From Quantum Bäcklund Transforms to Topological Quantum Field Theory

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Abstract

We derive the quantum analogue of a Bäcklund transformation for the quantised Ablowitz-Ladik chain, a space discretisation of the nonlinear Schrödinger equation. The quantisation of the Ablowitz-Ladik chain leads to the q-boson model. Using a previous construction of Baxter's Q-operator for this model by the author, a set of functional relations is obtained which matches the relations of a one-variable classical Bäcklund transform to all orders in $\hbar$. We construct also a second Q-operator and show that it is closely related to the inverse of the first. The multi-Bäcklund transforms generated from the Q-operator define the fusion matrices of a 2D TQFT and we derive a linear system for the solution to the quantum Bäcklund relations in terms of the TQFT fusion coefficients.

1 Introduction

In the context of classical integrable systems the main interest in the construction of Bäcklund and Darboux transformations is their application in the construction of soliton solutions [26, 20]. Compared to the classical Bäcklund transformations the discussion of their quantum cousins started more recently. In 1992 Gaudin and Pasquier [24] constructed for the quantum Toda chain an analogue of Baxter's Q-operator [2, Ch. 9-10] and showed that in the semi-classical limit $\hbar \to 0$ the similarity transformation $\mathcal{O} \to Q(u)\mathcal{O}Q(u)^{-1}$ is a Bäcklund transform of the Toda chain. An introductory account to this and further results can be found in Sklyanin's lecture notes [27].

1.1 The Ablowitz-Ladik chain and its quantisation

In this article we are interested in the quantum analogue of Bäcklund transforms for another integrable system: the Ablowitz-Ladik chain [1],

$$\left\{ \begin{array}{l} \partial_t \psi_j = \psi_{j+1} - 2\psi_j + \psi_{j-1} - \psi_j^* \psi_j (\psi_{j+1} + \psi_{j-1}) \\ \partial_t \psi_j^* = -\psi_{j+1}^* + 2\psi_j^* - \psi_{j-1}^* + \psi_j^* \psi_j (\psi_{j+1}^* + \psi_{j-1}^*) \end{array} \right. , \tag{1.1}$$

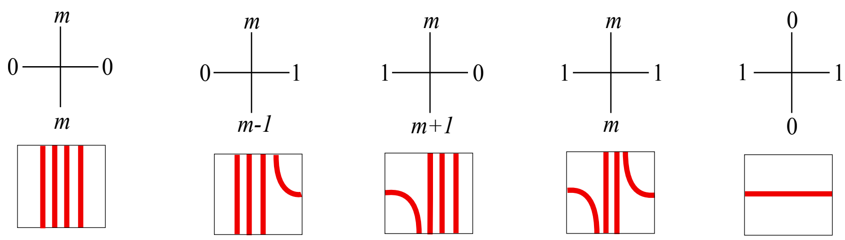

Figure 1: Vertex configurations for the T-operator. Consider a square lattice where we assign a nonnegative integer to each lattice edge, the statistical variable. At each vertex we only allow for particular configurations, for the T-operator the horizontal edges can only take values 0 or 1, below are the vertex configurations in terms of non-intersecting paths.

where we consider periodic boundary conditions $\psi_{j+n} = \psi_j$ and $\psi^*_{j+n} = \psi^*_j$. The equations (1.1) are a space discretisation of the following system of coupled PDEs

$$\left\{ \begin{array}{c} \partial_t \psi = \partial_x^2 \psi - 2\psi^* \psi^2 \\ \partial_t \psi^* = -\partial_x^2 \psi^* + 2\psi^{*2}\psi \end{array} \right. .$$

After changing to imaginary time, $t \to \imath t$ with $\imath = \sqrt{-1}$, this system of PDEs allows for a reduction, $\psi^* = \pm\bar{\psi}$ with $\bar{\psi}$ denoting the complex conjugate of ψ, to the nonlinear Schrödinger (NLS) equation, $-\imath\partial_t\psi = \partial_x^2\psi \mp 2\psi|\psi|^2$. Bäcklund and Darboux transformations for the system (1.1) and its reduction to the discrete NLS system can be found in e.g. [7, 25, 28, 32] and references in *loc. cit.*

Kulish considered in [17] a particular quantisation of the Ablowitz-Ladik chain leading to the q-boson model with Hamiltonian

$$H = -\sum_{j=1}^{n} (\beta_j \beta^*_{j+1} + \beta^*_j \beta_{j+1} - 2(1-q^2)N_j), \tag{1.2}$$

see also [4, 5, 6] as well as references therein. Here the β_j, β^*_j's are the generators of a q-deformed version of the oscillator or Heisenberg algebra with q being the quantisation parameter (see the definition (3.2) in the text) and N_j is the particle number operator at site j.

In the Fock space representation and with periodic boundary conditions on the lattice, $\beta_{j+n} = \beta_j$ and $\beta^*_{j+n} = \beta^*_j$, the model (1.2) can be solved via the quantum inverse scattering method [9]. Analogous to the classical case, one defines a quantum Lax operator $L(u)$ and the Hamiltonian (1.2) can then be understood as a particular element in a commutative algebra generated from the commuting transfer matrices $T(u) = \operatorname{Tr} L(u)$ of an exactly solvable lattice model in the sense of Baxter [2]; see Figure 1 for the vertex configurations defining the model.

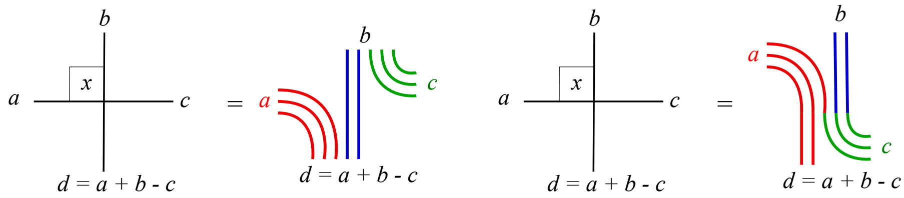

Figure 2: Vertex configurations for the $Q^{\pm}$-operators. Consider a square lattice where we assign a nonnegative integer to each lattice edge, the statistical variable. At each vertex we only allow for particular configurations, for the Q^{+}-operator (bottom left) we require that $b \geq c$ and $a+b=c+d$. For Q^{-} (bottom right) we impose instead the conditions $a+b \geq c$ and $a+b=c+d$. On the right of each vertex is an interpretation of these conditions in terms of non-intersecting lattice paths.

1.2 Baxter's Q-operators and quantum Bäcklund maps

In this article we discuss two analogues $Q^{\pm}$ of Baxter's Q-operator for the quantised Ablowitz-Ladik chain (1.2) with periodic boundary conditions. The construction of these $Q^{\pm}$-operators corresponds to the definition of two additional exactly solvable statistical mechanics models defined on a square lattice; the corresponding vertex configurations are depicted in Figure 2. We show that the transfer matrices $T(u), Q^{\pm}(u)$ of all three models are related via functional relations, Baxter's famous TQ-equation and a quantum Wronskian relation for $Q^{\pm}$. The Q^{+}-operator has been constructed earlier by the author, the construction of Q^{-} and the resulting functional relations (4.8), (4.13) in the text are new. We show that in the limit $n \to \infty$ the second solution Q^{-} becomes the inverse of Q^{+}.

Employing the Q^{+}-operator we consider the associated similarity transformations of the q-boson fields,

$$\beta_j \mapsto \tilde{\beta}_j(v) = Q^{+}(v)\beta_j Q^{+}(v)^{-1} \quad \text{and} \quad \beta_j^* \mapsto \tilde{\beta}_j^*(v) = Q^{+}(v)\beta_j^* Q^{+}(v)^{-1} \,.$$

The first main result of this article is the proof that the transformed fields $\tilde{\beta}_j(v)$ and $\tilde{\beta}_j^*(v)$ obey the functional relations (c.f. Theorem 5.2 in the text)

$$\frac{\tilde{\beta}_j - \beta_j}{v} = \left(1 - \beta_j^* \tilde{\beta}_j\right) \tilde{\beta}_{j-1} \qquad \text{and} \qquad \frac{\tilde{\beta}_j^* - \beta_j^*}{v} = \beta_{j+1}^* (\beta_j^* \tilde{\beta}_j - 1), \tag{1.3}$$

which allow one to compute $\tilde{\beta}_j(v)$ and $\tilde{\beta}_j^*(v)$ recursively via a power series expansion in v. This result is physically significant, since this Bäcklund map (and the one induced by the adjoint of the Q^{+}-operator) describes for small time steps $0 < v = \Delta t \ll 1$ the discrete time evolution of the quantum Ablowitz-Ladik chain. Surprisingly, the quantum relations (1.3) match *exactly* (i.e. to all orders in $\hbar$) the relations of the classical fields ψ_j, ψ_j^* under the analogous classical one-variable Bäcklund map $\mathcal{B}^+(v) : (\psi_j, \psi_j^*) \mapsto (\tilde{\psi}_j(v), \tilde{\psi}_j^*(v))$ considered by Suris in [28] for the system (1.1).

In comparison, the match between Baxter's Q-operator and the classical Bäcklund map for the Toda chain is established by formulating Q as an integral operator and then identifying its integral kernel in the limit $\hbar \to 0$ with the generating function of the Bäcklund transform; see [24]. Our approach avoids the generating function altogether and directly arrives at the functional relation (1.3) which allows one to compute the image of the quantum transform, the quantum variables $\tilde{\beta}_j$, $\tilde{\beta}_j^*$, via recurrence; see Section 5.1.

1.3 Multivariate Bäcklund maps and 2D TQFT

Once a one-variable Bäcklund map is constructed one can consider multivariate transforms via composition, $\mathcal{B}^+(x_1)\circ\mathcal{B}^+(x_2)\circ\cdots\circ\mathcal{B}^+(x_{n-1})$. These multivariate transforms are central to Sklyanin's separation of variables approach; see e.g. [18] and, for a discussion of the quantum separation of variables approach, [22] as well as references therein. Here we wish to highlight a novel aspect of these multivariate transforms: they generate the fusion matrices of a 2D topological quantum field theory (TQFT) constructed in [15].

Consider matrix elements of a product of Q^+-operators which is the quantum analogue of the above multivariate Bäcklund transform,

$$\langle\lambda|\prod_{i=1}^{n-1} Q^+(x_i)|\mu\rangle = \sum_{\nu}(-1)^{|\nu|} N_{\mu\nu}^{\lambda}(q) P_\nu(x_1,\ldots,x_{n-1};q) , \tag{1.4}$$

where λ, μ, ν are partitions labelling particle configurations of the q-boson model (see Eqn (3.15) in the text) and the P_λ's are a special basis in the ring of symmetric functions, the so-called q-Whittaker functions which are a special case of Macdonald's functions [19, Ch. VI]. Since the Q^+-operators for different x_i's commute, reflecting the analogous property of the classical Bäcklund transforms, the above expansion is well-defined. The expansion coefficients $N_{\mu\nu}^{\lambda}(q)$ are polynomials in q with integer coefficients and define the fusion in a 2D TQFT, i.e. they are the values of the pair of pants cobordism shown in Figure 3 which can be seen as a 2D analogue of a Feynman diagram describing the fusion of particles in a QFT. The TQFT defined via (1.4) is a q-deformation of the $SU(n)$-WZNW fusion ring, that is the constant term $N_{\mu\nu}^{\lambda}(0)$ in $N_{\mu\nu}^{\lambda}(q)$ is given by the operator product expansion of primary fields in WZNW conformal field theory.

The second main result of this article is that the matrix elements of the Bäcklund transformed q-boson fields $\tilde{\beta}_j(v)$ and $\tilde{\beta}_j^*(v)$ can be computed in terms of the TQFT fusion coefficients, thus relating the quantum Bäcklund transform of the Ablowitz-Ladik chain to fusion in a 2D TQFT; see Prop 6.4 in the text.

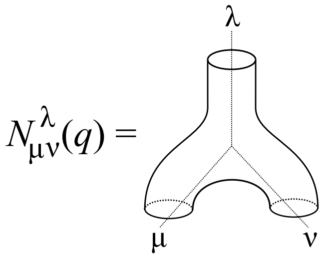

Figure 3: Depiction of the 'pair-of-pants' cobordism which fixes the fusion product in a 2D TQFT. The dotted lines inside are the corresponding conventional Feynman diagram.

1.4 Outline of the article

The outline of the article is as follows. In Section 2 we review some aspects of the classical Ablowitz-Ladik chain, in particular we recall a one-variable family of Bäcklund transforms considered by Suris in [28] which is quantised in terms of the Q^+-operator in Section 4. Before that Section 3 reviews the necessary background on the quantisation of the Ablowitz-Ladik chain, the q-boson algebra and the algebraic Bethe ansatz solution of the q-boson model. Section 4 gives the construction of the $Q^{\pm}$-operators and the derivation of Baxter's TQ-equation as well as the resulting quantum Wronskian relation. Section 5 contains the main result, the discussion of the quantum Bäcklund transform and the derivation of the functional relations (1.3). In Section 6 we then relate these relations to the 2D TQFT defined in terms of the Q-operator. Section 7 contains a discussion of the results and outlook towards future work.

2 The classical Ablowitz-Ladik chain

Interpreting the discrete fields $\{\psi_j\}_{j\in\mathbb{Z}_n}$ and $\{\psi_j^*\}_{j\in\mathbb{Z}_n}$ as components of vectors in $\mathbb{R}^{2n}$, the classical Ablowitz-Ladik chain (1.1) allows for the following Poisson structure (see e.g. [17] and [28])

$$\{\psi_i, \psi_j^*\} = \delta_{ij}(1 - \psi_j^*\psi_j) \qquad \text{and} \qquad \{\psi_i, \psi_j\} = \{\psi_i^*, \psi_j^*\} = 0\,. \tag{2.1}$$

Note that the discrete fields ψ_j, ψ_j^* are not a pair of canonical variables. The time evolution (1.1) can be described as Hamiltonian flows, $\partial_t\psi_j = \{H, \psi_j\}$ and $\partial_t\psi_j^* = \{H, \psi_j^*\}$, where (assuming $|\psi_j^*\psi_j| < 1$)

$$H = -\sum_{j\in\mathbb{Z}_n} \left(\psi_j^*\psi_{j+1} + \psi_j\psi_{j+1}^* + 2\ln(1 - \psi_j^*\psi_j)\right)\,. \tag{2.2}$$

Initially, one can consider ψ_j and ψ_j^* as independent variables, but as explained in the introduction, the system reduces to the NLS model when changing to imaginary time, $t \to \imath t$, and identifying ψ_j^* as the complex conjugate of ψ_j.

The integrals of motion of the system (1.1) are obtained from the spectral curve $\det(L(u)-\lambda)=0$, where the Lax operator $L(u)$ is given as a product of the following local Lax matrices,

$$L(u)=L_n(u)\cdots L_2(u)L_1(u),\qquad L_j(u)=\begin{pmatrix}1 & u\psi_j^* \\ \psi_j & u\end{pmatrix}. \tag{2.3}$$

Here we have chosen not to take the standard form of the Lax matrices, see e.g. [1, 17, 28] and references therein, but instead match the conventions in [15] which turn out to be convenient to discuss the quantum case. There are two spectral invariants, the determinant

$$\det L(u)=u^n\prod_{j=1}^n(1-\psi_j^*\psi_j) \tag{2.4}$$

and the trace

$$T(u)=\operatorname{Tr}L(u)=\sum_{r=0}^n u^r T_r\ . \tag{2.5}$$

Using the classical analogue of the Yang-Baxter equation one proves the following [17]:

Proposition 2.1 $\{T_r,T_{r'}\}=0$ *for all* $r,r'=0,1,\dots,n$.

The integrals of motion T_r create higher flows and their existence implies Liouville integrability of the system (1.1). The Hamiltonian (2.2) creating the physical time flow (1.1) is given by

$$H=-T_1-T_{n-1}+2\operatorname{Tr}(\ln L(1)) \tag{2.6}$$

Thus, we note that the physical time flow separates into three different commuting flows generated by $T_1=\sum_{j=1}^n\psi_j\psi_{j+1}^*$, $T_{n-1}=\sum_{j=1}^n\psi_j^*\psi_{j+1}$ and the determinant (2.4); see the discussion in [28]. For instance, the flow generated from $-T_1$ gives rise to the differential-difference equations [28, Eqns (2.7) and (2.8)]

$$\begin{cases}\partial_t\psi_j=\psi_{j-1}(1-\psi_j^*\psi_j)\\ \partial_t\psi_j^*=-\psi_{j+1}^*(1-\psi_j^*\psi_j)\end{cases}. \tag{2.7}$$

The classical Bäcklund transforms which we discuss in the next section can be understood as time-discretisation of this particular flow.

2.1 A one-variable family of Bäcklund transforms

In this article we are interested in one-variable families of Bäcklund transformations $\mathcal{B}^\pm(v):(\psi,\psi^*)\mapsto(\tilde\psi,\tilde\psi^*)$ which correspond to the flows discussed by Suris in [28, Prop 1 and Prop 4] with v playing the role of the discrete time parameter $v\mathbb{Z}$. We summarise here some of the results from *loc. cit.* for the case of periodic boundary conditions to keep this article self-contained and to introduce our notation. The novel aspect in this article is that we relate these discrete time flows to the $Q^{\pm}$-operator which we introduce in a subsequent section.

The transform $\mathcal{B}^+(v)$ is defined implicitly in the standard manner using a *Darboux transformation*: one considers the following local gauge transformation of the Lax operator (2.3)

$$D^+_{j+1}(u,v)L_j(u;\psi,\psi^*) = L_j(u;\tilde\psi,\tilde\psi^*)D^+_j(u,v)\ , \tag{2.8}$$

where the so-called *Darboux matrices* D^+_j are assumed to be of the following form,

$$D^+_j(u,v) = \left(\begin{array}{cc} v-ua_j & -ub_j \\ vc_j & -u \end{array}\right)\ . \tag{2.9}$$

Here v is an additional variable related to the spectrality property of the Bäcklund transform $\mathcal{B}^+(v)$; see [27] and references therein for an explanation in the context of the Toda chain. N.B. the dressing matrix is singular for $u=v$, $\det D^+_j(v,v)=0$, if we require that $a_j = 1-b_jc_j$ in close analogy with the case of the Toda chain discussed in *loc. cit.* The equality (2.8) together with the explicit form of the Darboux matrix (2.9) then fixes the map $\mathcal{B}^+(v):(\psi,\psi^*)\mapsto(\tilde\psi,\tilde\psi^*)$; compare with [28, Eqn (3.1)].

Proposition 2.2 *The transformed variables $(\tilde\psi_j,\tilde\psi^*_j)$ obey the following functional relations*

$$\left\{\begin{array}{l} \tilde\psi_j-\psi_j = v\ \tilde\psi_{j-1}(1-\psi^*_j\tilde\psi_j) \\ \tilde\psi^*_j-\psi^*_j = v\ \psi^*_{j+1}(\psi^*_j\tilde\psi_j-1) \end{array}\right.\ . \tag{2.10}$$

In particular, the matrix elements in (2.9) are $a_j = 1+v\psi^*_j\tilde\psi_{j-1}$, $b_j=-v\psi^*_j$, $c_j=\tilde\psi_{j-1}$.

Proof. A straightforward computation. Inserting (2.9) into (2.8) and comparing coefficients of powers in the spectral variable u yields the asserted equalities. ■

The identities (2.10) allow one to compute the transformed variables $(\tilde\psi,\tilde\psi^*)$ via recurrence upon expanding them in power series with respect to the variable v. That is, if we set

$$\tilde\psi_j(v)=\sum_{r\geq 0}v^r\tilde\psi_{j,r}\qquad\text{and}\qquad \tilde\psi^*_j(v)=\sum_{r\geq 0}v^r\tilde\psi^*_{j,r} \tag{2.11}$$

then the above relations imply the recurrence identities

$$\left\{\begin{array}{l} \tilde\psi_{j,r} = \tilde\psi_{j-1,r-1}+\psi^*_j\sum\limits_{a+b=r-1}\tilde\psi_{j-1,a}\tilde\psi_{j,b} \\ \tilde\psi^*_{j,r} = \psi^*_j\psi^*_{j+1}\tilde\psi_{j,r-1} \end{array}\right.\ . \tag{2.12}$$

Note that $\mathcal{B}^+(0)=\mathrm{Id}$ which fixes the initial conditions $\tilde\psi_{j,0}=\psi_j$ and $\tilde\psi^*_{j,0}=\psi^*_j$. By construction the map $\mathcal{B}^+(v)$ leaves the Hamiltonians (2.5) as well as (2.4) invariant, $T_r(\psi,\psi^*)=T_r(\tilde\psi,\tilde\psi^*)$. Furthermore, one has [28, Prop 5]:

Proposition 2.3 *$\mathcal{B}^+(v)$ preserves the Poisson structure (2.1).*

Therefore the transform $\mathcal{B}^+(v)$ satisfies the defining conditions of an integrable map [33] and it then follows from a general argument [27] that the transforms $\mathcal{B}^+(x)$ and $\mathcal{B}^+(y)$ commute,

$$\mathcal{B}^+(x) \circ \mathcal{B}^+(y) = \mathcal{B}^+(y) \circ \mathcal{B}^+(x) \; . \tag{2.13}$$

One can also consider the inverse of the Darboux matrix (2.9) and one then arrives at a second transform $\mathcal{B}^-(v) : (\psi, \psi^*) \mapsto (\hat{\psi}, \hat{\psi}^*)$ which in case of the infinite chain is related to the inverse of $\mathcal{B}^+(-v)$ defined via the relations [28, Prop 4, Eqns (3.13) and (3.14)]

$$\left\{ \begin{array}{c} \hat{\psi}_j - \psi_j = -v\psi_{j-1}(1 - \psi_j \hat{\psi}^*_j) \\ \hat{\psi}^*_j - \psi^*_j = v(1 - \psi_j \hat{\psi}^*_j)\hat{\psi}^*_{j+1} \end{array} \right. . \tag{2.14}$$

The proofs are analogous to the previous case. Both systems of equations, (2.10) and (2.14), are approximations of the flow (2.7). Similarly, one can investigate the transforms for the flow generated by $-T_{n-1}$; see [28, Prop 2 and Prop 3]. These turn out to be related to the hermitian adjoints of the $Q^{\pm}$-operators, we therefore restrict our discussion to (2.10) and (2.14).

3 Quantisation of the Ablowitz-Ladik system

The following quantisation of the Ablowitz-Ladik model was first discussed by Kulish [17],

$$\left\{ \begin{array}{c} \partial_t \beta_j = \beta_{j+1} - 2\beta_j + \beta_{j-1} - \beta^*_j \beta_j (\beta_{j+1} + \beta_{j-1}) \\ \partial_t \beta^*_j = -\beta^*_{j+1} + 2\beta^*_j - \beta^*_{j-1} + \beta^*_j \beta_j (\beta^*_{j+1} + \beta^*_{j-1}) \end{array} \right. , \tag{3.1}$$

where $\{\beta_j, \beta^*_j\}_{j \in \mathbb{Z}_n}$ are now noncommutative variables satisfying the defining relation of the q-boson or q-Heisenberg algebra $\mathcal{H}_n(q)$; c.f. [17, Eqn (14)] and [3]. Throughout this article we assume q to be an indeterminate.

Definition 3.1 *Let $\mathcal{H}_n(q)$ be the $\mathbb{C}(q)$-algebra generated by $\{\beta_i, \beta^*_i, q^{\pm N_i}\}_{i=1}^n$ subject to the relations*

$$\left\{ \begin{array}{cc} q^{N_i}\beta_j = q^{-\delta_{ij}}\beta_j q^{N_i}, & q^{N_i}\beta^*_j = q^{\delta_{ij}}\beta^*_j q^{N_i} \\ \beta_i \beta^*_j - \beta^*_j \beta_i = \delta_{ij}(1-q^2)q^{2N_i}, & \beta_i \beta^*_i - q^2 \beta^*_i \beta_i = 1 - q^2 \end{array} \right. . \tag{3.2}$$

N.B. the notation q^{N_i} is purely formal, it does *not* mean that q^{N_i} is given by exponentiating another generator N_i although we will often write $q^a(q^{N_i})^m$ as q^{mN_i+a}. We have also made a small change to the usual definition of the q-boson algebra, see e.g. [11], by multiplying one of the generators with an extra factor, $\beta_j \to (1-q^2)\beta_j$. This allows us to derive from the second set of relations in (3.2) the identity $\beta_j^*\beta_j = 1 - q^{2N_j}$ and, thus, we obtain the familiar quantisation formula $[\cdot,\cdot] = -\imath\hbar\{\cdot,\cdot\} + O(\hbar^2)$ for the Poisson structure (2.1) when evaluating the indeterminate as $q \to \exp(\imath\hbar\gamma/2)$,

$$[\beta_i, \beta_j^*] = \delta_{ij}(1-q^2)(1-\beta_i^*\beta_i) = -\delta_{ij}\ \imath\hbar\gamma(1-\beta_i^*\beta_i) + O(\hbar^2)\ , \tag{3.3}$$

compare with [17, Eqn (13)]. Similar to the classical case, β_j^* is initially an independent generator, but we may consider representations of the q-boson algebra where β_j^* is the hermitian adjoint of β_j; see the Fock representation (3.14) below. This corresponds to the above mentioned reduction of the Ablowitz-Ladik chain to the NLS model and due to the change to imaginary time, $t \to \imath t$, one then must choose $\gamma = \imath c$ with $c \in \mathbb{R}$ being the coupling constant in the QNLS Hamiltonian.

The form of the Hamiltonian closely resembles the classical one,

$$H = -\sum_{j=1}^{n}(\beta_j\beta_{j+1}^* + \beta_j^*\beta_{j+1} - 2(1-q^2)N_j), \tag{3.4}$$

where the N_j's are a set of additional generators not contained in the original q-boson algebra obeying

$$N_i q^{N_j} = q^{N_j} N_i, \quad \beta_i(N_j - \delta_{ij}) = N_j\beta_i, \quad \beta_i^*(N_j + \delta_{ij}) = N_j\beta_i^*\ . \tag{3.5}$$

If we identify β_i^* with the creation and β_i with the annihilation of a q-boson at site i, then we easily recognise that N_i plays the role of a number operator on this site. We shall denote the total particle number operator by $N = \sum_{j=1}^n N_j$.

In close analogy to the classical case (2.3) one defines the monodromy matrix of the quantised system as a product of the local Lax matrices

$$L(u) = L_n(u)\cdots L_1(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}, \qquad L_j(u) = \begin{pmatrix} 1 & u\beta_j^* \\ \beta_j & u \end{pmatrix} \tag{3.6}$$

and one then obtains the quantum versions of the spectral invariants (2.5) of the classical system,

$$T(u) = \operatorname{Tr} L(u) = A(u) + z\ D(u) = \sum_{r=0}^{n} u^r T_r\ , \tag{3.7}$$

where z is an additional quasi-periodicity parameter which we will need later on. The explicit form of the quantum integrals of motion T_r has been derived in [15, Prop 3.10, Eqn (3.43)]: set $a_i = \beta_i\beta_{i+1}^*$ for $i = 1,\ldots,n-1$ and $a_n = z\beta_n\beta_1^*$ (right or clockwise hopping) then

$$T_r = \sum_{w=i_1\cdots i_r} \frac{[a_{i_1},[a_{i_2},\ldots[a_{i_{r-1}},a_{i_r}]_{q^2}\ldots]_{q^2}}{(1-q^2)^{r-1}} \tag{3.8}$$

with $[X,Y]_{q^2} = XY - q^2YX$ and the sum is running over all cyclically ordered words w with letters $1 \le i_j \le n$ occurring at most once. Moreover, $T_0 = 1$,

$T_n = z$ and we set $T_r = 0$ for all $r > n$ and $r < 0$. Note that $T_1 = \sum_{j=1}^n \beta_j \beta_{j+1}^*$ and that T_n simplifies to $T_n = \sum_{j=1}^n \beta_j^* \beta_{j+1}$. Thus, as in the classical case one has a splitting of the quantum Hamiltonian (3.4) into 'right movers' and 'left movers'.

One now proves quantum integrability of the system (3.1) via the following solution of the Yang-Baxter equation [17],

$$R_{12}(u/v)L_{13}(u)L_{23}(v) = L_{23}(v)L_{13}(u)R_{12}(u/v), \tag{3.9}$$

where the R-matrix in $\mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2) \cong \mathrm{End}\,\mathbb{C}^4$ with the conventions used in the definition (3.4) of L reads

$$R(u) = \begin{pmatrix} \frac{uq^2-1}{u-1} & 0 & 0 & 0 \\ 0 & q^2 & \frac{q^2-1}{u-1}u & 0 \\ 0 & \frac{q^2-1}{u-1} & 1 & 0 \\ 0 & 0 & 0 & \frac{uq^2-1}{u-1} \end{pmatrix} . \tag{3.10}$$

By the standard argument one obtains as an immediate consequence the following proposition.

Proposition 3.2 *The subalgebra $\mathcal{A}_n \subset \mathcal{H}_n$ generated by the T_r is commutative, $[T_r, T_{r'}] = 0$.*

Proof. One verifies that (3.10) is invertible. Using the cyclicity of the trace in (3.7) one obtains $[T(u), T(v)] = 0$. This proves the assertion for $z = 1$. The case of general z follows by the same argument using that $R(u/v)$ commutes with $\left(\begin{smallmatrix} 1 & 0 \\ 0 & z \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 1 & 0 \\ 0 & z \end{smallmatrix}\right)$. ■

We now consider a particular representation of the q-boson algebra which allows one to apply the quantum inverse scattering method or algebraic Bethe ansatz to construct a common eigenbasis.

3.1 The Fock representation of q-bosons

For the sake of completeness we recall the following representation of the q-boson algebra in terms of q-difference operators, which can be found in [11]. Consider the ring of polynomials $\mathfrak{R}^n = \mathbb{C}(q)[\xi_1, \ldots, \xi_n]$ with rational coefficients in the indeterminate q, where the ξ_i's are some auxiliary variables. Let τ_i be the shift operator with respect to ξ_i,

$$(\tau_i^{\pm 1} f)(\xi) = f(\xi_1, ..., \xi_i q^{\pm 1}, ..., \xi_n), \qquad f \in \mathfrak{R}^n \tag{3.11}$$

and define $\mathcal{D}_i$ to be the following q-derivative with respect to ξ_i,

$$(\mathcal{D}_i f)(\xi) = \frac{f(\xi) - f(\xi_1, ..., q^2\xi_i, ..., \xi_n)}{\xi_i - q^2\xi_i} \ . \tag{3.12}$$

In the limit $q \to 1$ one recovers the ordinary partial derivative ∂_i. We introduce a hermitian bilinear form $\mathfrak{R}^n \times \mathfrak{R}^n \to \mathbb{C}(q)$ by setting

$$\langle f, g\rangle = \bar{f}(\mathcal{D}_1, ..., \mathcal{D}_n) g(\xi_1, ..., \xi_n)|_{\xi_1=\cdots=\xi_n=0} \tag{3.13}$$

and define $\mathcal{F}^n = \overline{\mathfrak{R}^n}$ to be the completion with respect to this inner product. We will simply write $\mathcal{F}$ for $\mathcal{F}^1$ and we have that $\mathcal{F}^n = \mathcal{F}^{\otimes n}$. The following result is contained in [11].

Proposition 3.3 (Fock representation) *The map $\mathcal{H}_n \to \operatorname{End}\mathcal{F}^n$ defined via*

$$\beta_i \mapsto (1-q^2)\mathcal{D}_i, \qquad \beta_i^* \mapsto \hat{\xi}_i, \qquad q^{\pm N_i} \mapsto \tau_i^{\pm 1}\ , \tag{3.14}$$

where $\hat{\xi}_i$ is the multiplication operator with ξ_i, defines a simple module for $\mathcal{H}_n$ for all $n \geq 1$.

There is an alternative formulation of the same module used in [15, Prop 3.3]: let $\mathcal{I} \subset \mathcal{H}_n$ be the *left* ideal generated by the elements β_i and $(1 - q^{\pm N_i})$. Define a highest weight vector $|0\rangle = 1+\mathcal{I}$, the "pseudo-vacuum", in the quotient $\mathcal{H}_n/\mathcal{I}$ with 1 the identity element. For any partition λ with at most n parts set

$$|\lambda\rangle = \prod_{j=1}^{n} \frac{(\beta_j^*)^{m_j(\lambda)}}{(q^2)_{m_j(\lambda)}}\, |0\rangle, \qquad (q^2)_m := \prod_{j=1}^{m}(1-q^{2j}), \tag{3.15}$$

where $m_j(\lambda)$ is the multiplicity of columns of height j in the Young diagram of λ. It is easy to verify that the map $|\lambda\rangle \mapsto \prod_{j=1}^n \xi_j^{m_j(\lambda)}/(q^2)_{m_j(\lambda)}$ provides a module isomorphism $\mathcal{H}_n/\mathcal{I} \tilde{\to} \mathcal{F}^n$ and we shall henceforth identify both modules. We denote by $\langle\lambda|$ the dual basis of (3.15) and the map $\tilde{\mathcal{F}}^n \to \mathcal{F}^n$ given by $\langle\lambda| \mapsto b_\lambda(q^2)|\lambda\rangle$ with $\tilde{\mathcal{F}}^n$ denoting the dual space and $b_\lambda = \prod_{j\geq 1}(q^2)_{m_j(\lambda)}$, then introduces an inner product on $\mathcal{F}^n$. Note that if q is evaluated at a root of unity the module ceases to be simple.

Remark 3.4 *The eigenspaces of the particle number operator $N = \sum_{j=1}^n \hat{\xi}_j \partial_{\xi_j}$ are the subspaces $\mathcal{F}_k^n \subset \mathcal{F}^n$ spanned by the vectors $\{|\lambda\rangle\}_{\lambda_1=k}$ for $k \in \mathbb{Z}_{\geq 0}$. In [15] it has been shown that $\mathcal{F}_k^n$ carries an $U_q(\widehat{sl}_n)$-action and can be identified with the Kirillov-Reshetikhin module of highest weight $k\omega_1$ with ω_1 being the first fundamental weight. The basis (3.15) is then Lusztig's canonical or Kashiwara's global crystal basis for this module.*

3.2 Completeness of the Bethe ansatz

The existence of the highest weight representation (3.14) allows for the application of the quantum inverse scattering method [9] or algebraic Bethe ansatz and one obtains the following important result.

Denote by $\Bbbk = \mathbb{C}\{\{q\}\}$ the algebraically closed field of Puiseux series in the indeterminate q.

Theorem 3.5 *In the Fock representation the difference operators given by (3.8) possess a common eigenbasis* $\{|y_\lambda\rangle\}_\lambda$ *where*

$$|y_\lambda\rangle = B(y_1^{-1})\cdots B(y_{\lambda_1}^{-1})|0\rangle \tag{3.16}$$

and λ *ranges over the partitions with at most* n *parts. The partition* λ *labels the solutions* $y_\lambda = (y_1,\dots,y_{\lambda_1}) \in \Bbbk^{\lambda_1}$ *of the following coupled set of equations*

$$y_i^n = z\prod_{j\neq i}\frac{y_iq^2-y_j}{y_i-y_jq^2}, \qquad i=1,2,\dots,\lambda_1 \ . \tag{3.17}$$

The eigenvalue equation reads,

$$T(u)|y_\lambda\rangle = \left(\prod_{j=1}^{\lambda_1}\frac{1-uq^2y_j}{1-u\ y_j} + zu^n\prod_{j=1}^{\lambda_1}\frac{q^2-uy_j}{1-u\ y_j}\right)|y_\lambda\rangle \ . \tag{3.18}$$

Remark 3.6 *The quantum inverse scattering method for the* q*-boson model was discussed in [17], [5] and [6] where the Bethe ansatz equations (3.17) can be found. A discussion of the coordinate Bethe ansatz for the discrete QNLS model arriving at the same set of equations appeared also in [30], where a proof of the completeness of the Bethe ansatz solutions can be found for* $q=\varepsilon$ *with* $-1<\varepsilon<1$*. Completeness of the Bethe ansatz for* q *an indeterminate together with a coordinate ring description of their solutions as quotient of the spherical Hecke algebra was obtained in [15, Section 7]. In order to find solutions of (3.17) one then needs to work over the algebraically closed field of Puiseux series in* q*.*

4 Two Q-operators for the q-boson model

In order to construct the quantum analogue of the Bäcklund transform (2.10) and its inverse one considers solutions $D^\pm$, $L^\pm$ to the Yang-Baxter equation

$$D_{12}^\pm(u,v)L_{13}(u)L_{23}^\pm(v) = L_{23}^\pm(v)L_{13}(u)D_{12}^\pm(u,v), \tag{4.1}$$

where $L(u)$ is the local quantum Lax matrix in (3.6) and $D^\pm(u,v), L^\pm(v)$ are respectively $2\times\frac{\infty}{2}$ and $\frac{\infty}{2}\times\frac{\infty}{2}$ matrices with entries in the q-boson algebra. As explained in [27] for the Toda chain, the Yang-Baxter equation (4.1) should be seen as a quantum analogue of the classical relation (2.8) for the Darboux matrix: due to the noncommutative matrix elements in the quantum case an extra matrix $L^\pm$ is required. Thus, in the quantum system one faces the problem of finding *two* matrices, $D^\pm$ and $L^\pm$, to construct the quantum analogue of the Bäcklund transform (2.10).

Define the following half-infinite matrices with entries in the q-boson algebra $\mathcal{H}_n$ (c.f. [15, Eqn (3.20)]),

$$L_i^+(v) = \left((-v)^m\frac{(\beta_i^*)^m\beta_i^{m'}}{(q^2)_m}\right)_{m,m'\geq 0} \tag{4.2}$$

and

$$L_i^-(v) = \left(v^m q^{m(m+1)} \frac{\beta_i^{m'}(\beta_i^*)^m}{(q^2)_m} \right)_{m,m'\geq 0} . \tag{4.3}$$

The operators $L^\pm$ which are not present in the classical equation enter in the definition of the $Q^\pm$-operators

$$Q^\pm(v) = \operatorname{Tr} z^N L_n^\pm(v) \cdots L_2^\pm(v) L_1^\pm(v) = \sum_{r\geq 0} Q_r^\pm v^r , \tag{4.4}$$

where $z^N = (z^{m\ \delta_{mm'}})$ and the trace is formally defined with $Q^\pm(v) \in \mathbb{C}[\![v]\!] \otimes \mathcal{H}_n$. That is, $Q^\pm(v)$ should be thought of as current operators: as formal power series in the variable v with the coefficients $Q_r^\pm$ being elements in the q-boson algebra $\mathcal{H}_n$.

Lemma 4.1 *The coefficients in the expansions (4.4) are*

$$Q_r^+ = (-1)^r \sum_{\alpha\vdash r} z^{\alpha_n} \frac{(\beta_1^*)^{\alpha_n}(\beta_1\beta_2^*)^{\alpha_1}\cdots(\beta_{n-1}\beta_n^*)^{\alpha_{n-1}}\beta_n^{\alpha_n}}{(q^2)_{\alpha_1}\cdots(q^2)_{\alpha_{n-1}}(q^2)_{\alpha_n}} \tag{4.5}$$

$$Q_r^- = \sum_{\alpha\vdash r} z^{\alpha_n} \frac{\beta_n^{\alpha_n}(\beta_{n-1}\beta_n^*)^{\alpha_{n-1}}\cdots(\beta_1\beta_2^*)^{\alpha_1}(\beta_1^*)^{\alpha_n}}{(q^2)_{\alpha_1}\cdots(q^2)_{\alpha_{n-1}}(q^2)_{\alpha_n}} \prod_{i=1}^n q^{\alpha_i(\alpha_i+1)} , \tag{4.6}$$

where the sums range over all compositions $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_{\geq 0}^n$ of $r \geq 0$.

Proof. Via induction in n one verifies that the monodromy matrices $T^\pm(v) = L_n^\pm(v)\cdots L_2^\pm(v)L_1^\pm(v)$ are given by (c.f. [15, Eqn (3.48)])

$$T_{m'm}^+(v) = z^m \sum_\alpha \frac{(-1)^{m+|\alpha|}v^{m+|\alpha|}}{(q^2)_m} \frac{(\beta_1^*)^m(\beta_1\beta_2^*)^{\alpha_1}\cdots(\beta_{n-1}\beta_n^*)^{\alpha_{n-1}}\beta_n^{m'}}{(q^2)_{\alpha_1}\cdots(q^2)_{\alpha_{n-1}}}$$

and

$$T_{m'm}^-(v) = z^m \sum_\alpha \frac{v^{m+|\alpha|}q^{m(m+1)}}{(q^2)_m} \frac{\beta_n^{\alpha_n}(\beta_{n-1}\beta_n^*)^{\alpha_{n-1}}\cdots(\beta_1\beta_2^*)^{\alpha_1}(\beta_1^*)^{\alpha_n}}{(q^2)_{\alpha_1}\cdots(q^2)_{\alpha_{n-1}}(q^2)_{\alpha_n}} \prod_{i=1}^{n-1} q^{\alpha_i(\alpha_i+1)}$$

where the sums range over all compositions $\alpha = (\alpha_1, \ldots, \alpha_{n-1}) \in \mathbb{Z}_{\geq 0}^{n-1}$. Summing over the diagonal matrix elements with $m = m'$ and fixing the degree of v the assertion now follows. ∎

4.1 Functional relations in the q-boson algebra

The characteristic property which prompts us to identify (4.4) as the analogue of Baxter's Q-operator for the q-boson model is the following set of functional relations. The first was already proved in [15, Prop 3.12], we repeat it here for completeness, the second identity (4.8) is new.

Theorem 4.2 *We have the following identities in the q-boson algebra*

$$T(u)Q^+(u) = Q^+(uq^2) + \Delta(u)Q^+(uq^{-2}), \tag{4.7}$$
$$Q^-(u)T(u) = Q^-(uq^{-2}) + \Delta(uq^2)Q^-(uq^2)\ , \tag{4.8}$$

where the coefficient is given by $\Delta(u) = zq^{2N}u^n$.

Proof. For the proof of the first equation we refer the reader to [15, Prop 3.12]. The proof of the second equation follows along similar lines. Namely, we consider the action of $L^-_{13}(u)L_{23}(u)$ on the tensor product $V \otimes \mathbb{C}^2 \otimes \mathcal{H}_1$ with $V = \bigoplus_{m\geq 0} \mathbb{C}v_m$, where we identify the basis vectors in $\mathbb{C}^2$ as v_0, v_1. Let X be any element in $\mathcal{H}_1$. Then a straightforward computation yields

$$L^-_{13}(u)L_{23}(u)v_m \otimes v_0 \otimes X =$$
$$\frac{u^m q^{m(m+1)}}{(q^2)_m} \sum_{m'\geq 0} v_{m'} \otimes v_0 \otimes \beta^{m'}(\beta^*)^m X + v_{m'} \otimes v_1 \otimes \beta^{m'}(\beta^*)^m \beta X$$

and for $m > 0$,

$$L^-_{13}(u)L_{23}(u)v_{m-1} \otimes v_1 \otimes X =$$
$$\frac{u^m q^{m(m-1)}}{(q^2)_{m-1}} \sum_{m'\geq 0} v_{m'} \otimes v_1 \otimes \beta^{m'}(\beta^*)^{m-1} f + v_{m'} \otimes v_0 \otimes \beta^{m'}(\beta^*)^m X$$

Consider now the subspace $W \subset V \otimes \mathbb{C}^2$ spanned by the vectors $\{w_m = v_m \otimes v_0 + v_{m-1} \otimes v_0 \mid m > 0\}$ and $w_0 = v_0 \otimes v_0$. Employing the commutation relation

$$(\beta^*)^m \beta = (\beta^*)^{m-1}(1 - q^{-2m}) + q^{-2m}\beta(\beta^*)^m,$$

which is easily verified by induction, one then finds from the two identities above that on $W \otimes \mathcal{H}_1$ we have

$$L^-_{13}(u)L_{23}(u)w_m \otimes X = \frac{u^m q^{-2m} q^{m(m+1)}}{(q^2)_{m-1}} \sum_{m'\geq 0} w_{m'} \otimes \beta^{m'}(\beta^*)^m X\ .$$

Next we consider the action on the quotient space $W' = V \otimes \mathbb{C}^2/W$. A basis is given by the vectors $\{v_m \otimes v_1 \mid m \geq 1\}$ and one arrives at

$$L^-_{13}(u)L_{23}(u)v_m \otimes v_1 \otimes X =$$
$$\frac{u^{m+1} q^{m(m+1)}}{(q^2)_m} \sum_{m'\geq 0} v_{m'} \otimes v_1 \otimes (\beta^{m'}\beta^{*\ m} - \beta^{m'+1}\beta^{*\ m+1})X + \cdots\ ,$$

where the omitted terms lie in $W \otimes \mathcal{H}_1$. Since

$$\beta^{m'}\beta^{*\ m} - \beta^{m'+1}\beta^{*\ m+1} = q^{2m+2}\beta^{m'}\beta^{*\ m} q^{2N}$$

we deduce that the product of L-operators block-decomposes as follows,

$$L_{13}^{-}(u)L_{23}(u) = \begin{pmatrix} L^{-}(uq^{-2}) & * \\ 0 & uq^{2N+2}L^{-}(uq^{2}) \end{pmatrix}$$

and the assertion (4.8) is now easily obtained by taking the trace of the product

$$T_0^{-}(u)T_{0'}(u) = L_{0n}^{-}(u)L_{0'n}(u)\cdots L_{02}^{-}(u)L_{0'2}(u)L_{01}^{-}(u)L_{0'1}(u)$$

of the monodromy matrices and applying the above block decomposition to each factor $L_{0j}^{-}(u)L_{0'j}(u)$. ■

In addition to the functional equations (4.7) and (4.8) one needs to prove that

$$[T(u),Q^{\pm}(v)] = [Q^{\pm}(u),Q^{\pm}(v)] = [Q^{\pm}(u),Q^{\mp}(v)] = 0 \tag{4.9}$$

for arbitrary values of the variables u,v. The first and second relation for the Q^{+} operator has been proved in [15, Props 3.7 and 3.8] by constructing explicit solutions to the Yang-Baxter equation (4.1) with [15, Eqn (3.23)]

$$D^{+}(u,v) = \begin{pmatrix} 1-\frac{u}{v}q^{2N} & -\frac{u}{v}\beta^{*} \\ \beta & -u/v \end{pmatrix} \tag{4.10}$$

as well as finding an additional solution of the equation $R_{12}^{+}(u,v)L_{13}^{+}(u)L_{23}^{+}(v) = L_{23}^{+}(v)L_{13}^{+}(u)R_{12}^{+}(u,v)$. The above expression for D^{+} yields the desired $2\times\frac{\infty}{2}$ matrix in (4.1) if the q-boson algebra elements in D^{+} are evaluated in the Fock space representation (3.14). Note the close resemblance of the quantum Darboux matrix (4.10) with the classical one (2.9). In principle one can proceed in the same manner for Q^{-} and one then finds

$$D^{-}(u,v) = \begin{pmatrix} q^{2N} & \beta^{*} \\ \frac{v}{u}\beta & 1-\frac{v}{u}q^{2N+2} \end{pmatrix} \tag{4.11}$$

which gives a second quantum Darboux matrix which closely resembles the inverse of (4.10) [15, Eqn (3.25)] in accordance with the classical case. To establish (4.9) one needs to find yet two other solutions of the Yang-Baxter equations $R_{12}^{-}(u,v)L_{13}^{-}(u)L_{23}^{-}(v) = L_{23}^{-}(v)L_{13}^{-}(u)R_{12}^{-}(u,v)$ and $R'_{12}(u,v)L_{13}^{+}(u)L_{23}^{-}(v) = L_{23}^{-}(v)L_{13}^{+}(u)R'_{12}(u,v)$. Here we shall instead take advantage of the already established functional relations (4.7), (4.8) as well as Prop 3.2 to give a much shorter and less computational proof of (4.9).

Corollary 4.3 *The operator coefficients of the formal power series $T(u),Q^{\pm}(u)$ all commute, i.e. we have $[T_r,Q_s^{\pm}] = [Q_r^{+},Q_s^{-}] = 0$ for all $r,s\geq 0$.*

Proof. Rewriting the functional equations (4.7) and (4.8) in terms of coefficients we find that

$$\begin{aligned}(1-q^{2r})Q_r^{+} &= \sum_{s=1}^{r}(-1)^{s-1}T_sQ_{r-s}^{+} + (-1)^n zq^{2(N+r-n)}Q_{r-n}^{+}\ , \\ (q^{-2r}-1)Q_r^{-} &= \sum_{s=1}^{r}Q_{r-s}^{-}T_s + zq^{2(N+r)}Q_{r-n}^{-},\end{aligned} \tag{4.12}$$

where we set $Q_r^{\pm} = 0$ for $r < 0$. Noting that $Q_0^{\pm} = 1$ by definition and that for each $r > 0$ the terms on the right hand side of both identities only involve $Q_s^{\pm}$ with $s < r$ we find that the $Q_r^{\pm}$ are polynomials in the T_r's. But since the latter commute among themselves, $[T_r, T_s] = 0$, the assertions now easily follow. ∎

Corollary 4.4 *The transfer matrix $T(u)$ and $Q^{\pm}(u)$ obey the additional relations*

$$1 = Q^+(u)Q^-(uq^{-2}) - zu^n q^{2N} Q^+(uq^{-2})Q^-(u) \tag{4.13}$$

and

$$T(u) = \begin{vmatrix} Q^+(uq^2) & \Delta(u)Q^+(uq^{-2}) \\ \Delta(uq^2)Q^-(uq^2) & Q^-(uq^{-2}) \end{vmatrix} . \tag{4.14}$$

As explained above these equalities should be understood as identities in terms of $T_r, Q_r^{\pm} \in \mathcal{H}_n$.

Proof. Consider the triple product $T(u)Q^+(u)Q^-(u)$. From (4.7) and (4.8) one deduces that the formal power series $W(u) = \sum_{r\geq 0} u^r W_r = Q^+(u)Q^-(uq^{-2}) - zu^n q^{2N} Q^+(uq^{-2})Q^-(u)$ obeys $W(u) = W(uq^2)$ and, hence, $W_r = 0$ for $r > 0$ since q is an arbitrary indeterminate. Setting $u = 0$ we find $W(0) = 1$ and the first assertion follows. The second equality (4.14) is now easily deduced from (4.13) by using once more (4.7) and (4.8). ∎

4.2 Specialisation to the Fock space representation

It is worth emphasising that so far we have worked on the level of the q-boson algebra, that is, the functional relations derived hold regardless of the representation chosen. We now specialise to the Fock space representation (3.14).

In the Fock representation the coefficients (4.5), (4.6) of the Q-operators become the following q-difference operators,

$$Q_r^+ = (-1)^r(1-q^2)^r \sum_{\alpha\vdash r} z^{\alpha_n} \frac{\hat{\xi}^{\alpha\prime} D^{\alpha}}{(q^2)_{\alpha}} , \tag{4.15}$$

$$Q_r^- = (1-q^2)^r \sum_{\alpha\vdash r} z^{\alpha_n} \frac{D^{\alpha}\hat{\xi}^{\alpha\prime}}{(q^2)_{\alpha}} \prod_{i=1}^{n} q^{\alpha_i(\alpha_i+1)} , \tag{4.16}$$

where $\alpha\prime = (\alpha_n, \alpha_1, \alpha_2, \ldots, \alpha_{n-1})$. We can now take matrix elements of the $Q^{\pm}$-operators and the $L^{\pm}$-operators in the Fock space representation. For the local Lax matrix (3.6) we obtain for the vertex configurations in Figure 1 the Boltzmann weights [15, Fig 3]

$$\langle 0, m|L(u)|0, m\rangle = \langle 1, m-1|L(u)|0, m\rangle = 1 \tag{4.17}$$

and

$$\langle 0, m+1|L(u)|1, m\rangle = u(1 - q^{2m+2}), \qquad \langle 1, m|L(u)|1, m\rangle = u . \tag{4.18}$$

From the $L^{\pm}$-operators we find the following Boltzmann weights for the vertex configurations in Figure 2,

$$\langle c,d|L^{+}(u)|a,b\rangle = (-u)^{a}\begin{bmatrix} d \\ a \end{bmatrix}_{q^2} \tag{4.19}$$

and

$$\langle c,d|L^{-}(u)|a,b\rangle = u^{a}q^{a(a+1)}\begin{bmatrix} a+b \\ b \end{bmatrix}_{q^2} \tag{4.20}$$

with $\begin{bmatrix} m \\ n \end{bmatrix}_{q^2} = \frac{(q^2)_m}{(q^2)_n(q^2)_{m-n}}$ for $m > n$ and zero otherwise. The matrix elements of the $Q^{\pm}$-operators then yield the weighted sums over the vertex configurations of a single lattice row, the row-to-row transfer matrices. According to (4.9) these define exactly solvable lattice models. As we have interpreted $Q^{\pm}(u)$ so far as formal power series with coefficients in the q-boson algebra we briefly explain how these give rise to well-defined operators in Fock space.

For fixed particle number k the formal power series $Q^{+}(u)|_{\mathcal{F}_k^n}$ is a well-defined operator as $Q_r^{+}|_{\mathcal{F}_k^n} = 0$ for $r > k$. In fact, from (4.5) one deduces that $Q_k^{+}|_{\mathcal{F}_k^n}$ is the discrete translation operator: the sum in (4.5) for $r = k$ is the sum over all possible k-particle configurations shifting each particle by one site forward. If $r > k$ the corresponding operator $Q_r^{+}|_{\mathcal{F}_k^n}$ would shift more particles forward than are in the system, hence its matrix elements vanish.

In contrast, the coefficients of $Q^{-}(u)|_{\mathcal{F}_k^n}$ are in general all nonzero on the k-particle space, since according to (4.3) and (4.8) particles are first created at the neighbouring site before they are annihilated at their place of origin. Thus, under the action of $Q^{-}(u)|_{\mathcal{F}_k^n}$ a single particle can travel several times around the lattice picking up a factor z each time it completes a round-trip. We therefore interpret the power of the quasi-periodicity variable z in (3.7) as a winding number. For any finite winding number $p \geq 0$ the corresponding coefficient of z^p when expanding $Q^{-}(u)|_{\mathcal{F}_k^n}$ as a power series in z, is a well-defined operator, since we have now limited the number of round trips a particle can make and there are only a finite number of particles in the system.

Corollary 4.5 *In the Fock space representation we have that the Bethe vectors (3.16) form a common eigenbasis of* $\{T_r, Q_r^{\pm}\}$ *with*

$$Q^{+}(u)|y_\lambda\rangle = \prod_{j=1}^{\lambda_1}(1-u\ y_j)\ |y_\lambda\rangle\ . \tag{4.21}$$

In particular, the eigenvalues of Q_r^{+} *are the elementary symmetric functions in the Bethe roots* y_i*. The eigenvalues of* Q_r^{-} *are then derived via (4.13).*

Proof. From (4.12) it follows that the Bethe vectors (3.16) are eigenvectors of $Q_r^{\pm}$ as the latter are polynomial in the T_r's. The identity (4.7) together with (3.18) gives the eigenvalue $\mathcal{Q}^{+}(u, y_\lambda) = \prod_{j=1}^{\lambda_1}(1-u\ y_j)$ in (4.21). ■

Naively one might expect that $Q^{-}(u)$ has analytic eigenvalues as well. Let us evaluate u in a neighbourhood $U \subset \mathbb{C}$ of $u = 0$ and for a fixed number of lattice sites n and particle number $k=\lambda_1$ set z in (3.7) to $z=\varepsilon(n,\lambda_1)$ with $0<\varepsilon\ll 1$. If one can show that the following expression converges,

$$F(u,y_\lambda)=\mathcal{Q}^+(u,y_\lambda)\sum_{r\geq 0}\frac{\varepsilon^r u^{rn}q^{nr(r+1)}q^{2r\lambda_1}}{\mathcal{Q}^+(uq^{2r},y_\lambda)\mathcal{Q}^+(uq^{2r+2},y_\lambda)} \tag{4.22}$$

then the Bethe ansatz equations (3.17) imply that the residues at $u=q^{-2m}/y_j$ vanish and, hence, that F is analytic in u. Via analytic continuation in ε one then defines F outwith the region of convergence. The identity (4.8) would then imply that $F(u,y_\lambda)=\mathcal{Q}^-(u,y_\lambda)$ is the eigenvalue of $Q^-(u)$. However, the expression (4.22) is also a power series in q with the Bethe roots being potentially Puiseux series in q. In order to control the convergence of the expression (4.22) one needs to know the dependence of the Bethe roots on q which is a difficult and technical issue. We hope to address this problem by different means, which go beyond the discussion in this article, in future work.

Remark 4.6 *After the construction of Q^+ in [15, Section 3] an alternative expression for a Q-operator has been put forward [35] in the Fock representation. In* loc. cit. *the Q-operator is stated in terms of a kernel function for the Jackson integral (Eqns (54) and (62)) which requires special convergence conditions,* $0<q<1$, *and an upper bound on the growth of functions in the state space $\mathcal{F}^n$. The derivation of this kernel function in* loc. cit. *postulates the existence of a q-analogue of the δ-function for the Jackson integral and I was unable to verify whether the constructed operator is either of the two operators constructed here.*

N.B. in the proof of (4.21) we have reversed the usual logic and derived the spectrum of the Q^+-operator from the spectrum (3.18) of the transfer matrix and the functional relation (4.7). As the system is solvable via the Bethe ansatz and completeness has been proved [15, Section 7], the significance of the Q^+-operator in the present context is *not* that of a mere 'auxiliary matrix' as in Baxter's original work [2] (and references therein) where it is used to find the physically relevant eigenvalues of T. Instead the Q^+-operator acquires in our setting a *direct* physical significance as it describes the discretised time evolution (2.7) of the quantum Ablowitz-Ladik chain, as we will see next.

5 Quantum Bäcklund transformation

We define the quantum analogue of the Bäcklund transformation $\mathcal{B}^+(v)$ in the Fock space representation by setting $\mathcal{B}^+(v):\operatorname{End}\mathcal{F}^n\to\mathbb{C}[\![v]\!]\otimes\operatorname{End}\mathcal{F}^n$ with $\mathcal{O}\mapsto Q^+(v)\mathcal{O}Q^+(v)^{-1}$. The quantum Bäcklund map $\mathcal{B}^+$ is well-defined because of the following result.

Proposition 5.1 *In the Fock representation the inverse of the Q^+-operator exists and is given by*

$$Q^+(v)^{-1}=\sum_{r\geq 0}v^r\det((-1)^{1-i+j}Q^+_{1-i+j})_{1\leq i,j\leq r}\ , \tag{5.1}$$

where the coefficients for $r < n$ simplify according to

$$\det((-1)^{1-i+j}Q^+_{1-i+j})_{1\le i,j\le r} = q^{-2r}Q^-_r\,, \qquad r<n\,. \tag{5.2}$$

Proof. The first identity for the inverse operator follows from the Bethe ansatz result: in the eigenbasis (3.16) the inverse of $Q^+(v)$ is given by the diagonal matrix with entries

$$\prod_{j=1}^{\lambda_1}\frac{1}{1-vy_j}=\sum_{r\ge 0}v^r h_r(y_1,\ldots,y_{\lambda_1})$$

with h_r denoting the complete symmetric functions [19, Ch. I]. Using the known determinant relation between elementary and complete symmetric functions, $h_r = \det(e_{1-i+j})_{1\le i,j\le r}$, the first assertion follows from (4.21). The second assertion is now an immediate consequence of (4.13) when expanding in the variable v. ■

We are particularly interested in the image of the q-boson algebra generators under the quantum Bäcklund transform,

$$\tilde\beta_j(v) = Q^+(v)\beta_j Q^+(v)^{-1}=\sum_{r\ge 0}v^r\tilde\beta_{j,r}\,. \tag{5.3}$$

We define $\tilde\beta^*_j(v)$ and $\tilde\beta^*_{j,r}$ in an analogous fashion. N.B. $\tilde\beta^*_j(v)\neq(\tilde\beta_j(v))^*$ since Q^+ is not unitary in the Fock space representation. For ease of notation we will often suppress the explicit dependence on the variable v but the reader should keep in mind that $\tilde\beta_j(v)$ and $\tilde\beta^*_j(v)$ are power series in v with coefficients in $\operatorname{End}\mathcal{F}^n$ as we are now working in the Fock space representation. Clearly, the transformed quantum variables $\{\tilde\beta_j,\tilde\beta^*_j\}$ still obey the q-boson algebra relations and, thus, we obtain a one-variable family of representations of $\mathcal{H}_n$. Moreover, by construction and because of (4.9) the quantum integrals of motion are left invariant,

$$T_r(\tilde\beta_j,\tilde\beta^*_j) = Q^+(v)T_r(\beta_j,\beta^*_j)Q^+(v)^{-1}=T_r(\beta_j,\beta^*_j)\,. \tag{5.4}$$

5.1 The image of the quantum Bäcklund map

We now derive a set of functional relations for the transformed quantum variables $\{\tilde\beta_j,\tilde\beta^*_j\}$ and show that they are an *exact* match of the classical relations (2.10) which describe the discretised time flow (2.7).

Theorem 5.2 *The quantum Bäcklund transformed variables $\{\tilde\beta_j(v),\tilde\beta^*_j(v)\}$ obey the functional relations*

$$\begin{cases}\tilde\beta_j-\beta_j = v(1-\beta^*_j\tilde\beta_j)\tilde\beta_{j-1}\\ \tilde\beta^*_j-\beta^*_j = v\beta^*_{j+1}(\beta^*_j\tilde\beta_j-1)\end{cases}. \tag{5.5}$$

These relations determine $\{\tilde\beta_j(v),\tilde\beta^*_j(v)\}$ via recurrence when expanding in the variable v.

Proof. A somewhat lengthy but straightforward computation. We give the intermediate steps and leave some details to the reader to verify. Set $a_j = \beta_j\beta^*_{j+1}$. Via induction in m one establishes the identities

$$\begin{aligned}
[a^m_{j-1}, \beta_j] &= -(1-q^{2m})a^{m-1}_{j-1}\beta_{j-1}q^{2N_j} \ , \\
[a^m_j, \beta^*_j] &= (1-q^{2m})q^{2N_j}\beta^*_{j+1}a^{m-1}_j \ .
\end{aligned}$$

From the explicit expression (4.5) one then derives the recurrence relations

$$\begin{aligned}
[Q^+_r, \beta_j] &= q^{2N_j}Q^+_{r-1}\beta_{j-1} - \beta^*_j[Q^+_{r-1}, \beta_j]\beta_{j-1} \ , \\
[Q^+_r, \beta^*_j] &= -\beta^*_{j+1}Q^+_{r-1}q^{2N_j} - \beta^*_{j+1}[Q^+_{r-1}, \beta^*_j]\beta_j \ .
\end{aligned}$$

Multiplying with v^r and subsequently summing over r on both sides of these two equalities produces

$$\begin{aligned}
[Q^+(v), \beta_j] &= -vq^{2N_j}Q^+(v)\beta_{j-1} - v\beta^*_j[Q^+(v), \beta_j]\beta_{j-1} \\
[Q^+(v), \beta^*_j] &= -v\beta^*_{j+1}Q^+(v)q^{2N_j} - v\beta^*_{j+1}[Q^+(v), \beta^*_j]\beta_j \ .
\end{aligned}$$

After multiplying with $Q^+(v)^{-1}$ from the right we obtain (recall that $q^{2N_j} = 1 - \beta^*_j\beta_j$)

$$\begin{cases}
\tilde\beta_j - \beta_j = v(1-\beta^*_j\beta_j)\tilde\beta_{j-1} - v\beta^*_j(\tilde\beta_j - \beta_j)\tilde\beta_{j-1} \\
\tilde\beta^*_j - \beta^*_j = v\beta^*_{j+1}(\tilde\beta^*_j\tilde\beta_j - 1) - v\beta^*_{j+1}(\tilde\beta^*_j - \beta^*_j)\tilde\beta_j
\end{cases}$$

from which the desired equalities in (5.5) now easily follow. ■

We demonstrate that (5.5) allows one to compute the image of the q-boson generators via recurrence. Namely, making an analogous power series expansion $\tilde\beta_j(v) = \sum_{r\geq 0} v^r\tilde\beta_{j,r}$ as in the classical case (2.11) we find for the first few coefficients $\tilde\beta_{j,0} = \beta_j$ and

$$\begin{aligned}
\tilde\beta_{j,1} &= \beta_{j-1}(1-\beta^*_j\beta_j) \\
\tilde\beta_{j,2} &= \beta_{j-2}(1-\beta^*_{j-1}\beta_{j-1})(1-\beta^*_j\beta_j) - \beta^2_{j-1}\beta^*_j(1-\beta^*_j\beta_j) \\
\tilde\beta_{j,3} &= \beta_{j-3}(1-\beta^*_{j-2}\beta_{j-2})(1-\beta^*_{j-1}\beta_{j-1})(1-\beta^*_j\beta_j) \\
&\quad -2\beta_{j-2}\beta_{j-1}\beta^*_j(1-\beta^*_{j-1}\beta_{j-1})(1-\beta^*_j\beta_j) \ .
\end{aligned}$$

Similarly, we find for $\tilde\beta^*_j$ that $\tilde\beta^*_{j,0} = \beta^*_j$ and

$$\begin{aligned}
\tilde\beta^*_{j,1} &= -(1-\beta^*_j\beta_j)\beta^*_{j+1} \\
\tilde\beta^*_{j,2} &= \beta_{j-1}\beta^*_j\beta^*_{j+1}(1-\beta^*_j\beta_j) \\
\tilde\beta^*_{j,3} &= -\beta^2_{j-1}\beta^{*\,2}_j\beta^*_{j+1}(1-\beta^*_j\beta_j) + \beta_{j-2}\beta^*_j\beta^*_{j+1}(1-\beta^*_{j-1}\beta_{j-1})(1-\beta^*_j\beta_j) \ .
\end{aligned}$$

We can interpret the above formulae as describing discrete time steps under the evolution (2.7): since the quantum relations (5.5) and classical relations (2.10) are the same, these expressions yield the corresponding coefficients $\tilde{\psi}_{j,r}$ and $\tilde{\psi}^*_{j,r}$ in (2.12) upon replacing $\tilde{\beta}_{j,r} \to \tilde{\psi}_{j,r}$ and $\tilde{\beta}^*_{j,r} \to \tilde{\psi}^*_{j,r}$.

Naturally, one wants to extend the discussion to include the Q^--operator and one then needs to show that its inverse exists. However, when proceeding along the same lines as for the Q^+-operator one must first prove convergence of the supposed eigenvalues (4.22). Since the latter is currently an open question, we state the following preliminary result.

Lemma 5.3 *We have the following commutation relations between the generators of the q-boson algebra and $Q^-(u)$*

$$\begin{cases} Q^-(v)\beta_j - \beta_j Q^-(v) = -v\beta_{j-1}\left[Q^-(v) + \beta_j Q^-(v)\beta_j^*\right] \\ Q^-(v)\beta_j^* - \beta_j^* Q^-(v) = v\left[Q^-(v) - \beta_j Q^-(v)\beta_j^*\right]\beta_{j+1}^* \end{cases} \tag{5.6}$$

Proof. Via a similar computation as in the case of Q^+ one shows the commutation relations

$$\begin{aligned} [Q_r^-, \beta_j] &= -\beta_{j-1}Q_{r-1}^- q^{2N_j+2} - \beta_{j-1}[Q_{r-1}^-, \beta_j]\beta_j^* \ , \\ [Q_r^-, \beta_j^*] &= q^{2N_j+2}Q_{r-1}^-\beta_{j+1}^* - \beta_j[Q_{r-1}^-, \beta_j^*]\beta_{j+1}^* \ . \end{aligned}$$

after multiplying with v^r and summing over r the assertion follows. ■

Define a one-variable family of operators $\{\hat{\beta}_j(v), \hat{\beta}^*_j(v)\}$ via the functional relations (compare with (2.14))

$$\begin{cases} \hat{\beta}_j - \beta_j = -v\beta_{j-1}(1 - \beta_j\hat{\beta}^*_j) \\ \hat{\beta}^*_j - \beta^*_j = v\ (1 - \beta_j\hat{\beta}^*_j)\hat{\beta}^*_{j+1} \end{cases} . \tag{5.7}$$

As in the previous case (5.5) the relations (5.7) also determine $\{\hat{\beta}_j(v), \hat{\beta}^*_j(v)\}$ via recurrence when making the analogous power series expansions $\hat{\beta}_j(v) = \sum_{r\geq 0} v^r \hat{\beta}_{j,r}$. Under the assumption that $Q^-(v)^{-1}$ exists, it then follows from (5.6) that $\hat{\beta}_j(v) = Q^-(v)\beta_j Q^-(v)^{-1}$ and $\hat{\beta}^*_j(v) = Q^-(v)\beta^*_j Q^-(v)^{-1}$.

6 From Bäcklund transformations to 2D TQFT

We now link the quantum Bäcklund transform (5.5) to the 2D TQFT constructed in [15, Section 7]. Consider multivariate deformations of the q-boson algebra by setting

$$\tilde{\beta}_j(x_1, \ldots, x_\ell) := \left(\prod_{i=1}^{\ell} Q^+(x_i)\right)\beta_j\left(\prod_{i=1}^{\ell} Q^+(x_i)^{-1}\right) \tag{6.1}$$

with $\ell \leq n$. We define $\tilde{\beta}^*_j(x_1, \ldots, x_\ell)$ analogously. We shall concentrate on the Q^+-operator because both solutions $Q^\pm$ are related via the functional relation (4.13), which shows that the transformation induced by Q^- can be constructed from Q^+. Since the $Q^+(x_i)$'s in (6.1) commute with each other - reflecting the analogous property (2.13) of the classical Bäcklund transform - one can expand the product of Q^+-operators in any basis in the ring of symmetric functions in the variables x_i. Set $P_\lambda(x_1,\dots,x_\ell;q) = P_\lambda(x_1,\dots,x_\ell;q^2,0)$ where $P_\lambda(x_1,\dots,x_\ell;q,t)$ are the celebrated Macdonald functions [19]. Then the expansion

$$\prod_{i=1}^{\ell} Q^+(x_i) = \sum_\lambda (-1)^{|\lambda|} Q_\lambda P_\lambda(x_1,\dots,x_\ell;q) \tag{6.2}$$

with the sum running over all partitions with at most ℓ parts defines uniquely a set of commuting polynomials $\{Q_\lambda\}$ in the q-boson algebra $\mathcal{H}_n \otimes \mathbb{C}[\![z]\!]$, where z is the "quasi-periodicity parameter" in (3.7) which we treat as formal variable. Note that the operators Q_λ can be defined explicitly as polynomials in the Q_r^+'s, see [15, Def 3.3]. For the discussion in this article their implicit definition via (6.2) suffices[1].

In the Fock representation it follows from the results in [15, Section 7] that if λ has a column of height n then $Q_\lambda = z^{m_n(\lambda)} Q_{\tilde\lambda}$ where $\tilde\lambda$ is the partition obtained from λ by removing all columns of height n and $m_n(\lambda)$ is the multiplicity of these columns. Setting $z = 1$ (periodic boundary conditions) we therefore restrict ourselves to $\ell = n-1$. We recall the following results from [15, Thm 7.2, Lem 7.7 and Cor 7.3].

Theorem 6.1 *(1) The $\{Q_{\tilde\lambda}\}$ form a basis in the ring $\mathcal{A}_n$ of quantum integrals of motion generated by the T_r's. (2) Let λ, μ, be partitions with at most n parts and $\mu_1 = \nu_1 = \lambda_1 = k \in \mathbb{Z}_{\geq 0}$. Then*

$$Q_{\tilde\lambda} Q_{\tilde\mu} = \sum_{\nu_1 = k} N_{\tilde\lambda\tilde\mu}^{\tilde\nu}(q) Q_{\tilde\nu}, \qquad N_{\tilde\lambda\tilde\mu}^{\tilde\nu}(q) = \langle \nu | Q_{\tilde\lambda} | \mu \rangle\ , \tag{6.3}$$

where the expansion coefficients are the fusion coefficient of a 2D TQFT with $N_{\tilde\lambda\tilde\mu}^{\tilde\nu}(0)$ being the $SU(n)$ WZNW fusion coefficient at level k. If $\mu_1 \neq \nu_1$ or $\mu_1 \neq \lambda_1$ the matrix element is zero.

6.1 A brief summary of 2D TQFT

For the sake of completeness and to make this article accessible to a wider audience we briefly summarise the definition of a 2D TQFT, for the precise definition we refer the interested reader to the abundant literature on the subject; see e.g. the text book [12].

In modern mathematical language a 2D TQFT, is a symmetric monoidal functor $Z : (2\mathrm{Cob}, \sqcup) \to (\mathrm{Vect}_{\Bbbk}, \otimes)$ from the category of 2-cobordisms into the category of finite-dimensional vector spaces over some base field $\Bbbk$. The objects in the category 2Cob are circles $\mathbb{S}^1$, closed strings, and we define a

[1]The reader should note that the definition of the Q^+-operator in this article corresponds to the operator $\boldsymbol{G}'(-u)$ in [15, Prop 3.11, Eqn (3.46)] which explains the extra sign factor in (6.2) compared to Eqn (3.56) in *loc. cit.*

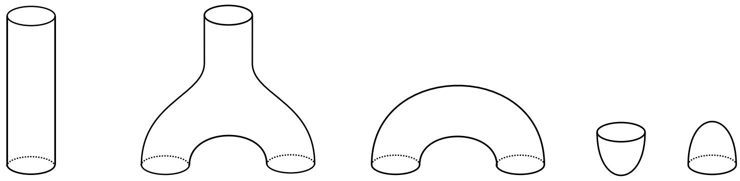

Figure 4: Depiction of elementary 2-cobordisms. From left to right: the identity map $\mathrm{Id} : Z(\mathbb{S}^1) \to Z(\mathbb{S}^1)$, the 'pair of pants' $m : Z(\mathbb{S}^1) \otimes Z(\mathbb{S}^1) \to Z(\mathbb{S}^1)$, the 'elbow' : $Z(\mathbb{S}^1) \otimes Z(\mathbb{S}^1) \to \Bbbk$, the 'cup' $\mathbf{1} : \Bbbk \to Z(\mathbb{S}^1)$ and the 'cap' $\mathrm{tr} : Z(\mathbb{S}^1) \to \Bbbk$.

product on them via the disjoint union $\sqcup$. The functor Z maps each closed string $\mathbb{S}^1$ onto a vector space $Z(\mathbb{S}^1)$ while preserving the product structure, i.e. $Z(\mathbb{S}^1 \sqcup \cdots \sqcup \mathbb{S}^1) = Z(\mathbb{S}^1) \otimes \cdots \otimes Z(\mathbb{S}^1)$ where $\otimes$ denotes the ordinary tensor product of vector spaces. One includes the case where the circle shrinks to a point pt and sets $Z(\mathrm{pt}) = \Bbbk$. This explains the monoidal property of Z. Asking Z to be *symmetric* refers to the fact that each circle carries an orientation: let $\mathbb{S}^1$ be anti-clockwise oriented and denote by $\bar{\mathbb{S}}^1$ the clockwise oriented circle. Then one demands that a change in orientation in 2Cob corresponds to taking the dual space in $\mathrm{Vect}_\Bbbk$, i.e. $Z(\bar{\mathbb{S}}^1) = Z(\mathbb{S}^1)^*$.

The morphisms between objects in 2Cob are two-dimensional compact manifolds which interpolate between two sets of circles, one representing the "in-states" $\underbrace{\mathbb{S}^1 \sqcup \cdots \sqcup \mathbb{S}^1}_{m}$ and the other the "out-states" $\underbrace{\mathbb{S}^1 \sqcup \cdots \sqcup \mathbb{S}^1}_{n}$ in physics terminology. The morphisms in 2Cob are called 2-cobordisms and can be thought of as a 2-dimensional analogue of Feynman diagrams with m incoming particles and n outgoing ones. Each such 2-cobordism is mapped under Z to an element in $\mathrm{Hom}(Z(\mathbb{S}^1)^{\otimes m}, Z(\mathbb{S}^1)^{\otimes n})$, i.e. a linear map between two tensor products of the vector space $Z(\mathbb{S}^1)$.

Because of the functorial property Z is fixed by specifying its values on certain "elementary" 2-cobordisms shown in Figure 4 which introduce on $Z(\mathbb{S}^1)$ a product $m : Z(\mathbb{S}^1) \otimes Z(\mathbb{S}^1) \to Z(\mathbb{S}^1)$, an identity element $\mathbf{1} : \Bbbk \to Z(\mathbb{S}^1)$ and an invariant bilinear form $\langle \cdot | \cdot \rangle : Z(\mathbb{S}^1) \otimes Z(\mathbb{S}^1) \to \Bbbk$. Instead of the latter, one often considers the trace functional given by $\mathrm{tr}(\cdot) = \langle \mathbf{1} | \cdot \rangle$ which defines a map $Z(\mathbb{S}^1) \to \Bbbk$. These maps endow $Z(\mathbb{S}^1)$ with the structure of a *Frobenius algebra* and the latter is called *symmetric* if the product is commutative.

6.2 From Q-operators to TQFT fusion matrices

Hence, in order to prove that (6.3) defines a 2D TQFT we need to endow the algebra $\mathcal{A}_n$ generated from the quantum integrals of motion (3.8) with the structure of a symmetric Frobenius algebra. From (3.8) and (4.5), (4.6) we infer that $\mathcal{A}_n$ block decomposes over the subspaces $\mathcal{F}_k^n$ of fixed particle number,

$$\mathcal{A}_n = \bigoplus_{k\geq 0} \mathcal{A}_{n,k}, \qquad \mathcal{A}_{n,k} \subset \operatorname{End} \mathcal{F}_k^n$$

with $\dim \mathcal{F}_k^n = \dim \mathcal{A}_{n,k} = \binom{n+k-1}{k}$. Fix $k \in \mathbb{N}$ and set $\Bbbk = \mathbb{C}\{\{q\}\}$, the field of Puiseux series in q. Consider the vector space $Z_k(\mathbb{S}^1) = \mathcal{A}_{n,k}$ together with the matrix product (6.3) in $\mathcal{A}_{n,k} \subset \operatorname{End} \mathcal{F}_k^n$ and

$$\mathbf{1}_k := Q_\emptyset \qquad \text{and} \qquad \langle Q_{\tilde{\lambda}} | Q_{\tilde{\mu}} \rangle = \delta_{\tilde{\lambda}^* \tilde{\mu}} / b_{\tilde{\lambda}}(q) \;, \tag{6.4}$$

where $\tilde{\lambda}^* = (k - \tilde{\lambda}_{n-1}, \ldots, k - \tilde{\lambda}_2, k - \tilde{\lambda}_1)$ and $b_{\tilde{\lambda}} = \prod_{j\geq 1} (q^2)_{m_j(\tilde{\lambda})}$ with $m_j(\tilde{\lambda})$ being the multiplicity of columns of height j in the reduced partition $\tilde{\lambda}$.

Theorem 6.2 ([15, Thm 7.2 & Cor 7.3]) *The maps (6.4) together with (6.3) define a commutative Frobenius algebra, that is $Z_k : (2\mathrm{Cob}, \sqcup) \to (\mathrm{Vect}_\Bbbk, \otimes)$ with $Z_k(\mathbb{S}^1) = \mathcal{A}_{n,k}$ is a symmetric monoidal functor.*

Remark 6.3 *The fusion coefficients (6.3) can be explicitly computed using the algorithm in [15, Section 7.3.1]. The latter is based on an algebra isomorphism which maps $\mathcal{A}_{n,k}$ on a quotient of the spherical Hecke algebra; see [15, Thm 7.3] for details. Alternatively, one can extract $N_{\tilde{\lambda}\tilde{\mu}}^{\tilde{\nu}}(q)$ from the combinatorial definition of cylindric generalisations of skew q-Whittaker functions: taking matrix elements in (6.2) one obtains symmetric functions $P_{\lambda/d/\mu}$ via*

$$\sum_{d\geq 0} z^d P_{\lambda/d/\mu}(x_1, \ldots, x_{n-1}; q) = \langle \lambda | \prod_{i=1}^{n-1} Q^+(x_i) | \mu \rangle \;.$$

Here $\lambda/d/\mu$ denotes a cylindric shape, a periodically continued skew diagram, and each $P_{\lambda/d/\mu}$ equals a sum over q-weighted cylindric semi-standard tableaux; see [15, Section 6.2].

6.3 The quantum Bäcklund transform in terms of fusion

The following result connects the fusion coefficients (6.3) for the simplest, one-variable case, with the quantum Bäcklund transform (5.5).

Proposition 6.4 *The solution of the quantum Bäcklund transform (5.5) satisfy the following linear systems involving the fusion coefficients (6.3),*

$$\sum_{a+b=r} \sum_\rho (-1)^a N_{(a)\tilde{\mu}}^{\tilde{\rho}}(q) \; \langle \lambda | \tilde{\beta}_{j,b} | \rho \rangle \;=\; N_{(r)\widetilde{\beta_j \mu}}^{\tilde{\lambda}}(q) \tag{6.5}$$

$$\sum_{a+b=r} \sum_\rho \frac{(-1)^a N_{(a)\tilde{\mu}}^{\tilde{\rho}}(q)}{1 - q^{2m_j(\mu)+2}} \; \langle \lambda | \tilde{\beta}_{j,b}^* | \rho \rangle \;=\; N_{(r)\widetilde{\beta_j^* \mu}}^{\tilde{\lambda}}(q) \;, \tag{6.6}$$

where the notation $\beta_j \mu$ and $\beta_j^* \mu$ stand for the partitions obtained by respectively deleting and inserting a column of height j in the Young diagram of μ.

Proof. This is an immediate consequence of the definition (6.2) of the fusion matrices and (6.3) which yields $N^{\tilde{\lambda}}_{(r)\tilde{\mu}} = (-1)^r \langle\lambda|Q^+_r|\mu\rangle$. On the other hand the definition (5.3) implies $\tilde{\beta}_j(v)Q^+(v) = Q^+(v)\beta_j$. Taking matrix elements in the latter equality and then expanding in the variable v yields (6.5). The proof of (6.6) follows along the same lines observing that $\beta^*_j|\mu\rangle = (1-q^{2m_j(\mu)+2})|\beta^*_j\mu\rangle$ according to (3.15). ■

For small particle numbers k there is a third way to compute the fusion coefficients (6.3) using the following "operator-state correspondence" in the Fock space [15, Cor 7.3]: let $|k^n\rangle = \frac{(\beta^*_n)^k}{(q)_k}|0\rangle$ denote the basis vector (3.15) with k particles sitting at site n and all other sites being empty. Then the basis vectors (3.15) with $\lambda_1 = k$ are obtained by $|\lambda\rangle = Q_{\tilde{\lambda}}|k^n\rangle$, where $\tilde{\lambda}$ is the partition obtained from λ by removing all columns of height n. Thus, we can introduce a fusion product on the k-particle space $\mathcal{F}^n_k$ by setting $|\lambda\rangle * |\mu\rangle = Q_{\tilde{\lambda}}|\mu\rangle$ and the resulting algebra is isomorphic to the TQFT (6.3). In order to obtain the fusion coefficient we then need to explicitly compute $Q_{\tilde{\lambda}}$ as a polynomial in the Q^+_r's and let them act on the state vector $|\mu\rangle$. In the final step one then removes all columns of height n in the partitions in the expansion of $Q_{\tilde{\lambda}}|\mu\rangle$ as these correspond to the trivial representation of $SU(n)$. We demonstrate this on a small example.

Example 6.5 *Set $n = 3$. We wish to relate the fusion coefficients (6.3) for particle numbers $k = 2, 3$ via the relation (6.6). In the latter equation choose $j = 1$ and $\mu = (2,2,1) = \ydiagram{2,2,1}$. Thus, $\beta^*_1\mu = \ydiagram{3,2,1}$ and the reduced partition with columns of height $n = 3$ deleted is simply $\tilde{\mu} = \ydiagram{1,1}$. The state $|\mu\rangle$ is a 2-particle state with one particle sitting at site 2 and the other at site 3. The state $\beta^*_1|\mu\rangle = (1-q^2)|3,2,1\rangle$ is a 3-particle state, with one particle sitting at each site. According to (6.6) we have the identity*

$$(1-q^2)N^{\tilde{\lambda}}_{(2)(2,1)} = \langle\lambda|\tilde{\beta}^*_{1,2}|\mu\rangle - \sum_\rho \langle\lambda|\tilde{\beta}^*_{1,1}|\rho\rangle N^{\tilde{\rho}}_{(1)\tilde{\mu}} + \sum_\rho \langle\lambda|\tilde{\beta}^*_{1,0}|\rho\rangle N^{\tilde{\rho}}_{(2)\tilde{\mu}} \tag{6.7}$$

with $\tilde{\beta}^*_{1,0} = \beta^*_1$, $\tilde{\beta}^*_{1,1} = -\beta^*_2 q^{2N_1}$ and $\tilde{\beta}^*_{1,2} = \beta^*_1\beta^*_2\beta_3 q^{2N_1}$. The first term on the right hand side in the above equation gives

$$\tilde{\beta}^*_{1,2}|\mu\rangle = \beta^*_1\beta^*_2\beta_3 q^{2N_1}|\ydiagram{2,2,1}\rangle = (1-q^2)(1-q^4)|\ydiagram{3,2}\rangle \; .$$

That is, the first matrix element in (6.7) is nonzero only for $\lambda = (3,2)$. For the other terms we need to compute the 2-particle fusion coefficients $N^{\tilde{\rho}}_{(1)\tilde{\mu}}$ and $N^{\tilde{\rho}}_{(2)\tilde{\mu}}$ first. This can be done using the operator-state correspondence $|\lambda\rangle = Q_{\tilde{\lambda}}|3,3\rangle$ or the algorithm in [15]. One finds (we simply write the Young diagrams of the partitions instead of the $Q_{\tilde{\lambda}}$'s)

$$\ydiagram{2} * \ydiagram{1,1} = \ydiagram{1} \qquad \text{and} \qquad \ydiagram{1} * \ydiagram{1,1} = \ydiagram{2,1} + (1+q^2)\,\emptyset \; .$$

Setting $q = 0$ one can verify that these give the correct WZNW fusion coefficients *of $SU(3)$ at level $k=2$. Thus, we obtain*

$$\begin{aligned}
\sum_\rho \tilde{\beta}^*_{1,1}|\rho\rangle N^{\tilde{\rho}}_{(1)\tilde{\mu}} &= -(1-q^2)(1+q^2)|\,\ydiagram{3,3,2}\,\rangle - (1-q^4)q^2|\,\ydiagram{3,2}\,\rangle \\
\sum_\rho \tilde{\beta}^*_{1,0}|\rho\rangle N^{\tilde{\rho}}_{(2)\tilde{\mu}} &= (1-q^4)|\,\ydiagram{3,1,1}\,\rangle
\end{aligned}$$

for the second and third term on the right hand side of (6.7). Adding all three terms we obtain the 3-particle fusion coefficients $N^{\tilde{\lambda}}_{(2)(2,1)}$ and, thus, arrive at the 3-particle fusion product (after deleting columns of height $n=3$ in all partitions)

$$\ydiagram{2} * \ydiagram{2,1} = (1+q^2)\ \left(\ydiagram{1} + \ydiagram{1,1} + \ydiagram{3,2}\right)$$

Again one can check that in the limit $q=0$ these are the correct $SU(3)$-WZNW fusion coefficients at level $k=3$. As demonstrated on this simple example one can use the linear system (6.6) to compute fusion coefficients recursively.

7 Conclusions

In this article we have added new insight to the quantisation of the Ablowitz-Ladik chain (1.1) by using Baxter's concept of the Q-operator.

7.1 Summary of results

The classical system (1.1) possesses a set of integral of motions, a Poisson commutative subalgebra $\mathcal{A}_n^{\hbar=0}$, generated by the trace of the Lax operator (2.3), $\{T_r, T_{r'}\}=0$. These integrals of motion generate additional "elementary" flows which can be used to construct the physical time flow given by the Hamiltonian (2.2). Suris constructed in [28] discrete approximations of these elementary flows in terms of Bäcklund transformations given in terms of a set of functional relations (2.10), (2.14). In [17] the q-deformed oscillator or q-boson algebra $\mathcal{H}_n$ was considered as a quantisation of the Poisson algebra (2.1) underlying the classical chain.

In this article we have discussed the quantum analogues of the classical Bäcklund transformations using the q-boson algebra. We have stated the explicit construction of two infinite families $\{Q_r^\pm\}_{r\in\mathbb{N}}$ of elements in the q-boson algebra, see (4.5) and (4.6), which commute among themselves and with the quantum integrals of motion $\mathcal{A}_n$ generated from the higher Hamiltonians $\{T_r\}_{r=1}^n$ of the quantised Ablowitz-Ladik chain. Introducing the associated current operators $Q^\pm(v)$ we derived their algebraic dependence in terms of a set of functional equations: Baxter's TQ-equation (4.7) and (4.8), which is a second order difference equation, and the related Wronskian (4.13) showing that Q^- is closely related to the inverse of Q^+ in the limit of large lattice sites. We showed that these relations hold on the level of the q-boson algebra without the need to specify a representation. Once we fixed the Fock space representation (3.14) we were able to show in addition that $Q^+(v)$ is invertible for the periodic chain by employing the completeness of the Bethe ansatz for the q-boson model proved independently in [30] for $-1 < q < 1$ and in [15] for q generic. This allowed us to consider the one-variable family of similarity transformations $\mathcal{O} \mapsto Q^+(v)\mathcal{O}Q^+(v)^{-1}$. Applying these transformations to the generators of the (noncommutative) q-boson algebra (3.2) we have shown that their image obey a set of functional relations (5.5) which match exactly the relations (2.10) of the classical Bäcklund transformation. This is an unexpected result: the discrete time flow (2.7) of the classical and the quantum variables is the same.

An open question is the extension of this result to the $Q^-(u)$-operator for which further investigations are needed to establish the existence of its inverse. As a preliminary result we showed that assuming the existence of $Q^-(v)^{-1}$ a second set of functional relations (5.7) follows which again match the classical relations (2.14) associated with the inverse of the classical Bäcklund transformation.

In the final section we then considered multivariate versions of the Bäcklund transformation via composition. The novel result is that they generate the fusion matrices of a 2D TQFT constructed in [15]. This TQFT is a q-deformed version of the $SU(n)_k$-WZNW fusion ring or Verlinde algebra which is recovered in the strong coupling limit $q \to 0$; the related combinatorial description of the WZNW fusion ring and its relation to quantum cohomology can be found in [16] and [13]. Because of this limit, it was suggested in [15, Section 8] that this TQFT might be related to the one considered by Teleman [29] in the context of twisted K-theory.

7.2 Integrability and topological quantum field theory

We explain how our findings differ from previously known connections between classical and quantum integrable systems and TQFTs starting with the quantum case.

Nekrasov and Shatashvili developed in a series of works the following connection between TQFTs and the Bethe ansatz equations of quantum integrable models [21]: starting from a 4D Yang-Mills theory one can consider its vacua in the infrared limit. The moduli space of vacua is described by a TQFT in terms of a set of equations matching the Bethe ansatz equations of a quantum integrable model using the Yang-Yang functional. For the q-boson model this approach has been followed recently by string theorists in [23] and [10] who put forward generalised versions of WZNW and Chern-Simons field theories.

In contrast our construction of a TQFT using Baxter's idea of a Q-matrix gives an *operator construction* of the TQFT in terms of q-difference operators, so-called quantum D-modules. That this operator construction of a TQFT is identical to the one obtained via the Bethe ansatz equations is a non-trivial statement: it requires the proof of the existence of an algebra isomorphism between the operator version of the TQFT (6.2), (6.3) and the coordinate ring presentation in terms of the Bethe ansatz equation as well as the completeness of the Bethe ansatz; see [15, Section 7 and 8]. As a result one obtains an explicit algorithm which allows one to compute the fusion coefficients, see [14] and [15, Sec. 7.3.1]. We showed in this article that the fusion coefficients satisfy linear systems (6.5) and (6.6) with the image of the q-boson algebra generators under the quantum Bäcklund transform.

In the context of classical integrable systems the connection with TQFT rests on the associativity condition of the underlying Frobenius algebra, the Witten-Dijkgraaf-Verlinde-Verlinde equations; see Dubrovin's lecture notes [8]. In contrast, associativity is built in from the start in the construction of the TQFT in [15] as the fusion matrices are realised as endomorphisms over the Fock space of the q-boson algebra, that is, the product in the TQFT is the ordinary matrix product. The crucial issue here is the commutativity of the product which follows from the construction of the commutative subalgebra $\mathcal{A}_n$ generated by the quantum integrals of motion of the quantised Ablowitz-Ladik chain, $\{T_r\}$ and $\{Q_r^\pm\}$. In other words, for fixed particle number k the 2D TQFT is the quantised Poisson subalgebra of classical integrals of motion of the Ablowitz-Ladik chain, $\lim_{\hbar\to 0}\mathcal{A}_{n,k} = \mathcal{A}_{n,k}^{\hbar=0}$. In this article we showed that a basis of the quantum integrals of motions, the fusion matrices of the TQFT, is obtained from the quantum analogue of Bäcklund transformations.

7.3 Open questions

As pointed our earlier in the text, the main physical significance of our investigation of the Q^+-operator for the quantum Ablowitz-Ladik chain is the result that its adjoint action describes the discrete time evolution (2.7) of the system. To obtain the full time evolution one needs in addition to consider the adjoint of the Q^+-operator. We wish to exploit this fact in a forthcoming publication where we discuss the discrete dynamics of the system and connect it to the TQFT.

An interesting question resulting from our discussion is a possible extension of our construction of quantum Bäcklund transformations to the continuum limit, where one obtains the quantum nonlinear Schrödinger model, as well as to different boundary conditions; see e.g. [31], [34] for a discussion of the q-boson system with more exotic boundaries. We hope to address these and related questions in future work.

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