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# On the natures of the spin and orbital parts of optical angular momentum

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## Abstract

The modern field of optical angular momentum began with the realisation by Allen *et al* in 1992 that, in addition to the spin associated with polarisation, light beams with helical phase fronts carry orbital angular momentum. There has been much confusion and debate, however, surrounding the intricacies of the field and, in particular, the separation of the angular momentum into its spin and orbital parts. Here we take the opportunity to state the current position as we understand it, which we present as six perspectives: (i) we start with a reprise of the 1992 paper in which it was pointed out that the Laguerre–Gaussian modes, familiar from laser physics, carry orbital angular momentum. (ii) The total angular momentum may be separated into spin and orbital parts, but neither alone is a true angular momentum. (iii) The spin and orbital parts, although not themselves true angular momenta, are distinct and physically meaningful, as has been demonstrated clearly in a range of experiments. (iv) The orbital part of the angular momentum in the direction of propagation of a beam is not simply the azimuthal component of the linear momentum. (v) The component of spin in the direction of propagation is not the helicity, although these are related quantities. (vi) Finally, the spin and orbital parts of the angular momentum correspond to distinct symmetries of the free electromagnetic field and hence are separately conserved quantities.

Keywords: optical angular momentum, spin angular momentum, orbital angular momentum, helicity

(Some figures may appear in colour only in the online journal)

## 1. Introduction

Most articles on optical angular momentum start with the paper by Allen *et al* in which it was shown that the helical phase fronts associated with Laguerre–Gaussian beams of light carry orbital angular momentum [1]. Both the helical phase fronts and the orbital angular momentum may be associated with the presence, on the beam axis, of a phase

singularity, or vortex, and it is the charge of this vortex that determines the quantity of orbital angular momentum carried by each photon. The term ‘phase singularity’ is an indication of the existence of a point at which the phase of the field is undefined [2]. These exotic points are, in fact, far from unusual, being present in the great majority of optical fields [3].

The study of the orbital angular momentum of light has its origins in optical vortices and the studies of each have been intertwined since the publication of the first papers by the Leiden group [1, 4]. Indeed early experimental contributions mostly had titles that emphasised the phase singularity [5–11], with the orbital angular momentum appearing perhaps



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less often [4, 12–15], at least initially. These papers are reprinted in [16] and some of the more recent developments are reviewed in [17, 18].

We shall discuss the link between the orbital angular momentum of light and optical vortices in more depth in the following sections, but it may be helpful to give at least an indication of the physical origin of the idea. To this end, let us consider a monochromatic complex scalar wave, with amplitude  $u$ . In the eikonal approximation, the local flux of energy, given by Poynting’s vector, has the form [19]

$$\mathbf{g} \propto \mathcal{I}(u^* \nabla u), \quad (1)$$

where the constant of proportionality depends on the manner in which we choose to normalise the amplitude  $u$ . This means that within the eikonal approximation, the energy flows perpendicular to the local phase fronts. If all of these phase fronts are nearly perpendicular to an optical axis then we recover the paraxial approximation. This form is strongly reminiscent of the current or probability-flux formulation of quantum mechanics [20, 21]. It is also similar to the momentum density for a superfluid, as introduced by Landau [22, 23]:

$$\left\langle \frac{1}{2} [\mathbf{p} \delta(\mathbf{r} - \mathbf{R}) + \delta(\mathbf{r} - \mathbf{R}) \mathbf{p}] \right\rangle = \hbar \mathcal{I}[\psi^*(\mathbf{R}) \nabla \psi(\mathbf{R})], \quad (2)$$

where  $\psi$  is the superfluid wavefunction. It is straightforward to obtain a corresponding local density for any conserved mechanical property [24] and this leads us to a density of the  $z$ -component of the orbital angular momentum in the form

$$L_z(\mathbf{R}) = \hbar \mathcal{I} \left[ \psi^*(\mathbf{R}) \frac{\partial}{\partial \phi} \psi(\mathbf{R}) \right], \quad (3)$$

where  $\phi$  is the azimuthal coordinate in cylindrical polars. If our wave function has the azimuthal dependence  $e^{i\ell\phi}$ , corresponding to a vortex of strength  $\ell$ , then we can associate an orbital angular momentum of  $\ell\hbar$  with each quantum in the superfluid. The observation that optical fields prepared in Laguerre–Gaussian modes carry orbital angular momentum has its origin in this idea.

The electromagnetic field, being a vector field, has the potential to carry both orbital and spin angular momenta.<sup>4</sup> The identification of these parts, and their physical natures is complicated, however, by the requirement that both the electric and the magnetic fields must be transverse, a property embodied in the first two Maxwell equations. It is this complication that has led to much of the debate about optical spin and orbital angular momenta. We address, in this paper, six aspects of this debate. We work throughout in a natural system of units in which  $\varepsilon_0 = \mu_0 = 1$ , so that the speed of light in vacuum is also unity. All results are general and exact for freely propagating light, unless otherwise stated.

<sup>4</sup> That a beam of circularly polarised light carries spin angular momentum was inferred by Poynting, who worked by analogy with a revolving shaft [26]. This is perhaps the first publication on optical angular momentum.

## 2. Beginnings: the 1992 paper of Allen *et al*

The key components of the original analysis in [1] are well-known, but they will, we may hope, bear a brief restatement. The authors first introduced the angular momentum of the field as the integral of the cross-product of the position with Poynting’s vector:

$$\mathbf{J} = \int \mathbf{r} \times (\mathbf{E} \times \mathbf{B}) \, dV. \quad (4)$$

While there cannot be any component of angular momentum in the case of transverse plane waves (the linear momentum,  $\mathbf{E} \times \mathbf{B}$ , is in the direction of propagation,  $\mathbf{r}$ , and hence there cannot be a component  $\mathbf{r} \times (\mathbf{E} \times \mathbf{B})$  in the same direction), this is not the case for the fields of laser modes, which being of finite transverse extent diffract, and so have components of  $\mathbf{E} \times \mathbf{B}$  perpendicular to the direction of propagation.

A uniformly polarised paraxial laser mode can be represented in the Lorenz gauge using the vector potential

$$\mathbf{A} = \Re[\tilde{\mathbf{A}} \exp(-i\omega t)] \quad (5)$$

for

$$\tilde{\mathbf{A}} = \tilde{\mathbf{e}} u \exp(ikz) \quad (6)$$

where  $\omega$  is the angular frequency,  $k = \omega/c$  is the wavenumber and  $\tilde{\mathbf{e}}$  is the complex polarisation vector, with  $\tilde{\mathbf{e}} = \hat{\mathbf{x}}$ , say, for linear polarisation and  $(\hat{\mathbf{x}} + i\sigma_z \hat{\mathbf{y}})/\sqrt{2}$ , say, for circular polarisation. Here  $\sigma_z$  is  $\pm 1$  for left or right circular polarisation, respectively. The complex field amplitude  $u$  satisfies the paraxial wave equation. The field amplitudes for the normalised Laguerre–Gaussian modes in particular satisfy this equation and have the form, in cylindrical co-ordinates [25]:

$$\begin{aligned} \text{LG}_p^\ell(\rho, \phi) &= \sqrt{\frac{2p!}{\pi(p+|\ell|)!}} \frac{1}{w(z)} \left( \frac{\rho\sqrt{2}}{w(z)} \right)^{|\ell|} \\ &\times \exp\left( \frac{-\rho^2}{w(z)^2} \right) L_p^{|\ell|} \left[ \frac{2\rho^2}{w(z)^2} \right] \\ &\times \exp(i\ell\phi) \exp\left[ -ik \frac{\rho^2 z}{2(z_R^2 + z^2)} \right] \\ &\times \exp[-i(2p + |\ell| + 1)\eta(z)], \end{aligned} \quad (7)$$

where  $L_p^{|\ell|}$  is a generalised Laguerre polynomial,  $z_R = kw_0^2/2$  is the Rayleigh range,  $w(z) = w_0 \sqrt{1 + z^2/z_R^2}$  is the beam width and  $\eta(z) = \tan^{-1}(z/z_R)$  is related to the Gouy phase. The azimuthal variation of the phase fronts is dictated by the integer  $\ell$  while the number of radial nodes is dictated by  $p$ , which can be any whole number.

Allen *et al* used this description to consider the form of Poynting’s vector. For a monochromatic, linearly polarised, laser mode propagating in the  $z$ -direction they found the form

$$\mathbf{g} = \Re(\tilde{\mathbf{E}}^* \times \tilde{\mathbf{B}}) = \omega \mathcal{I}(u^* \nabla u), \quad (8)$$

valid within the paraxial regime<sup>5</sup>, where  $\tilde{\mathbf{E}}$ ,  $\tilde{\mathbf{B}}$  are the complex fields

$$\begin{aligned}\tilde{\mathbf{E}} &= i\omega \left[ \tilde{\mathbf{A}} + \frac{1}{k^2} \nabla(\nabla \cdot \tilde{\mathbf{A}}) \right] \\ \tilde{\mathbf{B}} &= \nabla \times \tilde{\mathbf{A}}.\end{aligned}\quad (9)$$

Applying this to the Laguerre–Gaussian modes above, this Poynting vector acquires an azimuthal component in the form

$$g_\phi = \frac{\omega \ell}{k\rho} |u|^2. \quad (10)$$

When multiplied by the distance from the  $z$ -axis,  $\rho$ , and integrated over the beam this leads us to an orbital angular momentum of  $\ell \hbar$  per photon. It is this azimuthal component of the Poynting vector that gives rise to the physical picture of skew-rays and with it an orbital angular momentum associated with helical phase fronts [27].

Spin is incorporated into the analysis by switching from a linearly polarised beam to one with circular polarisation. This leads to an additional term in Poynting’s vector, acting in the azimuthal direction:

$$\mathbf{g} = \omega \mathcal{I}(u^* \nabla u) - \frac{\omega \sigma_z}{2} \frac{\partial |u|^2}{\partial \rho} \hat{\phi}, \quad (11)$$

where  $\sigma_z = \mp 1$  corresponds to right-handed or left-handed circular polarisation. It follows that a circularly polarised beam prepared in a Laguerre–Gaussian mode will have an angular momentum density per unit power of the form:

$$M_z = \frac{\ell}{\omega} |u|^2 - \frac{\sigma_z \rho}{2\omega} \frac{\partial |u|^2}{\partial \rho}. \quad (12)$$

Integrating this over the beam then gives the famous total angular momentum of  $(\ell + \sigma_z) \hbar$  per photon.

It will be helpful in the ensuing development to present the above argument in a different and slightly more general form. We return to the full vector potential,  $\mathbf{A}$ , and then use a simple vector identity to rewrite equation (4) in the form

$$\mathbf{J} = \int \sum_j [E_j(\mathbf{r} \times \nabla)A_j - \mathbf{r} \times \nabla_j(E_j \mathbf{A})] dV. \quad (13)$$

We leave, for the moment, the question of gauge-dependence, to which we return in the next section, but note here the relationship between the first and second terms in the integrand and the angular momentum density given in equation (12). In particular, the combination  $\mathbf{r} \times \nabla$  is reminiscent of the orbital angular momentum operator in quantum theory and the second term includes a derivative with respect to position and a multiplication by a position, in analogy with the second, spin-like, term in equation (12).

<sup>5</sup> We note that in [1] equation (6) appears to be the negative of the one given here. This is simply because in that paper  $u$  had the time dependence  $e^{i\omega t}$ . Here, to retain the close analogy with quantum wavefunctions, we give  $u$  the time-dependence  $e^{-i\omega t}$ .

### 3. Darwin’s separation

In the years following Dirac’s quantum theory of the electron there were a number of attempts to adapt his theory to light [28–30]. It was by this means that Darwin presented his analysis of the mechanical properties of light and, in particular, the separation of its angular momentum into orbital and spin parts [31]. We do not follow Darwin’s analysis in detail but can retain the spirit of it by performing an integration by parts on the second term in equation (13). If the field falls off sufficiently quickly to allow us to drop the surface term then we are left with

$$\mathbf{J} = \int \left[ \sum_j E_j(\mathbf{r} \times \nabla)A_j + \mathbf{E} \times \mathbf{A} \right] dV. \quad (14)$$

We stress that this form is physically identical to that in equation (13) and is, perhaps, the most common form in which to find the separation written, with the orbital and spin angular momenta *tentatively* assigned to be

$$\begin{aligned}\mathbf{L} &= \int \sum_j E_j(\mathbf{r} \times \nabla)A_j dV \\ \mathbf{S} &= \int \mathbf{E} \times \mathbf{A} dV.\end{aligned}\quad (15)$$

There remain some subtleties to be addressed which bring into question the validity of this separation. We address these below.

#### 3.1. Gauge dependence?

Perhaps the first thing to strike one about Darwin’s separation is the appearance of the vector potential. This quantity is not unique and it is unexpected, therefore, to find it appearing in physical properties. To make this point more clearly, consider the effect on our spin and orbital parts, in equation (15), of making the gauge transformation  $\mathbf{A} \rightarrow \mathbf{A} - \nabla\chi$ . We find that our previously assigned orbital and spin angular momenta become

$$\begin{aligned}\mathbf{L} &= \int \sum_j E_j(\mathbf{r} \times \nabla)A_j dV + \int \dot{\mathbf{B}}\chi dV \\ \mathbf{S} &= \int \mathbf{E} \times \mathbf{A} dV - \int \dot{\mathbf{B}}\chi dV.\end{aligned}\quad (16)$$

The *total* angular momentum is unchanged, but the separation into spin and orbital parts appears to depend on the choice of gauge! This feature, together with the fact that the photon does not have a rest frame, has led some authors to question the validity of a separate existence of spin and orbital angular momenta for light [32–38].

It is possible, however, to extract gauge-independent expressions for the spin and orbital parts of the angular momentum. To do this we note that the transverse or divergenceless part of the vector potential,  $\mathbf{A}^\perp$ , is a gauge-invariant field. If we use  $\mathbf{A}^\perp$  in place of the gauge-dependent  $\mathbf{A}$  then we arrive at the orbital and spin angular momenta as they are

usually written [37, 39–42]

$$\begin{aligned} \mathbf{L} &= \int \sum_j E_j (\mathbf{r} \times \nabla) A_j^\perp dV \\ \mathbf{S} &= \int \mathbf{E} \times \mathbf{A}^\perp dV. \end{aligned} \quad (17)$$

These expressions correspond, superficially, to expressions written in the Coulomb gauge, for which  $\nabla \cdot \mathbf{A} = 0$ , but this should *not* be interpreted as meaning that the expressions in equation (17) correspond to a choice of gauge. They are gauge-independent quantities and as such hold for any choice of gauge. We should note that these quantities, under suitable conditions, give the total orbital and spin angular momenta  $\ell\hbar$  and  $\sigma_z\hbar$  for the Laguerre–Gaussian modes.

It may be instructive to note that in reciprocal space

$$\mathbf{L} = \int \sum_j i\hbar \tilde{\alpha}_j (\mathbf{k} \times \nabla_{\mathbf{k}}) \tilde{\alpha}_j^* dV_{\mathbf{k}}, \quad (18)$$

$$\mathbf{S} = \int i\hbar \tilde{\alpha} \times \tilde{\alpha}^* dV_{\mathbf{k}}, \quad (19)$$

where  $\tilde{\alpha}$  is a complex vector field comprised of normal variables that is everywhere perpendicular to the wavevector  $\mathbf{k}$  and  $\nabla_{\mathbf{k}}$  is the gradient with respect to  $\mathbf{k}$  [35, 37, 43]. Evidently, the integrand of  $\mathbf{L}$  is everywhere perpendicular to  $\mathbf{k}$  while the integrand of  $\mathbf{S}$  is everywhere parallel to  $\mathbf{k}$ . A closely related approach towards this separation, based upon the idea of a photon wavefunction, can be found in [44].

### 3.2. ‘Spin’ and ‘orbital’ angular momenta?

It is important to realise that the quantities  $\mathbf{L}$  and  $\mathbf{S}$ , introduced in equation (17), are not themselves angular momenta. This point was made eloquently by van Enk and Nienhuis early in the development of optical angular momentum as a research field [39, 40]. They showed, in particular, that the cartesian components of the quantum operators  $\mathbf{S}$  mutually commute and inferred that this means they cannot represent a true angular momentum. As the sum of  $\mathbf{L}$  and  $\mathbf{S}$  is an angular momentum, it necessarily follows that  $\mathbf{L}$ , like  $\mathbf{S}$ , is not an angular momentum. It would perhaps be best to refer to  $\mathbf{L}$  and  $\mathbf{S}$  as the orbital and spin *parts* of the angular momentum, although common practice almost requires us to refer to them as the orbital and spin angular momenta.

The fact that  $\mathbf{L}$  and  $\mathbf{S}$  are not true angular momenta means that they are not the generators of simple rotations [39–41]. We might expect the spin angular momentum (SAM) to rotate the orientations of the fields but leave the field amplitudes unchanged, and the orbital angular momentum to have no effect on the field orientations but to rotate the amplitudes. Let us consider the infinitesimal form of such a rotation. We introduce the vector  $\boldsymbol{\theta}$ , the direction of which gives the axis of rotation and the, small, magnitude of which is the angle of the rotation. If  $\mathbf{L}$  and  $\mathbf{S}$  were true angular momenta then we would expect  $\boldsymbol{\theta} \cdot \mathbf{L}$  and  $\boldsymbol{\theta} \cdot \mathbf{S}$  to generate the transformations [41]

$$\begin{aligned} \mathbf{E} &\rightarrow \mathbf{E} - \boldsymbol{\theta} \cdot (\mathbf{r} \times \nabla) \mathbf{E} \\ \mathbf{E} &\rightarrow \mathbf{E} + \boldsymbol{\theta} \times \mathbf{E} \end{aligned} \quad (20)$$

respectively, with similar relations for the magnetic field. The difficulty is that the fields rotated in this way will, in general, acquire a component for each plane wave that is parallel to the wavevector, thus violating the first Maxwell equation. It is clear, therefore, that these transformations are unphysical. This does not imply that  $\mathbf{L}$  and  $\mathbf{S}$  are themselves unphysical, however, as the transformations that they actually generate are [39–41]

$$\begin{aligned} \mathbf{E} &\rightarrow \mathbf{E} - [\boldsymbol{\theta} \cdot (\mathbf{r} \times \nabla) \mathbf{E}]^\perp \\ \mathbf{E} &\rightarrow \mathbf{E} + (\boldsymbol{\theta} \times \mathbf{E})^\perp. \end{aligned} \quad (21)$$

The transformations of the fields are the closest approximation to the expected rotations that are consistent with the requirement of transversality, so that  $\nabla \cdot \mathbf{E} = 0$  and  $\nabla \cdot \mathbf{B} = 0$  for the fields transformed either by the action of  $\mathbf{L}$  or  $\mathbf{S}$ .

### 3.3. Electric-magnetic symmetry

There is one further subtlety concerning the forms of  $\mathbf{L}$  and  $\mathbf{S}$  that should be addressed and this is the electric-magnetic symmetry, or democracy [45], due to Heaviside and Larmor [46, 47]. This symmetry is a consequence of the fact that the free-field Maxwell equations retain their form under the duality rotation

$$\begin{aligned} \mathbf{E} &\rightarrow \cos \theta \mathbf{E} + \sin \theta \mathbf{B} \\ \mathbf{B} &\rightarrow \cos \theta \mathbf{B} - \sin \theta \mathbf{E}, \end{aligned} \quad (22)$$

where  $\theta$  is any pseudoscalar angle. We may expect, given this symmetry, that any physically significant property of the field should be unchanged by a transformation of this form [48] and it is immediately clear, for example, that the densities of energy and momentum,  $\frac{1}{2}(E^2 + B^2)$  and  $\mathbf{E} \times \mathbf{B}$ , satisfy this condition.

It is not at all obvious that our formulae for the orbital and spin parts of the angular momentum satisfy the Heaviside–Larmor symmetry. The most elegant way to show that they do is to introduce a second potential, the transverse part of which is represented by  $\mathbf{C}^\perp$  [49, 50], such that

$$\begin{aligned} \mathbf{E} &= -\nabla \times \mathbf{C}^\perp \\ \mathbf{B} &= -\dot{\mathbf{C}}^\perp. \end{aligned} \quad (23)$$

By writing  $\mathbf{L}$  and  $\mathbf{S}$  in terms of the two potentials and showing that they are unchanged by the transformations in equation (22) and the corresponding transformation of the potentials,

$$\begin{aligned} \mathbf{A}^\perp &\rightarrow \cos \theta \mathbf{A}^\perp + \sin \theta \mathbf{C}^\perp \\ \mathbf{C}^\perp &\rightarrow \cos \theta \mathbf{C}^\perp - \sin \theta \mathbf{A}^\perp, \end{aligned} \quad (24)$$

we can show that  $\mathbf{L}$  and  $\mathbf{S}$  do indeed satisfy the required Heaviside–Larmor symmetry. It is straightforward to confirm that this is indeed the case and that we can write our orbital

and spin parts of the angular momentum in the required form:

$$\mathbf{L} = \frac{1}{2} \int \sum_j [E_j(\mathbf{r} \times \nabla)A_j^\perp + B_j(\mathbf{r} \times \nabla)C_j^\perp] dV$$

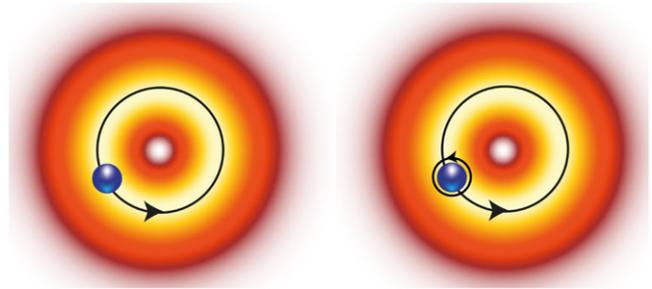
$$\mathbf{S} = \frac{1}{2} \int (\mathbf{E} \times \mathbf{A}^\perp + \mathbf{B} \times \mathbf{C}^\perp) dV. \quad (25)$$

We can take the integrands to be Heaviside–Larmor symmetric densities of these quantities, but the total quantities, given by the integrals, are the same whether or not we introduce the second potential form [41]. It is interesting to note that this separation of the angular momentum into spin and orbital parts, which satisfy the Heaviside–Larmor symmetry, emerges directly from the optical Dirac equation, a representation of Maxwell’s equations in Dirac form [52]. This is especially pleasing given Darwin’s original proposal to study light using the Dirac equation.

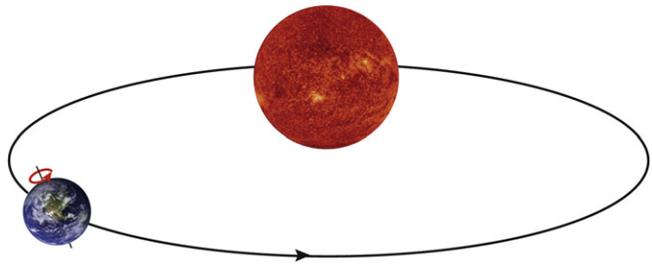
#### 4. What experiments tell us

There is a large body of work that has studied and used optical angular momentum in a variety of applications ranging from micro-manipulation to communications systems and sensing [16–18, 51]. An interesting and recurrent theme in this work has been the extent to which the spin and orbital parts are found to be similar or distinct.

The earliest series of experiments, which fuelled much of the current interest in optical angular momentum, are those involving the use of angular momentum carrying beams in optical tweezers [53]. Optical tweezers use tightly-focussed light beams to move micron-sized particles that are trapped in a beam focus by the change in optical linear momentum arising from refraction of the light beam by the particle [54]. Use of a light beam that carries angular momentum makes it possible to convert these tweezers into optical spanners and so to cause the trapped particles to rotate. The first experiment used a linearly-polarised, helically-phased light beam to transfer the orbital angular momentum to an absorbing particle and so induce a rotation [12]. Subsequent work showed that when the helically-phased beam was also circularly polarised the orbital and spin components could add or subtract, causing the particle rotation to speed up or to slow down [13] or, indeed, to stop completely [15]. The observation that the spin and orbital parts of the angular momentum, although both non-zero, could combine to give a total angular momentum of zero indicates that, in this case, they are equal and opposite in magnitude, a conclusion that is consistent with both being quantised in units of  $\hbar$  per photon. In the early experiments the focussed optical beam was much smaller than the particle and this meant that the transfer of spin and orbital angular momentum were indistinguishable. We should note, however, that this indistinguishability is only true when the transfer of angular momentum is based on absorption of light by the particle. If the particle is transparent then the mechanisms of angular momentum transfer will be different. If we use a transparent but birefringent particle, in



**Figure 1.** An object trapped in optical tweezers is rotated about the beam axis by the transfer of orbital angular momentum (OAM) from a high-order Laguerre–Gaussian beam (left). If the beam is circularly polarised, and hence carries spin angular momentum (SAM), then the particle also rotates about its own axis (right).

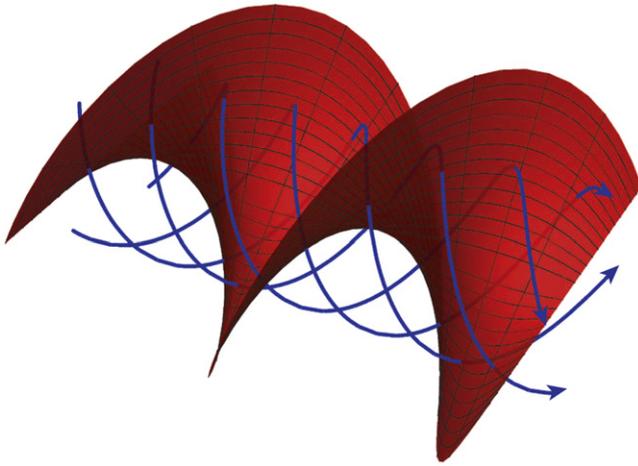


**Figure 2.** The motion of the Earth which spins on its axis as it orbits the Sun is analogous to that of a particle with both spin and orbital angular momentum. (Images provided by NASA.)

particular, then the polarisation state of the light and hence its SAM is changed, but its orbital angular momentum is not. In this situation, it is only the spin part of the angular momentum that can induce a rotation of the particle [55].

By expanding the laser beam so that it is significantly larger than the particle, the particle is trapped to lie within the high-intensity ring of the helically-phased beam, and so is displaced from the beam axis. In this configuration the particle is subject to an azimuthal force arising from a local inclination of the phasefronts [73], so that the transfer of orbital angular momentum is manifest as an orbital motion of the particle about the *beam* axis, as depicted in figure 1. If the particle is also birefringent then, in addition to this orbital motion, any SAM in the beam will cause the particle to also spin about its *own* axis [56, 57]. This separation of spin and orbital effects is reminiscent of the orbit of a planet about the Sun which has both orbital angular momentum, associated with the yearly cycle, and also SAM, which gives the days, as in figure 2. The spinning and orbiting of the particle arise, respectively, from the spin and orbital components of the optical angular momentum. In this case the separation into spin and angular orbital components is clear, but there is evidence that under extreme focussing there may also be an interchange between these [58].

Another class of experiment in which both the spin and orbital parts play important roles is that involving rotationally induced frequency shifts. It is well-established that when a circularly-polarised beam (that is one carrying SAM) is rotated about its axis then a frequency shift results [18, 59].



**Figure 3.** Stylistic representation of the off-axis skew rays for a helical mode with orbital angular momentum of  $\ell = 2$ .

This effect may be thought of as arising from a dynamical geometric phase shift, but it can also be understood in terms of the rotation of the electric field as viewed from a rotating frame: the observer sees the field rotate at a rate given by the sum of the optical frequency and the frequency of rotation. This is analogous to watching the hands of a clock move from a rotating frame; the hands move faster or slower depending on the sense of rotation. An analogous rotational frequency shift was predicted for beams carrying orbital angular momentum [60, 61] and this was subsequently observed in the millimetre-wave regime [62]. In both the spin and orbital angular momentum cases the resulting frequency shift is equal to the angular momentum per photon multiplied by the relative rotation rate between the frame of the source that of the observer [63]. The energy change associated with these frequency shifts can be linked to the reversal of the azimuthal component of the Poynting vector that occurs within the component used to induce the beam rotation [64]. These rotationally-induced frequency shifts are most significant in the context of this paper when considering a beam with both spin and orbital components of the angular momentum. In this case the observed frequency shift is proportional to the *total angular momentum* of the beam rather than to the individual spin and orbital components [65]. Rotational frequency shifts have been observed at the level of individual molecules [66] and atoms [67].

Most recently, rotational frequency shifts associated with the orbital angular momentum have been observed in the light back-scattered from rotating objects, both in the macroscopic [68] and microscopic [69] regimes. It is worth noting that in neither regime is the size of this shift dependent upon the polarisation state of the light and hence it is independent of the optical SAM.

Finally, it is perhaps interesting to note that both the micromanipulation and the rotational frequency shifts discussed above are observable either with monochromatic laser light or with white light [70, 71].

The accumulated experimental evidence covering both micromanipulation and rotational frequency shifts leaves little

room for doubt that the separation of optical angular momentum, about the direction of propagation, into spin and orbital parts is a genuine and quantifiable phenomenon.

## 5. Orbital angular momentum or the azimuthal component of linear momentum?

The original derivation by Allen *et al*, reviewed in section 2, obtained the orbital angular momentum as a consequence of the fact that a field with a vortex has an azimuthal component of Poynting's vector associated with it, as given in equation (10). If we associate this with a local momentum then we are led to a physical picture of off-axis skew rays leading to the angular momentum [72], see figure 3. This behaviour of the Poynting vector has been observed experimentally by using a Shack Hartmann wavefront sensor [73]. Poynting's vector gives the density of linear momentum and its volume integral is the total momentum. It is interesting to ask whether this means that the azimuthal component of this, upon which we have based so much, is truly a momentum. We find that it is not [74].

It suffices to consider only a scalar wave in this section, as we are addressing only the orbital part of the angular momentum. From equation (1), valid within the eikonal approximation, we can write our momentum density, at a position  $\mathbf{R}$ , in the form

$$\mathbf{P}(\mathbf{R}) = \Im[\psi^*(\mathbf{R})\nabla\psi(\mathbf{R})], \quad (26)$$

where we have absorbed the constant of proportionality required in equation (1) into the normalisation of  $\psi$ . It follows that the azimuthal component of this is

$$P_\phi = \Im\left(\psi^*\frac{1}{\rho}\frac{\partial}{\partial\phi}\psi\right), \quad (27)$$

where we have dropped the implicit argument ( $\mathbf{R}$ ). It follows that the density of the  $z$ -component of the orbital angular momentum is

$$L_z = \rho P_\phi = \Im\left(\psi^*\frac{\partial}{\partial\phi}\psi\right), \quad (28)$$

as it should be. Note that the existence of a density of orbital angular momentum necessarily implies a corresponding density of azimuthal momentum as we could have written

$$P_\phi = \frac{L_z}{\rho}. \quad (29)$$

Let us consider a field with a vortex of strength  $\ell$  so that the azimuthal dependence of  $\psi$  is  $e^{i\ell\phi}$ . This means that the total azimuthal momentum and orbital angular momentum are both proportional to  $\ell$ :

$$\begin{aligned} p_\phi &= \ell \int \frac{|\psi|^2}{\rho} dV \\ L_z &= \ell \int |\psi|^2 dV. \end{aligned} \quad (30)$$

The total azimuthal momentum for such a field is non-zero even though the total *cartesian* momentum in the  $x$ – $y$  plane is zero. This is because the azimuthal momentum points in different cartesian directions on different sides of the vortex. The nature of the problem becomes clear when we realise that we associate momentum with the wavevector  $\mathbf{k}$ , for example the momentum of a single photon in a momentum eigenstate is  $\hbar \mathbf{k}$ . If we consider the amplitude of a plane wave, however, we find

$$e^{i\mathbf{k}\cdot\mathbf{r}} = e^{i(k_\rho\rho+k_z z)}. \quad (31)$$

Here  $k_\rho$  and  $k_z$  are the radial- and  $z$ -components of the wavevector and we note that there is no  $\phi$  component of the wavevector and hence no azimuthal momentum. So where does the azimuthal momentum come from?

The answer to this question comes with the realisation that the azimuthal momentum is *not* a linear momentum. This is readily apparent in quantum theory, where we can write an azimuthal momentum operator (for units in which  $\hbar=1$ ):

$$\hat{P}_\phi = -i\frac{1}{\rho}\frac{\partial}{\partial\phi}, \quad (32)$$

the expectation value of which is the total azimuthal momentum given above. This operator is not a linear momentum, however, as is clear from the fact that it does not commute with the operators for the  $x$  and  $y$  components of the linear momentum [74]. There is no difficulty, however, in working with the momentum *density* as represented, for light, by the Poynting vector. To see this we can write an operator for the azimuthal momentum density in the form

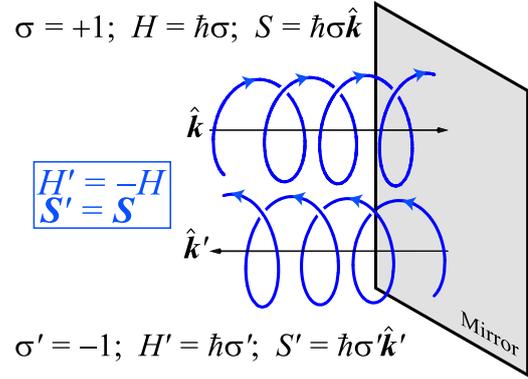
$$\hat{p}_\phi(\mathbf{R}) = \left\{ \frac{1}{2}(\cos\phi\hat{p}_y - \sin\phi\hat{p}_x), \delta(\hat{\mathbf{r}} - \mathbf{R}) \right\}, \quad (33)$$

where  $\{\dots,\dots\}$  represents an anticommutator. This quantity is both the density of the azimuthal momentum at  $\mathbf{R}$  but *also* it is the density of the *linear* momentum in the azimuthal direction. The density of the azimuthal momentum is a density of linear momentum, but the total azimuthal momentum is not itself a linear momentum; a subtle but important distinction [74].

It is tempting, and often insightful to think of light beams carrying orbital angular momentum in terms of rays directed by the local Poynting vector but we must be careful not to push this idea too far. If we attempt rigorously to map the field onto a set of rays using the Wolf function, for example, then we are led to ‘negative’ rays near to the vortex core [75].

## 6. Spin or helicity?

In texts on relativistic quantum mechanics and quantum field theory the helicity of a massless particle, like a photon, is usually defined as the value of the particle’s spin in the direction of propagation [76, 77]. In optics, however, it is more natural to work with a helicity defined in terms of the



**Figure 4.** Helicity  $\mathcal{H}$  is a pseudoscalar quantity but spin,  $\mathbf{S}$ , is a pseudovector. Consider a right-handed circularly polarised photon incident upon a mirror, as shown. After reflection, both the handedness (and therefore helicity) and the direction of propagation of the photon are inverted. This leaves the spin of the photon unchanged. Figure courtesy of [83].

two potentials,  $\mathbf{A}$  and  $\mathbf{C}$  [78–84]:

$$\begin{aligned} \mathcal{H} &= \frac{1}{2} \int (\mathbf{A} \cdot \mathbf{B} - \mathbf{C} \cdot \mathbf{E}) dV \\ &= \frac{1}{2} \int (\mathbf{A}^\perp \cdot \mathbf{B} - \mathbf{C}^\perp \cdot \mathbf{E}) dV. \end{aligned} \quad (34)$$

The global conservation of this quantity is a manifestation of the Heaviside–Larmor symmetry [84, 85]. Note that this quantity is manifestly gauge invariant as only the transverse parts of  $\mathbf{A}$  and  $\mathbf{C}$  contribute. We shall demonstrate the link between this expression and the idea of helicity in particle physics towards the end of this section.

We note that the helicity  $\mathcal{H}$  is a scalar quantity (more precisely a pseudoscalar) but our spin,  $\mathbf{S}$ , is a pseudovector. It is clear that they are of very different character. They also have fundamentally different properties in reflection: a circularly polarised plane wave on reflection retains the same SAM but reverses the sign of its helicity [83], as in figure 4.

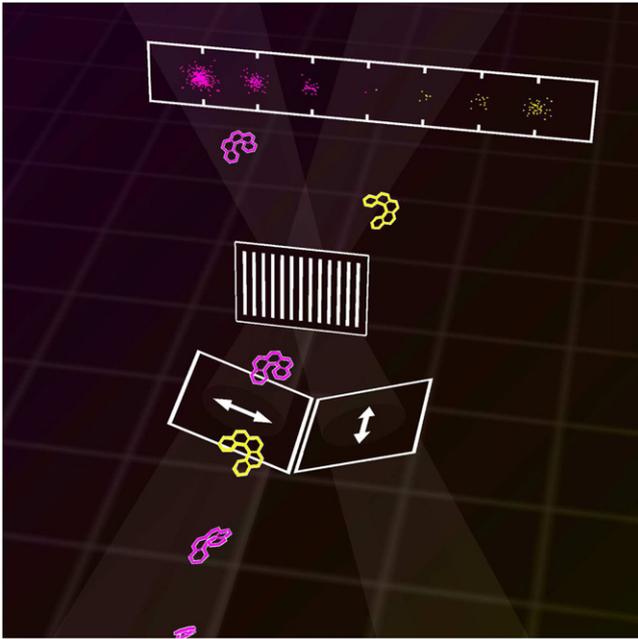
There is a simple relationship between the density of the helicity and the Heaviside–Larmor symmetric density of the spin in the form a local conservation law:

$$\begin{aligned} &\frac{\partial}{\partial t} \frac{1}{2}(\mathbf{A}^\perp \cdot \mathbf{B} - \mathbf{C}^\perp \cdot \mathbf{E}) \\ &+ \nabla \cdot \frac{1}{2}(\mathbf{E} \times \mathbf{A}^\perp + \mathbf{B} \times \mathbf{C}^\perp) = 0. \end{aligned} \quad (35)$$

The spin density plays the role of flux of helicity in much the same way that Poynting’s vector is both the density of linear momentum and also the flux of energy for the field [83, 84].

Further evidence in favour of adopting  $\mathcal{H}$  as the helicity comes from the quantum theory in which the helicity operator may be written in terms of the difference between the total number of left and right circularly polarised photons [83, 84]

$$\begin{aligned} \hat{\mathcal{H}} &= \frac{1}{2} \int (\hat{\mathbf{A}}^\perp \cdot \hat{\mathbf{B}} - \hat{\mathbf{C}}^\perp \cdot \hat{\mathbf{E}}) dV \\ &= \sum_{\mathbf{k}} \hbar (\hat{n}_{\mathbf{k},L} - \hat{n}_{\mathbf{k},R}). \end{aligned} \quad (36)$$



**Figure 5.** Schematic representation of a chiral grating based on optical helicity for discriminating between molecular enantiomers. Figure courtesy of [89].

The total spin operator has a similar form, but is a vector quantity in which the difference between the numbers of photons in each mode is multiplied by the direction of the corresponding wavevector:

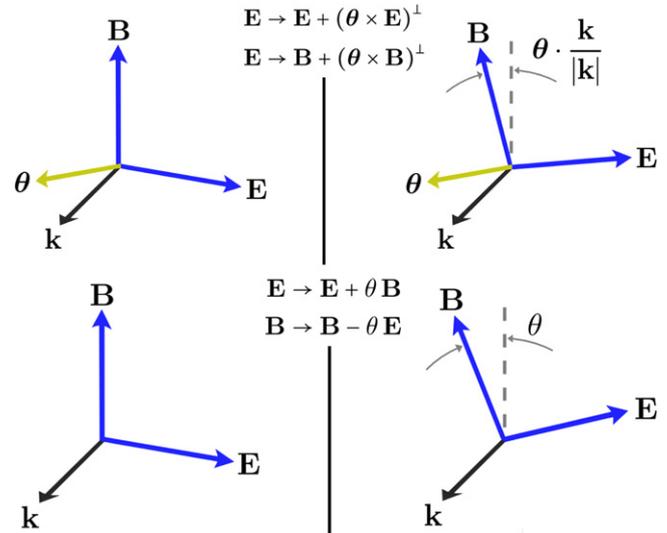
$$\begin{aligned} \hat{\mathbf{S}} &= \frac{1}{2} \int (\hat{\mathbf{E}} \times \hat{\mathbf{A}}^\perp + \hat{\mathbf{B}} \times \hat{\mathbf{C}}^\perp) dV \\ &= \sum_{\mathbf{k}} \hbar (\hat{n}_{\mathbf{k},L} - \hat{n}_{\mathbf{k},R}) \frac{\mathbf{k}}{k}. \end{aligned} \quad (37)$$

When written in this form, the connection with the particle physics idea of helicity is clear: if for each plane wave component we take the scalar product with the direction of propagation,  $\mathbf{k}/k$ , then we find the helicity.

Identifying the helicity of light is not a purely academic point as it plays an important part in the coupling between electromagnetic fields and chiral objects. In particular, manipulating this quantity [86] is the key to exerting different forces on objects with opposite chirality [87], including opposite molecular enantiomers [88]. A discriminatory chiral diffraction grating, for example, could be employed to measure the enantiomeric excess of a sample of chiral molecules, see figure 5 [89]. It is entirely possible that realistic techniques for the separation of chiral molecules of biological and pharmaceutical significance may emerge from these fledgling studies.

## 7. Noether's theorem and symmetries

We have arrived at a situation in which there are four angular-momentum related quantities each of which, for the free electromagnetic field, is a conserved quantity [85, 90]. These



**Figure 6.** (Top) spin generates the closest approximation to an infinitesimal rotation of the field vectors about  $\theta$  that is consistent with the requirement that the field remains transverse. For a plane wave, the field vectors are rotated in the transverse plane through the angle  $\theta \cdot \mathbf{k}/k$ , as depicted. (Bottom) helicity generates an infinitesimal duality rotation. For a plane wave, the field vectors are rotated about the wavevector  $\mathbf{k}$ , as depicted. Note that for clarity, the magnitudes of these rotations have been exaggerated. Reproduced from [84]. © IOP Publishing & Deutsche Physikalische Gesellschaft. CC BY-NC-SA.

are the spin and orbital parts of the angular momentum, the helicity and the total angular momentum<sup>6</sup>. Noether's theorem tells us that each of these should be associated with a symmetry of the electromagnetic field [92, 93], the forms of which can be derived from the Lagrangian. The application of Noether's theorem tells us something else that is important, which is that the densities and fluxes associated with each of our conserved quantities are not unique; the local conservation laws that emerge do so only up to an arbitrary four-divergence. Although the total helicity, angular momentum and its spin and orbital parts are fixed, the densities of these are not and we have some freedom in choosing their forms.

It is interesting to ask which are the symmetries that correspond to our four conserved quantities. The first point that we should address is that the Lagrangian itself is not unique. This non-uniqueness has no physical consequences, however, as *any* Lagrangian that gives the correct Maxwell equations will lead to our conserved quantities. This should not be a surprise as the conserved quantities may also be derived directly from the Maxwell equations. It is certainly the case, however, that the choice of Lagrangian density may make some symmetries and conserved quantities easier to extract. For our purposes a Lagrangian density that is manifestly Heaviside–Larmor symmetric has much to recommend it [80, 85, 94]. We find that the four conserved quantities, the spin and the orbital parts of the angular momentum, the

<sup>6</sup> In fact we can add to these several more including, in particular, the boost-inducing quantities associated with the angular momentum tensor in the four dimensions of space plus time [43, 91].

helicity and the total angular momentum correspond to invariance under the following symmetries: (i) conservation of the total spin part corresponds to the symmetry of the  $\mathbf{E}$  and  $\mathbf{B}$  fields about any chosen rotation axis and retaining only the transverse parts of the resulting fields. (ii) Conservation of total orbital part corresponds to rotating only the field amplitudes, keeping the directions of  $\mathbf{E}$  and  $\mathbf{B}$  fixed, and retaining only the transverse parts of the resulting fields. (iii) Conservation of the total angular momentum corresponds to symmetry under rotations of the fields, both the amplitudes and the directions. (iv) Finally, conservation of the helicity corresponds to the Heaviside–Larmor symmetry. The transformations of the  $\mathbf{E}$  and  $\mathbf{B}$  fields associated with the SAM and helicity are depicted in figure 6. The relationship between these four conserved quantities and distinct electromagnetic symmetries lends further weight, if needed, to our observation that all four have physical significance.

The non-uniqueness of the density of a conserved quantity allows us to make an interesting connection between the orbital part of the angular momentum and the *linear* momentum. If we apply Noether's theorem to the translational symmetry for our dual symmetric Lagrangian we are led to an expression for the momentum density in the form [95]

$$\tilde{\mathbf{g}} = \mathbf{E} \times \mathbf{B} + \sum_i \nabla_i \frac{1}{2} (\mathbf{A}^\perp E_i + \mathbf{C}^\perp B_i). \quad (38)$$

The first term is simply the familiar Poynting vector and the second is a spatial divergence. It is normal practice to remove any term of this form and, in the process, obtain an energy-momentum tensor that is both independent of the potentials and also that is symmetric [96]. Yet there is no necessity to do this and the commonly stated justification that it is essential that the energy–momentum tensor be symmetric in order to conserve angular momentum is not correct<sup>7</sup>. It is interesting to note that the azimuthal component of  $\tilde{\mathbf{g}}$  gives only the *orbital* part of the angular momentum and not the spin [95]. It is, in essence, equivalent to working in the eikonal approximation with the azimuthal component of the approximated Poynting vector in the form given in equation (1). The azimuthal component of the full Poynting vector  $\mathbf{E} \times \mathbf{B}$ , however, is the *total* angular momentum. The difference between  $\tilde{\mathbf{g}}$  and Poynting's vector amounts only to a divergence that makes no contribution to the total momentum obtained by integration over space, but it is in the subtleties of such partial integrations that the problem of separating out the spin and orbital parts of the angular momentum resides [31].

## 8. Conclusion

The analogy between optics and quantum theory makes it natural to associate the presence of optical vortices with an orbital angular momentum. The situation is made more

<sup>7</sup> We do note, however, that 'the' symmetric and traceless energy–momentum tensor serves naturally and uniquely as the gravitational source in conventional general relativity.

complicated, however, by the fact that the electric and magnetic fields are vector quantities and, moreover, that these are constrained to be transverse. There has been, as we have noted, considerable confusion over the years about the natures of the spin and orbital angular momenta of light and even whether such a separation could be considered to be physical. We contend that it is indeed physical and that the relevant quantities are the  $\mathbf{S}$  and  $\mathbf{L}$  given above. This separation is not without its subtle features, however, and it may be worthwhile, in this regard, to repeat the points made in the abstract: (i) we have presented a reprise of the 1992 paper in which it was pointed out that the Laguerre–Gaussian modes, familiar from laser physics, carry orbital angular momentum. (ii) The total angular momentum may be separated into spin and orbital parts, but neither alone is a true angular momentum. (iii) The spin and orbital parts, although not themselves true angular momenta, are distinct and physically meaningful, as has been demonstrated clearly in a range of experiments. (iv) The orbital part of the angular momentum in the direction of propagation of a beam is not simply the azimuthal component of the linear momentum. (v) The component of the total spin parallel to the total linear momentum is not the total helicity<sup>8</sup>, although spin and helicity are related quantities. (vi) Finally, the spin and orbital parts of the angular momentum correspond to distinct symmetries of the free electromagnetic field and hence are separately conserved quantities. As a final point, we note that the electromagnetic field is best understood in terms of relativity and it would be best to explore the useful analogy between optics and quantum theory using relativistic quantum mechanics. This suggests comparing Maxwell's equations with the Dirac equation for the electron [28–30]. Indeed, as we have already noted this was the basis of Darwin's approach [31], and the analogy continues to be a useful one [52, 97–99]. The analogy is useful here as the Dirac equation reveals that for an electron the orbital and spin angular momenta are not separately conserved, indeed, it was the non-conservation of the orbital angular momentum that led Dirac to infer the form of the spin contribution [100]. The spin and orbital parts of the angular momentum and also the helicity emerge quite naturally, moreover, from Maxwell's equations written as a Dirac equation [52]. It seems clear that the subtle natures of the spin and orbital angular momenta for light are intrinsically connected with the fact that Maxwell's theory is a relativistic one.

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