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Reply

Reply to Comment on 'Energy conservation and the constitutive relations in chiral and non-reciprocal media'

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Abstract

We respond to the comment by Mackay and Lakhtakia on our paper. These authors have missed the simple point that our chirality and non-reciprocity parameters are *real*. The 'inconsistency' claimed by them emerges from their incorrect attempt to apply our results instead to *complex* chirality and non-reciprocity parameters.

In our paper [1] we considered two generalised constitutive relations which we combine here as

$$\mathbf{D} = \epsilon\,(\mathbf{E} + \beta_d \nabla \times \mathbf{E}) + \alpha_d \mathbf{H},$$
$$\mathbf{B} = \mu\,(\mathbf{H} + \beta_b \nabla \times \mathbf{H}) + \alpha_b \mathbf{E} \qquad (1)$$

in the time domain, for the simple case in which the fields are restricted to a sufficiently narrow frequency bandwidth that the medium properties may be approximated by constants (that is, independent of the frequency, ω). Let us re-iterate from [1] that we are considering a non-absorbing region of the medium and that β_d, β_b, α_d and α_b are *strictly real*, as should be clear physically from the fact that $\mathbf{D}$, $\mathbf{B}$, $\mathbf{E}$ and $\mathbf{H}$ are themselves real. We neglect terms of second order in β_d, β_b, α_d and α_b, to keep our presentation brief and also simple. This is justifiable as chirality and non-reciprocity parameters are usually small in practice.

Our analysis, in [1], showed that the appropriate form of Poynting's theorem for the local conservation of energy requires that our *real* chirality parameters satisfy the conditions $\beta_d = \beta_b$ and that our *real* non-reciprocity parameters satisfy the conditions $\alpha_d = \alpha_b$. In their comment [2] Mackay and Lakhtakia misinterpret this, however, in the belief that our equations apply equally well to *complex* chirality parameters and to *complex* non-reciprocity parameters. They do not and we have not claimed that they do.

To make the point clearly let us consider complex fields now by writing $\mathbf{E} = \Re(\boldsymbol{\mathcal{E}})$ etc, where $\boldsymbol{\mathcal{E}}$ has a quasi-monochromatic time-dependence of the form $\mathrm{e}^{-\mathrm{i}\omega t}$. Our generalised constitutive relations (1) together with Maxwell's equations then imply that

$$\boldsymbol{\mathcal{D}} = \epsilon\,(\boldsymbol{\mathcal{E}} + \mathrm{i}\omega\beta_d \boldsymbol{\mathcal{B}}) + \alpha_d \boldsymbol{\mathcal{H}},$$
$$\boldsymbol{\mathcal{B}} = \mu\,(\boldsymbol{\mathcal{H}} - \mathrm{i}\omega\beta_b \boldsymbol{\mathcal{D}}) + \alpha_b \boldsymbol{\mathcal{E}}. \qquad (2)$$

If we make use in addition of the fact that our β_d, β_b, α_d and α_b are *real*, we can rewrite these as

$$\boldsymbol{\mathcal{D}} = \epsilon \boldsymbol{\mathcal{E}} + \tilde{\alpha}_d \boldsymbol{\mathcal{H}},$$
$$\boldsymbol{\mathcal{B}} = \mu \boldsymbol{\mathcal{H}} + \tilde{\alpha}_b \boldsymbol{\mathcal{E}}, \qquad (3)$$

where

$$\tilde{\alpha}_d = \alpha_d + \mathrm{i}\omega\epsilon\mu\beta_d,$$
$$\tilde{\alpha}_b = \alpha_b - \mathrm{i}\omega\epsilon\mu\beta_b. \qquad (4)$$

It follows *from* the equivalences $\alpha_d = \alpha_b$ and $\beta_d = \beta_b$ established in our paper [1] that $\tilde{\alpha}_d$ and $\tilde{\alpha}_b$ are *complex conjugates* of each other; *not* that they are equal as thought by Mackay and Lakhtakia [2].

Similarly, we can rewrite (2) as

$$\boldsymbol{\mathcal{D}} = \epsilon\,(\boldsymbol{\mathcal{E}} + \tilde{\beta}_d \nabla \times \boldsymbol{\mathcal{E}}),$$
$$\boldsymbol{\mathcal{B}} = \mu\,(\boldsymbol{\mathcal{H}} + \tilde{\beta}_b \nabla \times \boldsymbol{\mathcal{H}}), \qquad (5)$$

where

$$\tilde{\beta}_b = \beta_d - \mathrm{i}\frac{\alpha_d}{\omega\epsilon\mu},$$
$$\tilde{\beta}_d = \beta_b + \mathrm{i}\frac{\alpha_b}{\omega\epsilon\mu}. \tag{6}$$

Once again it follows *from* the equivalences $\alpha_d = \alpha_b$ and $\beta_d = \beta_b$ established in our paper [1] that $\tilde{\beta}_b$ and $\tilde{\beta}_d$ are complex conjugates of each other; *not* that they are equal as thought by Mackay and Lakhtakia [2].

Let us emphasise here once more that, as was stated explicitly in our paper and above, our attention is focused upon a 'non-absorbing chiral medium' [1]. It is true, of course, that the Kramers–Kronig relations require there to be absorption *somewhere* in the spectrum but there is no problem in finding a region in which the medium is, to a sufficient approximation, transparent. Indeed many optical experiments, including those aimed at measuring optical rotation, are designed in precisely this manner. We consider only frequencies where material damping can be neglected to fair approximation. Thus, we might write (3) for example, more explicitly as

$$\boldsymbol{\mathcal{D}}(\omega) = \epsilon(f_\omega)\boldsymbol{\mathcal{E}}(\omega) + [\alpha_d(f_\omega) + \mathrm{i}\alpha'_d(f_\omega)]\boldsymbol{\mathcal{H}}(\omega),$$
$$\boldsymbol{\mathcal{B}}(\omega) = \mu(f_\omega)\boldsymbol{\mathcal{H}}(\omega) + [\alpha_b(f_\omega) - \mathrm{i}\alpha'_b(f_\omega)]\boldsymbol{\mathcal{E}}(\omega), \tag{7}$$

where f_ω denotes a dispersive lineshape and we have introduced $\alpha'_d(f_\omega)$ and $\alpha'_b(f_\omega)$ as indicated for the sake of simplicity. If we move to a small collection of frequencies for which material damping is instead *not* negligible and make the appropriate replacements [3]

$$\begin{aligned}
\epsilon(f_\omega) &\to \epsilon(f_\omega) + \mathrm{i}\epsilon(g_\omega),\\
\mu(f_\omega) &\to \mu(f_\omega) + \mathrm{i}\mu(g_\omega),\\
\alpha_d(f_\omega) &\to \alpha_d(f_\omega) + \mathrm{i}\alpha_d(g_\omega),\\
\alpha'_d(f_\omega) &\to \alpha'_d(f_\omega) + \mathrm{i}\alpha'_d(g_\omega),\\
\alpha_b(f_\omega) &\to \alpha_b(f_\omega) + \mathrm{i}\alpha_b(g_\omega),\\
\alpha'_b(f_\omega) &\to \alpha'_b(f_\omega) + \mathrm{i}\alpha'_b(g_\omega),
\end{aligned} \tag{8}$$

where g_ω denotes an absorptive lineshape, then (7) becomes

$$\boldsymbol{\mathcal{D}}(\omega) = [\epsilon(f_\omega) + \mathrm{i}\epsilon(g_\omega)]\boldsymbol{\mathcal{E}}(\omega) + \mathcal{A}_d(f_\omega, g_\omega)\boldsymbol{\mathcal{H}}(\omega),$$
$$\boldsymbol{\mathcal{B}}(\omega) = [\mu(f_\omega) + \mathrm{i}\mu(g_\omega)]\boldsymbol{\mathcal{H}}(\omega) + \mathcal{A}_b(f_\omega, g_\omega)\boldsymbol{\mathcal{E}}(\omega), \tag{9}$$

with

$$\mathcal{A}_d(f_\omega, g_\omega) = \alpha_d(f_\omega) - \alpha'_d(g_\omega) + \mathrm{i}[\alpha'_d(f_\omega) + \alpha_d(g_\omega)],$$
$$\mathcal{A}_b(f_\omega, g_\omega) = \alpha_b(f_\omega) + \alpha'_b(g_\omega) - \mathrm{i}[\alpha'_b(f_\omega) - \alpha_b(g_\omega)]. \tag{10}$$

Blind application here of the results presented in our paper [1] and in the preceding paragraphs would suggest that $\mathcal{A}_d(f_\omega, g_\omega) = \mathcal{A}_b^*(f_\omega, g_\omega)$ or

$$\alpha_d(f_\omega) - \alpha'_d(g_\omega) = \alpha_b(f_\omega) + \alpha'_b(g_\omega),$$
$$\alpha'_d(f_\omega) + \alpha_d(g_\omega) = \alpha'_b(f_\omega) - \alpha_b(g_\omega). \tag{11}$$

This is at odds with the situation in reality, where $\alpha_d(f_\omega) = \alpha_b(f_\omega)$, $\alpha'_d(f_\omega) = \alpha'_b(f_\omega)$, $\alpha_d(g_\omega) = \alpha_b(g_\omega)$ and $\alpha'_d(g_\omega) = \alpha'_b(g_\omega)$. There is *no* contradiction or error here, however. Rather, it is not appropriate to demand that the energy of light propagating through an *absorbing* medium is locally conserved, as, naturally enough, the medium extracts energy from the light. Indeed, $\alpha'_d(g_\omega) = \alpha'_b(g_\omega)$ gives rise to circular dichroism.

In summary the results presented in our paper [1] and above remain true, in spite of the claims by Mackay and Lakhtakia to the contrary [2]. In particular, for our constitutive relations (1), with *real* fields and *real* material parameters, our proof based on local energy conservation stands and the *real* parameters do indeed satisfy the equations

$$\beta_d = \beta_b,$$
$$\alpha_d = \alpha_b \tag{12}$$

as stated.

Finally, we should note that the equalities $\beta_d = \beta_b$ and $\alpha_d = \alpha_b$ may also be derived from a microscopic treatment of the quantum mechanical interaction between the molecules forming the medium and the electromagnetic field [3]. In this sense, our paper did not present a new result but rather a new derivation of a familiar one.

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