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The bandwidth of optimized nonlinear vibration-based energy harvesters

A Cammarano[1], S A Neild[1], S G Burrow[2] and D J Inman[3]

[1] Department of Mechanical Engineering, University of Bristol, UK
[2] Department of Aerospace Engineering, University of Bristol, UK
[3] Department of Aerospace Engineering, University of Michigan, MI, USA

E-mail: andrea.cammarano@bristol.ac.uk

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Abstract

In an attempt to improve the performance of vibration-based energy harvesters, many authors suggest that nonlinearities can be exploited to increase the bandwidths of linear devices. Nevertheless, the complex dependence of the response upon the input excitation has made a realistic comparison of linear harvesters with nonlinear energy harvesters challenging. In a previous work it has been demonstrated that for a given frequency of excitation, it is possible to achieve the same maximum power for a nonlinear harvester as that for a linear harvester, provided that the resistance and the linear stiffness of both are optimized. This work focuses on the bandwidths of linear and nonlinear harvesters and shows which device is more suitable for harvesting energy from vibrations. The work considers different levels of excitation as well as different frequencies of excitation. In addition, the effect of the mechanical damping of the oscillator on the power bandwidth is shown for both the linear and nonlinear cases.

Keywords: energy harvesting, nonlinear dynamics, optimization, continuation, electrical load, bandwidth

(Some figures may appear in colour only in the online journal)

1. Introduction

Energy harvesting from vibrations has attracted much attention in the literature for more than a decade. With this technology it is possible to harvest energy from vibrating structures that would otherwise be dissipated as heat. The literature provides a great variety of devices featuring different designs and different transduction mechanisms to allow for the conversion of kinetic energy into electrical energy.

One of the most common techniques for harvesting of energy from vibration uses resonating oscillators tuned to the frequency of excitation, allowing the input vibration to be amplified in the device and therefore more easily harvested by a transducer. A key advantage of this technique is that it can be easily adapted to different transduction mechanisms (magnetic, piezoelectric and electrostatic); however, there are also disadvantages.

The narrow bandwidth of a device designed following this approach is of concern. Many methods of addressing this limitation have been pursued. One is to control the geometrical characteristics of the oscillator to change its natural frequency [1, 2]. Another is to use external forces to change the equivalent stiffness of the oscillator and therefore its resonant frequency: Challa *et al* [3] suggest use of the force between an oscillating cantilever beam and external magnets; Roundy and Zhang [4] show how to achieve the same result with a piezoelectric actuator. Although valid, there are some drawbacks to consider in these approaches; for example, the geometry of the oscillator cannot be adapted easily to the frequency of excitation. Control of the natural frequency by influencing external forces requires a substantial amount of energy, potentially defeating the purpose of the device.

Another possibility to widen the frequency bandwidth of a harvester involves the use of multiple degrees of freedom oscillators. This idea has been investigated in many works, including [5], but harvesting of energy from several modes efficiently is not easy: in [6] the use of several decoupled oscillators, each tuned at a specific natural frequency, is suggested.

A similar concept is electrical tuning of the harvester; here, one extra degree of freedom is introduced in the electrical circuitry. By manipulating the phase between the electric force and the velocity of the moving mass by changing the reactance of the electrical circuit, Cammarano *et al* [7] and Renno *et al* [8] showed that it is possible to tune a harvester to the frequency of excitation. Several works, aimed at implementing this concept practically, have been published recently, for example [9].

Alongside these methods to tune or extend the bandwidth of a linear harvester, in the last decade the idea of exploiting the effect of nonlinearities on the frequency responses of the oscillator (see for example [10, 11]) has become popular. In [12], Erturk *et al* show the possible advantages of the bandwidth of a nonlinear harvester in comparison with a linear harvester. In [13], using a configuration similar to that in [4], the possibility of having both a hardening-like and a softening-like response with one single device is discussed. A similar device is presented in [14], where a strong nonlinearity is obtained via the interaction of the oscillator with the magnetic field. In this case the magnets are housed on the armature and they move with respect to a ferromagnetic core housed on the stator. This configuration is used primarily to enhance the electromechanical coupling coefficient. The resulting nonlinear system can be either hardening or bistable; see [15] for a discussion of the latter.

In many works the nonlinearity of the device is presented as a design characteristic [16–18], in others it is the result of particular geometries or arrangement [3, 10]. Cammarano *et al* [19] addressed the problem of how these compare with linear devices and how to optimize the electrical load and the mechanical characteristics.

The work presented here builds on the findings of [19]. In [19] it has been demonstrated that under optimal conditions a nonlinear energy harvester can deliver as much power as a linear harvester to a purely resistive load. For this to be achieved both the resistive load and the underlying linear frequency ω_n must be selected carefully. Although this result is promising, it does not, on it own, justify the use of a nonlinear energy harvester over a linear one. In the literature many authors have claimed that the main motivation for using a energy harvester featuring nonlinear components is that such a device exhibits good performance over a wider frequency range than an equivalent linear harvester. Although this motivation has been discussed in several works [12, 13, 20], a comparison between the performance of comparable linear and nonlinear devices in which the load has been optimized for maximum power harvested has not been reported. Other works have addressed the problem of the bandwidth in nonlinear harvesters, but in different contexts. Daqaq [21], for example, considered the bandwidth of a harvester exited with a random input. He concluded that for random excitations stiffening nonlinearities should be avoided in the design of the device. In his work, though, the electrical load is not optimized and the dissipation

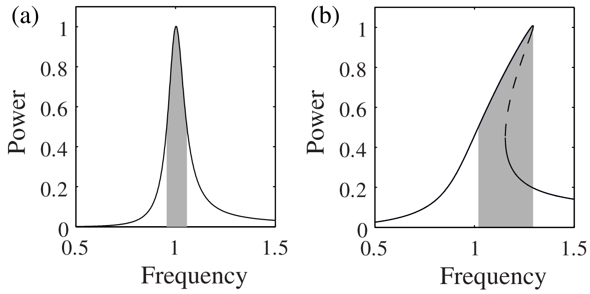

Figure 1. 3-dB-bandwidth schematics: application of the definition to the linear (a) and nonlinear (b) cases. The frequency axis in the schematics has been normalized with respect to the linear natural frequency ω_n, whereas the power axis is normalized with respect to the maximum power.

effects of the electrical load are included in the total damping of the system.

In this work we consider how the bandwidth of a nonlinear energy harvester compares with the bandwidth of a linear harvester when both are optimized for maximum power harvested at the frequency of excitation. Answering this question, using the optimization technique developed in [19], allows us to assess when the use of a nonlinear harvester is potentially more beneficial than a linear one. In section 2 the definition of linear and nonlinear bandwidth is provided and the effect of the mechanical parameter of the harvester on the bandwidth is shown. Then, in section 3.1, different definitions of bandwidth are considered; this requires a careful analysis of the linear harvester used for the comparison. In the same section, the strategy used for a fair comparison is discussed and the effects on the bandwidth are highlighted. Conclusions are drawn in section 4.

2. Bandwidth assuming upper branch solutions

The definition of bandwidth is derived from signal processing, where the bandwidth is commonly used as a tool to identify the sampling frequency necessary to reconstruct a signal with good accuracy [22]. More precisely, the bandwidth used in this work is the *3-dB bandwidth*, as we consider the frequency range over which the spectrum of the power response is greater than or equal to half of the maximum power achieved.

Figure 1 shows how the definition applies to the linear (1(a)) and nonlinear (1(b)) cases. In contrast to a linear harvester, which has a power response that is almost symmetric around the peak, the maximum power of a nonlinear single DOF device occurs in the proximity of an unstable region (assuming that the energy level is sufficient for a region of multiple solutions). For this reason, only one stable point exists where the power is equal to half the maximum power and the bandwidth is not symmetric about the maximum. Note that, for a given frequency of excitation, more than one stable solution exists. The power produced from low amplitude oscillations is far lower and therefore a far less desirable solution. The bandwidth of the nonlinear harvester, shown in figure 1(b), does not take into account the existence of these low energy oscillations but is based on the assumption that the harvester is always operated on the high energy branch. The case where this does not occur is discussed in section 3.1.

2.1. Calculation of the harvester bandwidth

The power bandwidth for a linear energy harvester can be easily derived using the relation between the quality factor Q, the damping ratio ζ and the bandwidth $\Delta\omega$

$$Q = \frac{1}{2\,\zeta} = \frac{\omega_n}{\Delta\omega} \tag{1}$$

with ω_n the natural frequency of the harvester. Equation (1) is only valid when the damping is small ($\zeta < \sqrt{2}/2$). In this case the maximum power occurs in the proximity of the natural frequency ω_n.

The damping ratio has to be evaluated using the total damping of the system (c_T), i.e. the sum of the mechanical and the electrical damping (c_m and c_e respectively). For vibration-based magnetic energy harvesters, the electrical damping can be written as

$$c_e = \frac{\theta^2}{R_L} \tag{2}$$

where θ is the electromechanical coupling coefficient and R_L is the resistance of the electric load. In equation (2) the electrical characteristics of the coil as well as the reactance of the load are neglected. Note that the effects of those parameters have been reported in previous works on linear energy harvesters (see, for example, [3, 7]). These works show that the coil resistance can be considered by augmenting the mechanical damping. The reactive part of the electrical impedance is often very small at the frequencies of interest, and hence negligible [7].

Using equation (1), $\Delta\omega$ can be written as

$$\Delta\omega = 2\omega_n\,\zeta = \frac{c_m + c_e}{m} \tag{3}$$

where m is the moving mass of the oscillator in the harvester. Equation (3) shows that the bandwidth of a linear harvester depends only on the total damping of the system. In order to deliver maximum power to the load, it was shown in [23] that the electrical damping must be equal to the mechanical damping.

The envelope of the maximum power that a linear energy harvester can deliver via an optimized resistive load is

$$P_{opt} = \frac{1}{8}\frac{\Omega_o^4 Y_o^2 m^2}{c_m} \tag{4}$$

where Y_o and Ω_o are the amplitude and frequency of excitation, respectively. Hence, the maximum power rapidly decreases as the damping increases. Thus, for the linear case, when neglecting the electrical load (and hence c_e), there is a compromise between bandwidth and maximum obtainable power.

It is reported in the literature that one possible method to increase the bandwidth without reducing the power harvested is via nonlinearity. Here, we consider an energy harvester with cubic stiffness described by

$$m\ddot{x} + \left(c_m + \frac{\theta^2}{R_L}\right)\dot{x} + kx + k_{nl}\,x^3 = -m\ddot{y} \tag{5}$$

where m is the moving mass and k is the linear stiffness. The ratio $\omega_n = \sqrt{k/m}$ is the natural frequency associated with the underlying linear system. As with the linear harvester, the total damping is given by the sum of the mechanical damping c_m and the electrical damping θ^2/R_L. The equivalent elastic force is $k\,x + k_{nl}x^3$ and includes both the mechanical stiffness of the system and the component of the magnetic force in phase with the displacement, see for example [14].

With nonlinearity present, both the shape and the maximum of the power response are altered. Here, a comparison between the bandwidths of linear and nonlinear harvesters with cubic elastic characteristics is performed considering that both the devices deliver the same power to a resistive load for one desired frequency of excitation. In other words both the linear and the nonlinear harvester are optimized at a specific frequency of excitation Ω_o.

Using the results developed in [19], it is possible to find the optimal resistance

$$R_{opt} = \frac{\theta^2}{m}\left(\frac{2\,(\Omega_o^2 - \omega_n^2)}{\Omega_o\,Y_0\sqrt{3\,\alpha\,\left(\Omega_o^2 - \omega_n^2\right)} - 2\,\zeta_m\,\omega_n\left(\Omega_o^2 - \omega_n^2\right)}\right) \tag{6}$$

and the optimal value of linear stiffness k_{opt}

$$k_{opt} = m\Omega_o^2\left(1 - \frac{3}{16}\frac{Y_o^2 m^2\alpha}{c_m^2}\right) \tag{7}$$

which allow the nonlinear harvester to produce the same power P_{opt} as a linear one at frequency Ω_o. Since this can be carried out for every Ω_o, the envelope of all the maxima leads to the same equation as obtained for the linear case, that is equation (4). Note that equations (6) and (7) follow the same nomenclature as that used in [19]: $\alpha = k_{nl}/m$. Also note that by evaluating the underling linear frequency ω_n using the expression of the optimal stiffness k_{opt} provided in equation (7) and substituting in equation (6), the expression for the optimal resistance becomes

$$R_{opt} = \frac{\theta^2}{c}. \tag{8}$$

This is the same expression as that of the optimal resistive load to be used to maximize the power output of a linear energy harvester. This implies that for a nonlinear device to achieve the optimal power for a given excitation frequency Ω_o only the linear stiffness must be tuned, while the resistance can remain unaltered.

To evaluate the frequency bandwidth, we have to evaluate the frequency at which the nonlinear harvester produces $P_{opt}/2$. This can be carried out numerically from the equation that relates the excitation frequency Ω to the power P

$$2\left(\left(\omega_n^2 - \Omega^2\right) + \frac{3}{2}\frac{\alpha\,R_L P}{\theta^2\Omega^2}\right)^2 \frac{R_L P}{\theta^2\Omega^2} + 2\left(\frac{c}{m} + \frac{\theta^2}{R_L m}\right)^2 \frac{R_L P}{\theta^2} = F^2 \tag{9}$$

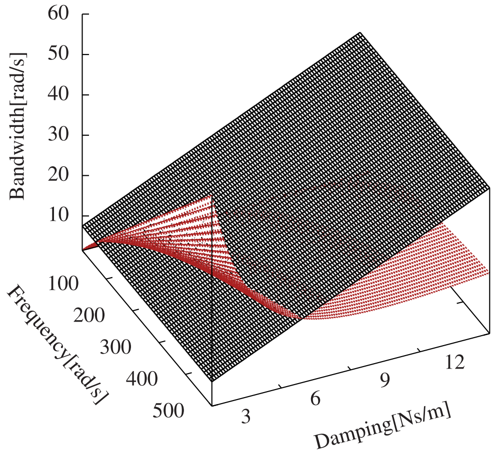

Figure 2. Effect of the damping on the bandwidth: the black surface is the bandwidth of the linear harvester whereas the red surface is representative of the bandwidth of the nonlinear harvester.

Table 1. Mechanical and electrical parameters.

Linear stiffness	300	N m^{-1}
Nonlinear stiffness	1.02×10^8	N m^{-3}
Mechanical damping	6	N s m^{-1}
Seismic mass	80	g
Electromechanical coefficient	8.9	V s m^{-1}
Base displacement	0.1	mm

substituting P with the value $P_{\mathrm{opt}}/2$ and solving for Ω. See the discussion around equation (9) in [19] for the derivation of this relationship and its use to derive the maximum power.

Here, the crucial step is that we use this equation to consider the bandwidth rather than the maximum power. In fact, the difference $\Delta\Omega = \Omega_{P_{\mathrm{opt}}} - \Omega_{P_{\mathrm{opt}}/2}$ defines the bandwidth of the nonlinear harvester (see figure 1(b)). The procedure can be repeated by varying different parameters, so that the influence of a given parameter on the 3-dB bandwidth is highlighted.

We now examine the influence of various key parameters on the bandwidth. To do this, we use parameters taken from an experimental characterization of a real device [24], which are listed in table 1. The parameters used in this work are considered with no uncertainties. For the effect of uncertainties in the physical parameters on the performance of the harvester see [25]. The device considered has an underlying linear frequency of approximately 62 rad s^{-1}.

We then select optimal values for the linear stiffness and the resistance according to (6) and (7) respectively and proceed to examine the bandwidth.

2.2. Damping

First we consider the effect of mechanical damping on the bandwidth. Figure 2 shows that for high values of damping the bandwidth of the linear harvester is larger than that of the nonlinear one.

As previously discussed, this condition is not particularly desirable for energy harvesters, since high values of the damping are detrimental for the efficiency of the harvester. Figure 2 shows that for small values of the damping, there exists a region in which the bandwidth of the nonlinear device is greater. This region is influenced by the frequency at which the device has to be tuned. Even if the system is lightly damped, at low frequency the bandwidth of the linear harvester is larger than that of the nonlinear device. In fact, the excitation considered has a fixed amplitude of displacement (0.1 mm). At low frequency the energy input to the harvester is limited and therefore the nonlinear behavior is hardly encountered. As the tuning frequency increases more and more energy is delivered to the harvester. This results in a greater bending of the response peak and therefore in a broader bandwidth.

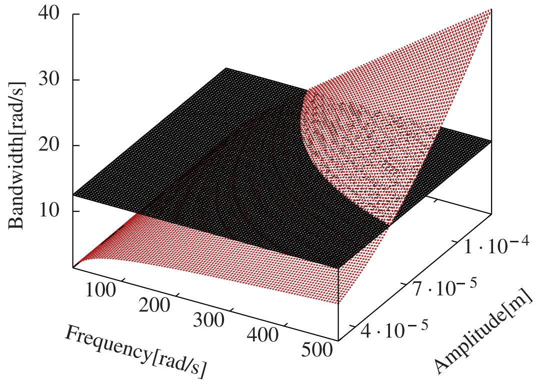

Figure 3. Effect of the amplitude of excitation on the bandwidth: the black surface is the bandwidth of the linear harvester whereas the red surface is the bandwidth of the nonlinear harvester.

2.3. Amplitude of excitation

The amplitude of excitation has no effect on the bandwidth of the linear harvester, but is highly influential on the response of the nonlinear harvester, see figure 3. In fact the amplitude of excitation determines the amplitude of the oscillation and therefore the magnitude of the nonlinear forces loading the oscillator. If the amplitude of excitation is very small, the nonlinear harvester behaves as if it were linear since the nonlinear component of the elastic force is small in comparison with the linear component. The approximation used in the optimization process does not allow calculation of the optimum resistance if the nonlinear component of the elastic force is very small, resulting in no region of multiple solution in figure 1(a). For this reason it is not possible to extend the surface in figure 3 to low amplitude of excitation. However, this case is not of particular interest since at a low level of excitation the linear and nonlinear harvesters behave very similarly and hence the bandwidth surfaces would converge.

In figure 3 the bandwidths of the linear and nonlinear harvesters are evaluated as a function of the amplitude of excitation and of the tuning frequency. All the other parameters are kept constant, included the damping of the two systems which is fixed to 6 N m s^{-1} and the nonlinear stiffness of the nonlinear device ($\sim 3 \times 10^9$ N m^{-3}).

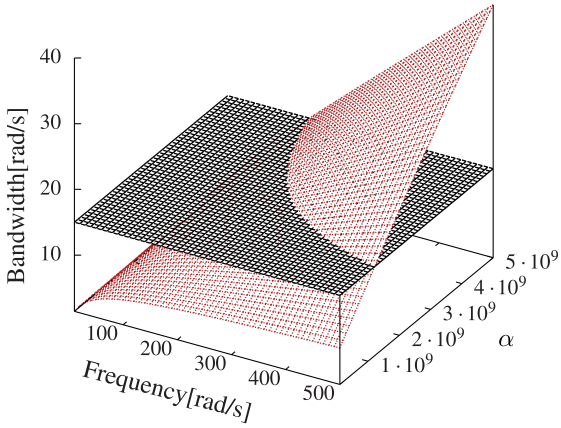

Figure 4. Effect of the nonlinear stiffness on the bandwidth: the black surface is the bandwidth of the linear harvester whereas the red surface is representative of the bandwidth of the nonlinear harvester.

2.4. Nonlinear stiffness

Finally, the influence of the nonlinear component of the equivalent elastic force on the bandwidth is evaluated. This is shown in figure 4 as a function of the tuning frequency. The surfaces are evaluated for a fixed amplitude of excitation (0.1 mm) and a constant damping coefficient of 6 N s m^{-1}. The surface representing the linear harvester (black) is parallel to the xy plane since it is independent of the nonlinear coefficient. For the nonlinear system (red), the higher the nonlinear coefficient, the wider the bandwidth becomes. The plot also shows that if the frequency of excitation is small, the coefficient of the nonlinear term has to be higher to ensure that the bandwidth of the response is larger than that of the linear harvester.

In summary, we have seen that using this definition of bandwidth and assuming that we remain on the upper solution branch in regions where multiple solutions exist, there are regions where the nonlinear device has larger bandwidth than the linear device.

2.5. Numerical validation

The analytical solution used for the evaluation of the bandwidth relies on the fact that the frequency response of the nonlinear device goes through a fold bifurcation. When the device is tuned so that its underlying linear natural frequency is very low, the response of the system does not present any folding points because the amplitude of the force transmitted to the device decreases with the square of the excitation frequency and the energy is not sufficient for the nonlinearity to induce folding of the frequency response and therefore the fold bifurcation to occur. To assess the region of validity of the analytical method and to investigate the behavior of the nonlinear harvester when the analytical method cannot be used a numerical investigation of the bandwidth is now performed via the following steps.

(1) A frequency $\Omega_{\rm opt}$ in the range 50–400 rad s^{-1} is chosen as the frequency at which maximum power occurs.
(2) The value of $\omega_{\rm n}$ that allows maximum power to be delivered at $\Omega_{\rm opt}$ is identified.

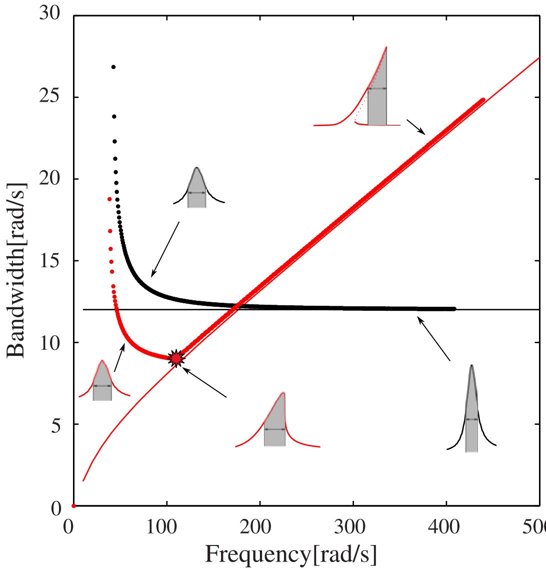

Figure 5. Comparison between the analytical and theoretical bandwidths for different values of $\omega_{\rm n}$: the bandwidths computed numerically are represented by dots whereas the analytical results are shown by solid lines. The linear bandwidth is shown in black and the red curves are representative of the nonlinear bandwidth. The red star shows the value of $\omega_{\rm n}$ for which the folding points coalesce.

(3) For the selected value of $\omega_{\rm n}$ the frequency response is evaluated.
(4) The bandwidth of the frequency response is found following the definition provided in section 2.1.
(5) A new value of $\Omega_{\rm opt}$ is selected and steps 1–4 are repeated.

The procedure has been applied to both the linear and the nonlinear devices considering a constant damping coefficient of 6 N s m^{-1}. The results are shown in figure 5. As expected, for high frequency there is a good matching between the analytical and the numerical results, but as the frequency decreases the folding points disappear (red star in figure). Even the formula used for the linear bandwidth is not valid at low frequency. This is because when excited with constant base displacement, the amplitude of the force changes over the spectrum and this influences the bandwidth of the oscillator. This influence becomes less and less important as the resonant frequency of the device moves toward higher frequencies. When the frequency is particularly low, the bandwidths of the linear and nonlinear devices converge. It can be seen that despite the limit that a fold is required for the analytical technique to work (a fold is observed for frequencies above ~110 rad s^{-1}), the analytical approach is able to identify correctly the regions where the nonlinear harvester has a larger bandwidth than the linear device, i.e. for frequencies higher than 190 rad s^{-1}. As the damping ratio increases, the amplitude of oscillation decreases and the folding points in the frequency response become closer to each other until they coalesce and vanish. This means that the analytical approach does not work for high values of damping. A similar numerical technique to the one used for the frequency is used to check the validity of the analytical approach as the damping changes. Here, the underlying natural frequency of the harvester has been kept

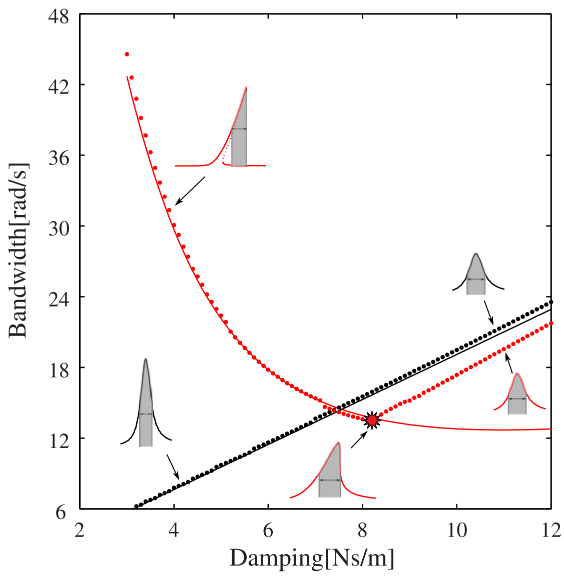

Figure 6. Comparison between the analytical and theoretical bandwidths at different values of the damping coefficient: the bandwidths computed numerically are represented by dots whereas the analytical results are shown by solid lines. The linear bandwidth is shown in black and the red curves are representative of the nonlinear bandwidth. The red star shows the value of c for which the folding points coalesce.

constant at 300 rad s^{-1}. The results are shown in figure 6. As expected, for low values of damping, the numerical and analytical results show good agreement, until the folding points coalesce (red star). For higher values of damping, the analytical approach is not able to predict the bandwidth of the nonlinear device. The nonlinear harvester behaves almost linearly and its bandwidth tends to the bandwidth of the linear device.

3. Bandwidth and multiple solution response

3.1. Further definitions of nonlinear bandwidth

All the results shown so far are based on the definition of nonlinear bandwidth provided in section 2, which differs from the classical definition for a linear device in two main ways:

- at some frequencies, within the bandwidth, the response has multiple stable solutions;
- the bandwidth is not centered about the maximum point, instead maximum power occurs at one of its extremes.

The first point is arguably of greater concern. At the frequencies where multiple solutions exist, the higher amplitude oscillations have a finite basin of attraction. Therefore it is possible that the system is attracted to a low energy solution. If this occurs, then the bandwidth calculated using the definition shown in figure 1(b) is misleading. A conservative definition for the bandwidth which guarantees that the response is on a high energy branch is to limit the bandwidth to frequencies at which only one branch exists. The maximum power achievable using this definition will be referred to as the *maximum power/unique solution* (*MPUS*). For oscillators with cubic elastic characteristics this frequency corresponds to the frequency at which the frequency response exhibits the first fold. This point separates the region where only one solution exists from the region with multiple solutions. Figure 7 shows the difference between the bandwidth defined in section 2 (panel (a)) and the bandwidth obtained following this definition (panel (b)).

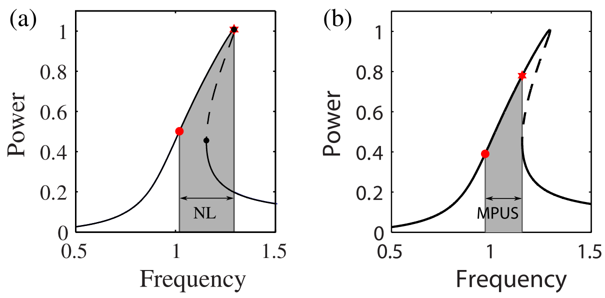

Figure 7. Schematic comparison of 3-dB-bandwidth definitions: bandwidth based on maximum power (a) and bandwidth based on the MPUS (b). Note from (a) that the folding points (black dots) coincide with a local maximum and minimum of the function $\Omega(P)$.

To evaluate the bandwidth according this new definition the following steps are necessary.

- Find the frequency at which the lower fold occurs. This can be achieved by considering that the multiple solution region extends in between the fold points and that these coincide with the minimum and the maximum of the function $\Omega(P)$ (see figure 7(a)).
- Evaluate the power corresponding to the only stable solution at this frequency[4].
- Find the frequency at which half of the power previously evaluated is delivered to the load.
- Evaluate the bandwidth.

The advantage of this definition is that the harvester does not require any control to ensure that high amplitude (and therefore high power) oscillations are maintained. Nevertheless, there are also disadvantages: the harvester does not deliver the maximum power to the load, and hence a linear device is able to harvest more energy.

3.2. Comparison with the linear harvester

The introduction of the MPUS-bandwidth raises questions about the correct comparison of this bandwidth with the linear bandwidth. In the first comparison both the harvesters produce, for a given tuning frequency, the same maximum power. In the MPUS-bandwidth the maximum power is determined by the point where the fold occurs and it is always less than (or at the best equal to) the peak power. In order to achieve a fairer comparison between the two devices, several cases have been considered. Figure 8 provides a graphical aid for the comparisons taken into account.

[4] At the frequency where the folds occurs, there are two possible solutions, one corresponding to the fold and the other exhibiting a higher amplitude of oscillation. The solution corresponding to the fold is not stable and, therefore, if the harvester is excited at this frequency it responds with high amplitude oscillations.

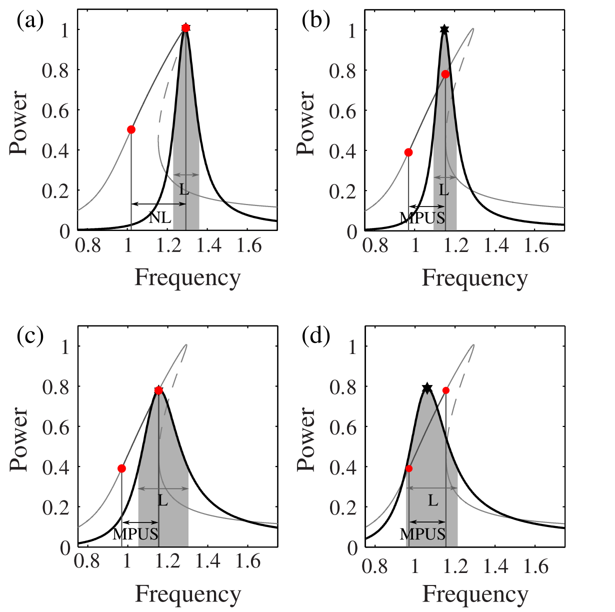

Figure 8. Schematic comparison between linear and nonlinear bandwidths. Panel (a) shows the bandwidth used in the previous analysis: the maximum power for both the linear (black star) and the nonlinear (red dot) devices is the same as well as the amount of mechanical damping. The nonlinear (NL) and linear (L) bandwidths are highlighted with lines and shading respectively. Panel (b) shows the case in which the linear bandwidth (L) is compared with the *maximum power unique solution* bandwidth (MPUS). The comparison is made with a linear device having the same mechanical damping and tuned so that the maximum power occurs at the same frequency for both devices. Panels (c) and (d) show comparisons of the MPUS-bandwidth with the bandwidths of linear devices harvesting the same maximum power as the nonlinear device. The peak power of the linear device has been decreased by increasing the electrical damping. In panel (c) both the linear and the nonlinear device generate the maximum power at the same frequency whereas in panel (d) the linear device is tuned so that its peak power point occurs at the center of the MPUS-bandwidth.

For convenience, figure 8(a) shows the linear and nonlinear bandwidths built around the maximum achievable power (the definition considered in section 2). Figure 8(b) shows the power obtained when the linear harvester is tuned at Ω_{MPUS}. The comparison in figure 8(b) is rather unfair, since the maximum powers achieved in the bandwidths of the two devices are different. A fairer comparison is shown in figure 8(c). Here, the damping of the linear harvester is increased so that its maximum power equals P_{MPUS}. Consequently the bandwidth of the linear harvester increases.

Although the MPUS-bandwidth addresses the problem of the multiple solutions, the bandwidth is not centered around the frequency at which maximum power occurs. Given the shape of the frequency response, this problem cannot be avoided. Nevertheless, the power and the bandwidth of the nonlinear harvester can be compared with a linear harvester tuned at the central frequency of the MPUS-bandwidth. In figure 8(d), the linear harvester has been tuned to the central frequency and the damping is such that the maximum power of the linear harvester is the same as the maximum power in the MPUS-bandwidth. The bandwidth in this case is approximately the same as the bandwidth obtained in

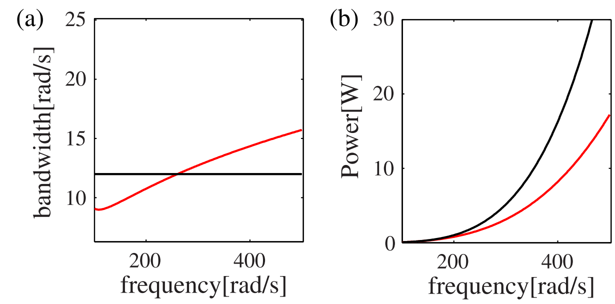

Figure 9. Comparison between linear and nonlinear bandwidth MPUS: the linear bandwidth is evaluated following the definition shown in figure 8(b). In panel (a) the values of linear and MPUS-bandwidth are shown. For each tuning frequency, the maximum power delivered to the load in the linear (black) and nonlinear (red) cases is shown in panel (b).

figure 8(c), but unlike that shown in figure 8(c), the linear and the nonlinear harvester achieve the same maximum power at different frequencies.

3.3. Bandwidth based on the MPUS definition

From the previous section it can be seen that the MPUS-bandwidth definition is not particularly favorable for the nonlinear energy harvester. In fact, following this definition, either the nonlinear harvester produces less power than the linear one, or the damping of the linear harvester has to be increased to reduce its power output. The latter results in an increase in the linear bandwidth which consequently might remove any advantage of using a nonlinear device. Both these cases, relating to the definitions shown in figure 8 (panels (b) and (c), respectively), are analyzed in this section.

If the mechanical damping coefficient is the same in both the linear and the nonlinear harvesters, than the linear device produces more power at Ω_{MPUS} than the nonlinear one. In fact, whereas the load of the linear harvester is optimized at this frequency, it is not in the nonlinear harvester. For the nonlinear harvester the optimized load for maximum power is in the region where multiple solutions exist, a region that, by definition, is not considered in the MPUS case. Since at Ω_{MPUS} different levels of power are achieved, a direct comparison of the bandwidth alone would be unfair. A fairer comparison has to take into account the difference in power output. For this reason, in figures 9(a) and (b) both the bandwidth and the power achieved with the linear (black) and nonlinear (red) devices are shown. The linear solutions are computed using the formula presented in [4]. The nonlinear solutions are found using the procedure described step by step in section 3.1.

In the figure the linear stiffness is changed to obtain maximum power at the tuning frequency. The results follow from the comparison criterion shown in figure 8(b), i.e. the linear harvester is tuned so that it achieves maximum power at Ω_{MPUS}. The damping is equal for both the linear and the nonlinear device. From figure 9(a) it can be seen that the bandwidth of the nonlinear harvester increases with the tuning frequency. However, the bandwidth of the linear harvester, depending only on the mechanical damping, remains unchanged. For this configuration, the MPUS-bandwidth is

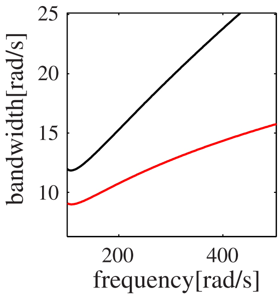

Figure 10. Comparison between the linear and nonlinear bandwidth MPUSs following the comparison criterion shown in figure 8(c). The maximum power delivered to the load in the linear (black) and nonlinear (red) cases is the same.

greater than that of the linear device for tuning frequencies higher than ∼43 Hz. Nevertheless, as the tuning frequency increases, the power of the linear harvester increases faster than the power harvested by the nonlinear device. Therefore, although the nonlinear harvester has a wider bandwidth above ∼43 Hz, the maximum power it is able to harvest in the MPUS region is much lower—see figure 9(b).

If the damping of the linear device is changed so that at Ω_{MPUS} both the linear and the nonlinear device produce the same amount of power, adopting the definition shown in figure 8(c), the linear harvester always has a broader bandwidth than the nonlinear one, as shown in figure 10. This is due to the fact that the increase of damping in the linear device leads to an increase in its bandwidth.

This shows that it is necessary to ensure that a nonlinear harvester can operate in the multiple solution region (on the upper solution branch) for the bandwidth to be comparable, and in some cases larger, than that for the equivalent power of the linear device.

4. Conclusions

This work provides a comparison between the bandwidths of linear and nonlinear optimized energy harvesters. After the definitions of linear and nonlinear bandwidth have been provided a study of the bandwidth as a function of the design parameters and the input excitation is presented. The results demonstrate that there are some regions in which the nonlinear harvester has a larger bandwidth than that for the linear one, as suggested by many authors in the literature. However, this finding is based on a bandwidth definition that assumes that the device remains on the upper solution branch in the region where multiple solutions exist.

Definition of a different bandwidth which avoids operation in the multiple solution region, i.e. effectively assuming that the lower branch is always observed, has been considered as a more conservative definition of bandwidth. Using this definition, if the bandwidth comparison is made with a linear system having its maximum power at Ω_{MPUS} and the same mechanical damping as the nonlinear device, then for some set of parameters the bandwidth of the nonlinear harvester is wider. However, in this case the comparison is biased, as the maximum power of the linear system is always significantly higher.

In order to maintain the same maximum power for both the linear and the nonlinear system, the damping of the linear device has to be increased. The result of this is that the bandwidth of the linear device increases and it is shown that the bandwidth of the nonlinear device is always narrower than that of the linear device.

Finally, a comparison between the nonlinear bandwidth and the bandwidth of a linear harvester exhibiting maximum power at the central frequency of the nonlinear bandwidth has been shown. To have the maximum power the same for both the devices the damping of the linear harvester must be almost equal to the previous case, and therefore the bandwidth of the linear harvester is, again, always wider than that of the nonlinear one.

In conclusion, the nonlinear harvester exhibits a wider bandwidth only if it operates on the upper branch in the multiple solution region. To ensure that this occurs, a controller which constrains the system to oscillate at high amplitude is needed. This is beyond the scope of this work but will be addressed in the future.

Note that, even in the case where the multiple solution region can be exploited, the nonlinear device exhibits wider bandwidth only when the damping is small enough and the nonlinearity is sufficiently high. Also, the bandwidth, as well as the solution of a nonlinear harvester, depends on the amplitude of excitation, and therefore a careful investigation of the operational conditions is highly recommended.

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