z,0) = f(Z)$, where $f \in \mathrm{C}^0[0,H]$ satisfies the compatibility conditions $(31)_2$ and $(31)_3$. Then, $\|u-\bar{u}\| \to 0$ as $t \to +\infty$.*

Proof. Under the new hypotheses on the initial datum f, the solution (45)–(48) to the IBVP (44) is *classical*. Thus, it follows that

$$\|w(\cdot,t)\| = \max_{Z\in[0,H]} |w(Z,t)| \le \sum_{n=1}^{+\infty} |a_n N_n(t)| \to 0 \quad \text{as } t \to +\infty. \tag{50}$$ ◻

5. Solving the IBVP. Due to the linearity of equation (27), the solution to the IBVP (27)–(30) can be written as

$$u(Z,t) = \frac{(H-Z)g_0(t) + Zg_H(t)}{H} + u_0(Z,t) + \psi(Z,t), \tag{51}$$

where u_0 and ψ are the solutions to the following IBVPs

$$\begin{cases} \rho u_{0tt} = \mu u_{0ZZ} + \nu u_{0ZZt}, \\ u_0(Z,0) = 0, \quad u_{0t}(Z,0) = f(Z) - \dfrac{(H-Z)\dot{g}_0(0) + Z\dot{g}_H(0)}{H}, \\ u_0(0,t) = 0, \quad u_0(H,t) = 0, \end{cases} \tag{52}$$

and

$$\begin{cases} \rho \psi_{tt} = \mu \psi_{ZZ} + \nu \psi_{ZZt} - \dfrac{\rho}{H}[(H-Z)\ddot{g}_0(t) + Z\ddot{g}_H(t)], \\ \psi(Z,0) = \psi_t(Z,0) = 0, \\ \psi(0,t) = \psi(H,t) = 0, \end{cases} \tag{53}$$

respectively.

Solving the IBVP (52) by means of the method of separation of variables gives

$$u_0(Z,t) = \sum_{n=1}^{+\infty} \left[a_n N_n(t) \sin\left(\frac{n\pi Z}{H}\right) \right], \tag{54}$$

with a_n, $N_n(t)$ and λ_n as in (46), (47) and (48), respectively

As the IBVP (53) is concerned, in virtue of the completeness of the Hilbert basis $\mathscr{B}$ in the space $\mathcal{X}$ and since ψ meets homogeneous boundary conditions for all $t \ge 0$, we may expand ψ as follows

$$\psi(Z,t) = \sum_{n=1}^{+\infty} \sqrt{\frac{2}{H}}\, \Phi_n(t) \sin\left(\frac{n\pi Z}{H}\right), \tag{55}$$

where $\Phi_n(t) = \sqrt{\dfrac{2}{H}} \displaystyle\int_0^H \psi(Z,t) \sin\left(\frac{n\pi Z}{H}\right) \mathrm{d}Z$ ($n \in \mathbb{N}$) are the finite Fourier transforms of ψ.

To proceed, we multiply $(53)_1$ by $\sqrt{\dfrac{2}{H}}\sin\left(\dfrac{n\pi Z}{H}\right)$ and integrate over the interval $[0,H]$. Then, by taking into account the initial and boundary conditions satisfied by ψ, we obtain a hierarchy of Cauchy problems for Φ_n:

$$\left\{\begin{array}{l}\ddot{\Phi}_n(t)+\dfrac{n^2\pi^2}{\rho H^2}\left[\nu\dot{\Phi}_n(t)+\mu\Phi_n(t)\right]=\dfrac{\sqrt{2H}}{n\pi}\left[(-1)^n\,\ddot{g}_H(t)-\ddot{g}_0(t)\right],\\ \Phi_n(0)=\dot{\Phi}_n(0)=0.\end{array}\right. \tag{56}$$

Therefore, solving (56) yields

$$\psi(Z,t)=\sum_{n=1}^{+\infty}\tilde{N}_n(t)\sin\left(\frac{n\pi Z}{H}\right), \tag{57}$$

where

$$\tilde{N}_n(t)=\frac{\sqrt{2H}}{n\pi}\int_0^t[(-1)^n\,\ddot{g}_H(\tau)-\ddot{g}_0(\tau)]N_n(t-\tau)\mathrm{d}\tau. \tag{58}$$

Obviously, this approach makes sense if and only if $\psi(\cdot,t)\in\mathcal{X}$ for any $t\geq 0$, *i.e.*, if and only if

$$\sum_{n=1}^{+\infty}\frac{2H}{n^2\pi^2}\left\{\int_0^t[(-1)^n\,\ddot{g}_H(\tau)-\ddot{g}_0(\tau)]N_n(t-\tau)\mathrm{d}\tau\right\}^2<+\infty\quad\forall t\geq 0. \tag{59}$$

Condition (59) is satisfied if g_0 and g_H are continuously differentiable functions with piecewise continuous second derivatives.

Finally, if f is continuous, g_0 and g_H are continuously differentiable functions with piecewise continuous second derivatives, and f, g_0 and g_H satisfy the compatibility conditions (31), then the series in (54) and (57) and their term-by-term derivatives $\dfrac{\partial^2}{\partial t^2}$, $\dfrac{\partial^2}{\partial Z^2}$ and $\dfrac{\partial^3}{\partial Z^2\partial t}$ converge uniformly. Thus, in such a case

$$\begin{aligned}u(Z,t)=&\sum_{n=1}^{+\infty}\left[a_nN_n(t)\sin\left(\frac{n\pi Z}{H}\right)\right]+\frac{(H-Z)g_0(t)+Zg_H(t)}{H}\\&+\sum_{n=1}^{+\infty}\tilde{N}_n(t)\sin\left(\frac{n\pi Z}{H}\right),\end{aligned} \tag{60}$$

with a_n, $N_n(t)$ and $\tilde{N}_n(t)$ as in (46), (47) and (58), is a *classical* solution to the IBVP (27)–(30). If the initial datum f is not continuous but of class $\mathrm{L}^2[0,H]$, then (60) represents a *generalized* solution to the IBVP (27)–(30).

6. Oscillating boundaries. We now assume that the boundary $Z=0$ is at rest (*i.e.*, $g_0\equiv 0$) whereas the upper boundary oscillates with period $2\pi/\omega$ ($\omega>0$) according to the law

$$g_H(t)=A\sin(\omega t). \tag{61}$$

Now, it is convenient to non-dimensionalize equations (27)–(30) by introducing the following dimensionless quantities

$$Z^*=\frac{Z}{H},\quad t^*=\omega t,\quad u^*=\frac{u}{A}. \tag{62}$$

By dropping the asterisks for simplicity of notation, the IBVP (27)–(30) reduces to the dimensionless form

$$\begin{cases} \varepsilon u_{tt} = \delta u_{ZZ} + u_{ZZt} & \forall (Z,t) \in [0,1] \times]0, +\infty[, \\ u(Z,0) = 0, \quad u_t(Z,0) = F(Z) & \forall Z \in [0,1] \\ u(0,t) = 0, \quad u(1,t) = \sin t & \forall t \geq 0, \end{cases} \tag{63}$$

where

$$\varepsilon = \frac{\rho\omega H^2}{\nu} = \frac{\mathrm{Re} H}{A}, \quad \delta = \frac{\mu}{\nu\omega} = \mathrm{Wi}^{-1}, \quad F = \frac{f}{A\omega}, \tag{64}$$

and $\mathrm{Re} = \rho\omega AH/\nu$ and $\mathrm{Wi} = \nu\omega/\mu$ are the Reynolds and Weissenberg numbers, respectively. In the present case the compatibility conditions (31) read

$$F(0) = 0, \quad F(1) = 1. \tag{65}$$

Solving the IBVP (63) as indicated in the previous section gives

$$u(Z,t) = Z \sin t + \sum_{n=1}^{+\infty} [b_n M_n(t) \sin(n\pi Z)] + \sum_{n=1}^{+\infty} \tilde{M}_n(t) \sin(n\pi Z), \tag{66}$$

where

$$b_n = \sqrt{2} \int_0^1 [F(Z) - Z] \sin(n\pi Z) \mathrm{d}Z, \tag{67}$$

$$M_n(t) = \begin{cases} \sqrt{2} \exp\left(-\frac{n^2\pi^2}{2\varepsilon} t\right) \frac{\sinh(\hat{\lambda}_n t)}{\hat{\lambda}_n} & \text{if } \varepsilon\delta < \frac{n^2\pi^2}{4}, \\ \sqrt{2} t \exp(-2\delta t) & \text{if } \varepsilon\delta = \frac{n^2\pi^2}{4}, \\ \sqrt{2} \exp\left(-\frac{n^2\pi^2}{2\varepsilon} t\right) \frac{\sin(\hat{\lambda}_n t)}{\hat{\lambda}_n} & \text{if } \varepsilon\delta > \frac{n^2\pi^2}{4}, \end{cases} \tag{68}$$

$$\hat{\lambda}_n = \frac{n\pi}{2\varepsilon} \sqrt{|n^2\pi^2 - 4\varepsilon\delta|}, \tag{69}$$

$$\begin{aligned} \tilde{M}_n(t) = {} & \frac{2(-1)^n \varepsilon^2}{n\pi[\varepsilon^2 - 2\varepsilon\delta n^2\pi^2 + (1+\delta^2) n^4\pi^4]} \\ & \times \left[\left(1 - \frac{\delta n^2\pi^2}{\varepsilon}\right) \sin t + \frac{n^2\pi^2}{\varepsilon} \cos t - \exp\Big(- \frac{n^2\pi^2}{2\varepsilon} t\Big) \varphi_n(t) \right] \end{aligned} \tag{70}$$

and

$$\varphi_n(t) = \begin{cases} \left(\frac{n^4\pi^4}{2\varepsilon^2} - \frac{\delta n^2\pi^2}{\varepsilon} + 1\right) \frac{\sinh(\hat{\lambda}_n t)}{\hat{\lambda}_n} + \frac{n^2\pi^2}{\varepsilon} \cosh(\hat{\lambda}_n t) & \text{if } \varepsilon\delta < \frac{n^2\pi^2}{4}, \\ \left(4\delta^2 + 1\right) t + 4\delta & \text{if } \varepsilon\delta = \frac{n^2\pi^2}{4}, \\ \left(\frac{n^4\pi^4}{2\varepsilon^2} - \frac{\delta n^2\pi^2}{\varepsilon} + 1\right) \frac{\sin(\hat{\lambda}_n t)}{\hat{\lambda}_n} + \frac{n^2\pi^2}{\varepsilon} \cos(\hat{\lambda}_n t) & \text{if } \varepsilon\delta > \frac{n^2\pi^2}{4}. \end{cases} \tag{71}$$

If F is a continuous function satisfying the compatibility conditions (65), then (66)–(71) yield the classical solution to the IBVP (63). If the initial datum F is only of class $\mathrm{L}^2[0,1]$ or it does not satisfy the compatibility conditions (65), then (66)–(71) yield instead the generalized solution to the IBVP (63).

6.1. Short-time approximation. For short times, from (66)–(71) we deduce that if the initial datum F is a function of class $\mathrm{C}^2[0,1]$ satisfying (65) and $F''(0) = F''(1) = 0$ (where the prime denotes differentiation with respect to Z), then

$$u(Z,t) = F(Z)t + \frac{\delta}{2\varepsilon}F''(Z)t^2 + O(t^3) \quad \text{as } t \to 0 \tag{72}$$

for all $Z \in [0,1]$. Proceeding with the approximation as $t \to 0$, if F is of class $\mathrm{C}^4[0,1]$, satisfies (65) and is such that $F''(0) = F''(1) = F^{IV}(0) = F^{IV}(1) = 0$, then

$$u(Z,t) = F(Z)t + \frac{\delta}{2\varepsilon}F''(Z)t^2 + \frac{\delta^2 F^{IV}(Z) + \varepsilon F''(Z)}{6\varepsilon^2}t^3 + O(t^4) \quad \text{as } t \to 0 \tag{73}$$

for all $Z \in [0,1]$.

6.2. Large-time approximation. If F is a continuous function satisfying the compatibility conditions (65), from (66)–(71) we deduce that $\|u - u_\infty\| \to 0$ as $t \to +\infty$, where

$$u_\infty(Z,t) = \alpha(Z)\sin t + \frac{\alpha''(Z)}{\varepsilon}(\delta \sin t - \cos t), \tag{74}$$

$$\begin{aligned} \alpha(Z) &= Z + \sum_{n=1}^{+\infty} \frac{2(-1)^n \varepsilon^2}{n\pi[\varepsilon^2 - 2\varepsilon\delta n^2\pi^2 + (\delta^2+1)n^4\pi^4]} \sin(n\pi Z) \\ &= \frac{\delta \sinh\lambda \cos\varpi + \cosh\lambda \sin\varpi}{\cosh^2\lambda - \cos^2\varpi} \cosh(\lambda Z)\sin(\varpi Z) \\ &\quad - \frac{\delta \cosh\lambda \sin\varpi - \sinh\lambda \cos\varpi}{\cosh^2\lambda - \cos^2\varpi} \sinh(\lambda Z)\cos(\varpi Z), \end{aligned} \tag{75}$$

$$\lambda = \sqrt{\frac{\varepsilon\left(\sqrt{\delta^2+1} - \delta\right)}{2(\delta^2+1)}}, \quad \varpi = \sqrt{\frac{\varepsilon\left(\sqrt{\delta^2+1} + \delta\right)}{2(\delta^2+1)}}. \tag{76}$$

If F satisfies the milder conditions stated at the end of Section 6, then the generalized solution given by (66)–(71) tends in the mean to u_∞ as $t \to +\infty$. In both cases, one can readily check that u_∞ is a solution of $(63)_1$ and satisfies the boundary conditions $(63)_3$.

Figure 1 shows the non-zero component of displacement u, the strain $\gamma = u_Z$ and the (dimensionless) shear stress

$$\sigma \equiv \frac{H\sigma_{13}}{\nu A\omega} = \underbrace{\delta\gamma}_{\sigma^E} + \underbrace{\gamma_t}_{\sigma^D} \tag{77}$$

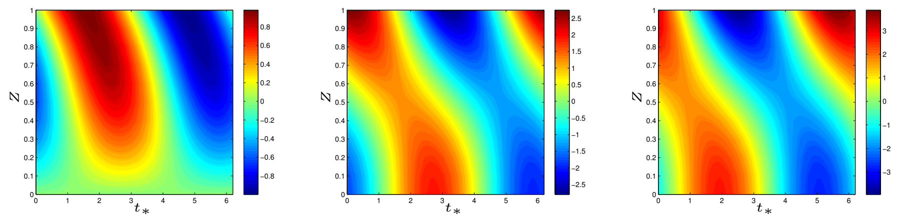
at large times. The strain and shear stress fields at large times (denoted γ_∞ and σ_∞, respectively) are

$$\gamma_\infty = \left[\alpha'(Z) + \frac{\delta}{\varepsilon}\alpha'''(Z)\right]\sin t - \frac{\alpha'''(Z)}{\varepsilon}\cos t \tag{78}$$

and

$$\sigma_\infty = \sigma_\infty^E + \sigma_\infty^D = \left[\delta\alpha'(Z) + \frac{\delta^2+1}{\varepsilon}\alpha'''(Z)\right]\sin t + \alpha'(Z)\cos t, \tag{79}$$

with α as in (75). The fields u_∞, γ_∞ and σ_∞ are periodic in time with the same period as the oscillating upper boundary and for this reason in Figure 1 they are plotted for $t_* = t - 2n\pi \in [0, 2\pi]$ $(n \in \mathbb{N},\ n \gg 1)$.



(a) u_∞. (b) γ_∞. (c) σ_∞.

FIG. 1. *Dimensionless displacement, strain and shear stress fields at large times* $t_* = t - 2n\pi$ $(n \in \mathbb{N},\ n \gg 1)$ *for* $\varepsilon = 10$ *and* $\delta = 1$. *For this value of* δ *the phase lag between* σ_∞ *and* γ_∞ *is* $\Theta = \pi/4$.

Clearly, σ_∞^E is in phase with the strain γ_∞, whereas σ_∞^D is 90° out of phase with it. Furthermore, from (78) and (79) the phase lag Θ between the shear stress and the strain, also known as the *mechanical loss angle* [10], is

$$\Theta = \arctan\delta^{-1} = \arctan(\mathrm{Wi}). \tag{80}$$

Integrating the in-phase and out-of-phase components separately, the mechanical work $\mathscr{W}_\infty$ done per loading cycle is

$$\begin{aligned} \mathscr{W}_\infty &= \int_0^1 \mathrm{d}Z \int_0^{2\pi} (\sigma_\infty^E + \sigma_\infty^D)\gamma_{\infty_t}\mathrm{d}t_* \\ &= \frac{\delta}{2}\int_0^1 \left[\gamma_\infty^2\right]_{t_*=0}^{t_*=2\pi}\mathrm{d}Z + \int_0^1 \mathrm{d}Z\int_0^{2\pi}\gamma_{\infty_t}^2\mathrm{d}t_* = 0 + \pi\alpha'(1)(>0). \end{aligned} \tag{81}$$

Hence, the in-phase components produce no net work when integrated over a cycle, whereas the out-of-phase components result in a net dissipation per cycle equal to $\pi\alpha'(1)$. It is worth noting that the work done per loading cycle tends to π as $\delta \to +\infty$ like in the case of slowly oscillating upper boundary (Section 6.3), while

$$\mathscr{W}_\infty = \sqrt{\frac{\varepsilon}{2}}\frac{\sinh(\sqrt{2\varepsilon}) + \sin(\sqrt{2\varepsilon})}{\cosh(\sqrt{2\varepsilon}) - \cos(\sqrt{2\varepsilon})}, \tag{82}$$

for $\delta = 0$, that is for a Newtonian fluid.

6.3. Slowly oscillating upper boundary. We now assume that the upper boundary oscillates so slowly that the Reynolds number is very small compared to the ratio of the amplitude of oscillations of the upper boundary and the thickness of the block, *i.e.*,

$$\mathrm{Re} \ll \frac{A}{H}. \tag{83}$$

Under such an assumption $\varepsilon \ll 1$ and the asymptotic solution (74)-(75) approximates to

$$u_\infty = \underbrace{Z \sin t}_{u_\infty^{(0)}} + O(\varepsilon), \tag{84}$$

that is to the quasi-static solution widely used by experimentalists to study the material response at long times. At order $O(\varepsilon^0)$ the strain and the shear stress depend sinusoidally on time according to

$$\gamma_\infty^{(0)}(Z,t) = \sin t, \quad \sigma_\infty^{(0)}(Z,t) = \sqrt{\delta^2+1}\,\sin(t+\Theta), \tag{85}$$

with the phase lag Θ between them as in (80). Proceeding with the power series expansion of u_∞ in terms of the small parameter ε, at order $O(\varepsilon)$ we find that the time dependence of the strain $\gamma_\infty^{(1)}$ and the shear stress $\sigma_\infty^{(1)}$ is still sinusoidal but their amplitudes are not constant like at order $O(1)$ but vary with the height Z. More precisely,

$$u_\infty^{(1)}(Z,t) = \frac{Z(1-Z^2)}{6\sqrt{\delta^2+1}} \sin(t-\Theta), \tag{86a}$$

$$\gamma_\infty^{(1)}(Z,t) = \frac{1-3Z^2}{6\sqrt{\delta^2+1}} \sin(t-\Theta), \tag{86b}$$

$$\sigma_\infty^{(1)}(Z,t) = \frac{1-3Z^2}{6} \sin t, \tag{86c}$$

by which it is evident that the phase lag between $\sigma_\infty^{(1)}$ and $\gamma_\infty^{(1)}$ is Θ.

We finally observe that when the upper boundary oscillates slowly, from (81) the mechanical work done per loading cycle approximates to

$$\mathscr{W}_\infty = \pi + \frac{\pi}{45(\delta^2+1)}\varepsilon^2 + O(\varepsilon^3). \tag{87}$$

7. Nonlinear case. We now consider regimes which do not satisfy the restriction (25).

In a fully nonlinear (differential) theory the (dimensionless) equation governing shearing motions is of the form

$$u_{tt} = \left[\sigma^E(u_Z) + \sigma^D(u_Z, u_{Zt})\right]_Z. \tag{88}$$

A satisfactory qualitative study of equation (88) is still missing. Few results on the existence and uniqueness of the solution to (88) are thus far available in the literature. However, there is evidence that a global solution does not exists for a large class of analytic constitutive functions σ^D. Therefore, it makes no sense to consider large-time approximations for a general fully nonlinear differential model for σ^D. If the viscous part of the Cauchy stress is constitutively given by the Kelvin-Voigt model, *viz* $\sigma^D = u_{Zt}$, it has been shown by several authors (see, for instance, [1, 2, 5] and references therein) that the IBVPs for equation (88) admit global (weak) solutions under mild hypotheses on σ^E. For this reason we restrict our attention to the Kelvin-Voigt model for σ^D.

In this framework the IBVP governing the motion of a block whose upper plate oscillates sinusoidally is given by

$$\begin{cases} \varepsilon u_{tt} = \delta[Q(u_Z^2)u_Z]_Z + u_{ZZt}, \\ u(Z,0) = 0, \quad u_t(Z,0) = F(Z), \\ u(0,t) = 0, \quad u(1,t) = \sin t, \end{cases} \tag{89}$$

where

$$Q(u_Z^2) = \frac{2(W_1 + W_2)}{\mu} \tag{90}$$

is the dimensionless generalized shear modulus. When ε is small, that is the Reynolds number satisfies the inequality (83), the inertial term can be neglected at large enough times and thus the quasi-static solution $u(Z,t) = Z \sin t$ approximates the solution to (89) provided that the generalized shear modulus Q satisfies appropriate conditions. However, the inertial term cannot be neglected at small times. In fact, if one neglects the inertial term the initial conditions $(89)_2$ cannot be satisfied unless the initial velocity profile is $F(Z) = Z$. Therefore, a singular perturbation analysis in the time variable needs to be performed. We will distinguish two distinct approximations of the solution to the equation of motion $(89)_1$. One holds in the initial time interval $(0, \varepsilon)$ during which the inertial effects must be taken into account (*initial layer solution*), and the other is valid at large times and corresponds to the quasi-static regime (*outer solution*).

7.1. Initial layer solution. At short times $t = \varepsilon\tilde{t}$ ($\tilde{t} \in [0,1]$) the IBVP (89) becomes

$$\begin{cases} u_{\tilde{t}\tilde{t}} = \varepsilon\delta[Q(u_Z^2)u_Z]_Z + u_{ZZ\tilde{t}}, \\ u(Z,0) = 0, \quad u_{\tilde{t}}(Z,0) = \varepsilon F(Z), \\ u(0,\varepsilon\tilde{t}) = 0, \quad u(1,\varepsilon\tilde{t}) = \sin(\varepsilon\tilde{t}). \end{cases} \tag{91}$$

Expanding u as

$$u(Z,\varepsilon\tilde{t}) = \sum_{n=0}^{+\infty} \varepsilon^n u^{(n)}(Z,\tilde{t}), \tag{92}$$

and collecting terms of the same order in ε give the following hierarchy of approximations:

$$\begin{cases} u^{(0)}_{\tilde{t}\tilde{t}} = u^{(0)}_{ZZ\tilde{t}}, \\ u^{(0)}(Z,0) = 0, \quad u^{(0)}_{\tilde{t}}(Z,0) = 0, \\ u^{(0)}(0,\tilde{t}) = 0, \quad u^{(0)}(1,\tilde{t}) = 0 \end{cases} \tag{93}$$

at order $O(\varepsilon^0)$, and

$$\begin{cases} u^{(i)}_{\tilde{t}\tilde{t}} = \delta\left[Q\left({u_Z^{(i-1)}}^2\right)u_Z^{(i-1)}\right]_Z + u^{(i)}_{ZZ\tilde{t}}, \\ u^{(i)}(Z,0) = 0, \quad u^{(i)}_{\tilde{t}}(Z,0) = F_i(Z), \\ u^{(i)}(0,\tilde{t}) = 0, \quad u^{(i)}(1,\tilde{t}) = g_i(\tilde{t}) \end{cases} \tag{94}$$

at order $O(\varepsilon^i)$ $(i \in \mathbb{N})$, where

$$(95) \qquad F_i(Z) = \begin{cases} F(Z) & \text{if } i = 1, \\ 0 & \text{if } i \geq 2, \end{cases} \qquad g_i(\tilde{t}) = \begin{cases} \dfrac{(-1)^{(i-1)/2}}{i!}\tilde{t}^i & \text{if } i \text{ is odd}, \\ 0 & \text{if } i \text{ is even}. \end{cases}$$

By solving (93) and (94) we deduce that the effects due to the nonlinear generalized shear modulus do not manifest at orders $O(1)$ and $O(\varepsilon)$ and the solution to (89) approximates to

$$(96) \qquad u(Z,\varepsilon\tilde{t}) = \varepsilon\left[Z\tilde{t} + \sum_{n=1}^{+\infty}\frac{\sqrt{2}b_n}{n^2\pi^2}\left(1 - \mathrm{e}^{-n^2\pi^2\tilde{t}}\right)\sin(n\pi Z)\right] + O(\varepsilon^2) \quad \text{as } t \to 0,$$

with b_n as in (67) irrespective of the model for the strain energy function W. If the initial condition F is a continuous function satisfying the compatibility conditions (65), then the function between square brackets in (96) is the classical solution to (94) with $i = 1$. In the special case in which the initial velocity profile is $F(Z) = Z$, then the effects due to the nonlinearity of the model for the elastic strain energy become evident only at the fourth order because one can readily check that

(97)

$$\begin{aligned} u(Z,\varepsilon\tilde{t}) &= \varepsilon Z\tilde{t} \\ &+ \varepsilon^3\left[\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n^7\pi^7}\left(1 - n^2\pi^2\tilde{t} + \frac{n^4\pi^4}{2}\tilde{t}^2 - \mathrm{e}^{-n^2\pi^2\tilde{t}}\right)\sin(n\pi Z) - \frac{Z}{6}\tilde{t}\right] + O(\varepsilon^4). \end{aligned}$$

7.2. Outer solution. At large times $t = \hat{t}/\varepsilon$ $(\hat{t} \geq 1)$ the IBVP (89) reduces to the following boundary-value problem

$$(98) \qquad \begin{cases} \varepsilon^3 u_{\hat{t}\hat{t}} = \delta[Q(u_Z^2)u_Z]_Z + \varepsilon u_{ZZ\hat{t}}, \\ u(0,\hat{t}) = 0, \quad u(H,\hat{t}) = \sin\hat{t}. \end{cases}$$

As before, expanding u as

$$(99) \qquad u(Z,\hat{t}) = \sum_{n=0}^{+\infty}\varepsilon^n u^{(n)}(Z,\hat{t})$$

and collecting terms of the same order in ε yield the following hierarchy of approximations:

$$(100) \qquad \begin{cases} \left[Q\left({u_Z^{(0)}}^2\right)u_Z^{(0)}\right]_Z = 0, \\ u^{(0)}(0,\hat{t}) = 0, \quad u^{(0)}(1,\hat{t}) = \sin\hat{t} \end{cases}$$

at order $O(1)$,

$$(101) \qquad \begin{cases} \left[Q\left({u_Z^{(i)}}^2\right)u_Z^{(i)}\right]_Z + u_{ZZ\hat{t}}^{(i-1)} = 0, \\ u^{(i)}(0,\hat{t}) = 0, \quad u^{(i)}(1,\hat{t}) = 0 \end{cases}$$

at order $O(\varepsilon^i)$ $(i = 1, 2)$ and

$$\begin{cases} u^{(i-3)}_{\hat{t}\hat{t}} = \left[Q\left({u^{(i)}_Z}^2\right)u^{(i)}_Z\right]_Z + u^{(i-1)}_{ZZ\hat{t}}, \\ u^{(i)}(0,\hat{t}) = 0, \quad u^{(i)}(1,\hat{t}) = 0 \end{cases} \tag{102}$$

at order $O(\varepsilon^i)$ $(i \geq 3)$.

In solving (100) and (102), we observe that since the strain energy function W satisfies the strong ellipticity condition, $\mathcal{F}(\xi) \equiv Q(\xi^2)\xi$ is invertible (see Appendix A for details). Thus, if the domain of $\mathcal{F}$ contains the interval $[-1, 1]$, then the outer solution to (89) approximates to

$$u(Z,\hat{t}) = Z\sin\hat{t} + O(\varepsilon^3). \tag{103}$$

(If dom$\mathcal{F} \not\supseteq [-1, 1]$ equation $(98)_1$ does not admit a solution that satisfies the boundary conditions $(98)_2$, while if $\mathcal{F}$ is not invertible $(98)_1$ may not admit a unique solution satisfying $(98)_2$.) As a consequence of (103), up to terms of order $O(\varepsilon^3)$ the strain $\gamma(Z,\hat{t})$ is the same as in the linear regime, whereas the nonlinear stress response is not a perfect sinusoid (see Figures 2(a), 2(d) and 2(g)) as

$$\sigma(Z,\hat{t}) = \underbrace{\delta Q(\sin^2\hat{t})\sin\hat{t}}_{\sigma_E} + \underbrace{\cos\hat{t}}_{\sigma_D}. \tag{104}$$

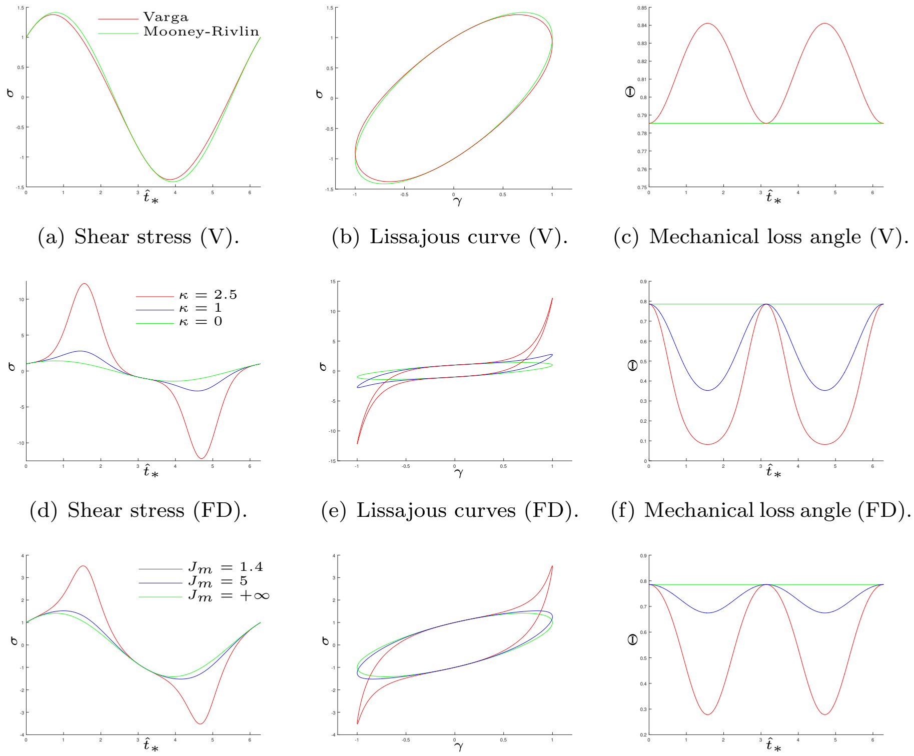
However, like in the linear case, the elastic part σ^E is in phase with the strain $\gamma = \sin\hat{t}$, whereas the viscous part σ^D is 90° out of phase with it. Unlike the linear case, the mechanical loss angle Θ is not constant but it is a continuous π-periodic function of time[1] (see Figures 2(c), 2(f) and 2(i)):

$$\Theta(\hat{t}) = \arctan\frac{\mathrm{Wi}}{Q(\sin^2\hat{t})}. \tag{105}$$

Like in the linear regime, at large times the mechanical work done per loading cycle is $\mathscr{W}_\infty = \pi$ irrespective of the model for W as the component of stress in phase with the strain does not produce work. Then, since the mechanical work done per loading cycle equals the area enclosed by the Lissajous curve - the curve in the $\gamma\sigma$-plane with parametric equations $(\gamma(\hat{t}), \sigma(\hat{t}))$ - the area enclosed by each Lissajous curve in Figures 2(b), 2(e) and 2(h) is equal to π. On the contrary, the relative dissipation - defined as the ratio between the net dissipation per loading cycle $\mathscr{W}^{dis}_\infty = \int_0^{2\pi}\sigma^D\gamma_{\hat{t}}\mathrm{d}\hat{t}$ and the maximum energy stored per loading cycle $\mathscr{W}^{st}_\infty = \int_0^{\frac{\pi}{2}}\sigma^E\gamma_{\hat{t}}\mathrm{d}\hat{t}$ [22] - depends on the nonlinear constitutive model for the elastic part of the Cauchy stress. More precisely, from $(64)_2$ and (90) we deduce that the relative dissipation is related to the strain energy function through

$$\frac{\mathscr{W}^{dis}_\infty}{\mathscr{W}^{st}_\infty} = \frac{\pi\mu}{\delta W(4,4)} = \frac{\pi\nu\omega}{W(4,4)}. \tag{106}$$

[1]Since the strain energy function W satisfies the strong ellipticity condition the dimensionless generalized shear modulus Q is positive (see Appendix A).



(a) Shear stress (V). (b) Lissajous curve (V). (c) Mechanical loss angle (V).

(d) Shear stress (FD). (e) Lissajous curves (FD). (f) Mechanical loss angle (FD).

(g) Shear stress (G). (h) Lissajous curves (G). (i) Mechanical loss angle (G).

FIG. 2. *Shear stress, Lissajous curves and mechanical loss angle for Varga (V), Fung-Demiray (FD) and Gent (G) models. The shear stresses and the mechanical loss angles are plotted against* $\hat{t}_* = \hat{t} - 1$*. The results predicted by the linear theory (SAOS) coincide with those for the Mooney-Rivlin model.*

8. Concluding Remarks. In this paper we have derived the usual quasi-static approximation that is widely used in dynamic oscillatory tests. In a parallel plate geometry and assuming that the lower plate is at rest while the upper one oscillates sinusoidally in time, we have derived the quasi-static approximation from the large-time behaviour of the exact solution to the equations governing shearing motions. We have shown that the quasi-static approximation is valid whenever the Reynolds number is much smaller than the ratio between the amplitude of the oscillation and the thickness of the sample. If the Reynolds number does not satisfy the aforementioned inequality, we have proved that the strain and the stress vary sinusoidally in time but their amplitudes vary with the height Z. The strain and stress are not in phase and the phase lag is constant and equal to that predicted by the quasi-static approximation.

In the nonlinear case we have shown that for strong elliptic strain-energies the same assumption on the Reynolds number guarantees the validity of the quasi-static approximation. Interestingly, the displacement and strain fields have the same expressions as in the linear case (up to terms of a certain order in the small parameter ε and under appropriate conditions on the generalized shear modulus). However, the stress is completely different as its elastic part is proportional to the generalized shear modulus which, at this order of approximation, is a nonlinear function of time. Finally, in the nonlinear regime the mechanical loss angle (that in the linear case is a constant depending on the Weissenberg number Wi) depends on the generalized shear modulus as well as on Wi. This is an important difference between the two regimes that can be used to investigate time dependent properties of soft materials using LAOS tests.

Appendix A. Invertibility of $\mathcal{F}$. We now show that if the strain energy function (4) satisfies the strong ellipticity condition then $\mathcal{F}$ is invertible. We start by noticing that the principal stretches in the motion (17) are

$$\lambda_1 = \sqrt{\frac{u_Z^2 + 2 + \sqrt{u_Z^2(u_Z^2+4)}}{2}} \equiv \lambda > 1, \quad \lambda_2 = \lambda^{-1}, \quad \lambda_3 = 1, \tag{107}$$

whence the principal invariants I_1 and I_2 in terms of the principal stretches read

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = \lambda^2 + \lambda^{-2} + 1 = \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2 = I_2. \tag{108}$$

In view of (108), we introduce the function $\hat{W}(\lambda) = W(I_1(\lambda), I_2(\lambda))$. As proved by Ogden [18], the strain energy function (4) satisfies the strong ellipticity condition if and only if

$$\frac{\lambda \hat{W}'(\lambda)}{\lambda^2 - 1} > 0, \quad \lambda^2 \hat{W}''(\lambda) + \frac{2\lambda \hat{W}'(\lambda)}{\lambda^2+1} > 0. \tag{109}$$

With the aid of (107) and (108), these inequalities can be rewritten as

$$W_1 + W_2 > 0 \quad \text{and} \quad W_1 + W_2 + 2(W_{11} + 2W_{12} + W_{22})u_Z^2 > 0. \tag{110}$$

Inequality $(110)_1$ implies the positivity of the generalized shear modulus, while $(110)_2$ yields the positivity of the first derivative (and hence the invertibility) of $\mathcal{F}$.

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